**Fixed Income Asset Pricing**
**Bus 35130 Spring 2024**
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**Homework 5**
Due at the beginning of Class 6

**Note 1:** For each section below, there are questions that require a pencil and paper (PP) answer, and questions that require actual computations using data and computer programs (CP). You are supposed to do both.


**Note 2:** As with past homework assignments there are "guides" for doing the homework in Excel, Matlab and Python. In each code provides partial solutions to the questions. To make the code run you are required to complete some formulas or to produce some of the results yourself. You are not required to use any of the guides, but use of one of them is recommended.
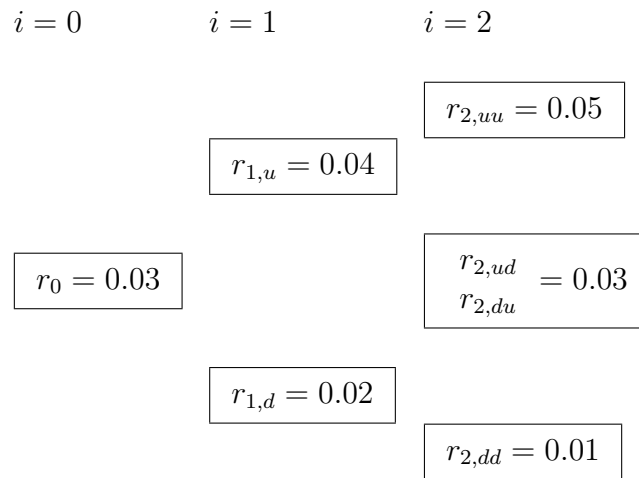
**Part I. Using Black's formula for Caps, Floors, and Swaptions**

1. (CP) Attached are two data screens from Bloomberg. Figure **??** quotes out to 30 years. In using this data use the mid-point quotes. The second, figure **??**, provides data on implied flat Black forward-volatility for Caps and Floors and implied Black volatility for Swaptions. This screen also provides the 1-month LIBOR rate needed for the exercise. (There may be some slight inconsistency in the quotes in the figures because they were not extracted from Bloomberg at exactly the same time.) Compute:

   - The dollar value of a two-year Cap (quoted caps are at-the-money, and therefore the strike rate equals the current swap rate). The only think you need to enter in the code is the proper volatility and swap rates (along with 3 month LIBOR) to produce this price and the swaption price below.

   - The dollar value of a 1-year swaption to enter into a 5-year swap.

2. (PP) In the exercise above, make sure to illustrate the option-pricing formulas used for (a) the first caplet in the cap (i.e. the one with 6-months to maturity); and (b) the swaption. Show your work. (Look at the code and match it to the notation from the notes class).

**Part II. Interest Rate Trees**

1. (PP) Consider the following interest rate tree, where each time interval is 1-year (that is, $\Delta = 1$).

$$i = 0 \qquad\qquad i = 1 \qquad\qquad i = 2$$

$$\boxed{r_{2,uu} = 0.05}$$

$$\boxed{r_{1,u} = 0.04}$$

$$\boxed{r_0 = 0.03} \qquad\qquad \boxed{\begin{array}{l} r_{2,ud} \\ r_{2,du} \end{array} = 0.03}$$

$$\boxed{r_{1,d} = 0.02}$$

$$\boxed{r_{2,dd} = 0.01}$$

The probability of an "up" movement can either be

$$p = 40\% \quad \text{or} \quad p = 70\%$$

You also know the current value of a 2-period zero coupon bond is

$$Z_0(2) = 0.94$$

(a) Find the risk neutral probability $\pi^*$ (see TN4, page 21) at time 0.

(b) You have to compute the value of a one-year call option on the interest rate $r_1$ at the strike rate $r_K = 3\%$ and notional $N = 1000$, that is, with payoff

$$\text{payoff}_{1u} = 1000 * max(r_{1u} - r_K, 0) \quad \text{in the "up" node; and}$$
$$\text{payoff}_{1d} = 1000 * max(r_{1d} - r_K, 0) \quad \text{in the "down" node}$$

Use the risk-neutral probability $\pi^*$ in previous point to compute the price for this option.

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(c) Does it matter which probability $p$ is the true one (i.e. whether $p = 40\%$ or $p = 70\%$)? Why?

(d) Find the portfolio of securities that replicates the payoff of the option in point (b). What is the value of this portfolio at time 0? How does it compare with the value of the option computed in point (b)? Show your work and comment.

(e) Assume the risk neutral probability $\pi^*$ computed in point (a) is constant throughout the binomial tree. Use the risk neutral methodology to compute

  i. the value of a 3-year zero-coupon bond

  ii. the value of a 3-year coupon bond with annual coupon equal to 3%.

  iii. Compute a replicating portfolio at time 0 that replicates the payoff of the coupon bond at time $i = 1$. That is:

    • Define a portfolio with $N_0$ units of the 3-period zero-coupon bond $Z_0(3)$ and $N_1$ units of the 1-period zero $Z_0(1)$

$$P_0 = N_0 Z_0(3) + N_1 Z_0(2)$$

    so that $P_{1,u}$ and $P_{1,d}$ corresponds to the total value of an investor in the coupon bond would receive (remember that such an investor is entitled to the coupon at time $i = 1$ in both nodes).

    • Check that the replicating portfolio "replicates", that is, that indeed $P_{1,u}$ and $P_{1,d}$ corresponds to the payoffs of the coupon bond.

# Figure 1: Swap Rates



# Figure 2: Cap, Floor and Swap Volatilities