Useful Formulas for Midterm Exam
BUS 35130

- Compounding:

$$V = \left(1 + \frac{r}{n}\right)^{n \times T}$$

- For n large, we converge to continuous compounding

$$V = \left(1 + \frac{r}{n}\right)^{n \times T} \longrightarrow e^{r \times T}$$

- Discounting:

$$Z(T) = \frac{1}{\left(1 + \frac{r}{n}\right)^{n \times T}}$$

- In the limit as $n \to \infty$ we obtain the usual continuous compounding formula

$$Z(T) = \frac{1}{\left(1 + \frac{r}{n}\right)^{n \times T}} \longrightarrow Z(T) = e^{-rT}$$

- Coupon Bond Pricing:

$$\begin{aligned} P(t, T_n) &= c/2 \times Z(t, T_1) + c/2 \times Z(t, T_2) + ... + (100 + c/2)\, Z(t, T_n) \\ &= \sum_{i=1}^{n} \frac{c/2}{\left(1 + r_2(t, T_i)/2\right)^{2 \times (T_i - t)}} + \frac{100}{\left(1 + r_2(t, T_n)/2\right)^{2 \times (T_n - t)}} \end{aligned}$$

- Forward Rates:

$$\left(1 + \frac{f_n(0, T_1, T_2)}{n}\right)^{n(T_2 - T_1)} = \frac{Z(0, T_1)}{Z(0, T_2)}$$

Taking the limit as $n \to \infty$ the continuously compounded forward rates is

$$\begin{aligned} f(0, T_1, T_2) &= \frac{\ln(Z(0, T_1)) - \ln(Z(0, T_2))}{T_2 - T_1} \\ &= \frac{r(0, T_2)T_2 - r(0, T_1)T_1}{T_2 - T_1} \end{aligned}$$

Instantaneous forward:

$$f(0, T_1, T_1) - r(0, T_1) = T_1 \times \frac{dr(0, T_1)}{dT_1}$$

- Bootstrapping discount rates from coupon bonds:

$$Z(0,T_i) = \frac{P^i(0,T_i) - c^i/2 \times \left(\sum_{j=1}^{i-1} Z(0,T_j)\right)}{1 + c^i/2}$$

- Duration:
  - Zero Coupon:

$$\begin{aligned} D_{Z,T} &= -\frac{1}{Z(t,T)}\frac{dZ(t,T)}{dr} \\ &= -\frac{1}{Z(t,T)} \times \left[-(T-t) \times e^{-r(t,T)\times(T-t)}\right] \\ &= -\frac{1}{Z(t,T)} \times [-(T-t) \times Z(t,T)] \\ &= T - t \end{aligned}$$

  - Portfolio:

$$D_\Pi = w_1 D_1 + w_2 D_2$$

    where

$$w_i = \frac{N_i P_i}{\Pi}$$

  - Coupon Bonds:

$$D = -\frac{1}{P(0,T)}\frac{dP(0,T)}{dr} = \sum_{i=1}^{n} w_i D_{Z,T_i} = \sum_{i=1}^{n} w_i \times T_i$$

- Convexity

$$C = \frac{1}{P}\frac{d^2\ P}{d\ r^2}$$

  - Zero Coupon:

$$C_Z = \frac{1}{Z(t,T)}\frac{d^2\ Z(t,T)}{d\ r^2} = (T-t)^2$$

– Coupon Bond:

$$C = \frac{1}{P_c}\frac{d^2\ P_c}{d\ r^2} = \sum_{i=1}^{n}\ w_i\ (T_i - t)^2$$

– Pricing Implications:

$$\frac{d\ P}{P} \approx -D\ dr + \frac{1}{2}\ C\ dr^2$$

– Hedging with Duration and Convexity of a coupon bond:

$$V = P_c + k_1\ P_1 + k_2\ P_2 \Longrightarrow d\ V = d\ P_c + k_1\ d\ P_1 + k_2\ d\ P_2$$

– Two Equations:

$$\begin{aligned} k_1\ D_1\ P_1 + k_2\ D_2\ P_2 &= -D\ P_c \qquad \text{(Delta Hedging)} \\ k_1\ C_1\ P_1 + k_2\ C_2\ P_2 &= -C\ P_c \qquad \text{(Convexity Hedging)} \end{aligned}$$

so that:

$$k_1 = -\frac{P_c}{P_1}\left(\frac{D\ C_2 - C\ D_2}{D_1\ C_2 - C_1\ D_2}\right); \quad k_2 = -\frac{P_c}{P_2}\left(\frac{D\ C_1 - C\ D_1}{D_2\ C_1 - C_2\ D_1}\right)$$

