title: Phase space integral
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Phase space integral in invariant form

Convention

$$
\begin{aligned}
\lambda(x,y,z) &= x^2+y^2+z^2-2xy-2xz-2yz \\
d\Phi_n(P;p_1,\cdots,p_n) &= (\prod_{i=1}^{n}\frac{d^3p_i}{(2\pi)^3 2E_i})(2\pi)^4\delta(P-\sum_{i=1}^{n}p_i) \\
&= \frac{1}{(2\pi)^{3n-4}}(\prod_{i=1}^{n}d^4p_i\delta(p_i^2-m_i^2))\delta(P-\sum_{i=1}^{n}p_i) \\
d\Gamma_{1\to n} &= \frac{1}{2M}|\mathcal{M}|^2 d\Phi_n(P;p_1,\cdots,p_n) \\
d\sigma_{2\to n} &= \frac{|\mathcal{M}|^2}{2\sqrt{\lambda(s,m_a^2,m_b^2)}}d\Phi_n(p_1+p_2,p_3,\cdots,p_{n+2})
\end{aligned}
$$

1->n

1->2

$$
\begin{aligned}
d\Gamma &= \frac{1}{32\pi^2}|\mathcal{M}|^2\frac{|\vec{p}_1|}{M^2}d\Omega \\
&= \frac{|\mathcal{M}|^2\sqrt{\lambda(M^2,m_1^2,m_2^2)}}{64\pi^2M^3}d\Omega
\end{aligned}
$$

	p	p_1	p_2
p	M^2	$\frac{M^2+m_1^2-m_2^2}{2}$	$\frac{M^2+m_2^2-m_1^2}{2}$
p_1		m_1^2	$\frac{M^2-m_1^2-m_2^2}{2}$
p_2			m_2^2

1->3

Scheme 1

$$
d\Gamma = \frac{1}{(2\pi)^3}\frac{1}{32M^3}|\mathcal{M}|^2 dm_{12}^2 dm_{23}^2
$$

The limits are determined by:$(m_1+m_2)^2 \le m_{12}^2 \le (M-m_3)^2$ and $(E_2^*+E_3^*)^2-(\sqrt{E_2^{*2}-m_2^2}+\sqrt{E_3^{*2}-m_3^2})^2 \le m_{23}^2 \le (E_2^*+E_3^*)^2-(\sqrt{E_2^{*2}-m_2^2}-\sqrt{E_3^{*2}-m_3^2})^2$,where $E_2^*=(m_{12}^2-m_1^2+m_2^2)/2m_{12}$and $E_3^*=(M^2-m_{12}^2-m_3^2)/2m_{12}$.

	p	p_1	p_2	p_3
p	m^2	$\frac{m^2+m_1^2-s_{23}}{2}$	$\frac{s_{12}+s_{23}-m_1^2-m_3^2}{2}$	$\frac{m^2+m_3^2-s_{12}}{2}$
p_1		m_1^2	$\frac{s_{12}-m_1^2-m_2^2}{2}$	$\frac{m^2+m_2^2-s_{23}-s_{12}}{2}$
p_2			m_2^2	$\frac{s_{23}-m_2^2-m_3^2}{2}$
p_3				m_3^2

Scheme 2

Define n-body decay phase space integral as:

$$
\begin{aligned}
D_n &= \prod_{i=1}^{n}[d^4p_i\delta(p_i^2-m_i^2)] \times \delta^4(Q-\sum_{i=1}^{n}p_i) \\
d\Phi_n(P;p_1\cdots p_n) &= \prod_{i=1}^{n}\frac{d^3p_i}{(2\pi)^3 2E_i}(2\pi)^4\delta^4(P-\sum_{i=1}^{n}p_i) = \frac{D_n}{(2\pi)^{3n-4}} \\
d\Gamma &= \frac{|\mathcal{M}|^2}{2M(2\pi)^{3n-4}}D_n
\end{aligned}
$$

Note that: $\int\frac{d^3p_i}{2E_i} = \int d^4p_i\delta(p_i^2-m_i^2)$.

