

# Assignment one: supervised learning

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# Task 1

## Introduction

The dataset used for this task is from Schularick and Taylor(2012, “Credit Booms Gone Bust”). It is an annual dataset covering 14 countries and 140 years. Among the variables they collected, the most important one is the yearly aggregate bank loans, which is the main source of predictive power. The variable of interest is the CrisisST which take a value of 1 when there is a financial crisis and 0 otherwise.

Following the guidance of Schularick and Taylor(2012), I explored the relationships between several macro variables within two eras of finance capitalism, tested the predictive power of different macro variables, and compared predictive power of different supervised learning methods. The last part of above is also inspired by the Fricke(2017, Financial Crisis Prediction: A Model Comparison“).

The methods used are logistic regression, classification tree, classification forest, and SVM. The major criteria used here is area under receiver operating curve. The secondary criteria is confusion matrix. The major validation method is the modified cross validation method as mentioned in the Fricke paper that takes time order into consideration.

## Data description and cleaning

To explore the changing features between two eras, firstly I created variables of interest: credit to GDP ratio, bank asset to GDP ratio, money to GDP ratio, credit to money ratio, and bank asset to money ratio. To see the distinctive trends in different historical periods, I regroup the whole dataset by years and take mean value of each variable each year. Plotting mean values of the ratios above against time, I recovered Figure 1 and 2 in Schularick and Taylor(2012).

Figure 1 shows that bank loans, bank asset and broad money supply remain steady related to the size of economy representing by GDP, before the WW2 period. After the war, the money to GDP ratio stays flat while the other two start to increase dramatically. In figure 2 we can see that the loan to money ratio and bank asset to money ratio start to take off after the destruction of WW2 implying that credit start to grow faster than broad money supply and no steady relationship between the two can be found in this period.

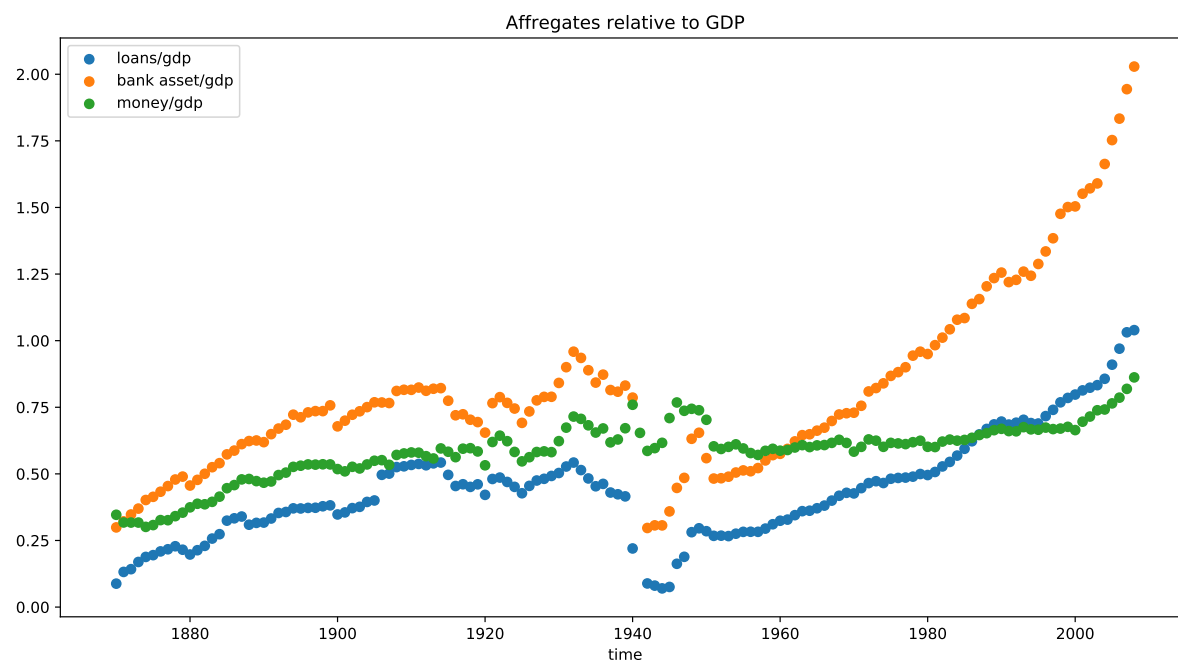


Figure 1: Aggregates relative to GDP (year effects)

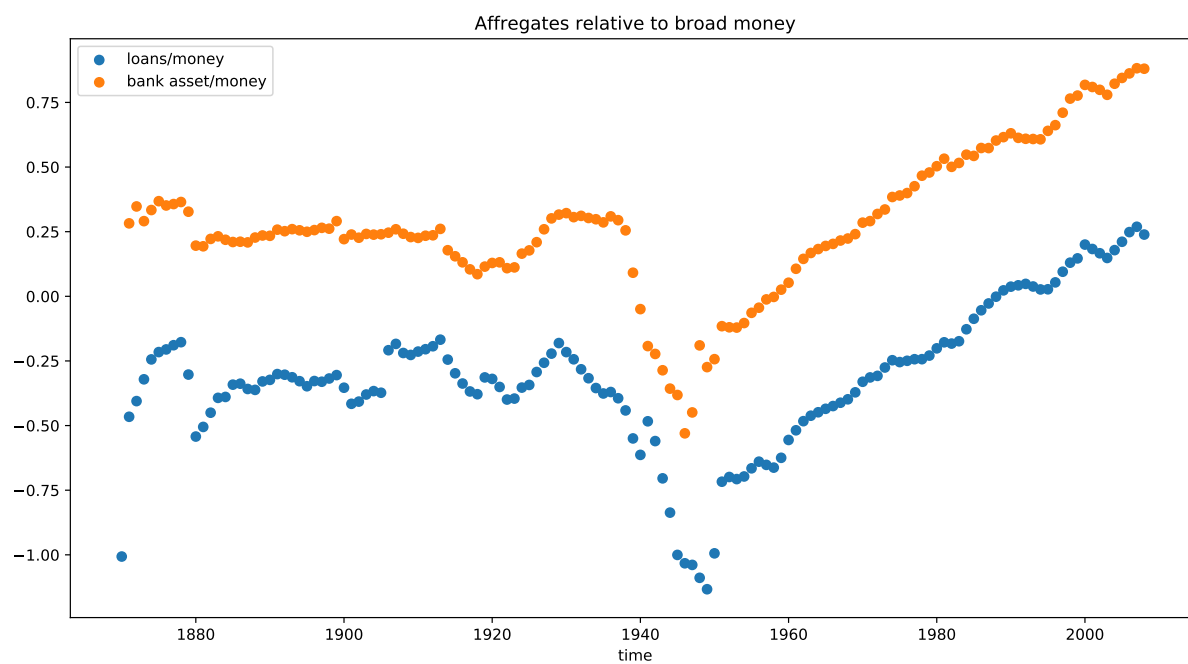


Figure 2: Aggregates relative to broad money (year effects)

To study financial crisis caused by the internal reasons from economic system, we need to exclude the crisis caused by the two world wars. As did in Schularick and Taylor paper, I excluded the war periods (1914 to 1919 and 1939 to 1947) and German crisis after WW1(1920 to 1925). Divided the cleaned up dataset into pre- and post-war periods, I recovered the upper panel of table 1 in the paper.

## Annual summary statistics pre- and post-war

	credit_to_GDP	bankAsset_to_GDP	money_to_GDP	credit_to_money	bank_asset_to_money
	Pre-war				
count	685	611	736	662	580
mean	0.408977	0.714051	0.533292	0.735337	1.282481
std	0.359888	0.447337	0.207534	0.449343	0.566104
	Post-war				
count	831	828	834	833	831
mean	0.546975	1.013497	0.645801	0.838012	1.575839
std	0.423878	0.668770	0.240497	0.494226	0.752540

More than half of the countries in the dataset are from Europe, so I also excluded all the observations in post WW1 period(1920 to 1925) since all the crises happened in that period could be caused by WW1 rather than economic system. I only kept the variables that have potentially strong predictive power according to the results and the robustness test from the paper.

After dropping any row that contains missing value, I have 1433 observations and 59 out of these are crisis events.

## Supervised learning methods for classification

### Logistic regression and choice of explanatory variables

As defined in Schularick and Taylor paper, I created CPI nomalized bank loan and take the difference of log values of this variable as the change of credit environment. This variable will be called credit change. I also take credit to GDP ratio as one potential explanatory variable base on robustness test of the paper. This variable lagged 1 year will be called credit size. After assign each country for each year its lagged 1 to 5 credit change and credit size, I sorted the dataset by time to make sure that when fitting a model, it is not trying to predict 1960s crisis with 1990s' data.

I started the analysis by using lag 1 to lag 5 credit change to fit a logistic regression model. The AUC is slightly higher than 0.5 for the whole dataset and for the pre-war dataset, and significantly higher than 0.5 for the post-war dataset. Since in the paper, lag 2 credit change is the only lagged variable that is significant, I also fitted logistic regression with only lag 2 data and the model fit slightly better for both whole set and pre-war period in respect of AUC. The change in post-war period is unambiguous and credit size seems add predictive power to the post-war period. The lag 2 credit change is indeed the main source of information.

#### In-sample AUC for logistic regression with lag 1 to 5 credit change

	whole set	pre-war	post-war
without credit size	0.5861360718870345	0.5762987012987013	0.7608543417366948
with credit size	0.5892169448010269	0.5892169448010269	0.615546218487395

Table 1: in-sample logistic regression trained with whole time period, pre-war period and post-war period with lag 1 to 5 credit change as major explanatory variable.

#### In-sample AUC for logistic regression with lag 2 credit change

	whole set	pre-war	post-war
without credit size	0.6373277827336704	0.6286424526999033	0.697533908754624
with credit size	0.4518906730102092	0.5177922018137459	0.7200369913686806

Table 2: in-sample logistic regression trained with whole time period, pre-war period and post-war period with lag 2 credit change as major explanatory variable.

