**Useful Formulas for Midterm Exam**
**BUS 35130**

- Compounding:

$$V = \left(1 + \frac{r}{n}\right)^{n \times T}$$

- For $n$ large, we converge to continuous compounding

$$V = \left(1 + \frac{r}{n}\right)^{n \times T} \longrightarrow e^{r \times T}$$

- Discounting:

$$Z(T) = \frac{1}{\left(1 + \frac{r}{n}\right)^{n \times T}}$$

- In the limit as $n \to \infty$ we obtain the usual continuous compounding formula

$$Z(T) = \frac{1}{\left(1 + \frac{r}{n}\right)^{n \times T}} \longrightarrow Z(T) = e^{-rT}$$

- Coupon Bond Pricing:

$$\begin{aligned} P(t, T_n) &= c/2 \times Z(t, T_1) + c/2 \times Z(t, T_2) + ... + (100 + c/2)\, Z\,(t, T_n) \\ &= \sum_{i=1}^{n} \frac{c/2}{\left(1 + r_2\,(t, T_i)\,/2\right)^{2 \times (T_i - t)}} + \frac{100}{\left(1 + r_2\,(t, T_n)\,/2\right)^{2 \times (T_n - t)}} \end{aligned}$$

- Forward Rates:

$$\left(1 + \frac{f_n\,(0, T_1, T_2)}{n}\right)^{n(T_2 - T_1)} = \frac{Z(0, T_1)}{Z(0, T_2)}$$

  Taking the limit as $n \to \infty$ the continuously compounded forward rates is

$$\begin{aligned} f(0, T_1, T_2) &= \frac{\ln(Z(0, T_1)) - \ln(Z(0, T_2))}{T_2 - T_1} \\ &= \frac{r(0, T_2)T_2 - r(0, T_1)T_1}{T_2 - T_1} \end{aligned}$$

  Instantaneous forward:

$$f(0, T_1, T_1) - r(0, T_1) = T_1 \times \frac{dr(0, T_1)}{dT_1}$$

- Bootstrapping discount rates from coupon bonds:

$$Z\left(0,T_i\right) = \frac{P^i\left(0,T_i\right) - c^i/2 \times \left(\sum_{j=1}^{i-1} Z\left(0,T_j\right)\right)}{1 + c^i/2}$$

- Duration:
  - Zero Coupon:

$$\begin{aligned} D_{Z,T} &= -\frac{1}{Z(t,T)}\frac{dZ(t,T)}{dr} \\ &= -\frac{1}{Z(t,T)} \times \left[-(T-t) \times e^{-r(t,T)\times(T-t)}\right] \\ &= -\frac{1}{Z(t,T)} \times \left[-(T-t) \times Z(t,T)\right] \\ &= T - t \end{aligned}$$

  - Portfolio:

$$D_\Pi = w_1 D_1 + w_2 D_2$$

    where

$$w_i = \frac{N_i P_i}{\Pi}$$

  - Coupon Bonds:

$$D = -\frac{1}{P(0,T)}\frac{dP(0,T)}{dr} = \sum_{i=1}^{n} w_i D_{Z,T_i} = \sum_{i=1}^{n} w_i \times T_i$$

- Convexity

$$C = \frac{1}{P}\frac{d^2\ P}{d\ r^2}$$

  - Zero Coupon:

$$C_Z = \frac{1}{Z(t,T)}\frac{d^2\ Z(t,T)}{d\ r^2} = (T-t)^2$$

– Coupon Bond:

$$C = \frac{1}{P_c} \frac{d^2\ P_c}{d\ r^2} = \sum_{i=1}^{n}\ w_i\ (T_i - t)^2$$

– Pricing Implications:

$$\frac{d\ P}{P} \approx -D\ dr + \frac{1}{2}\ C\ dr^2$$

– Hedging with Duration and Convexity of a coupon bond:

$$V = P_c + k_1\ P_1 + k_2\ P_2 \Longrightarrow d\ V = d\ P_c + k_1\ d\ P_1 + k_2\ d\ P_2$$