And for $1 \to 3$ case:

$$
\begin{aligned}
s_1 &= (Q-p_1)^2 \\
u_1 &= (Q-p_2)^2 \\
D_3 &= \frac{\pi^2}{4M^2}\int_{(m_2+m_3)^2}^{(M-m_1)^2}ds_1\int_{u_1-}^{u_1+}du_1F(s_1,u_1)
\end{aligned}
$$

$$
u_1\pm = M^2+m_2^2-\frac{(s_1+m_2^2-m_3^2)(M^2+s_1-m_1^2)}{2s_1} \pm \frac{\sqrt{\lambda(s_1,m_2^2,m_3^2)\lambda(M^2,s_1,m_1^2)}}{2s_1}
$$

	p	p_1	p_2	p_3
p	m^2	$\frac{m^2-s_1+m_1^2}{2}$	$\frac{m^2-u_1+m_2^2}{2}$	$\frac{s_1-m_1^2+u_1-m_2^2}{2}$
p_1		m_1^2	$\frac{m^2-s_1-u_1+m_3^2}{2}$	$\frac{u_1-m_1^2-m_3^2}{2}$
p_2			m_2^2	$\frac{s_1-m_2^2-m_3^2}{2}$
p_3				m_3^2

1->4

Define mandelstam variables as:

$$
\begin{aligned}
s_1 &= (p-p_1)^2 \\
u_1 &= (p-p_2)^2 \\
s_2 &= (p-p_1-p_2)^2 \\
u_2 &= (p-p_3)^2 \\
t_2 &= (p-p_2-p_3)^2
\end{aligned}
$$

And all scalar products can be expressed with the above ones as:

	p	p_1	p_2	p_3	p_4
p	m^2	$\frac{m^2-s_1+m_1^2}{2}$	$\frac{m^2-u_1+m_2^2}{2}$	$\frac{m^2-u_2+m_3^2}{2}$	$\frac{s_1+u_1+u_2-m^2-m_1^2-m_2^2-m_3^2}{2}$
p_1		m_1^2	$\frac{m^2-s_1-u_1+s_2}{2}$	$\frac{u_1-s_2-t_2+m_4^2}{2}$	$\frac{t_2-m_1^2-m_4^2}{2}$
p_2			m_2^2	$\frac{m^2+t_2-u_1-u_2}{2}$	$\frac{u_1+u_2+s_1-s_2-m^2-m_2^2-t_2}{2}$
p_3				m_3^2	$\frac{s_2-m_3^2-m_4^2}{2}$
p_4					m_4^2

And then the phase space integral can be evaluated:

$$
\begin{aligned}
s_2' &= s_2 + M^2 + (m_1^2 + m_2^2) - (s_1 + u_1) \\
\xi_2 &= \frac{(M^2 + s_2' - s_2)(M^2 + m_2^2 - u_1) - 2M^2(s_2' + m_2^2 - m_1^2)}{[\lambda(M^2, s_2, s_2')\lambda(M^2, u_1, m_2^2)]^{1/2}} \\
\eta_2 &= \frac{2M^2(s_2 + m_3^2 - m_4^2) - (M^2 + m_3^2 - u_2)(M^2 + s_2 - s_2')}{[\lambda(M^2, s_2, s_2')\lambda(M^2, m_3^2, u_2)]^{1/2}} \\
\omega_2 &= \frac{2M^2(u_1 + m_3^2 - t_2) - (M^2 + m_3^2 - u_2)(M^2 + u_1 - m_2^2)}{[\lambda(M^2, u_1, m_2^2)\lambda(M^2, m_3^2, u_2)]^{1/2}} \\
\zeta_2 &= \frac{\omega_2 - \xi_2 \cdot \eta_2}{\sqrt{(1-\xi_2^2)(1-\eta_2^2)}} \\
D_4 &= \frac{\pi^2}{4}\int_{(m_2+m_3+m_4)^2}^{(M-m_1)^2} ds_1 \int_{(m_3+m_4)^2}^{(\sqrt{s_1}-m_2)^2} ds_2 \int_{u_1-}^{u_1+} \frac{du_1}{\sqrt{\lambda(M^2, s_2, s_2')\lambda(M^2, m_2^2, u_1)(1-\xi_2^2)}} \\
&\int_{u_2-}^{u_2+} \frac{du_2}{\sqrt{\lambda(M^2, m_3^2, u_2)(1-\eta_2^2)}} \int_{t_2-}^{t_2+} \frac{dt_2}{\sqrt{1-\zeta_2^2}} F(s_1, s_2; u_1, u_2, t_2)
\end{aligned}
$$