I did a out-of-sample test for the choice of variable using 30% of the dataset as test set and 70% as training set. The results show that model fitted with lag 2 credit change have significantly better out-of-sample performance than model fitted with lag 1 to lag 5 and credit size doesn't seem to add predicting power to the model. The AUC are reported in the following tables.

#### Out-of-sample AUC for logistic regression with lag 1 to lag 5 credit change

	whole set	pre-war	post-war
without credit size	0.5861360718870345	0.5762987012987013	0.7608543417366948
with credit size	0.5892169448010269	0.5800865800865801	0.615546218487395

Table 3: out-of-sample logistic regression trained with 70% of each data set and tested on 30% of data with lag 1 to lag 5 credit change as major explanatory variable.

#### Out-of-sample AUC for logistic regression with lag 2 credit change

	whole set	pre-war	post-war
without credit size	0.7370988446726572	0.7976190476190476	0.7461484593837535
with credit size	0.531193838254172	0.6737012987012987	0.6355042016806722

Table 4: out-of-sample logistic regression trained with 70% of each data set and tested on 30% of data with lag 2 credit change as major explanatory variable.

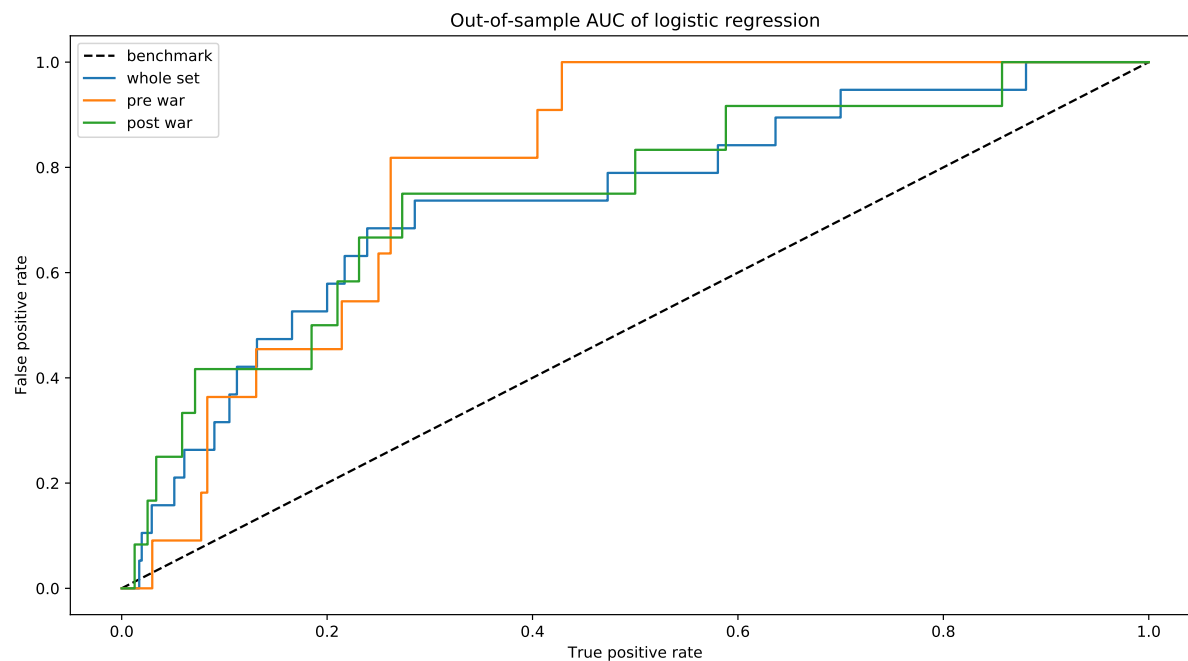


Figure 3: out-of-sample AUC of logistic fitted with 70% of data as training set and 30% as testing set. All three periods fitted with lag 2 credit change only.

Given result above, I only considered lag 2 credit change and credit size variables in the rest of this study. This also means all the models compared here will have same information as input and thus makes the comparison meaningful.

At last for logistic regression, I did a modified cross validation as mentioned in the Fricke paper. I divided the whole dataset into four equal folds. First, I use fold one to train and test on fold two. Then I use fold one and two to train and test on fold three, etc.. The average AUC is 0.56129 which is higher than 0.5. The reason for divide the dataset into 4 rather than 5 fold like did in the Fricke paper is that when the fold is too small, due to the sparseness of crisis events, there might be only one class in the whole training set or test set.

### Tree and forest

Next I fitted the data with a classification tree. With maximum depth equals to 3, here are result of in-sample prediction. It is obvious from the table that credit size add on predictive power for classification tree at least for in-sample test.

**AUC for classification tree**

	whole set	pre-war	post-war
without credit size	0.6937568143522649	0.7030106338903466	0.780209617755857
with credit size	0.7598684210526315	0.7878746029553929	0.8358199753390876

Table 5: in-sample classification tree fitted with lag 2 credit change as major explanatory variables

From the AUC plot we can see that there are much less point on each line for the tree compared with logistic regression. This is because for logistic regression, each observation will have its own estimated probability, but for a tree, the observations belong to the same leaf will have the same probability.

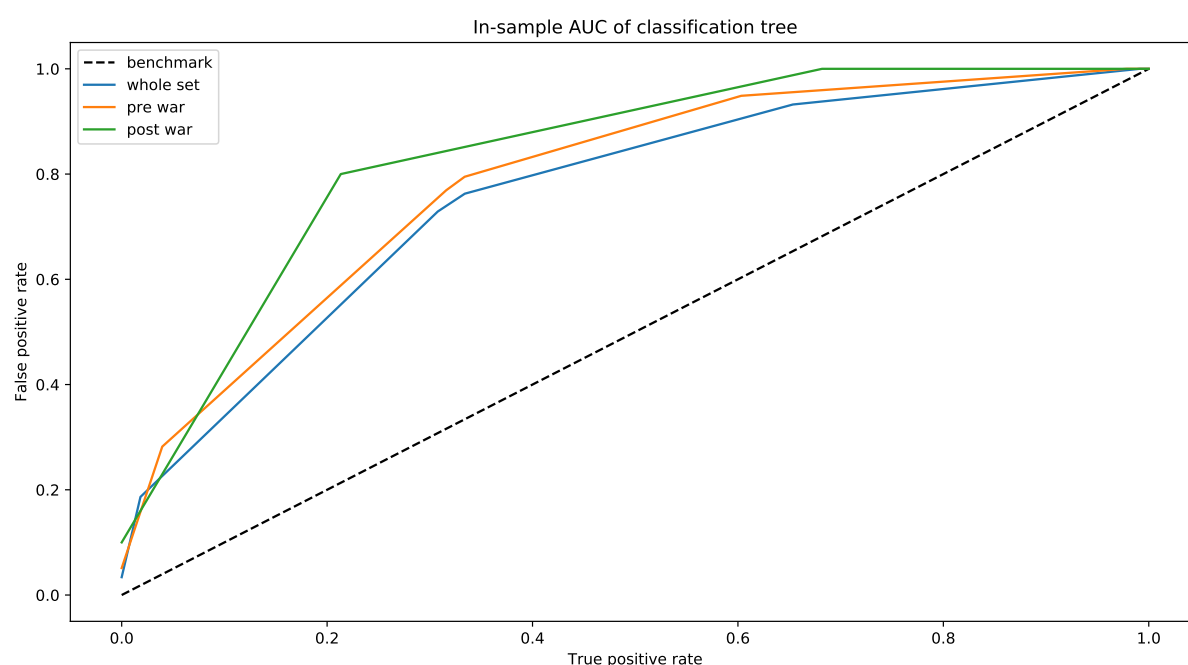


Figure 4: AUC of classification tree fitted with whole time period, pre-war and post-war period. All three period fitted with lag 2 credit change and credit size.

For the choice of maximum depth, we need a good balance between fully use all the information and avoid over fitting. To find the maximum depth that gives highest AUC for each tree, I performed an analysis with following steps: 1> for each of whole set, pre-war, and post-war period, set up a modified cross validation of 5 fold as mentioned in logistic regression part with certain maximum depth and record the average AUC for each model. 2> collect average AUC of each model for maximum depth from 2 to 50. 3> plot the average AUC against maximum depth for each model and pick up the depth that coincide with the highest AUC.

