MSiA 400 Lab Introduction to Markov Chain

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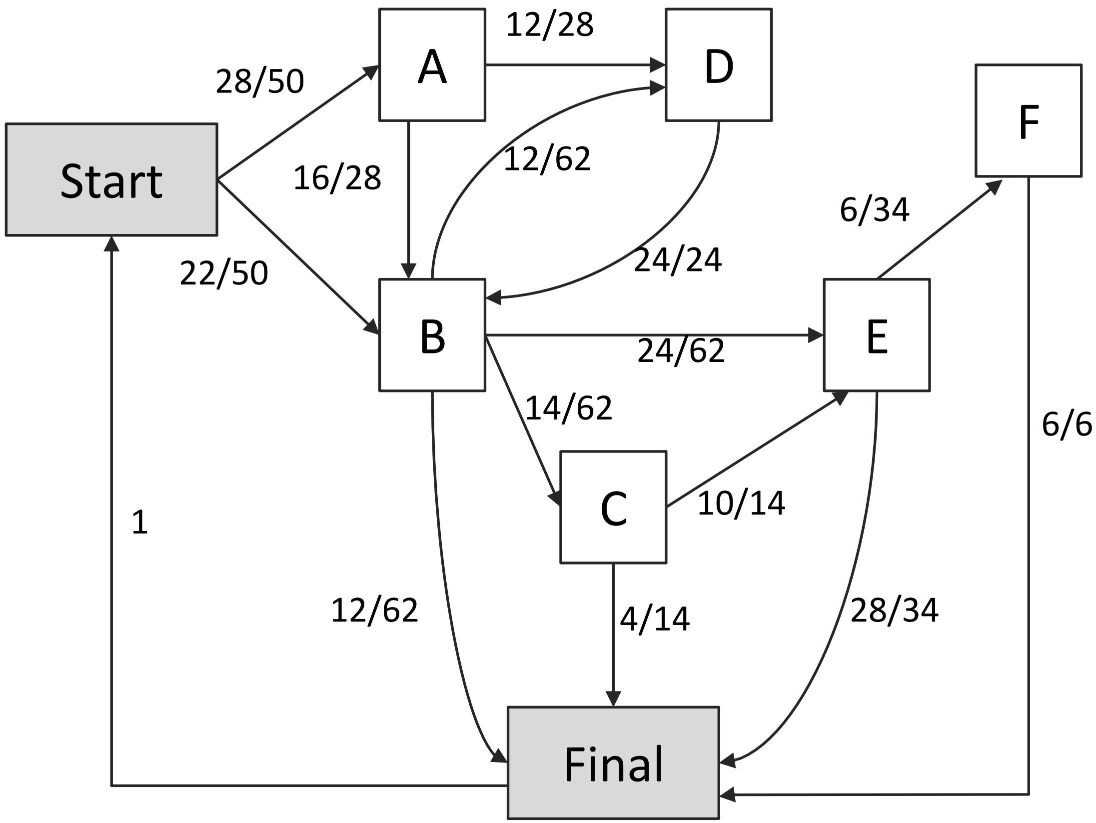
Example from Web Analytics

- Navigational trails as Markov Chain



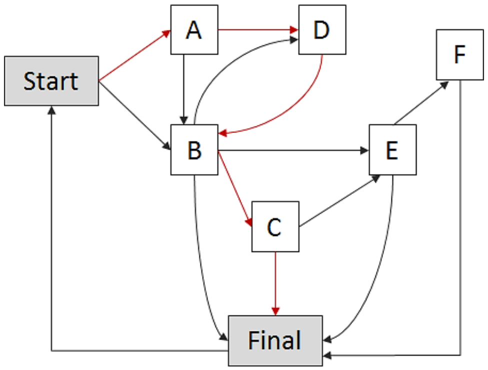
Definitions

- X_t : value of the system characteristic at time n
- Discrete time stochastic process $\{X_t : t = 0,1,2,3, \dots\}$
  - A sequence of random variables over time
  - A description of the relation between random variables $X_0, X_1, X_2, \dots$

Example

A page view history of user: A – D – B – C

$X_0 = start, X_1 = A, X_2 = D, X_3 = B, X_4 = C, X_5 = final$



- Markov property
  - The state at time $n + 1$ only depends on the state at time t

$$\mathrm{P}[X_{t+1} = i_{t+1} | X_t = i_t, X_{t-1} = i_{t-1}, \dots, X_1 = i_1] = \mathrm{P}[X_{t+1} = i_{t+1} | X_t = i_t]$$

Example

$$\mathrm{P}[X_5 = F | X_4 = C, X_3 = B, X_2 = D, X_1 = A] = \mathrm{P}[X_5 = F | X_4 = C\,]$$

- Markov chain is a stochastic process with Markov property

Definitions (Cont.)