- Factors:

– Interest-rate dynamics:

$$\begin{aligned} \Delta r_1(t) &= \alpha_1 + \beta_{11}\Delta\phi_1(t) + \beta_{12}\Delta\phi_2(t) + \beta_{13}\Delta\phi_3(t) + \varepsilon_1(t) \\ \Delta r_2(t) &= \alpha_2 + \beta_{21}\Delta\phi_1(t) + \beta_{22}\Delta\phi_2(t) + \beta_{23}\Delta\phi_3(t) + \varepsilon_2(t) \\ \vdots &= \vdots \\ \Delta r_n(t) &= \alpha_n + \beta_{n1}\Delta\phi_1(t) + \beta_{n2}\Delta\phi_2(t) + \beta_{n3}\Delta\phi_3(t) + \varepsilon_n(t) \end{aligned}$$

– Factors:

$$\Delta\phi_i(t) = a_{i1} \times \Delta r_1(t) + ... + a_{in} \times \Delta r_n(t)$$

– PCA approach results in $\beta_{ji} = a_{ij}$

– Factor Duration for a bond:

$$D_j = -\frac{1}{P_c}\frac{d\ P_c}{d\ \phi_j} = \sum_{i=1}^{n} w_i \tau_i \beta_{ij}$$

- Term Structure:

  – Expectations hypothesis:

$$\begin{aligned} \textit{The long term yield} &= \textit{forecasted average path of future rates} \\ y_t(n+1) &= E_t\left[\frac{1}{n+1}\sum_{i=0}^{n} y_{t+i}(1)\right] \end{aligned}$$

  Forward rate is equal to expected future short-term rate:

$$f_t(\tau_i, \tau_i + \Delta t) = E_t[y_{t+\tau_i}(\Delta t)]\ .$$

  – In general there are risk-premia:

$$y_t(n+1) = E_t\left[\frac{1}{n+1}\sum_{i=0}^{n} y_{t+i}(1)\right] + RP_t$$

  – TIPS:

$$P_c^{TIPS}(t;T) = \frac{Idx(t)}{Idx(0)} \times \left[\frac{c \times 100}{2}\sum_{i=1}^{n} Z^{real}(t;T_i) + Z^{real}(t;T)\right]$$

  – Link between real and nominal rates:

$$r(0,T) = r_{real}(0,T) + \overline{\pi} + \kappa - \frac{1}{2}\sigma_\pi^2$$

  – Economic model of nominal rate:

$$r(0,T) = \underbrace{\left(\rho + \gamma\overline{g} - \frac{\gamma^2}{2}\sigma_g^2\right)}_{\text{real rate } r_{real}(0,T)} + \underbrace{\overline{\pi} - \frac{1}{2}\sigma_\pi^2}_{\text{exp. infl. \& conv.}} - \underbrace{\gamma\sigma_\pi\sigma_g\rho_{g,\pi}}_{\text{risk premium}}$$

- Derivatives

  – Forward Rate Agreements.

-
$$\text{Net payment at } T_2 = N \times \Delta \times [r_n(T_1, T_2) - f_n]$$
- Forward Discount:
$$F(t, T_1, T_2) = \frac{Z(t, T_2)}{Z(t, T_1)}$$
- forward rate:
$$f_n\left(t, T_1, T_2\right) = n \times \left(\frac{1}{F\left(t, T_1, T_2\right)^{\frac{1}{n \times (T_2 - T_1)}}} - 1\right)$$
- Value of a forward rate agreement:
$$\begin{aligned} V^{FRA}(t) &= N \times Z(t, T_2) \times \left[M - \frac{Z(t, T_1)}{Z(t, T_2)}\right] \\ &= N \times Z(t, T_2) \times \Delta \times [f_n(0, T_1, T_2) - f_n(t, T_1, T_2)] \end{aligned}$$