The integral limits are determined by:

$$
\begin{aligned}
u_1\pm &= M^2 + m_2^2 - \frac{(s_1 + m_2^2 - s_2)(M^2 + s_1 - m_1^2)}{2s_1} \pm \frac{\sqrt{\lambda(s_1, m_2^2, s_2)\lambda(M^2, s_1, m_1^2)}}{2s_1} \\
u_2\pm &= M^2 + m_3^2 - \frac{(s_2 + m_3^2 - m_4^2)(M^2 + s_2 - s_2')}{2s_2} \pm \frac{\sqrt{\lambda(s_2, m_3^2, m_4^2)\lambda(M^2, s_2, s_2')}}{2s_2} \\
t_2\pm &= u_1 + m_3^2 - \frac{(M^2 + m_3^2 - u_2)(M^2 + u_1 - m_2^2)}{2M^2} + \frac{\sqrt{\lambda(M^2, m_3^2, u_2)\lambda(M^2, u_1, m_2^2)}}{2M^2} \\
&\times \{-\xi_2 \cdot \eta_2 \pm \sqrt{(1-\xi_2^2)(1-\eta_2^2)}\}
\end{aligned}
$$

1->5

Define mandelstam variables as:

$$
\begin{aligned}
s_1 &= (p - p_1)^2 \\
s_2 &= (p - p_1 - p_2)^2 \\
s_3 &= (p - p_1 - p_2 - p_3)^2 \\
u_1 &= (p - p_2)^2 \\
u_2 &= (p - p_3)^2 \\
u_3 &= (p - p_4)^2 \\
t_2 &= (p - p_2 - p_3)^2 \\
t_3 &= (p - p_2 - p_3 - p_4)^2 \\
s_{34} &= 2p_3 \cdot p_4 \\
s_0 &= m^2 \\
s_4 &= m_5^2 \\
u_0 &= s_1 \\
t_1 &= u_1
\end{aligned}
$$

All scalar-products are:

	p	p_1	p_2	p_3	p_4	p_5
p	m^2	$\frac{m^2-s_1+m_1^2}{2}$	$\frac{m^2-u_1+m_2^2}{2}$	$\frac{m^2-u_2+m_3^2}{2}$	$\frac{m^2-u_3+m_4^2}{2}$	$\frac{s_1+u_1+u_2+u_3-2m^2-m_1^2-m_2^2-m_3^2-m_4^2}{2}$
p_1		m_1^2	$\frac{s_2-s_1-u_1+m^2}{2}$	$\frac{s_3+u_1-s_2-t_2}{2}$	$\frac{m_5^2+t_2-s_3-t_3}{2}$	$\frac{t_3-m_1^2-m_5^2}{2}$
p_2			m_2^2	$\frac{m^2-u_1+t_2-u_2}{2}$	$\frac{m^2+t_3-t_2-s_{34}-u_3}{2}$	$\frac{s_1+u_1+u_2+u_3+s_{34}-s_2-m_2^2-2m^2-t_3}{2}$
p_3				m_3^2	$\frac{s_{34}}{2}$	$\frac{s_2-s_3-m_3^2-s_{34}}{2}$
p_4					m_4^2	$\frac{s_3-m_4^2-m_5^2}{2}$
p_5						m_5^2

phase space integral

$$D_5 = \frac{\pi^2 m^2}{4} \int_{s_{1-}}^{s_{1+}} ds_1 \int_{s_{2-}}^{s_{2+}} ds_2 \int_{s_{3-}}^{s_{3+}} ds_3 \int_{u_{1-}}^{u_{1+}} du_1 \int_{u_{2-}}^{u_{2+}} \frac{du_2}{\sqrt{\lambda(m^2, s_2, s_2')\lambda(m^2, m_3^2, u_2)}}$$

$$\int_{u_{3-}}^{u_{3+}} \frac{du_3}{\sqrt{\lambda(m^2, s_3, s_3')\lambda(m^2, m_4^2, u_3)}} \int_{t_{2-}}^{t_{2+}} \frac{dt_2}{\sqrt{\lambda(m^2, t_1, t_1')(1-\xi_2^2)(1-\eta_2^2)(1-\zeta_2^2)}}$$