- Stationary assumption
  - The transition probabilities are independent of time t

$$\mathrm{P}[X_{t+1} = j |\, X_t = i] = p_{ij} \text{ for any } t$$

Example

A page view history of user: A – D – B – D – B – C – F

$$\mathrm{P}[X_3 = B |\, X_2 = D] = \mathrm{P}[X_5 = B |\, X_3 = D] = p_{BD}$$

- Markov chain with Stationary assumption is stationary Markov chain

Transition Matrix

- Transition
  - The system moves from state i to state j
- Transition probabilities
  - Probability of transition from state i to j : $\mathrm{P}[X_{n+1} = j \,|\, X_n = i] = p_{ij}$
- Transition matrix

$$
P = \begin{array}{|cccccccc|}
\hline
p_{SS} & p_{Sa} & p_{Sb} & p_{Sc} & p_{Sd} & p_{Se} & p_{Sf} & p_{SF} \\
p_{as} & p_{aa} & p_{ab} & p_{ac} & p_{ad} & p_{ae} & p_{af} & p_{aF} \\
p_{bs} & p_{ba} & p_{bb} & p_{bc} & p_{bd} & p_{be} & p_{bf} & p_{bF} \\
p_{cs} & p_{ca} & p_{cb} & p_{cc} & p_{cd} & p_{ce} & p_{cf} & p_{cF} \\
p_{ds} & p_{da} & p_{db} & p_{dc} & p_{dd} & p_{de} & p_{df} & p_{dF} \\
p_{es} & p_{ea} & p_{eb} & p_{ec} & p_{ed} & p_{ee} & p_{ef} & p_{eF} \\
p_{fs} & p_{fa} & p_{fb} & p_{fc} & p_{fd} & p_{fe} & p_{ff} & p_{fF} \\
p_{Fs} & p_{Fa} & p_{Fb} & p_{Fc} & p_{Fd} & p_{Fe} & p_{Ff} & p_{FF} \\
\hline
\end{array}
$$

Transition Matrix

- Transition
  - The system moves from state i to state j
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  - Probability of transition from state i to j : $\mathrm{P}[X_{n+1} = j \mid X_n = i] = p_{ij}$
- Transition matrix

$$P = \begin{bmatrix} 0 & 28/50 & 22/50 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 16/28 & 0 & 12/28 & 0 & 0 & 0 \\ 0 & 0 & 0 & 14/62 & 12/62 & 24/62 & 0 & 12/62 \\ 0 & 0 & 0 & 0 & 0 & 10/14 & 0 & 4/14 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6/34 & 28/34 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

n-step Transition Probabilities

- We are interested in
  - Suppose we are at time k. If a Markov chain is in state i now, what is the probability that the Markov chain is in state j after n periods?

$$\mathrm{P}[X_{k+n} = j \mid X_k = i] = \mathrm{P}[X_n = j \mid X_0 = i] = p_{ij}(n)$$

  - $p_{ij}(n)$ is called n-step probability of a transition from state i to state j

- n-step transition probability $p_{ij}(n)$ = ij^{th} element of P^n

- Initial distribution a: probability distribution of initial state at time 0

- What is the probability of being in state j after n steps, given a?