– Swaps
- Cash flow:
$$\text{Net Cash Flow at } T_i = N \times \Delta \times [r_n(T_{i-1}) - c]$$
- Value at reset dates:
$$V^{swap}(T_i; c, T) = 100 - \left(\frac{c}{2} \times 100 \times \sum_{j=i+1}^{M} Z(T_i, T_j) + Z(T_i, T_M) \times 100\right)$$
- Swap value at initiation is zero so that swap rate is:
$$c = n \times \left(\frac{1 - Z\left(0, T_M\right)}{\sum_{j=1}^{M} Z\left(0, T_j\right)}\right)$$
- Bootstrapping yield curve from swap rates:
$$Z\left(t, T_1\right) = \frac{1}{1 + \frac{c(t, T_1)}{n}}$$
while for $i = 2, ..., M$
$$Z\left(t, T_i\right) = \frac{1 - \frac{c(t, T_i)}{n} \times \sum_{j=1}^{i-1} Z\left(t, T_j\right)}{1 + \frac{c(t, T_i)}{n}}$$

- Caps and Floors Cash

  A cap pays a stream of payments at T_i, $i = 1, .., n$ with $T_{i+1} = T_i + \Delta$, where

$$CF(T_{i+1}) = \Delta \times N \times \max\left(r_n(T_i, T_{i+1}) - r_K, 0\right)$$

  A floor pays

$$CF(T_{i+1}) = \Delta \times N \times \max\left(r_K - r_n(T_i, T_{i+1}), 0\right)$$

- Put-Call Parity (at the money caps and floors):

$$\text{cap} = \text{floor} + \text{swap}$$

- Payoff to a swaption at maturity:

$$\text{Payoff of Payer Swaption} = \sum_{j=i^*+1}^{n} Z(T_{i^*}, T_j)\,\Delta N \max\left(c(T_{i^*}, T) - r_K, 0\right)$$

  where $c(T_{i^*}, T)$ is the swap rate at time T_{i^*} to a swap that matures at time T.

- Binomial Models
  - Hedging:
    - Hedge Portfolio:

$$\Pi_0 = V_0 - \Delta Z_0(j)$$

    - Hedge Ratio:

$$\Delta = \frac{V_{1,u} - V_{1,d}}{Z_{1,u}(j) - Z_{1,d}(j)}$$

    - No arbitrage price of portfolio:

$$\Pi_0 = Z_0(1)\,\Pi_{1,u} \text{ or, equivalently, } \Pi_0 = Z_0(1)\,\Pi_{1,d}$$

  - No arbitrage pricing:
    - Any expectation:

$$V_0 = Z_0(1)\,E^{\tilde{p}}(V_1) - \Delta \times \left\{Z_0(1)\,E^{\tilde{p}}[Z_1(j)] - Z_0(j)\right\}$$

    - Reward for risk is equalized across securities:

$$\frac{E^{\tilde{p}}\left[\frac{V_1}{V_0}\right] - \frac{1}{Z_0(1)}}{\left[\frac{V_{1,u}}{V_0} - \frac{V_{1,d}}{V_0}\right]} = \frac{E^{\tilde{p}}\left[\frac{Z_1(j)}{Z_0(j)}\right] - \frac{1}{Z_0(1)}}{\left[\frac{Z_{1,u}(j)}{Z_0(j)} - \frac{Z_{1,d}(j)}{Z_0(j)}\right]}$$

- Present values with risk correction:

$$V_0 = Z_0\left(1\right)\left\{E\left[V_1\right] - \lambda\left[V_{1,u} - V_{1,d}\right]\right\}$$

where

$$\lambda = \frac{E\left[\frac{Z_1(2)}{Z_0(2)}\right] - \frac{1}{Z_0(1)}}{\left[\frac{Z_{1,u}(2)}{Z_0(2)} - \frac{Z_{1,d}(2)}{Z_0(2)}\right]}$$

– Risk-neutral probabilities set λ to zero

$$\pi^* = \frac{Z_0\left(j\right)/Z_0\left(1\right) - Z_{1,d}\left(j\right)}{Z_{1,u}\left(j\right) - Z_{1,d}\left(j\right)}$$

$$\boxed{V_0 = Z_0\left(1\right)E^*\left[V_1\right]}$$

1. **Short Answer/ True, False 25 points, 5 points each, points awarded for a complete explanation!)**. You should answer each question in 2 or 3 sentences.

   (a) True or False. We can use the relative prices of TIPS and nominal treasury bonds to back out the market's expectation of future inflation.

   (b) Short answer. Explain how and why an "at the money" (strike prices equal to swap rates) interest floor can be priced based on the prices of an interest rate cap with the same strike price and maturity.

   (c) True or False. Risk neutral models should be designed to help us forecast future interest rates.

   (d) True or False. The goal of hedging of with factors models is not to eliminate all risks, but to eliminate exposure to important risks.