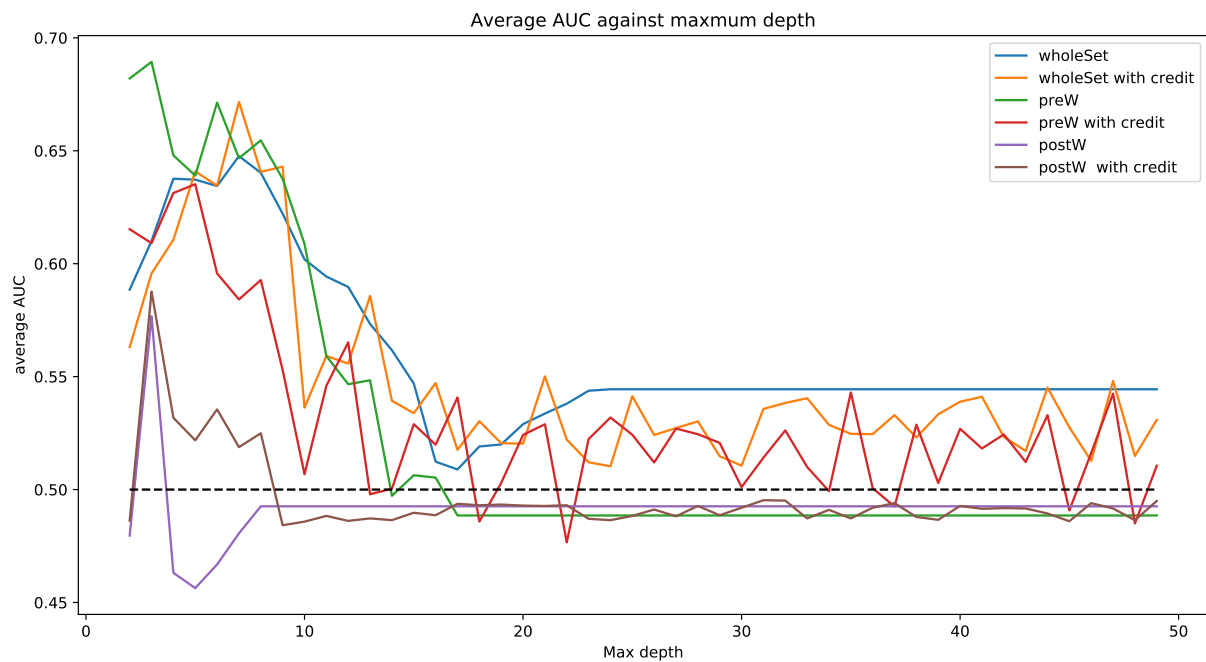


Figure 5: Average AUC for the whole, pre- and post-war dataset fitted with and without credit size plot against maximum depth

With optimal max-depth 3, 3, 5 respectively, I fitted 70% training set of whole period and pre-war dataset with lag 2 credit change and post-war with lag 2 credit change and credit size. With 30% of the total data as a test set, these models show AUC 0.62895, 0.69021 and 0.43697 respectively.

The dramatically different performance between in-sample and out-of-sample test is expected and indicates overfitting. Given a high enough max-depth, classification tree can achieve AUC 1.0 in an in-sample test. But overfitting will lead to very poor out-of-sample prediction. This is indicated by the flattening out in the figure above. Two trees with different maximum depth in the appendix demonstrate this point.

With exactly same idea, I performed analysis with classification forest model. The in-sample performance is better than the tree with same max-depth which in general indicate higher predictive power. However, forest is also vulnerable to overfitting.

**AUC for classification forest**

	whole set	pre-war	post-war
without credit size	0.7419033105362275	0.7823735211526954	0.9169852034525278
with credit size	0.7619994548518189	0.8678359342632234	0.8669852034525277

Table 6: in-sample classification forest fitted with lag 2 credit change as major explanatory variable.

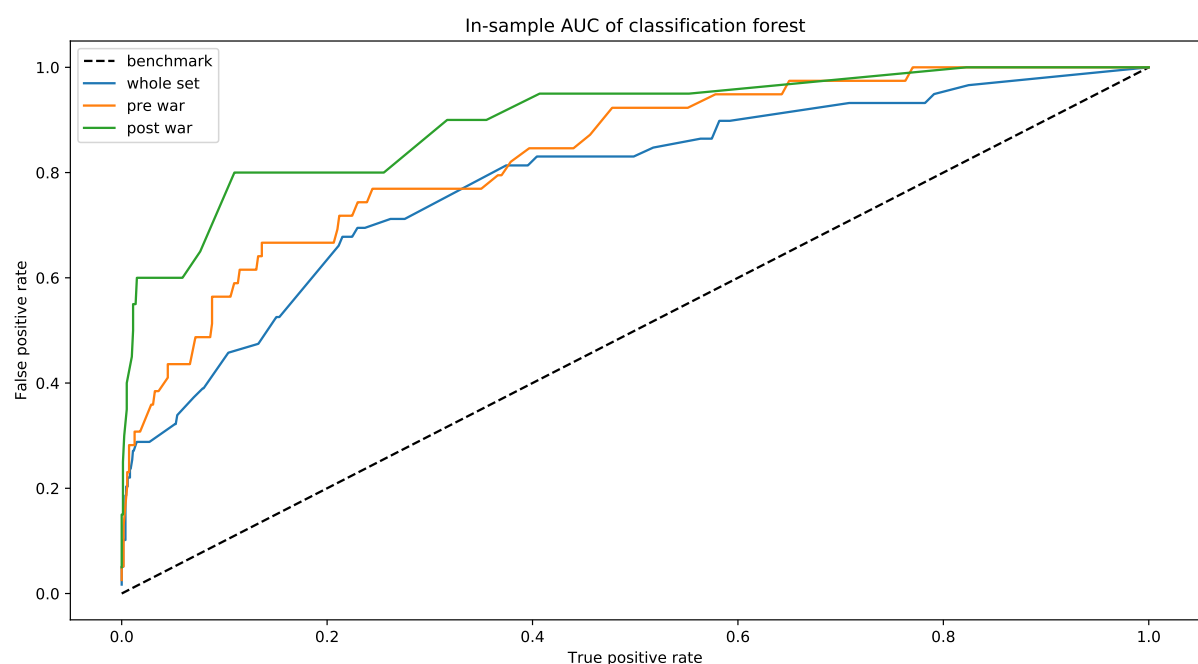


Figure 6: AUC of classification forest fitted with whole time period, pre-war and post-war period. First two period fitted with lag 2 credit change and credit size and post-war fitted with lag 2 credit change only.

With optimal max-depth picked up by the same method as in the tree analysis, datasets are fitted with lag 2 credit change and/or credit size. Tested with 30% data in the dataset, the AUC are 0.62426, 0.73593, and 0.71709 for whole, pre-war and post-war data.

Tree and forest learners perform brilliantly in-sample and are not much more impressive than logistic regression in out-of-sample test. Due to the advantage of averaging multiple trees, and reduce overfitting and variance, forest performs slightly better than tree method.

In both tree and forest analysis, I used gini index and entropy and there are little difference between the AUC.

## SVM

When fitting SVM model with the data, I used two kinds of kernels: rbf(Gaussian kernel) and sigmoid kernel. The results are both affected by randomness when fitting the model and sigmoid kernel in general has better out of sample performance.

## A few interesting facts and potential explanations

When taking a closer look at the results of tree and forest, I found a few interesting facts. First, for both tree and forest, I found the results are different when fitting models with exactly same data set and parameters, indicating randomness in the fitting procedure. Second, for the models fitted with both lag 2 credit change and credit size, the average AUC show fluctuation in the over fitting zone when plotted with maximum depth. In the plot of forest, all models show fluctuation in the over fitting parts but for models with two explanatory variables, the fluctuations have higher amplitudes. Last, for some trees and forests, the out of sample AUC do not drop to 0.5 even in the obviously over fitting zone.

To answer the first question, one needs to identify the source of randomness. I found two potential sources for tree and three for forest by reading the document. One obvious candidate is the random start point when searching for the optimal arguments to minimize cost function. However, as long as the problem is convex, the

searching results should be with in a relatively small interval with possible difference due to limited resolution. This doesn't match the observation. An argument called `max_features` in tree is defined as "The number of features to consider when looking for the best split(sklearn.tree.DecisionTreeClassifier document page)". When fitted with two variables, the model with max feature setted to 1 shows jumps while model with max features setted to 2 does not. For the forest, there is one extra source of randomness. To grow multiple trees, the sklearn library bootstrap observations using random selection with replacement. In both tree and forest functions from sklearn library, the argument `random state` controls all the randomness.

Carrying knowledge mentioned above, I try to decompose the fluctuations. Fixing random state eliminates fluctuation in AUC-max-depth plot of both tree and forests. For the forest, I first turn off the max feature limit, and the resulting plot doesn't show less fluctuation. Next, I add the limit on max feature and turn off bootstrap, and this reduced the fluctuation. And eventually, when I turn off both limit and bootstrap, the fluctuation was almost totally eliminated. These changes indicates that the forest fitting is extremely sensitive to the chang of training dataset. Another interesting observation here is that the AUC of model fitted with pre-war data and with both credit change and credit size as explanatory variables dropped below 0.5 benchmark after the limit and bootstrap was turn off. I cannot explain this change. Relavent plots are in the appendix.

I notice that the model that has predictive power even when over fitted are models fitted with the whole dataset. This is ture for both tree and forest analysis(with randomness turn off in forest). From the plot of AUC of trees fitted with different datasets, I notice that there is only one point in each of the three line. This indecates that in the out of sample test, only one split, presumably the first split take effect, most likely due to over fitting. The whole set has the largest number of observations, it is likely that even when over fitted, the first split still has some predictive power. Here is one exampl of AUC plot when over fitted. This also explains the same observation in the forest analysis.

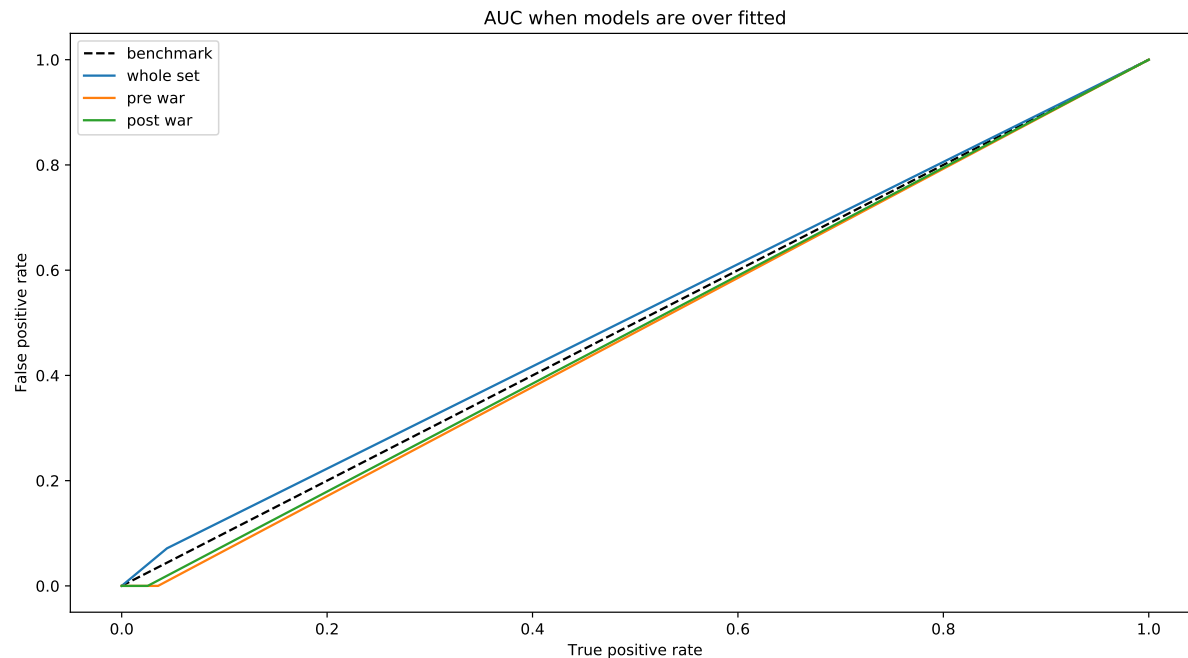


Figure 7: All three models are fitted with lag 2 credit change and credit size. These models are fitted with 80% of dataset and tested on 20% of set. To eliminate effect of randomness, the random state argument was fixed when fit these models.