$$\int_{t_{3-}}^{t_{3+}} \frac{dt_3}{\sqrt{\lambda(m^2, t_2, t_2')(1-\xi_3^2)(1-\eta_3^2)(1-\zeta_3^2)}} \frac{F(s_1, s_2, s_3; u_1, u_2, u_3, t_2, t_3, s_{34}^-) + F(s_1, s_2, s_3; u_1, u_2, u_3, t_2, t_3, s_{34}^+)}{2}$$

$$s_1' = m_1^2$$

$$s_2' = s_2 + m^2 + m_1^2 + m_2^2 - u_0 - u_1 = s_2 + m^2 + m_1^2 + m_2^2 - s_1 - u_1$$

$$s_3' = s_3 + 2m^2 + m_1^2 + m_2^2 + m_3^2 - u_0 - u_1 - u_2 = s_3 + 2m^2 + m_1^2 + m_2^2 + m_3^2 - s_1 - u_1 - u_2$$

$$t_1' = m_2^2$$

$$t_2' = t_2 + m^2 + m_2^2 + m_3^2 - u_1 - u_2$$

$$t_3' = t_3 + 2m^2 + m_2^2 + m_3^2 + m_4^2 - u_1 - u_2 - u_3$$

$$\xi_2 = \frac{(m^2 + s_2' - s_2)(m^2 + t_1' - t_1) - 2m^2(s_2' + t_1' - m_1^2)}{\sqrt{\lambda(m^2, s_2, s_2')\lambda(m^2, t_1, t_1')}}$$

$$\xi_3 = \frac{(m^2 + s_3' - s_3)(m^2 + t_2' - t_2) - 2m^2(s_3' + t_2' - m_1^2)}{\sqrt{\lambda(m^2, s_3, s_3')\lambda(m^2, t_2, t_2')}}$$

$$\eta_2 = \frac{2m^2(s_2 + m_3^2 - s_3) - (m^2 + m_3^2 - u_2)(m^2 + s_2 - s_2')}{\sqrt{\lambda(m^2, s_2, s_2')\lambda(m^2, m_3^2, u_2)}}$$

$$\eta_3 = \frac{2m^2(s_3 + m_4^2 - m_5^2) - (m^2 + m_4^2 - u_3)(m^2 + s_3 - s_3')}{\sqrt{\lambda(m^2, s_3, s_3')\lambda(m^2, m_4^2, u_3)}}$$

$$\omega_2 = \frac{2m^2(t_1 + m_3^2 - t_2) - (m^2 + m_3^2 - u_2)(m^2 + t_1 - t_1')}{\sqrt{\lambda(m^2, t_1, t_1')\lambda(m^2, m_3^2, u_2)}}$$

$$\omega_3 = \frac{2m^2(t_2 + m_4^2 - t_3) - (m^2 + m_4^2 - u_3)(m^2 + t_2 - t_2')}{\sqrt{\lambda(m^2, t_2, t_2')\lambda(m^2, m_4^2, u_3)}}$$

$$\zeta_2 = \frac{\omega_2 - \xi_2 \cdot \eta_2}{\sqrt{(1-\xi_2^2)(1-\eta_2^2)}}$$

$$\zeta_3 = \frac{\omega_3 - \xi_3 \cdot \eta_3}{\sqrt{(1-\xi_3^2)(1-\eta_3^2)}}$$

The integral limits are:

$$s_{1-} = (m_2 + m_3 + m_4 + m_5)^2$$
$$s_{1+} = (m - m_1)^2$$
$$s_{2-} = (m_3 + m_4 + m_5)^2$$
$$s_{2+} = (\sqrt{s_1} - m_2)^2$$
$$s_{3-} = (m_4 + m_5)^2$$
$$s_{3+} = (\sqrt{s_2} - m_3)^2$$

$$t_{2\pm} = t_1 + m_3^2 - \frac{(m^2 + m_3^2 - u_2)(m^2 + t_1 - t_1')}{2m^2} + \frac{\sqrt{\lambda(m^2, m_3^2, u_2)\lambda(m^2, t_1, t_1')}}{2m^2} \times \{-\xi_2\eta_2 \pm \sqrt{(1-\xi_2^2)(1-\eta_2^2)}\}$$
$$t_{3\pm} = t_2 + m_4^2 - \frac{(m^2 + m_4^2 - u_3)(m^2 + t_2 - t_2')}{2m^2} + \frac{\sqrt{\lambda(m^2, m_4^2, u_3)\lambda(m^2, t_2, t_2')}}{2m^2} \times \{-\xi_3\eta_3 \pm \sqrt{(1-\xi_3^2)(1-\eta_3^2)}\}$$