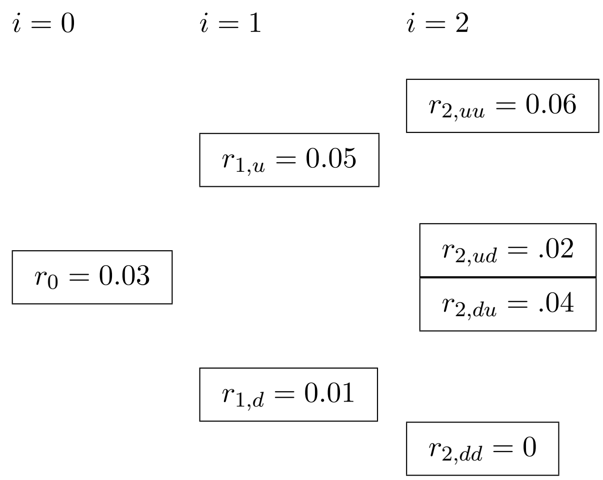
   (e) Short answer. Consider the Ho-Lee model binomial model with dynamics for short rates given by:

$$
\begin{aligned}
r_{i+1,j} &= r_{i,j} + \theta_i \times \Delta t + \sigma \times \sqrt{\Delta t} \quad \text{for a “up” movement} \\
r_{i+1,j+1} &= r_{i,j} + \theta_i \times \Delta t - \sigma \times \sqrt{\Delta t} \quad \text{for a “down” movement}
\end{aligned}
$$

   How are the values of θ_i calibrated? Are their features of the world that are at odds with this type of model (provide 2 and explain)?

2. **(65 points)**

Consider the following binomial model for 6-month continuously compounded risk-free rates reported in annualized units. Each period is six months. Notice that the tree is NOT RECOMBINING. The true or objective probability of an "up" move at each node is 50%.



(a) (10 points) At each node in period 1, what are the expected *changes* in short-term interest rates (difference between expected rates at time 2 and current rates)? Do the expected changes differ across nodes? What underlying model of interest rates would reflect your findings and why? (HINT: remember assignment #1 ...).

(b) (3 points) What is the time 0 price of a one-period (6-month) zero coupon bond with face value of \$1?

(c) (10 points) A 1-year zero coupon bond with face value \$1 is trading at time zero at $Z_0(2) = 0.972$ Create and report the evolution of the price of this bond at each node of the above interest rate tree.

(d) (5 points) Use the calculations in points (2b) and (2c) to compute the risk neutral probability π^* of moving from time 0 to node $\{1, u\}$.

(e) (17 points) Consider a coupon bond with *inverse floating and "snowballing" coupons*:

$$\begin{aligned}
\text{Coupon at time 1} &= max\{(\bar{c} - r_0)/2, 0\} && (1)\\
\text{Coupon at time 2} &= max\{(\text{Coupon at 1}) + (\bar{c} - r_1)/2, 0\} && (2)\\
\text{Coupon at time 3} &= max\{(\text{Coupon at 2}) + (\bar{c} - r_2)/2, 0\} && (3)
\end{aligned}$$

where $\bar{c} = 0.03$

Note that the cash flow at time i depends on the path of interest rates strictly before i. At maturity (time $i = 3$) the security also pays back the principal of 1. Assume the risk neutral probability of moving "up", π^*, computed in part 2d is constant throughout the tree. What is the value the snowball inverse floater? *(Tips: You should first calculate and report the cash flows to be paid in various interest rate* *scenarios. As you do the calculation, outline the binomial tree of prices for the bond, as it may be useful in the next question.)*

(f) (10 points) Assume that the bond from part 2e is in fact *callable*: it can be called back at par by the issuer any time before maturity. What is the value of the callable inverse floater? When would the bond be called by the issuer? How are the times of early exercise related to the level of rates and why?

(g) (10 points) Suppose the bond from part 2f is selling at part ($1). Is there an arbitrage opportunity? If so, how could you use the bonds of parts 2b and 2c to take advantage of this opportunity?

3. **(45 points)**

The current continuously-compounded yield on a 6-month T-Bill is 4.5%. Suppose that you have the following information about swap rates with maturities of 1 year, 1.5 year and 2 years with semi-annual payments where the floating rate is given by the 6-month T-Bill rate determined 0.5 years before each payment.

Maturity.	Swap rate (annualized)
1 year	5%
1.5 years	5.2%
2 years	5.5%

(a) (15 points). What are the implied prices of zero-coupon bonds with 1, 1.5 and 2 years to maturity?