## Another criteria

Except for AUC, a few other values from confusion matrix can also be good criterias for model comparison. In the tables below, I collected some out-of-sample results for logistic regression and forest fitted with 70% of whole data set regarding confusion matrix. False alarm is defined as  $M01 / (M01 + M11)$ , and can be interpreted as when model say crisis, how much of that are miss classified. Total flag is the percentage of total observation that has been flagged by the model as crisis.

To capture 10% of the crisis, logistic regression need to flag 2% of the total observation, and 80% of those flags are false alarm. On the other hand, forest need only flag less than 1% of data and only half of those are false alarm. However, if the goal is to capture half of the crisis, logistic regression only need to flag less than 20% of data with 87% false alarm while forest need to flag half of the data point and more than 90% of the flag are false alarm. The main idea here is that there is no sigle best criteria and the criteria selection should be based on the goal of analysis.

threshold	sensitiveity	falseAlarm	totalFlag
0.95845	0.05263	0.87500	0.01865
0.95842	0.10526	0.80000	0.02331
0.95829	0.15789	0.80000	0.03497
0.95817	0.21053	0.84000	0.05828
0.95814	0.26316	0.83333	0.06993
0.95805	0.31579	0.86047	0.10023
0.95801	0.36842	0.86000	0.11655
0.95799	0.42105	0.85185	0.12587
0.95796	0.47368	0.85714	0.14685
0.95793	0.52632	0.87179	0.18182
0.95791	0.57895	0.88172	0.21678
0.95790	0.63158	0.88119	0.23543
0.95789	0.68421	0.88288	0.25874
0.95786	0.73684	0.89313	0.30536
0.95773	0.78947	0.92823	0.48718
0.95767	0.84211	0.93701	0.59207
0.95764	0.89474	0.93885	0.64802
0.95759	0.94737	0.94098	0.71096
0.95745	1.00000	0.95000	0.88578

Table 7: Out-of-sample performance for logistic regression based on confusion matrix. Logistic regression fitted with 70% of the whole period and tested with 30% of whole period. Explanatory variable is lag 2 credit change only. AUC is 0.73709.

threshold	sensitiveity	falseAlarm	totalFlag
0.31088	0.05263	0.00000	0.00233
0.15587	0.10526	0.50000	0.00932
0.05381	0.15789	0.90909	0.07692
0.04785	0.36842	0.89394	0.15385
0.04160	0.73684	0.93665	0.51515
0.03808	0.84211	0.93822	0.60373
0.03177	0.89474	0.93885	0.64802
0.01070	0.94737	0.95000	0.83916
0.00829	1.00000	0.95571	1.00000

Table 8: Out-of-sample performance for forest based on confusion matrix. Forest fitted with 70% of the whole period and tested on 30% of the whole period. Variable is lag 2 credit change only, max depth is 3. The AUC of this model is 0.67079.

## Conclusion and potential further questions to answer

In this study, I implemented a few supervised learning models on the data. The major predicting power lies in lag 2 credit change and this conclusion alines with Schularick and Taylor paper. Predictive power of models are much stronger in sample. This confirms the result of Fricke paper. Tree and forest are particularly vulnerable to over fitting thus require carefully picked parameters like maximum depth or maximum number of leaves. They are also quite sensitive to the change of training set. Another issue for tree and forest is that when there is a dominating class in the training data, the model generated could be biasd. This issue can be reduced by balance the data prior fitting or assign similar weight to different classes. However, these are not explored in this report and can be interesting topic for futher study. Since forest shows great protential to predict crisis, and machine learning methods are not widly use in economic research, solving this issue could have practical meaning. All the models tested in this report show some level of predictive power. However, there is no single standard to judge which is the best model. The selecting criteria strongly depends on the purpose of the analysis and different models may have adventages in different tasks.

References

[1] Moritz Schularick & Alan M. Taylor *Credit Booms Gone Bust: Monetary Policy, Leverage Cycles, and Financial Crisis, 1870-2008*.  
[2] Daniel Fricke *Financial Crisis Prediction: A Model Comparison*.