$$u_{1\pm} = m^2 + m_2^2 - \frac{(s_1 + m_2^2 - s_2)(m^2 + s_1 - s_1')}{2s_1} \pm \frac{\sqrt{\lambda(s_1, m_2^2, s_2)\lambda(m^2, s_1, s_1')}}{2s_1}$$
$$u_{2\pm} = m^2 + m_3^2 - \frac{(s_2 + m_3^2 - s_3)(m^2 + s_2 - s_2')}{2s_2} \pm \frac{\sqrt{\lambda(s_2, m_3^2, s_3)\lambda(m^2, s_2, s_2')}}{2s_2}$$
$$u_{3\pm} = m^2 + m_4^2 - \frac{(s_3 + m_4^2 - m_5^2)(m^2 + s_3 - s_3')}{2s_3} \pm \frac{\sqrt{\lambda(s_3, m_4^2, m_5^2)\lambda(m^2, s_3, s_3')}}{2s_3}$$

Note that there is extra invariant s_{34} which does not appear in the integral limits but may appear in the amplitudes. Since n(n>4) four-momentum are not linearly independent(our time-space is 4-dimensional), the Gram determinant of any 5 momentum must vanish, i.e.

$$det[p_i \cdot p_j] = 0$$

however, this equation will give a two-fold solution of s_{23}:$s_{23}^{\pm} = s_{23}^r \pm s_{23}^s$, where $s_{23}^{r/s}$ refer to the rational and squared part of the solution of quadratic equation. Thus, the full phase integral should be:

$$\int d\Phi_5 |\mathcal{M}|^2 \to d\Phi_5 (\frac{1}{2}|\mathcal{M}|^2_{|s_{34}\to s_{34}^+} + \frac{1}{2}|\mathcal{M}|^2_{|s_{34}\to s_{34}^-})$$

This procedure applies to $2 \to 4$ process also.

2->n

Scheme 1

$$\begin{aligned} d\Phi_2 &= \frac{|\vec{p_1}|}{16\pi^2 M} d\Omega \\ &= \frac{\sqrt{\lambda(M^2, m_1^2, m_2^2)}}{32\pi^2 M^2} d\Omega \\ d\sigma_{2\to 2} &= \frac{|\mathcal{M}|^2}{64\pi^2 M^2} d\Omega \end{aligned}$$

Scheme 2(Byckling's)

$$
\begin{aligned}
t_i &= (Q - p_1 - \cdots - p_i)^2 \\
s_i &= (p_i + p_{i+1})^2 \\
\hat{s}_i &= (p_1 + \cdots + p_i)^2 \\
R_n(s) &= \int \prod_{i=1}^{n} d^4 p_i \delta(p_i^2 - m_i^2) \delta^4(p_a + p_b - p_1 - \cdots - p_n) \\
&= \int d\hat{s}_{n-1} dt_{n-1} d\phi \frac{1}{4\sqrt{\lambda(\hat{s_n}, t_n, m_a^2)}} \\
&\times \int d\hat{s}_{n-2} dt_{n-2} ds_{n-1} \frac{\Theta(-\Delta_4(n-1))}{8[-\Delta_4(n-1)]^{1/2}} \\
&\times \cdots \times \int d\hat{s}_2 dt_2 ds_3 \frac{\Theta(-\Delta_4(3))}{8[-\Delta_4(3)]^{1/2}} \times \int dt_1 ds_2 \frac{\Theta(-\Delta_4(2))}{8[-\Delta_4(2)]^{1/2}} \\
\Delta_4(i) &= \frac{1}{16} \begin{vmatrix} 2m_a^2 & m_a^2 + t_{i-1} - \hat{s}_{i-1} & m_a^2 + t_i - \hat{s}_i & m_a^2 + t_{i+1} - \hat{s}_{i+1} \\ m_a^2 + t_{i-1} - \hat{s}_{i-1} & 2t_{i-1} & t_i + t_{i-1} - m_i^2 & t_{i-1} + t_{i+1} - s_i \\ m_a^2 + t_i - \hat{s}_i & t_i + t_{i-1} - m_i^2 & 2t_i & t_i + t_{i+1} - m_{i+1}^2 \\ m_a^2 + t_{i+1} - \hat{s}_{i+1} & t_{i-1} + t_{i+1} - s_i & t_i + t_{i+1} - m_{i+1}^2 & 2t_{i+1} \end{vmatrix} \\
d\sigma &= \frac{|\mathcal{M}|^2 R_n(s)}{2(2\pi)^{3n-4}\sqrt{\lambda(s, m_a^2, m_b^2)}}
\end{aligned}
$$

