Fixed Income Pircing

Black and Scholes Formula

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1. Black and Scholes Formula

1.1 Dynamic Replication

1.2 Black and Scholes and the Binomial Trees

Option Premium

- Black, Scholes and Merton show that (under certain conditions) there exists a *trading strategy* involving only stocks and bonds that *replicate* the payoff at T of a call or a put option.
- Assume a stock S_t has constant expected (log) return μ and constant volatility σ.
- That is, if the log return during a small time interval h be $R_t = log(S_{t+h}/S_t)$, assume

$$E[R_t] = \mu \times h; \quad E[R_t^2] = \sigma^2 \times h$$

  - (μ and σ are the annualized expected log return and volatility)
- Consider now a put option with strike price K and maturity T.
- The following trading strategy **replicates** the final payoff $\max(K - S_T, 0)$.

Option Premium by Dynamic Replication

1. At time 0:

   (a) Short $\Delta_0 = -N(-d_{1,0})$ of stocks

   - Here $N(x)$ is the standard normal cumulative density function, and $d_{1,t}$ is

   $$d_{1,t} = \frac{\ln\left(S_t/K\right) + \left(r + \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}$$

   - r is the continuously compounded risk free rate; σ is the volatility of stock returns.

   (b) Buy an amount $B_0 = K \times e^{-r\times T} \times N(-d_{2,0})$ of Treasury Zero Coupon bonds.

   - Here $d_{2,0} = d_{1,t} - \sigma \times \sqrt{T}$.
   - The portfolio so constructed has value at time 0

   $$P_0 = B_0 + \Delta_0 S_0$$

   - (it can be shown $P_0 > 0$).

2. From now on, *rebalance* the portfolio, to make sure that at every t, the portfolio has a position in stocks given by $\Delta_t = -N(-d_{1,t})$

   - E.g. if $S_t \downarrow \Longrightarrow \Delta_t \downarrow \Longrightarrow$ short more stocks and put proceeds into bonds $\Longrightarrow B_t \uparrow$.
   - Or if $S_t \uparrow \Longrightarrow \Delta_t \uparrow \Longrightarrow$ buy back stocks by liquidating some bonds $\Longrightarrow B_t \downarrow$.

Option Premium by Dynamic Replication

- For instance, let $S = K = 100$, $T = 1$, $r = 5\%$, $\sigma = 20\%$. Then,

$$d_1 = .35; d_2 = .15; N(-d_1) = 0.3632; N(-d_2) = 0.4404 \Longrightarrow \Delta_0 = -N(-d_1) = -0.3632$$

  – Initial short position in stocks: $\Delta \times 100 = -N(d_1) \times 100 = -0.3632 \times 100 = -\$36.32$.
  – Initial bond position: $B_0 = Ke^{-rT}N(-d_2) = \$41.89$.
  – Initial value of the portfolio: $P_0 = B_0 + \Delta_0 S_0 = \$41.89 - \$36.32 = \$5.57$

- One day later ($h = 1/252 = 1$ day) the stock is $S_h = 99 \Longrightarrow \Delta_h = -N(-d_{1,h}) = -.3821$