Appendix

Figure 8: nomal depth tree

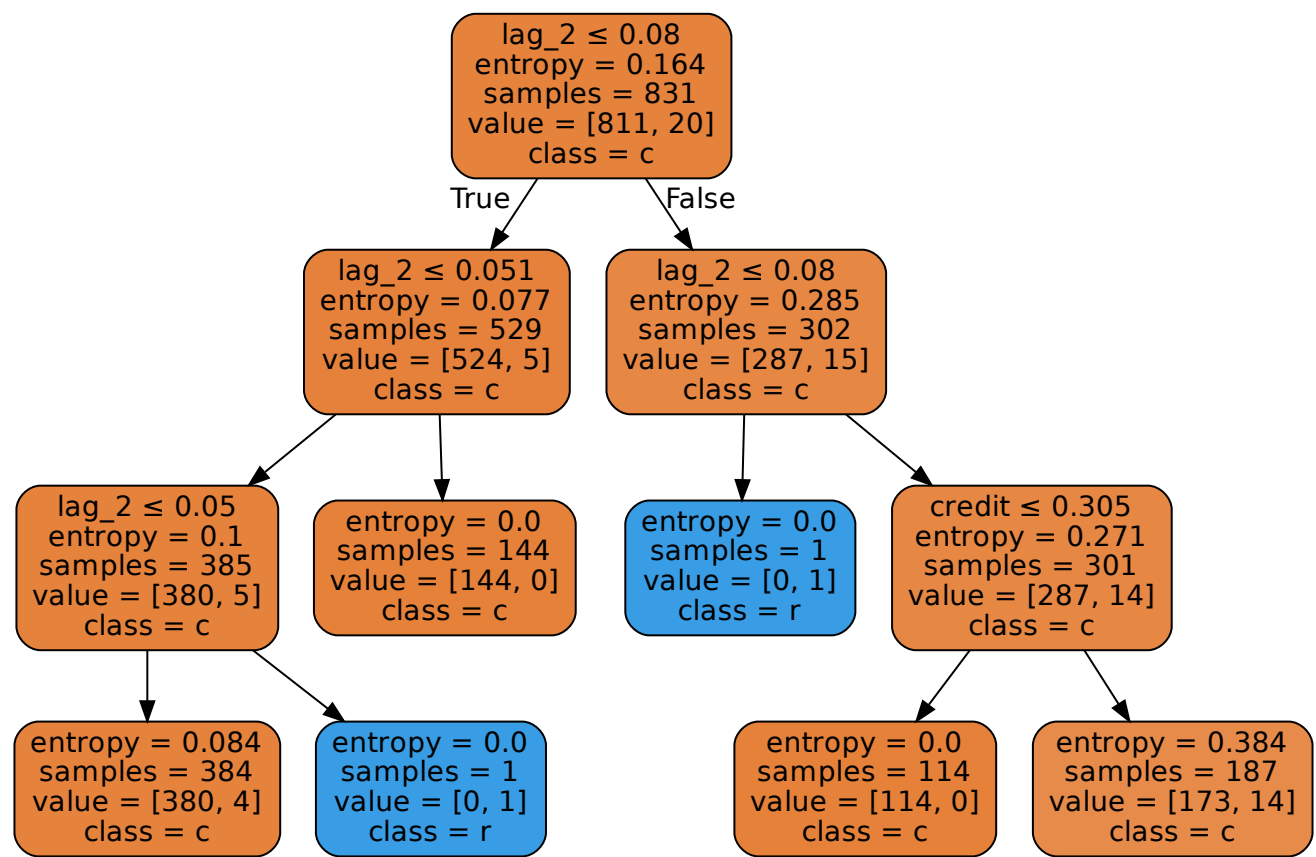




Figure 9: over fitting tree

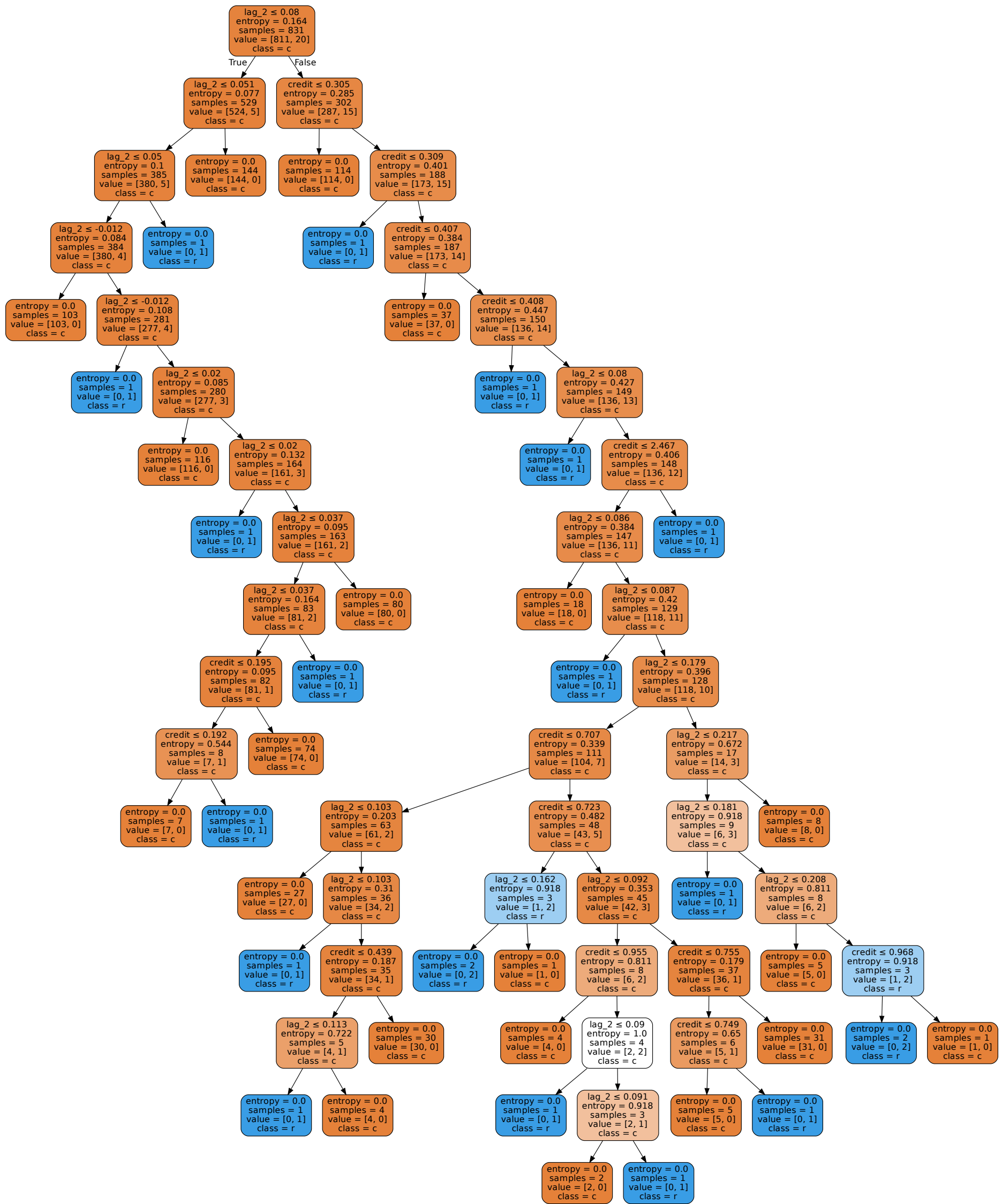


Figure 10: different trees generated by same data with limited max feature. We can see there is a tie in the cost function values.



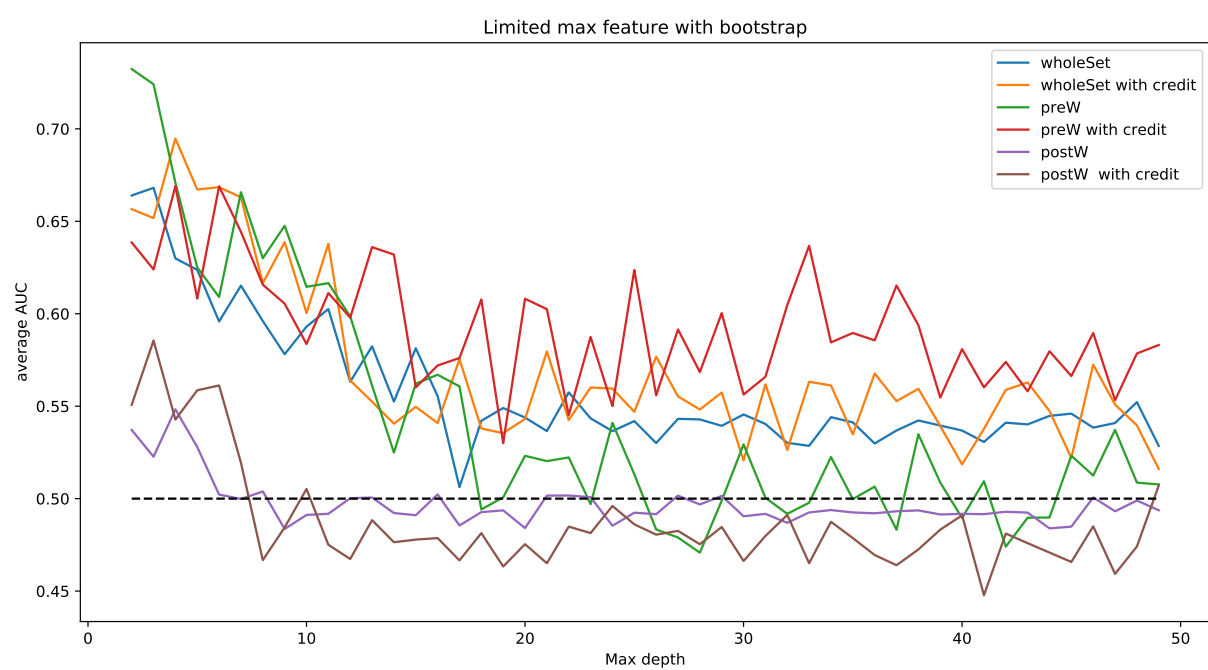
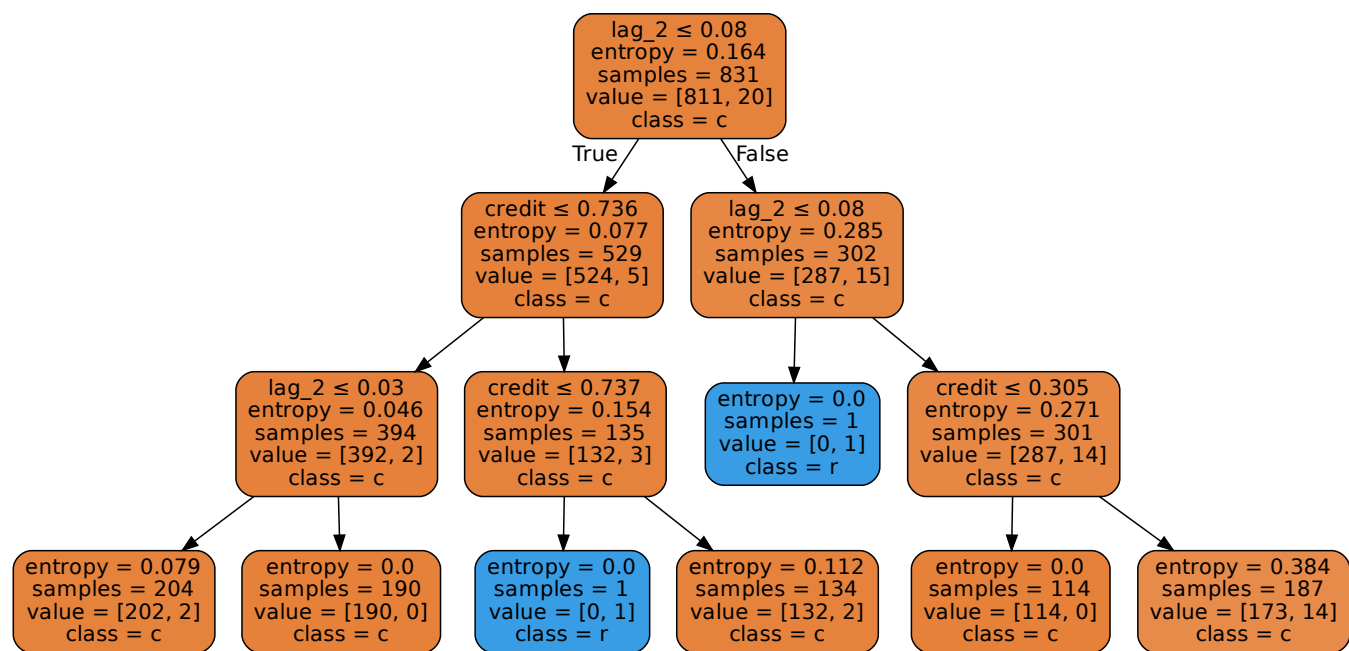
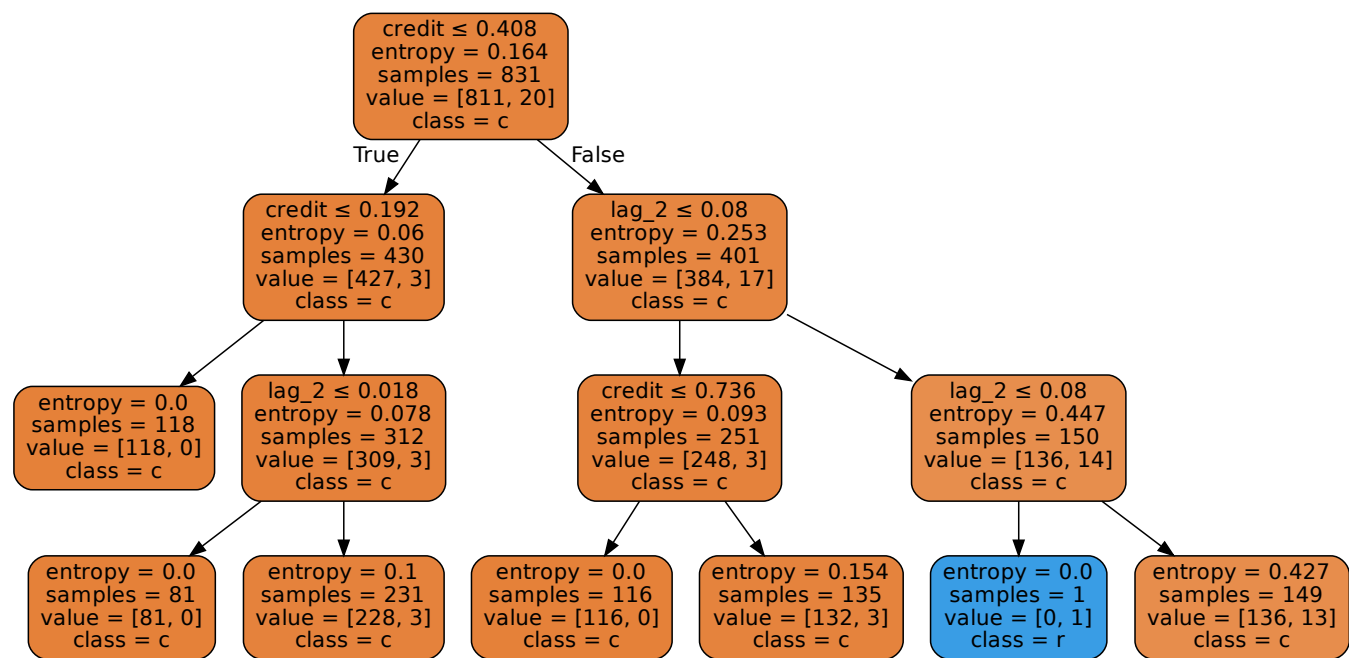