– Two Equations:

$$\begin{aligned} k_1\ D_1\ P_1 + k_2\ D_2\ P_2 &= -D\ P_c \qquad \text{(Delta Hedging)} \\ k_1\ C_1\ P_1 + k_2\ C_2\ P_2 &= -C\ P_c \qquad \text{(Convexity Hedging)} \end{aligned}$$

so that:

$$k_1 = -\frac{P_c}{P_1}\left(\frac{D\ C_2 - C\ D_2}{D_1\ C_2 - C_1\ D_2}\right); \qquad k_2 = -\frac{P_c}{P_2}\left(\frac{D\ C_1 - C\ D_1}{D_2\ C_1 - C_2\ D_1}\right)$$

- Factors:

– Interest-rate dynamics:

$$\begin{aligned} \Delta r_1(t) &= \alpha_1 + \beta_{11}\Delta\phi_1(t) + \beta_{12}\Delta\phi_2(t) + \beta_{13}\Delta\phi_3(t) + \varepsilon_1(t) \\ \Delta r_2(t) &= \alpha_2 + \beta_{21}\Delta\phi_1(t) + \beta_{22}\Delta\phi_2(t) + \beta_{23}\Delta\phi_3(t) + \varepsilon_2(t) \\ \vdots &= \vdots \\ \Delta r_n(t) &= \alpha_n + \beta_{n1}\Delta\phi_1(t) + \beta_{n2}\Delta\phi_2(t) + \beta_{n3}\Delta\phi_3(t) + \varepsilon_n(t) \end{aligned}$$

– Factors:

$$\Delta\phi_i(t) = a_{i1} \times \Delta r_1(t) + ... + a_{in} \times \Delta r_n(t)$$

– PCA approach results in $\beta_{ji} = a_{ij}$

– Factor Duration for a bond:

$$D_j = -\frac{1}{P_c}\frac{d\ P_c}{d\ \phi_j} = \sum_{i=1}^{n} w_i \tau_i \beta_{ij}$$

- Term Structure:

  – Expectations hypothesis:

$$\begin{aligned} \textit{The long term yield} &= \textit{forecasted average path of future rates} \\ y_t(n+1) &= E_t\left[\frac{1}{n+1}\sum_{i=0}^{n} y_{t+i}(1)\right] \end{aligned}$$

  Forward rate is equal to expected future short-term rate:

$$f_t(\tau_i, \tau_i + \Delta t) = E_t[y_{t+\tau_i}(\Delta t)]\ .$$

  – In general there are risk-premia:

$$y_t(n+1) = E_t\left[\frac{1}{n+1}\sum_{i=0}^{n} y_{t+i}(1)\right] + RP_t$$

  – TIPS:

$$P_c^{TIPS}(t;T) = \frac{Idx(t)}{Idx(0)} \times \left[\frac{c \times 100}{2}\sum_{i=1}^{n} Z^{real}(t;T_i) + Z^{real}(t;T)\right]$$

  – Link between real and nominal rates:

$$r(0,T) = r_{real}(0,T) + \overline{\pi} + \kappa - \frac{1}{2}\sigma_\pi^2$$

  – Economic model of nominal rate:

$$r(0,T) = \underbrace{\left(\rho + \gamma\overline{g} - \frac{\gamma^2}{2}\sigma_g^2\right)}_{\text{real rate } r_{real}(0,T)} + \underbrace{\overline{\pi} - \frac{1}{2}\sigma_\pi^2}_{\text{exp. infl. \& conv.}} - \underbrace{\gamma\sigma_\pi\sigma_g\rho_{g,\pi}}_{\text{risk premium}}$$

- Derivatives

  – Forward Rate Agreements.

* $$\text{Net payment at } T_2 = N \times \Delta \times [r_n(T_1, T_2) - f_n]$$
* Forward Discount:
$$F(t, T_1, T_2) = \frac{Z(t, T_2)}{Z(t, T_1)}$$
* forward rate:
$$f_n\left(t, T_1, T_2\right) = n \times \left(\frac{1}{F\left(t, T_1, T_2\right)^{\frac{1}{n \times (T_2 - T_1)}}} - 1\right)$$
* Value of a forward rate agreement:
$$\begin{aligned} V^{FRA}(t) &= N \times Z(t, T_2) \times \left[M - \frac{Z(t, T_1)}{Z(t, T_2)}\right] \\ &= N \times Z(t, T_2) \times \Delta \times [f_n(0, T_1, T_2) - f_n(t, T_1, T_2)] \end{aligned}$$