(b) (10 points) What should be the price of a coupon bond with face value $100, 6.5% coupon and maturity of 2 years with semi-annual coupon payments?

(c) (15 points) Suppose the bond in part 3b is selling at par? Is there a potential arbitrage trade assuming that 6-month repo rates are current equal to the 6-month T-bill yield? If so, how would you execute this trade?

(d) (5 points). Suppose that you execute the trade in part 3c with as much leverage as you can, what risk would you face in holding the trade to maturity?

4. **(45 points)** Consider a hedge fund with equity of $0.5 billion that would like to invest in a bond position with a market value of $5 billion.

(a) (10 points) Suppose that the bond position they would like to invest in has duration of 8 and convexity of 60. To finance this position they will borrow using a combination of cash borrowing (over-night borrowing), 5 year and 15 year zero coupon bonds. What combination of these instruments should the hedge-fund use?

(b) (5 points) Although the hedge fund is hedged using duration and convexity, discuss other potential sources of risk they might face? (Limit the discussion to yield curve variation.)

(c) (10 points) Discuss how principle component analysis could be used to potentially help with hedging.

(d) (15 points) Suppose that you run a PCA analysis as in the class with the following results:

Factor	1 month	6 months	1 year	2 year	3 year	4 year	5 year
				Maturity			
				Panel A: Level			
β_{i1}^{PCA}	0.4416	0.4167	0.4315	0.3762	0.3421	0.3169	0.2935
R^2	61.15%	79.52%	91.66%	88.61%	83.89%	75.06%	73.03%
				Panel B: Slope			
β_{i2}^{PCA}	-0.7307	-0.2774	0.0848	0.2425	0.2909	0.3559	0.3344
R^2	95.91%	86.83%	92.39%	96.25%	96.48%	94.72%	92.71%
				Panel C: Curvature			
β_{i3}^{PCA}	0.5186	-0.7099	-0.2869	-0.0231	0.1411	0.2267	0.2700
R^2	99.99%	97.99%	94.35%	96.27%	97.17%	96.58%	95.70%

Suppose further that you know the exposure of the bond strategy to the first two PCA components are 5 and 12 respectively. You would like to use the 3-year and 5-year zero coupon bonds in your borrowing. What combination should you use?

(e) (5 points) Would this hedging strategy work well if the factors where to make "large" movements? Why or why not? If not, what might you consider in addition?

1. (60 points, 10 points each) Short-Answer/True-False Questions. Grade depends on completeness of answer, however you should try to be short and to the point in your response. We're only looking for the obvious big point(s). We don't give credit for long winded answers and will take off points if you add things that are wrong or irrelevant.

   (a) The duration of zero coupon bonds is given by its time to maturity. Therefore, it is independent of current interest rates. It follows that the convexity of a zero coupon bond has to be zero.

   (b) Changes in the slope of the term structure of interest rates can be effectively hedged through the so called "Duration and Convexity" hedging.

   (c) The following quotes are from Bloomberg and are from a single day:

Maturity (T)	LIBOR	3 month Forward Rate $F(T, T+3\ \text{ months })$
3 months	1.18438%	$F(3\ month, 6\ month)$: 1.231 %
6 months	1.66 %	$F(6\ month, 9\ month)$: 1.281 %
9 months	1.82375%	$F(9\ month, 1\ year)$: 1.45 %
1 year	1.975 %	

   There is an arbitrage in the quoted forward rate $F(9\ months, 1\ year) = 1.45\%$ compared to LIBOR rates. Discuss.

   (d) The expectation hypothesis – the fact that the long-term yield equals the market forecast of future short-term rates – is strongly supported by the historical data.

   (e) The payoff of a receiver swaption is equivalent to the payoff of a call option on a coupon bond with coupon rate equal to the swap rate. Such call option has a strike price of par (= 100).

   (f) The duration of an inverse floater may be higher than the time to maturity of the floater.

2. (50 points) You asked your assistant to fit some bond prices, and he came back with the following binomial trees for *zero-coupon bonds*. The true probability to go up the tree is $p = 0.5$ and the time step is one year. Unfortunately, as you and your assistant were communicating, one entry ($Z_{1,u}(3)$) did not go through, hence the question marks ?? in the tree. Your assistant left for vacation and cannot be reached.