Figure 11: The maximum feature is limited and bootstrap is on, this is exactly the same plot used to choose optimal depth. We can see a lot of fluctuation here and at least three AUC do not approach 0.5 benchmark.

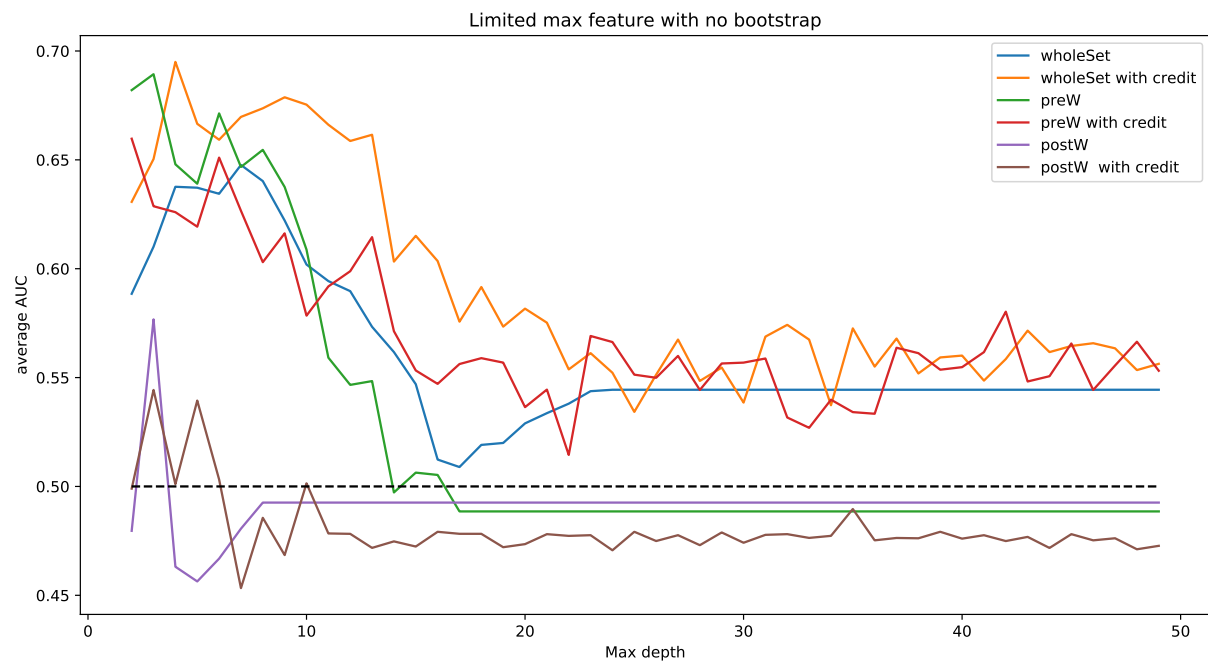


Figure 12: The maximum feature is limited and bootstrap is off, and we can see that fluctuation is reduced.

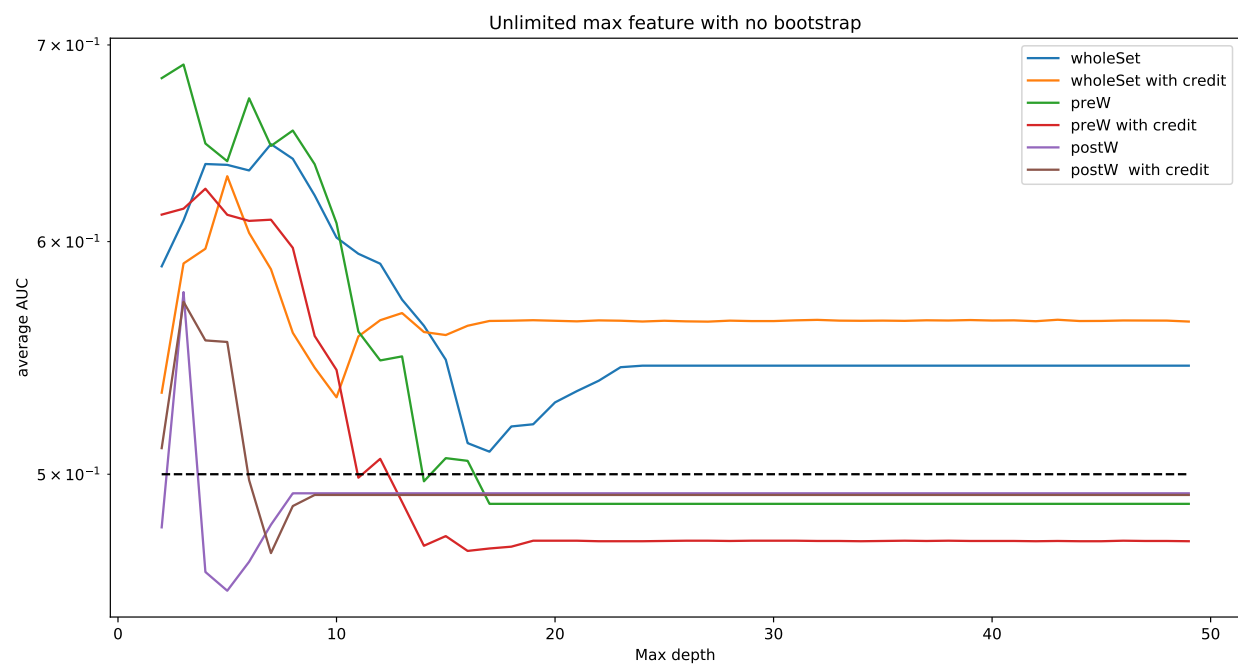


Figure 13: The maximum feature is unlimited and bootstrap is off, and we can see dramatical reduce of fluctuation and only two lines do not approach 0.5 benchmark.

## Task 2

### Introduction

Markovitz portfolio theory has always be a classic. However it is known that the optimal portfolio is not stable bringing issue on frequently rebalancing for investors. Following the lead of Fastrich, Paterlini, and Winker (Constructing optimal sparse portfolios using regularization methods, 2014), this study try to find global minimum variance portfolio under regularizations thus stablize the portfolio choice and reducing the rebalancing problem.

The regularizations used in this study are Lasso and Ridge. Resulting global minimum variance portfolios has lower variance than Markovitz portfolios and therefore confirmed the results from paper montioned above.

### Data description and cleaning

The data used is French 48 industry portfolios daily data. After comparing variance of each industry and of the whole dataset, I choose to use equal weighted data since it has higher variance and hopefully the different among regularizations will be more obvious. After deleting any row that contains missing value, I have about 44 years data left which is a good amount.

In figure 1, I plotted standard diviation of all 48 industies and we can see that except for a few outliers, most industries standard diviation of returns lies in a interval between 0.75 to 1.5.



Figure 1: std of 48 industries show that except for a few, all industries has std with a steady interval

Next I plotted mean returns for each of 48 industries together with minimum and maximum returns of each industry. The maximum returns seem to have more fluctuation than the minimum.

