Financial Instruments

Winter 2024

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Homework 5

Due at the beginning of class 6

1 Multiperiod binomial tree

In each of the next two periods ($i = 1$ and $i = 2$), a stock whose value in period $i = 0$ is $S_0 = 100$, can either rise or fall by 10% (using the notation of the teaching notes, $u = 1.1$ and $d = \frac{1}{u}$). Suppose the per-period risk-free rate is $r = 5\%$. A financial intermediary sold a European call option on the stock, with exercise price equal to $K = 100$.

1. What actions would the firm need to take in order to hedge its risk from writing the call? (*Tip: describe how you would set up a portfolio that the intermediary can use to mirror the payoffs of the option.* Note: answering that the firm can buy the same call will not get full credit!)

   (a) Using a two-period binomial tree, construct the replicating portfolio at each node. How much would you charge for the call option at period $i = 0$? What is the "delta" (the number of shares bought or sold in the replicating portfolio) and how much borrowing or lending is required at each node?

   (b) Let's see what happens if we change the initial stock price S_0 by a little. In particular I would like you to figure out how the value of the replicating portfolio changes when you change S_0. To do this compute the value of the replicated portfolio, V^{RP} when the initial stock price is equal to $S_0 + \triangle S$ with $\Delta S = 1, 2, ..., 10$. When doing this remember that future stock prices are also changed under the assumption that the stock price rises or falls by 10% going forward.

i. Is the change in value of the portfolio linear in ΔS? (*Tip: the change in value is linear in* ΔS *if* $V^{RP}(k \cdot \triangle S) = k \cdot V^{RP}(\triangle S)$ *).* Show your calculations.

ii. If your answer in the previous point was "No", would you describe the relationship between V^{RP} and S as concave or convex? What would be the sign of the second derivative of a function $V^{RP}(S)$?

(c) Use the initial assumption for S_0 for this and the remaining questions.

Is the portfolio "self-financing"? (*Tip: The portfolio is self-financing if you do NOT need any additional capital infusion between period* $i = 0$ *and the period of the final payoff* $T = 0$)

(d) What is the profit/loss on the replicating portfolio?

(e) What is the price of the call option if the stock pays a 5% dividend yield in period $i = 1$? Show the resulting trees for the stock price and for the option as well as your computations. (Hint: when the dividend is paid the stock price must decline in response. In the "up" state in period 1 before the dividend is paid, the stock prices is $S_0 \times u$. If y is the dividend yield, the stock price next period in the "up" state *after the payment of the dividend* is $S_0 \times u \times (1 - y)$. A similar adjustment must be made to the stock price in the "down" state after the dividend is paid.)

(f) What is the price of the call option if the stock pays a $ 5 dividend in period $i = 1$? In this case the dividend is a fixed dollar amount that is independent of the stock price. Show the resulting trees for the stock price and for the option as well as your computations. Comment on the difference with part (e).

2 Black and Scholes (and Merton) Formula

Today is January 4, 2022. The stock price of Verotende Inc. (a well known publicly traded investment bank in early 2022) is trading at at USD 42. In addition the annualized

stock return volatitiy is 20% and the annualized continuously compounded yield on 6 months T-bills is 10%.

1. Using the Black-Scholes formula, compute the price of a European call and a European put option, with identical exercise price $K^c = K^p = 40$ dollars, and six months to maturity. Show your calculations.

2. Compute the price of the call and put options using several initial stock prices values, from 10% in the money to 10% out of the money. Plot the price of the the two options as a function of the stock price in a chart. Would you describe the relationship between the two option prices and S as concave or convex?

3. Consider changing the inputs: compute the price of the call and of the put using different values for the parameters listed in the first column of Table (1). Report your findings on the relationship between options prices and inputs, by filling the white spaces in Table (1) (simply write: "increase", "decrease" or "uncertain" in each white space). Provide some intuition for your findings

	If input increases...	
Input	Change in call price:	Change in put price:
Stock Price		
Strike Price		
Volatility		
Maturity		
Risk free rate		

Table 1: Sensitivity of Black and Scholes option prices to inputs

4. If you sell an at-the-money put option on Verotende Inc. to a counterparty, and you want to hedge the short put using a position in stocks and bonds, what is going to be your position in stocks and bonds?

3 Big binomial tree

Use the file BinomialTree.xls available on chalk to confirm that as you increase the number of steps, the option price and delta from the Tree converges to the Black and Scholes ones. Report the prices and deltas in a table for $n = 2, 5, 10, 25, 50, 125, 250$ tree steps.