- Swaps
  * Cash flow:
$$\text{Net Cash Flow at } T_i = N \times \Delta \times [r_n(T_{i-1}) - c]$$
  * Value at reset dates:
$$V^{swap}(T_i; c, T) = 100 - \left(\frac{c}{2} \times 100 \times \sum_{j=i+1}^{M} Z(T_i, T_j) + Z(T_i, T_M) \times 100\right)$$
  * Swap value at initiation is zero so that swap rate is:
$$c = n \times \left(\frac{1 - Z\left(0, T_M\right)}{\sum_{j=1}^{M} Z\left(0, T_j\right)}\right)$$
  * Bootstrapping yield curve from swap rates:
$$Z\left(t, T_1\right) = \frac{1}{1 + \frac{c(t, T_1)}{n}}$$

    while for $i = 2, ..., M$
$$Z\left(t, T_i\right) = \frac{1 - \frac{c(t, T_i)}{n} \times \sum_{j=1}^{i-1} Z\left(t, T_j\right)}{1 + \frac{c(t, T_i)}{n}}$$

- Caps and Floors Cash

  A cap pays a stream of payments at $T_i$, $i = 1, .., n$ with $T_{i+1} = T_i + \Delta$, where

$$CF\left(T_{i+1}\right) = \Delta \times N \times \max\left(r_n\left(T_i, T_{i+1}\right) - r_K, 0\right)$$

  A floor pays

$$CF\left(T_{i+1}\right) = \Delta \times N \times \max\left(r_K - r_n\left(T_i, T_{i+1}\right), 0\right)$$

- Put-Call Parity (at the money caps and floors):

$$\text{cap} = \text{floor} + \text{swap}$$

- Payoff to a swaption at maturity:

$$\text{Payoff of Payer Swaption} \quad = \quad \sum_{j=i^*+1}^{n} Z\left(T_{i^*}, T_j\right) \Delta N \max\left(c\left(T_{i^*}, T\right) - r_K, 0\right)$$

  where $c(T_{i^*}, T)$ is the swap rate at time $T_{i^*}$ to a swap that matures at time $T$.

- Binomial Models
  - Hedging:
    - Hedge Portfolio:

$$\Pi_0 = V_0 - \Delta Z_0\left(j\right)$$

    - Hedge Ratio:

$$\Delta = \frac{V_{1,u} - V_{1,d}}{Z_{1,u}\left(j\right) - Z_{1,d}\left(j\right)}$$

    - No arbitrage price of portfolio:

$$\Pi_0 = Z_0\left(1\right)\Pi_{1,u} \text{ or, equivalently, } \Pi_0 = Z_0\left(1\right)\Pi_{1,d}$$

  - No arbitrage pricing:
    - Any expectation:

$$V_0 = Z_0\left(1\right) E^{\tilde{p}}\left(V_1\right) - \Delta \times \left\{Z_0\left(1\right) E^{\tilde{p}}\left[Z_1\left(j\right)\right] - Z_0\left(j\right)\right\}$$

    - Reward for risk is equalized across securities:

$$\frac{E^{\tilde{p}}\left[\frac{V_1}{V_0}\right] - \frac{1}{Z_0(1)}}{\left[\frac{V_{1,u}}{V_0} - \frac{V_{1,d}}{V_0}\right]} = \frac{E^{\tilde{p}}\left[\frac{Z_1(j)}{Z_0(j)}\right] - \frac{1}{Z_0(1)}}{\left[\frac{Z_{1,u}(j)}{Z_0(j)} - \frac{Z_{1,d}(j)}{Z_0(j)}\right]}$$

* Present values with risk correction:

$$V_0 = Z_0\left(1\right)\left\{E\left[V_1\right] - \lambda\left[V_{1,u} - V_{1,d}\right]\right\}$$

where

$$\lambda = \frac{E\left[\frac{Z_1(2)}{Z_0(2)}\right] - \frac{1}{Z_0(1)}}{\left[\frac{Z_{1,u}(2)}{Z_0(2)} - \frac{Z_{1,d}(2)}{Z_0(2)}\right]}$$

- Risk-neutral probabilities set $\lambda$ to zero

$$\pi^* = \frac{Z_0\left(j\right)/Z_0\left(1\right) - Z_{1,d}\left(j\right)}{Z_{1,u}\left(j\right) - Z_{1,d}\left(j\right)}$$

$$\boxed{V_0 = Z_0\left(1\right)E^*\left[V_1\right]}$$