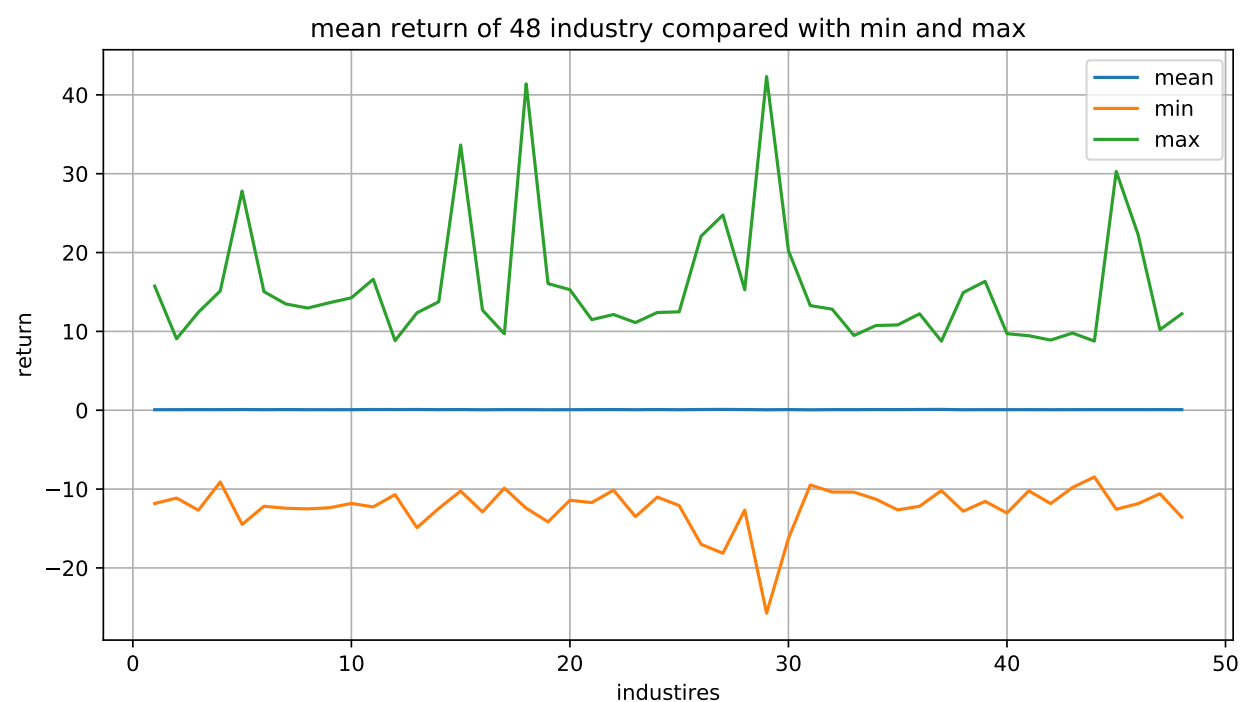


Figure 2: mean, minimum and maximum returns of each industry

The industry that has highest variance is the coal and utilities has the lowest variance. Here is a plot to demonstrate that.

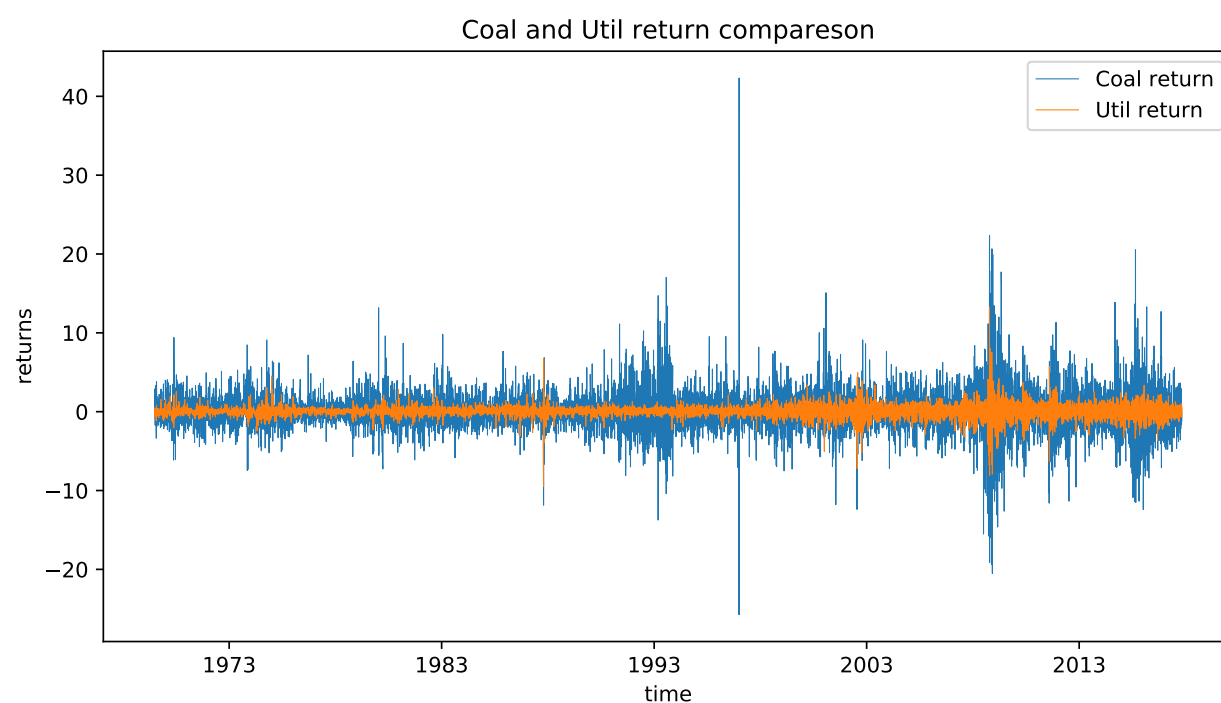


Figure 3: Variances of coal industry and utility industry.

## Global minimum variance portfolio with different regularizations

### Three methods to select portfolios

In this study, I used three methods for selecting minimum variance portfolios. The first one is Markovitz portfolio selection. This is a problem to minimize the variance of the portfolio under the constraint that the sum of all the weights equals to 1. I also performed selection for long-only portfolios to use together with long-short portfolios as benchmarks. To find the long-only portfolios, the minimization problem needs an extra constraint that all the weights are greater than or equal to 0. The second method I used is a Markovitz portfolio under Lasso regularization. This is a minimization problem that minimizes the sum of variance and a 1-norm penalty:

$$\mathbf{w}^T \Sigma \mathbf{w} + \lambda \sum_{n=1}^K |w_n|,$$

where  $\mathbf{w}$  is weight vector,  $\mathbf{\Sigma}$  is the covariance matrix and  $K$  is the number of stocks in your investing universe. Parameter  $\lambda$  controls how much penalty to add. To find a portfolio, one need to solve this minimizing problem under the same constrain that sum of weights equals to one. The other method in this report is a Markovitz under Ridge regularization. In this minimizing problem, the target function is a sum of variance and 2-norm penalty:

$$\mathbf{w}^T \mathbf{\Sigma} \mathbf{w} + \lambda \sum_{n=1}^K |w_n|^2,$$

and solving this under same budget constraint give a minimum variance portfolio under Ridge regularization.

### Selection $\lambda$ with cross validation

If  $\lambda$  equals to 0, both regularized method collasps to classic Markovitz. On the other hand, if  $\lambda$  is too large, which means the penalty term in the target function will dominate the variance term and thus impose to much bias. To select optimal  $\lambda$ , I performed a cross validation as did in the Fastrich paper. This cross validation is only meaningful under stationary assumption. First, I shuffle the whole training set and divide it into 10 equal folds. In each round, I take one of these 10 folds as test set and the rest 9 folds as training set. Then, in each round, for each value in a series of  $\lambda$  that I want to test, solve the minimizing problem under relative constraint and thus find a portfolio. Hold this portfolio in the test set and recieve a return series. Calculate the standard deviation of this return series since our aim is to find minimum variance portfolio. After 10 round, one will have 10 standard deviation for each  $\lambda$ , and taking mean of these gives the score of that  $\lambda$ . The optimal  $\lambda$  is the one with the lowest score. And a good searching interval should give a smile shape curve when plot all the score against  $\lambda$ . If the optimal  $\lambda$  is on the edge of searching interval, a change of interval is required to find a true minimum score. In the next to figure, I plotted scores against  $\lambda$  for a Lasso penalty and a Ridge penalty.

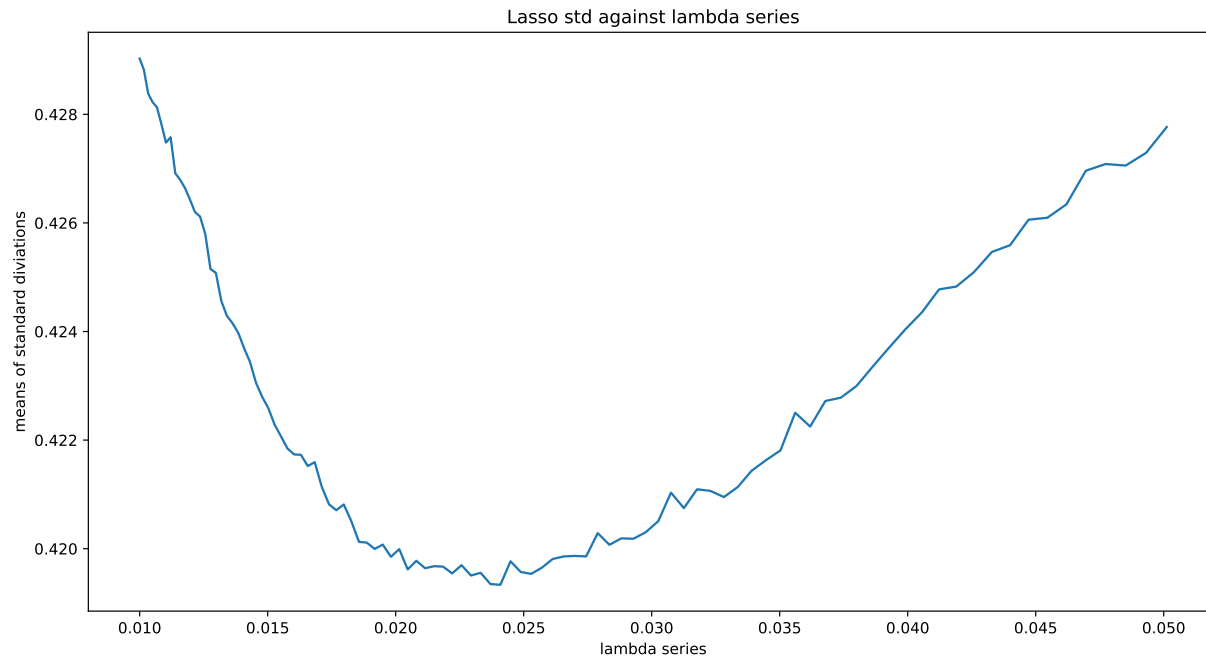


Figure 4: portfolio std with Lasso regularizaton. Training period 1 year. Searching interval  $10^{-2}$  to  $10^{-1.3}$  and searching steps 100.