The integral limits are determined by:

$$
\begin{aligned}
&\sqrt{\hat{s}_{i-1}} + m_i \leq \sqrt{\hat{s}_i} \leq \sqrt{\hat{s}_{i+1}} - m_{i+1} \\
&t_{i-1} = \hat{s}_{i-1} + m_a^2 + \frac{(-\hat{s}_i + t_i - m_a^2)(\hat{s}_i + \hat{s}_{i-1} - m_i^2)}{2\hat{s}_i} + \frac{\sqrt{\lambda(\hat{s}_i, t_i, m_a^2)\lambda(\hat{s}_i, \hat{s}_{i-1}, m_i^2)}}{2\hat{s}_i} \cos\theta_{i-1} \\
&s_i = \hat{s}_{i-1} + \hat{s}_{i+1} + \frac{2}{\lambda(\hat{s}_i, t_i, m_a^2)} \times \{4V + \cos\phi_{i-1}\sqrt{G(i)G(i-1)}\} \\
&V = -\frac{1}{8} \begin{vmatrix} 2\hat{s}_i & m_a^2 + \hat{s}_i - t_i & \hat{s}_i + \hat{s}_{i-1} - m_i^2 \\ m_a^2 + \hat{s}_i - t_i & 2m_a^2 & m_a^2 + \hat{s}_{i-1} - t_{i-1} \\ \hat{s}_{i+1} + \hat{s}_i - m_{i+1}^2 & m_a^2 + \hat{s}_{i+1} - t_{i+1} & 0 \end{vmatrix}
\end{aligned}
$$

where Gram determinant are defined as: $G(i) = G(t_i, \hat{s}_{i+1}, \hat{s}_i, t_{i+1}, m_{i+1}^2, m_a^2)$ and

$$
\begin{aligned}
G(x, y, z, u, v, w) = {} & x^2 y + x y^2 + z^2 u + z u^2 + v w^2 + v^2 w + x z w + x u v + y z v + y u w \\
& - x y(z + u + v + w) - z u(x + y + v + w) - v w(x + y + z + u)
\end{aligned}
$$

$2 \rightarrow 2$

	p_a	p_b	p_1	p_2
p_a	m_a^2	$\frac{s-m_a^2-m_b^2}{2}$	$\frac{m_a^2+m_1^2-t}{2}$	$\frac{m_a^2+m_2^2-u}{2}$
p_b		m_b^2	$\frac{s+t-m_a^2-m_2^2}{2}$	$\frac{m_b^2+m_2^2-t}{2}$
p_1			m_1^2	$\frac{s-m_1^2-m_2^2}{2}$
p_2				m_2^2

2->3

$$
d\sigma_{2->3} = \frac{|\mathcal{M}|^2 d\hat{s}_2 dt_2 dt_1 ds_2 d\phi}{8^2(2\pi)^5 \lambda(s, m_a^2, m_b^2)[-\Delta_4(2)]^{1/2}}
$$

	p_a	p_b	p_1	p_2	p_3
p_a	m_a^2	$\frac{s-m_a^2-m_b^2}{2}$	$\frac{m_a^2-t_1+m_1^2}{2}$	$\frac{\hat{s}_2-t_2+t_1-m_1^2}{2}$	$\frac{s+t_2-m_b^2-\hat{s}_2}{2}$
p_b		m_b^2	$\frac{s+t_1-s_2-m_a^2}{2}$	$\frac{s_2+t_2-t_1-m_3^2}{2}$	$\frac{t_2-m_3^2-m_b^2}{2}$
p_1			m_1^2	$\frac{\hat{s}_2-m_1^2-m_2^2}{2}$	$\frac{s-s_2-\hat{s}_2+m_2^2}{2}$
p_2				m_2^2	$\frac{s_2-m_2^2-m_3^2}{2}$
p_3					m_3^2