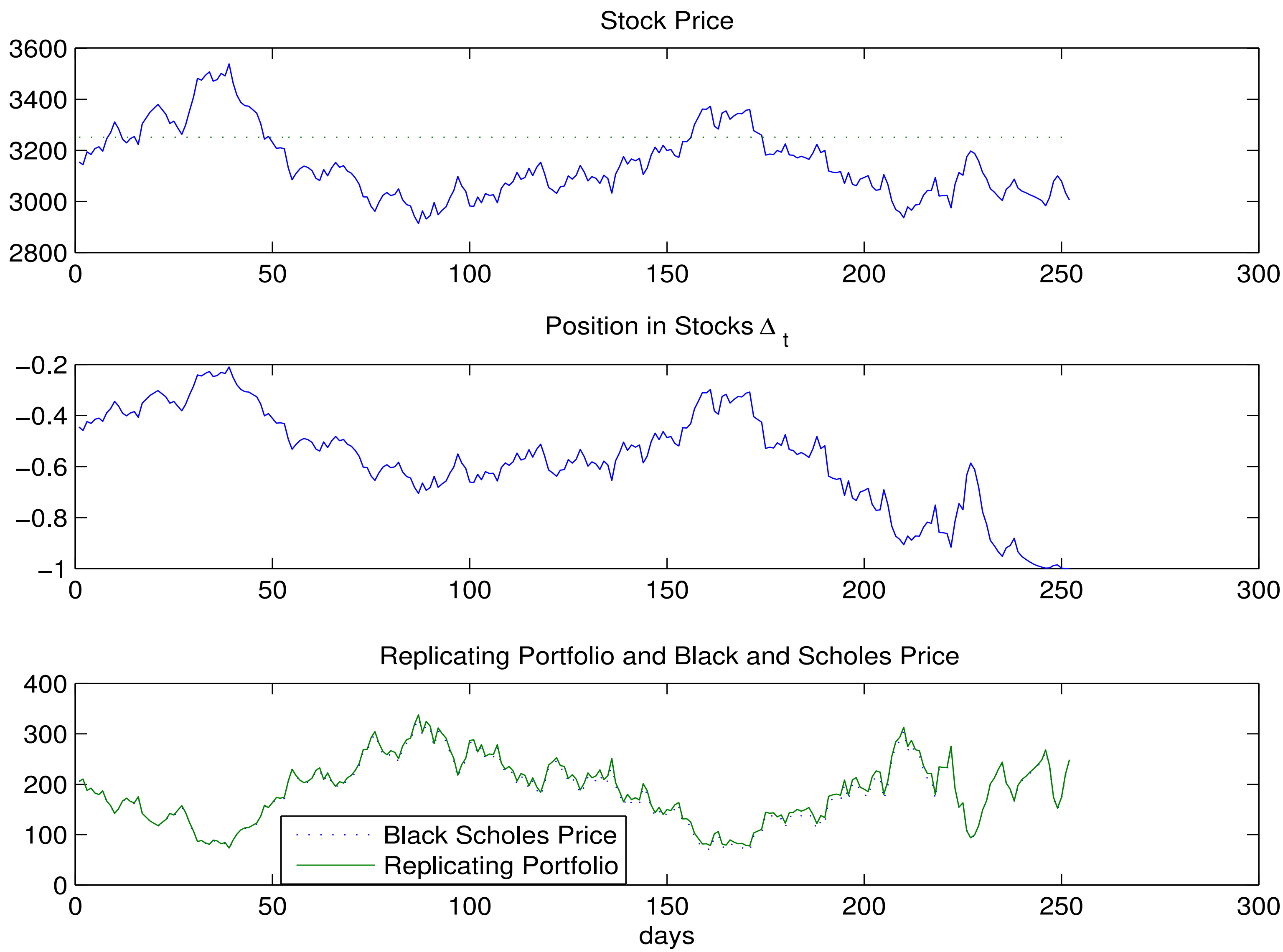
  – Need to short more, and thus sell $|\Delta_h - \Delta_0| = |-.3821 - (-0.3632)| = 0.0189$ shares.
  – Obtain cash $= 0.0189 \times 99 = \$1.879$, and put it in bonds:

$$\begin{aligned}\text{New Bond Position} &= B_h = B_0 \times e^{r\times h} + \$1.879 = \$41.89 - \$1.879 = \$43.777 \\ \text{New Portfolio Position} &= P_h = B_h + \Delta_h \times S_h = \$43.777 - .3821 \times 99 = \$5.941\end{aligned}$$