$i = 0$	$i = 1$	$i = 2$
		$Z_{2,uu}(2) = 1.00$ $Z_{2,uu}(3) = 0.91$
	$Z_{1,u}(1) = 1.00$ $Z_{1,u}(2) = 0.92$ $Z_{1,u}(3) =??$	
$Z_0(1) = 0.96$ $Z_0(2) = 0.90$ $Z_0(3) = 0.85$		$Z_{2,ud}(2) = Z_{2,du}(2) = 1.00$ $Z_{2,ud}(3) = Z_{2,du}(3) = 0.96$
	$Z_{1,d}(1) = 1.00$ $Z_{1,d}(2) = 0.97$ $Z_{1,d}(3) = 0.94$	
		$Z_{2,dd}(2) = 1.00$ $Z_{2,dd}(3) = 0.99$

(a) (5 points) Compute the risk neutral probability at time $i = 0$.

(b) (5 points) What is the value of $Z_{1,u}(3)$?

(c) *Expected Bond Returns in Excess of Risk Free Rate* (not risk neutral expectations here! Use the true ones for this part of the quesiton.)

i. (5 points) What is the expected *excess* return between $i = 0$ and $i = 1$ from an investment at time 0 in the 2-period bond $Z_0(2)$?

ii. (10 points) Is this expected excess return (over the same period) the same as for an investment at time 0 in the 3-period bond $Z_0(3)$? If not, why not? Be as precise as possible.

(d) Consider a forward contract to purchase a 1-period zero coupon bond at time $i = 1$ for a price F. The payoff of the forward contract at $i = 1$ is

$$\text{Payoff at } 1 = Z_1(2) - F$$

The value of a forward contract is always zero at time 0.

i. (15 points) What is the value of F? How does it compare with the risk neutral expected value of $E_0^*[Z_1(2)]$? How about with the forward discount $F(0, 1, 2)$ which can be computed from current bond prices? In each case explain any differences.

ii. (10 points) What is a replicating strategy for the payoff to the forward contract? Do you actually need the binomial tree to create this strategy? Why or why not?

3. (35 points) Using the history of changes in interest rates, you computed the following "betas" from principal component analysis.

	Maturity							
Factor	3 months	6 months	1 year	2 year	4 year	6 year	8 year	10 year
				Panel A: Level				
β_{i1}^{PCA}	0.26	0.28	0.28	0.26	0.22	0.18	0.16	0.15
R^2	75 %	88%	96%	96%	89%	81%	71%	61%
				Panel B: Slope				
β_{i2}^{PCA}	-0.40	-0.24	-0.08	0.08	0.12	0.23	0.24	0.24
R^2	98%	97%	97%	97%	99%	97%	90%	81%
				Panel C: Curvature				
β_{i3}^{PCA}	-0.17	0.07	0.20	0.19	0.01	-0.17	-0.30	-0.39
R^2	99%	97%	99%	99%	99%	100%	99%	98%

(a) (15 points) What are the factor durations of 3-months, 4-year, and 10-year zero-coupon bonds? Compute them for all three factors and provide an intuitive reason for their values.

(b) Your fund is long \$100 million in a particular fixed income portfolio. This portfolio has factor durations $D_{level} = 0$ and $D_{slope} = -15$. The current term structure is flat at 5% (continuously compounded).

i. (5 points) Suppose the level factor increases by 100bps and the slope factor also by 100bps. What is the impact on this portfolio?

ii. (10 points) Design a hedging strategy using the zero coupon bonds in 3a to hedge these positions. (You do not necessarily need to use all of them!) Discuss the choice of bonds in your answer. *(Note: Partial credit given to set up the problem right, even without full numerical solution.)*

iii. (5 points) Would the hedging strategy of part 3(b)ii leave the fund with factor and/or residual risk exposure? Why or why not?

4. (35 points) Freddie Mac is one of the key players in mortgage backed securities market.

   The following statements are from a Freddie Mac's Annual Report at a time when Freddie Mac was a privately owned company:

   > "Our primary interest-rate risk measures are PMVS [Portfolio Market Value Sensitivity] and duration gap. PMVS is an estimate of the change in the market value of our net assets and liabilities from an instantaneous 50 basis point shock to interest rates. [...] PMVS is measured in two ways, one measuring the estimated sensitivity of our portfolio market value to parallel movements in interest rates (PMVS-Level or PMVS-L) and the other to nonparallel movements (PMVS-YC).
   > [...]
   > The duration and convexity measures are used to estimate PMVS under the following formula:
   >
   > PMVS = -*[Duration]* multiplied by [rate shock] plus [0.5 multiplied by *Convexity*] multiplied by [rate shock]^2
   >
   > [...]
   > To estimate PMVS-L, an instantaneous parallel 50 basis point shock is applied to the yield curve, as represented by the US swap curve [...]. This shock is applied to the duration and convexity of all interest-rate sensitive financial instruments.
   > [...]
   > To estimate sensitivity related to the shape of the yield curve, a yield curve steepening and flattening of 25 basis points is applied to the duration of all interest-rate sensitive instruments. [...] The more adverse yield curve scenario is then used to determine the PMVS-yield curve."