2->4

$$d\sigma_{2->4}=\frac{|\mathcal{M}|^2 d\hat{s}_3 dt_3 d\hat{s}_2 dt_2 ds_3 dt_1 ds_2 d\phi}{8^3(2\pi)^8\lambda(s,m_a^2,m_b^2)[-\Delta_4(3)]^{1/2}[-\Delta_4(2)]^{1/2}}$$

	p_a	p_b	p_1	p_2	p_3	p_4
p_a	m_a^2	$\frac{s-m_a^2-m_b^2}{2}$	$\frac{m_a^2+m_1^2-t_1}{2}$	$\frac{\hat{s}_2+t_1-t_2-m_1^2}{2}$	$\frac{\hat{s}_3-\hat{s}_2+t_2-t_3}{2}$	$\frac{s-m_b^2-\hat{s}_3+t_3}{2}$
p_b		m_b^2	$\frac{s+t_1-m_a^2-s_{234}}{2}$	$\frac{t_2-t_1+s_{s234}-s_3}{2}$	$\frac{s_3+t_3-t_2-m_4^2}{2}$	$\frac{m_4^2+m_b^2-t_3}{2}$
p_1			m_1^2	$\frac{\hat{s}_2-m_1^2-m_2^2}{2}$	$\frac{\hat{s}_3-s_2-\hat{s}_2+m_2^2}{2}$	$\frac{s+s_2-\hat{s}_3-s_{234}}{2}$
p_2				m_2^2	$\frac{s_2-m_2^2-m_3^2}{2}$	$\frac{s_{234}-s_3-s_2+m_3^2}{2}$
p_3					m_3^2	$\frac{s_3-m_3^2-m_4^2}{2}$
p_4						m_4^2

Note that if s_{234} must appear in the amplitude, the (differential) cross section is:

$$d\sigma_{2->4}=\frac{(|\mathcal{M}|^2_{s_{234}=s_{234-}}+|\mathcal{M}|^2_{s_{234}=s_{234+}})}{2}\frac{d\hat{s}_3 dt_3 d\hat{s}_2 dt_2 ds_3 dt_1 ds_2 d\phi}{8^3(2\pi)^8\lambda(s,m_a^2,m_b^2)[-\Delta_4(3)]^{1/2}[-\Delta_4(2)]^{1/2}}$$

Where the $s_{234}\pm$ are solution of Gram determinant:

$$\begin{aligned}\Delta_5(p_a,p_b,p_1,p_2,p_3) &= |p_i\cdot p_j| \\ &= \begin{vmatrix} m_a^2 & \frac{s-m_a^2-m_b^2}{2} & \frac{m_1^2+m_a^2-t_1}{2} & \frac{\hat{s}_3-\hat{s}_2+t_2-t_3}{2} & \frac{s+t_3-\hat{s}_3-m_b^2}{2} \\ \frac{s-m_a^2-m_b^2}{2} & m_b^2 & \frac{s+t_1-s_{234}-m_a^2}{2} & \frac{s_3+t_3-t_2-m_4^2}{2} & \frac{m_4^2+m_b^2-t_3}{2} \\ \frac{m_1^2+m_a^2-t_1}{2} & \frac{s+t_1-m_a^2-s_{234}}{2} & m_1^2 & \frac{\hat{s}_3-\hat{s}_2+m_2^2-s_2}{2} & \frac{s-\hat{s}_3+s_2-s_{234}}{2} \\ \frac{\hat{s}_3-\hat{s}_2+t_2-t_3}{2} & \frac{s_3+t_3-m_4^2-t_2}{2} & \frac{\hat{s}_3+m_2^2-\hat{s}_2-s_2}{2} & m_3^2 & \frac{s_3-m_3^2-m_4^2}{2} \\ \frac{s+t_3-\hat{s}_3-m_b^2}{2} & \frac{m_4^2+m_b^2-t_3}{2} & \frac{s+s_2-\hat{s}_3-s_{234}}{2} & \frac{s_3-m_3^2-m_4^2}{2} & m_4^2 \end{vmatrix} \\ &= As_{234}^2+Bs_{234}+C \\ &= 0\end{aligned}$$

Reference

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