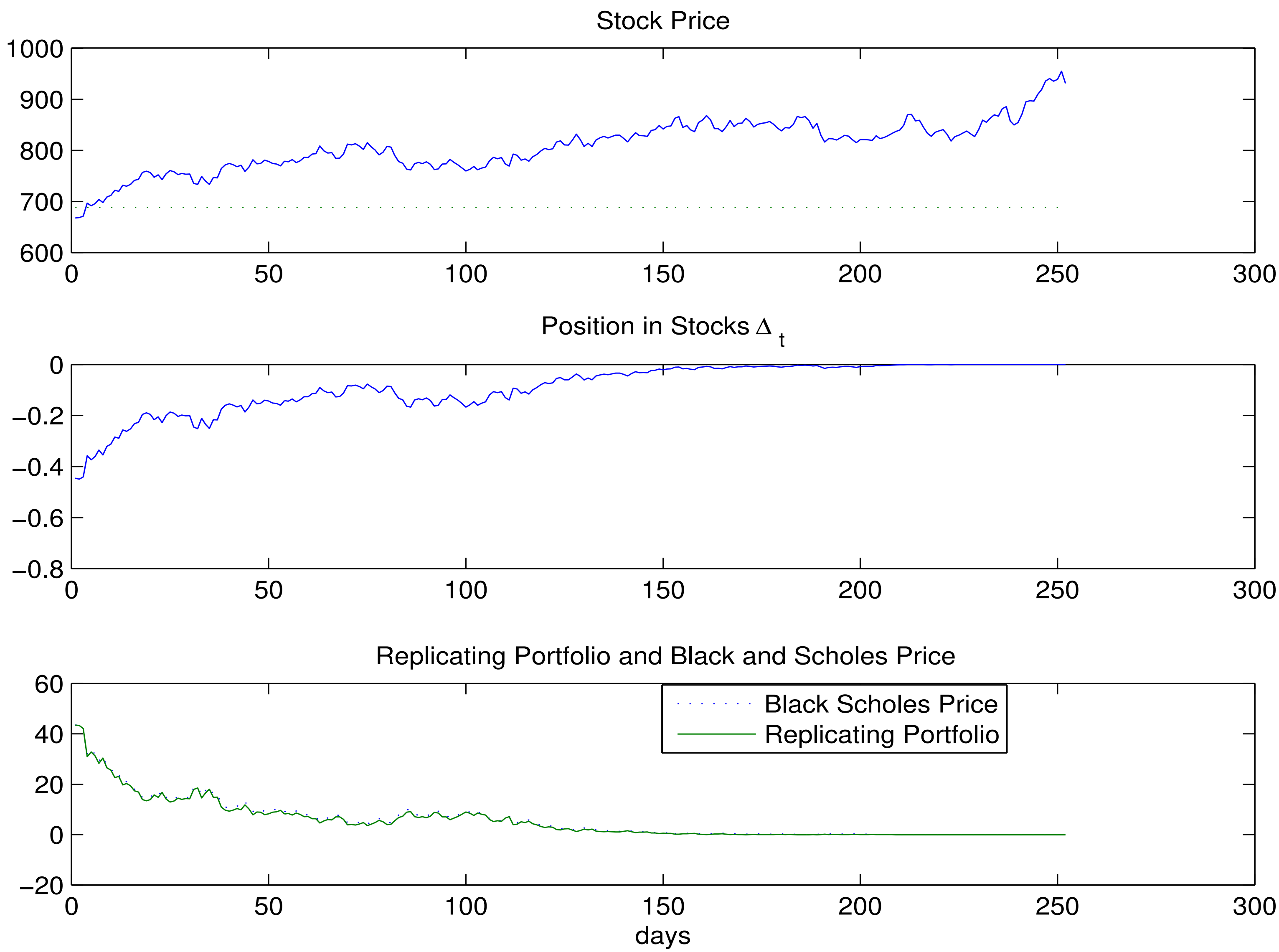
Option Premium by Dynamic Replication

- **Fact:** The portfolio $P_t = \Delta_t S_t + B_t$ obtained from the above trading strategy *replicates* the put option payoff.
  - That is, at maturity
  $$P_T = \Delta_T S_T + B_T = \max(K - S_T, 0)$$
- *Proof by simulation:* Next two figures shows that the strategy works, even when portfolio rebalancing is at daily interval ($h = 1/252$).
  - I simulate stock price paths. And then performed the above trading strategy.