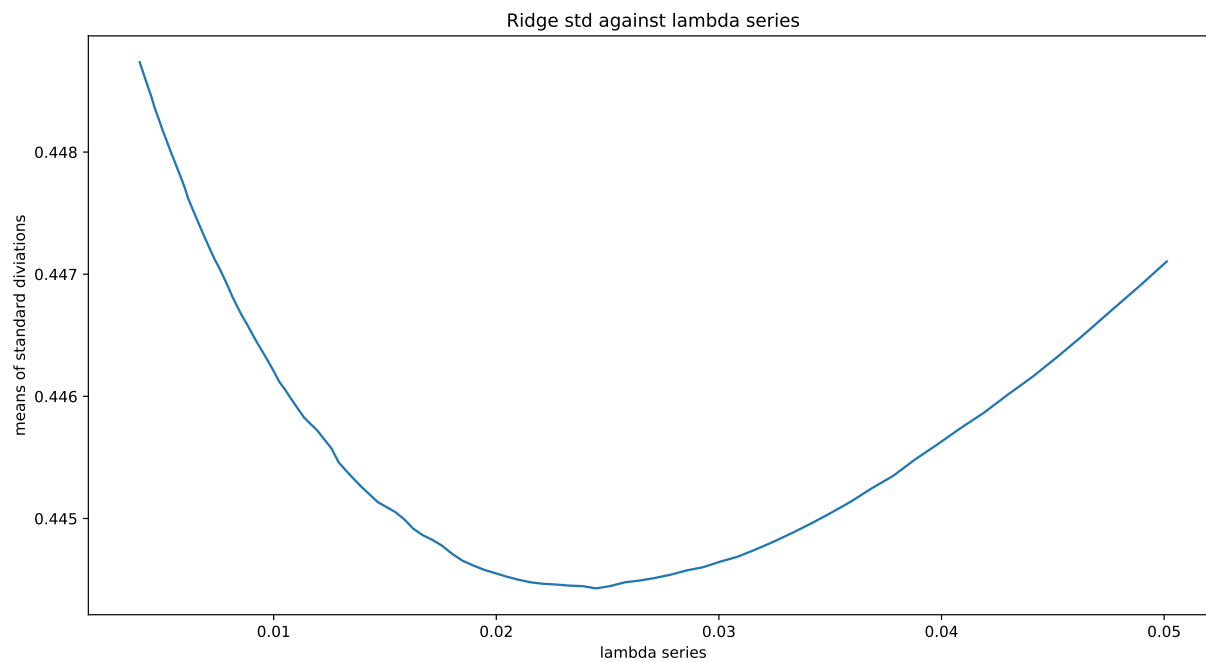


Figure 5: portfolio std with Ridge regularizaton. Training period 1 year. Searching interval  $10^{-2}$  to  $10^{-1.3}$  and searching steps 100.

Since there is randomness involved in the shuffling step, to find the reasonable searching interval, I run cross validation for both regularizatons multiple times to make sure all the optimal values are with in the interval. When using on the whold data set, if the optimal solution is on the edge, the program will through a warning, shift both boundaries of interval to the coresponding direction, and perform another search.

### Out-of-sample test

Out-of-sample tests are performed on the data with vary training windows and holding periods. In the plot below is return series trained on a one year window and hold for 3 months.

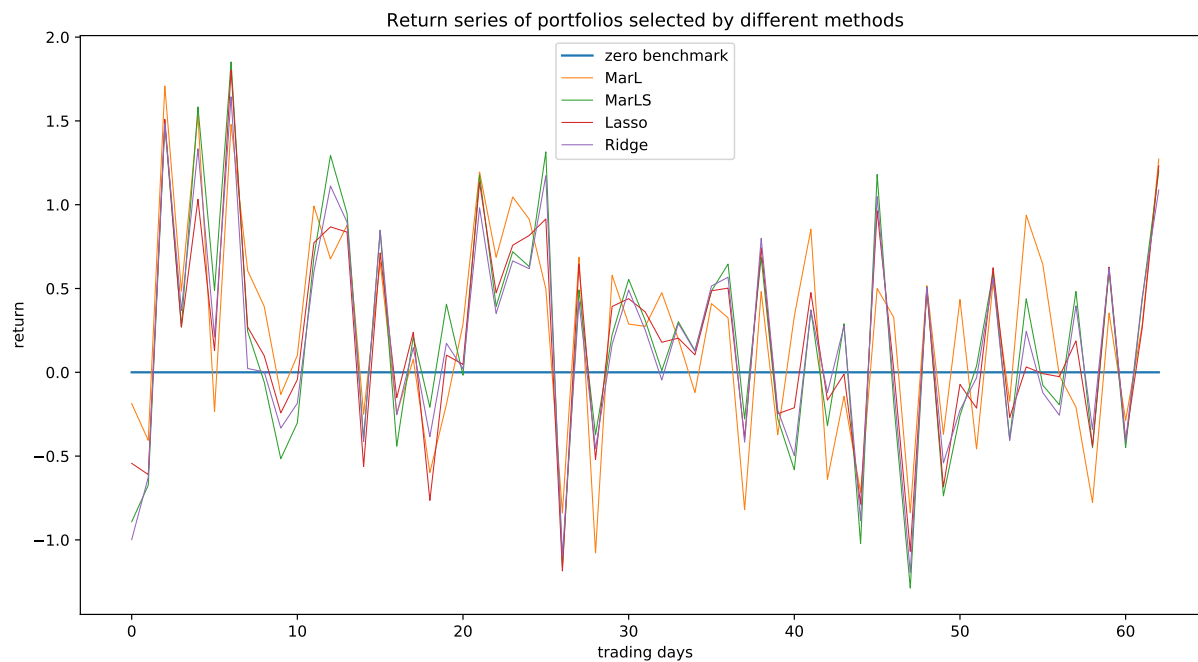


Figure 6: return series trained on 250 days and hold for 63 days. Cross validation interval, Lasso:  $10^{-2.1}$  to  $10^{-1.2}$ , Ridge:  $10^{-2.3}$  to  $10^{-1.3}$ . Searching steps: 100 for both.

The variance of portfolios are 4.5392, 3.9004, 3.6182 and 3.3498 in percentage respectively for Markovitz long-short, Markovitz long-only, Lasso, and Ridge. As expected, portfolio found under regularizations have lower variance during the holding period. This is due to the barrier set by penalty term blocks the noise in the data, thus in the variance covariance matrix, to enter the results freely. A small surprise here is that with more constraint, the long-only portfolio has lower variance than the long-short portfolio. This is not consistent across all the back tests I performed, but it indicates the instability of Markovitz solution.

Next, I performed a series of out-of-sample tests with moving windows. I tested different training window to see whether having longer training window affects any methods. I also test different holding periods to see whether some methods perform better when the selected portfolios were rebalanced more often. Since Fastrich paper only used 5 to 6 years of data, and considering running time, I decided to run these tests on the last 5 years from the whole data set.

#### Out-of-sample test result

Training window (days)	Holding window (days)	window number	Markovitz long-short	Markovitz long-only	Lasso	Ridge
500	63	11	3.5831	4.3738	3.4346	3.4377
250	63	15	3.4866	4.3458	3.2998	3.2627
125	63	17	4.1919	4.3288	3.4325	3.4712
250	125	8	3.9520	4.5060	3.6145	3.7762
250	63	15	3.4866	4.3459	3.2840	3.2682
250	21	47	3.3769	4.1968	3.1406	3.1526

Table 9: Out-of-sample test with moving windows. Data use are the last 1250 observations of the data set

From the whole table, we can see that the result is robust to the change of training table and holding period. In all the tests cross various training period and holding length, the moving window out-of-sample tests demonstrate that regularized portfolio constantly out perform the Markovitz portfolio. Among two bench marks, long-only portfolio has higher variance because it cannot take advantage of shorting thus has less ability to diverge the risk. Between Lasso and Ridge, there is no clear result about which one provide portfolios with lower variance. The first three row of the table seems to indicate that shorter training period make the regularized portfolio out perform even more. This is not enough to evidence to conclude, but one potential reason could be that in a short training period, the true information in the covariance matrix is more mingled with noise and thus the Markovitz method has more difficulty to select real optimal portfolio. And because of the penalty term, regularized methods has a more restrict standard for only the significant information to pass through, thus allow them to out perform the Markovitz even more. For the changing of holding period, there is no clear result about whether a certain length of holding benefit a particular method.

At last, I performed a out-of-sample test on the whole 44 years data. Resulting variance are 2.1290, 3.5056, 2.0711, and 2.1074 for Markovitz long-short, Markovitz long-only, Lasso, and Ridge. As usual, portfolios under regularizations out performed both benchmarks. The order of variances for the four portfolios and magnitude of differences match the results of Fastrich, Paterlini and Winker paper.

Both Fastrich paper and Brodie paper mentioned sparseness as another feature as portfolio chosen by regularized Markovitz methods, however, set threshold at 0, neither set of portfolios selected under regularizations show this feature. I did not explore more from here but it might related to the solver used. Also, a position with weight less than  $10^{-2}$  is practically non-active, so the sparse feature can also rely on the choice of threshold.

#### Conclusion and potential further questions to answer

Observed global minimum variance portfolios found under different regularizations, I confirmed that regularized portfolios out perform both Markovitz benchmarks in the out-of-sample tests thus found optimal portfolios has lower variance in various the holding periods. Markovitz solutions are not stable and contains noise from the data so regularization provides practical benefit to investors. Another benefit of regularizations proposed by both Fastrich paper and Brodie paper is it gives sparse portfolios thus reduce the transition cost. This is not observed here and can be a interesting further topic.

## References

- [1] B.Fastrich, S.Paterlini & P.Winker (2014) *Constructing optimal sparse portfolios using regularization methods*.
- [2] J.Brodie et al. (2009) *Sparse and stable Markovitz portfolios*.