   Suppose for simplicity that the portfolio of Freddie Mac's *assets* is given by 1 unit each of the 1-year and the 10-year zero-coupon bonds in the table below. In addition, its liabilities are given by 2 units of the 7-year zero coupon bond (I know all these bonds have positive convexity, and not negative convexity as mortgage backed securities, but for the sake of this exercise, this will do).

Table 1: yields and discounts

	1 year	7 year	10 year
$y(t)$	0.7125	1.7307	2.1456
$Z(0,t)$	99.29	88.59	80.69

(a) (15 points) Compute the duration and the convexity of the net assets and liabilities portfolio. *(That is, I am asking the duration and convexity of Equity = Assets minus Liabilities)*

(b) (10 points) Given the result in part 4a compute PMVS_L from a 50 basis point shock (positive or negative).

(c) (10 points) Compute PMVS_YC from a 25 basis point flattening and steepening of the term structure. *(Hint: This appears to be applied only to duration and not convexity of all interest rate sensitive securities. Hint 2: It is OK to assume plus/minus 25 bps change at the extremes, and 0 at the 7-year level).* What is larger in absolute value? PMVS_L or PMVS_YC?

Problem 1. (35 points) True/False Questions. Grade depends on completeness of answer.

(1.A) (7 points) [True/False] The duration of a floating rate note with semi-annual payments is at most equal to 0.5.

(1.B) (7 points) [True/False] If the spot curve is increasing, the forward rate is above the spot curve, while if the spot curve is decreasing, the forward rate is below the spot curve.

(1.C) (7 points) [True/False] The value of a fixed-for-floating swap is always equal to zero, both at inception of the contract and later.

(1.D) (7 points) [True/False] In binomial trees, risk neutral probabilities are crazy because they assume that investors are really risk neutral. Everyone knows that market participants are on average risk averse.

(1.E) (7 points) [True/False] Suppose the current yield-to-maturity on a one-year Treasury Bill is 4% and the market expect inflation of 2% over the next year. The yield-to-maturity on a one-year Treasury inflation protected security would necessarily be 2%.

Problem 2. (35 points) You fit the following model to the current term structure of interest rates, and you came up with the following binomial tree. The probability of moving up the tree is 1/2. (Risk-neutral). For simplicity, the time step is one year.

$i = 0$	$i = 1$	$i = 2$
		$r_{2,uu} = 0.1$
	$r_{1,u} = 0.07$	
$r_0 = 0.04$		$r_{2,ud}$, $r_{2,du}$ $= 0.06$
	$r_{1,d} = 0.03$	
		$r_{2,dd} = 0.02$

Compute the following: (Make sure you show all the computations at all nodes.)

(2.A) (7 points) Compute the price today ($i = 0$) of the one-, two-, three-period zero-coupon bonds.

(2.B) (5 points) Using the zero-coupon bonds prices from point 2.A, compute the one-period forward rates for $i = 1$ and $i = 2$. How do they compare with expected interest rates computed from the tree (using the risk-neutral probabilities)?

(2.C) (7 points) The market expects that because of a tightening of monetary policy, the one-period interest rate at time $i = 1$ will be 4%. What is the implied market price of risk? Comment on its sign and its economic meaning.

(2.D) (8 points) An investor would like to purchase a security with payoff given by

$$\text{Payoff}(i) = \begin{cases} 0 & i = 0 \\ 100 \times max(Average(r_0, r_1) - r_0, 0) & i = 1 \\ 100 \times max(Average(r_0, r_1, r_2) - r_0, 0) & i = 2 \end{cases}$$

Compute the value, at time $i = 0$ of this **American option**. (*Tip 1: At time $i = 0$ and $i = 1$, the investor can either exercise the option or wait to exercise the option. Tip 2: The calculations below may help speed up your own calculations.*)