Option Premium by Dynamic Replication



Option Premium by Dynamic Replication



Black and Scholes Formula

- Since portfolio P_t replicates the payoff of the put option, the value of the portfolio at any time must equal the value of the put option.
  - Why?
  - Arbitrage: "Buy Cheap / Sell Dear".
  - For instance, if $P_t <$ Put Option Premium $\Longrightarrow$
    1. Sell option and set up the replicating portfolio (which costs P_t)
    2. Today make (Put Option Premium $-P_t) > 0$.
    3. At maturity T the replicating portfolio provides the payoff, exactly.
- In particular, at time 0, the value of the option must be

$$\begin{aligned} \text{Put Premium at 0, } p_0 &= P_0 = B_0 + \Delta_0 \times S_0 \\ &= K \times e^{-rT} \times N(-d_{2,0}) - S_0 \times N(-d_{1,0}) \end{aligned}$$

  - This is the celebrated "Black and Scholes" formula for option pricing.
- Similarly, a call option formula is given by

$$\text{Call Premium at 0, } c_0 = S_0 \times N(d_{1,0}) - K \times e^{-rT} \times N(d_{2,0})$$

Delta Hedging and Dynamic Replication

- Why does the dynamic replication strategy work?
  - Suppose you sold a put option and decide to hedge using the replicating portfolio P.
  - Let Π be the portfolio short the put $(-p)$ and long the replicating portfolio

$$\begin{aligned}\Pi &= -p + P \\ &= -p + \Delta S + B\end{aligned}$$

  - What is the sensitivity of Π to *small* variations in stocks?

$$\frac{d\,\Pi}{d\,S} = -\frac{d\,p}{d\,S} + \Delta \times 1 + 0$$

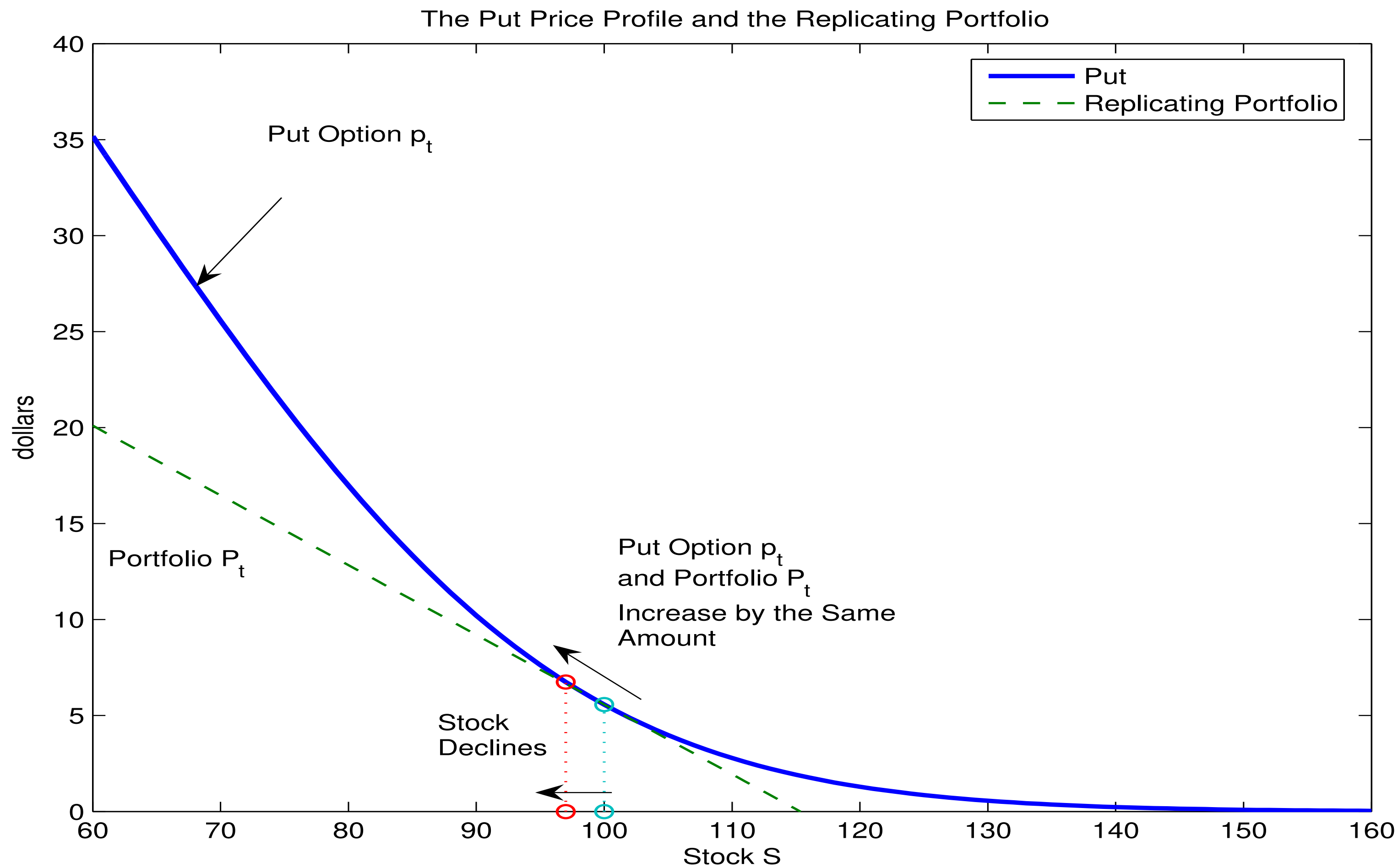
  - The portfolio Π is delta hedged $(d\,\Pi/d\,S = 0)$ if

$$\Delta = \frac{d\,p}{d\,S}$$

  - It can be shown that indeed $\Delta = -N(-d_1)$ is exactly $d\,p/d\,S$
- This implies that for every *small* variation in stock S, the portfolio and the option move exactly by the same amount.
  - $\Longrightarrow$ the dynamic replication works.

Delta Hedging and Dynamic Replication

- The next figure shows the dynamic replication at work in a graph:



Black and Scholes Formula

- Example: Consider an at-the-money option.
  - The stock price is $S = 100$, the strike price is $K = 100$, the (continuously compounded) interest rate is $r = 5\%$, maturity is $T = 1$, and the return volatility $\sigma = 30\%$.
- We then have

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\log\left(\frac{100}{100}\right) + (.05 + (0.30)^2/2) \times 1}{0.30\sqrt{1}} = 0.3167;$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.3167 - .3\sqrt{1} = 0.0167$$

- Therefore $N(d_1) = 0.62425$ and $N(d_2) = 0.50665$.
- The value of the call option is

$$c_0 = SN(d_1) - Ke^{-rT}N(d_2) = 100 \times 0.62425 - 100 \times e^{.05\times 1} \times 0.50665 = 14.2312$$

- The value of a put option can be computed from these data by recalling that

$$N(-d_1) = 1 - N(d_1) = 0.37575; \quad N(-d_2) = 1 - N(d_2) = 0.49335$$

- so that

$$p_0 = -SN(-d_1) + Ke^{-rT}N(-d_2) = -100 \times 0.37575 + 100 \times e^{.05\times 1} \times 0.49335 = 9.3542$$

The Binomial Tree and Black and Scholes Formula

- To interpret the Black and Scholes formula it is convenient to go back to binomial trees.
- Black and Scholes model assumes continuous trading
  - That is, traders can trade at any instant t
- Moreover, stock prices can take on any value, and not only the values on a tree.
- Both conditions are approximately met as we let the time interval binomial tree go to zero.