Quantity	Value
$\frac{1}{2} \times (r_0 + r_{1,u})$	0.055
$\frac{1}{2} \times (r_0 + r_{1,d})$	0.035
$\frac{1}{3} \times (r_0 + r_{1,u} + r_{2,uu})$	0.07
$\frac{1}{3} \times (r_0 + r_{1,d} + r_{2,uu})$	0.0567
$\frac{1}{3} \times (r_0 + r_{1,u} + r_{2,ud})$	0.0567
$\frac{1}{3} \times (r_0 + r_{1,d} + r_{2,ud})$	0.0433
$\frac{1}{3} \times (r_0 + r_{1,u} + r_{2,dd})$	0.0433
$\frac{1}{3} \times (r_0 + r_{1,d} + r_{2,dd})$	0.03

(2.E) (8 points) Compute the replicating portfolio for the security in 2.D *at each node along the tree.* What other securities do you use to replicate? Why? What weights do you get? Check that the replication in fact replicates the security's payoff at $i = 1$ and that the price of your replicating portfolio at $i - 0$ equals the price from 2.D. (*Tip: If you are unable to solve 2.D, you can still receive partial credit for a correct setup in terms of generic payoffs.*)

Problem 3. (**35 points**) The pension plan of BBB Corp has \$1,000,000 in liabilities and is 80% funded. Also there are three bonds available to invest in for the pension plan asset portfolio. The following table contains the main characteristics of the pension plan's assets, liabilities and available bonds:

	Price	Duration	Convexity
Liability	\$1,000,000	10	100
Assets	\$800,000		
Bond_1	\$100	14	150
Bond_2	\$100	8	50
Bond_3	\$100	2	4

(3.a) (5 points) If we believe that interest rate term structure will only move in a parallel fashion in the future, then what kind of bond portfolio will always outperform liability for all possible interest rate movements and by the greatest possible amount out of all bond portfolios based on only three available bonds? Using the available three bonds, determine what bond or bonds are required and determine the exact position or positions. Assume that only long positions can be entered into (no short positions) and you are free to use any set of bonds you wish along with Cash if needed. (You don't need to use all of the bonds)

(3.b) The plan sponsor would like to create a bond portfolio that exactly matches duration and convexity of the liability to that of the assets.

(3.b.i) (5 points) Discuss the pros and cons of this strategy.

(3.b.ii) (8 points) Select the two most appropriate bonds out of three available and determine the bond positions. Once the bond positions are determined, please illustrate that the total bond portfolio is not greater than the total assets available (it is okay to be smaller). As before, no short positions are allowed (i.e. bond weights can not be negative). Explain intuitively why any other combination of bonds out of three available will not be suitable.

(3.c) (4 points) Suppose now that you conduct a Principal Component Analysis of the yield curve and determine that three principal components are need to fit changes in the yield curve. Would you be comfortable with your hedge from part (3.b.ii)? Why or why not? If not, discuss how you might construct a better hedge. Would you need all three bonds in this case? Why or why not?

(3.d) XYZ insurance company has liability cash flows (aka benefit payouts), as summarized in the following table, for their annuity line of business. The table also contains continuously compounded spot rates, corresponding Z-factors and three PCA β's for Level, Slope and Curvature factors respectively. "Time" column in the table is the number of years from today until the payments are due, i.e. all the payments are made annually, with spot rates, Z-Factors, and PCA β's shown at the same annual frequency (TIP: this is a standalone question and not a continuation of questions 3a-3c)

Time	Spot Rates	Z-Factors	β_{Level}	β_{Slope}	$\beta_{Curvature}$	Liability CF
1	0.0300	0.9704	0.0968	0.3189	0.2003	\$1,000
2	0.0360	0.9305	0.1453	0.0802	(0.1099)	\$1,100
3	0.0432	0.8784	0.1298	(0.0500)	(0.1348)	\$1,200
4	0.0518	0.8129	0.1227	(0.0945)	(0.0257)	\$1,300
5	0.0622	0.7327	0.1216	(0.0988)	0.1314	\$1,400

(3.d.i) (8 points). Using only information from the table above (and not the information for questions 3a-3c), calculate factor sensitivities (i.e. factor duration) for Level, Slope and Curvature factors a) for XYZ insurance company's Liability, using liability cash flows provided in the table above, and b) for three hypothetical zero coupon bonds with 1, 3 and 5 years to maturity respectively and principal \$1000 each

(3.d.ii) (5 points) Using factor sensitivities for XYZ insurance company's liability and three bonds calculated in the previous question (3.d.i), discuss how you can establish individual bond positions in a three bond portfolio such that bond portfolio's factor sensitivities would match liability's factor sensitivities for all three factors. Write down all the necessary equations but there is no need to solve them, i.e. no need to calculate actual bond positions