- To see the similarity of Black and Scholes formula with the one stemming from a binomial tree, consider the following example.

The Binomial Tree and Black and Scholes Formula

$i = 0$

$i = 1$

$S_{1,u} = S_0 \times u$
$c_{1,u} = \max(S_1 - K, 0) = S_1 - K$

S_0
$c_0 = e^{-r \times T} E^* [\max(S_1 - K, 0)]$

$S_{1,d} = S_0 \times d$
$c_{1,d} = \max(S_1 - K, 0) = 0$

- Consider $i = 0$ and $i = 1$ with $S_{1,u} = S_0 \times u$ and $S_{1,d} = S_0 \times d$.
- Assume the price of the option has $S_{1,u} > K > S_{1,d}$, so that the payoffs from the tree above result.
- Let q^* be the risk neutral probability of going up in the tree.

The Binomial Tree and Black and Scholes Formula

- The price of the option at time 0 according to risk neutral pricing is the

$$\begin{aligned} c_0 &= e^{-r\times T} E^* \left[\max(S_1 - K, 0)\right] \\ &= e^{-r\times T} \times \left[q^* \times \max(S_{1,u} - K, 0) + (1 - q^*) \times \max(S_{1,d} - K, 0)\right] \\ &= e^{-r\times T} \times q^* \times (S_{1,u} - K) \\ &= S_0 \times e^{-r\times T} \times q^* \times u - e^{-r\times T} \times K \times q^* \\ &= S_0 \times N_1 - e^{-r\times T} \times K \times N_2 \end{aligned}$$

- where, defining by u_{cc} the annualized c.c. return from an up movement $S_{1,u}/S_0 = e^{u_{cc}\times T} = u$

$$N_1 = e^{-r\times T} \times q^* \times u = e^{(u_{cc}-r)\times T} \times q^* \quad \text{and} \quad N_2 = q^*$$

- The similarity with Black and Scholes formula is not coincidential

$$\text{Call} = S \times N(d_1) - K \times e^{-rT} \times N(d_2)$$

- Interpretation:
  - $N_2 = N(d_2)$ risk neutral probability to be in the money at maturity;
  - $N_1 = N(d_1)$ risk neutral expected excess return *conditional* on exercise at T.

The Binomial Tree and Black and Scholes Formula

- For a large number of binomial steps, n for a fixed maturity:

$$\begin{aligned} c_0 &= e^{-rT} E^* \left[\max(S_T - K, 0)\right] \\ &= e^{-r\times T} \sum_{j=0}^{n} \left(\frac{n!}{j!(n-j)!}\right) \max(S_{T,j} - K, 0) \end{aligned}$$

- where $S_{T,j} = S_0 \times u^{(n-j)} \times d^j$.
- Let a be the smallest integer for which $S_{T,j} > K$ for $j \geq a$, and $S_{T,j} < K$ for $j < a$.
- Putting all together:

$$\begin{aligned} c_0 &= e^{-r\times T} \sum_{j=a}^{n} \left(\frac{n!}{j!(n-j)!}\right) (S_{T,j} - K) \\ &= S_0 \times N_1 - K \times e^{-r\times T} \times N_2 \end{aligned}$$

$$\text{with} \quad N_1 = \left(e^{-r\times T} \sum_{j=a}^{n} \left(\frac{n!}{j!(n-j)!}\right) \times u^{(n-j)} \times d^j\right) \quad \text{and} \quad N_2 = \sum_{j=a}^{n} \left(\frac{n!}{j!(n-j)!}\right)$$

- It can be shown that $N_1 \to N(d_1)$ and $N_2 \to N(d_2)$ as $n \to \infty$
- The interpretation, though, is the same as in the simple 2-period model.