  Example
  We are given a=(1,0,0,0,0,0,0,0).
  The probability of being in state B after three steps?

n-step Transition Probabilities with R

- Import web.txt from Blackboard (folder “Lab Nov 2” under Documents)

```
> web = read.table(“../web.txt", header=T);
> P = as.matrix(web);
```

- Calculate prob distribution after 3 steps given initial vector a=(1,0,0,0,0,0,0,0)

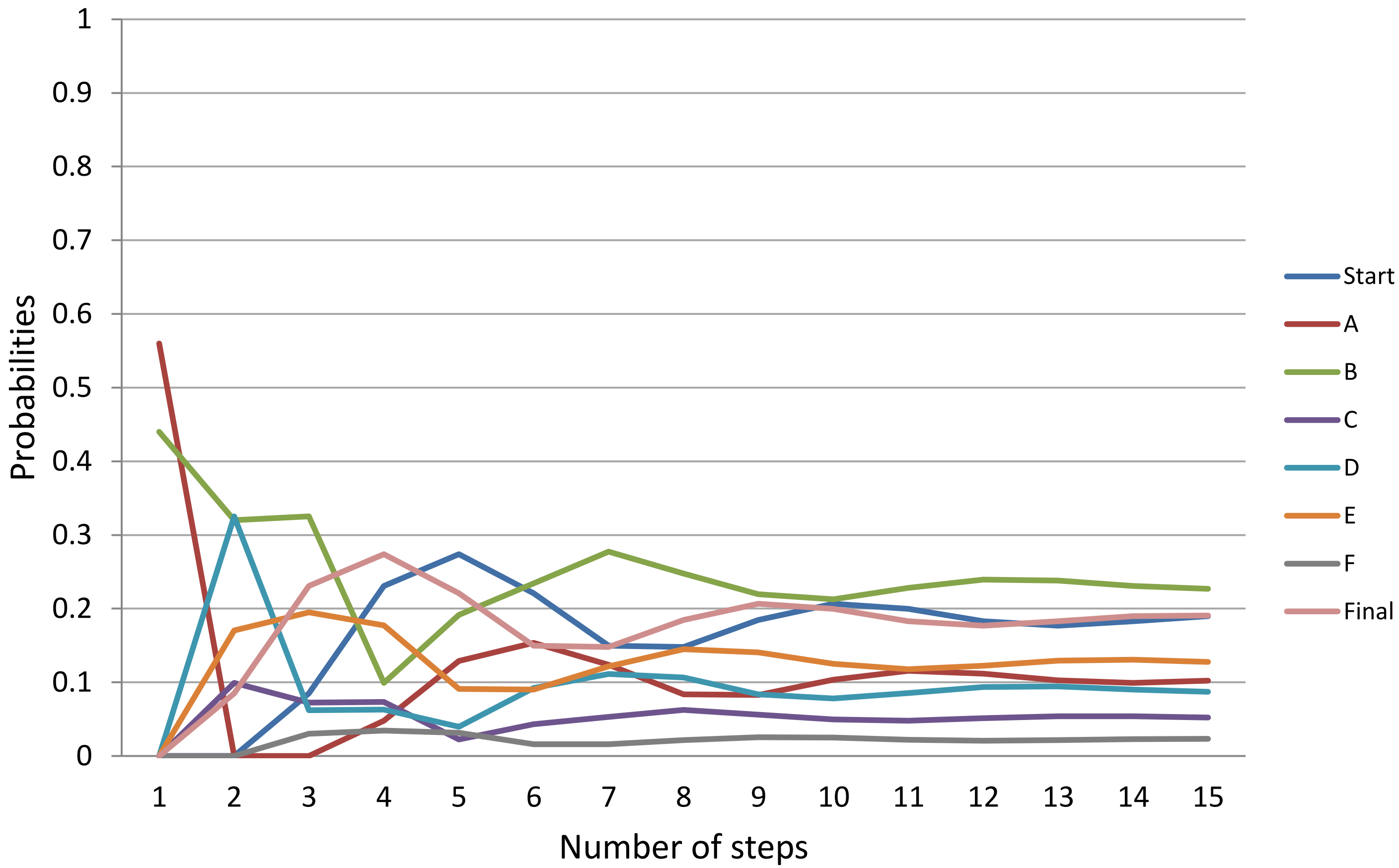
```
> a = c(1,0,0,0,0,0,0,0);
> a %*% P %*% P %*% P
```

Note

- For matrix multiplication, use %*% instead of *
- BTW, then what does * do?
- Or, there exists an alternative!

```
> library(expm);
> a = c(1,0,0,0,0,0,0,0);
> a %*% (P %^%3)
```

Example: n-step Transition Probabilities



Q: What do we observe?

Steady-state Probabilities

- Let P be the transition matrix for an s-state ergodic Markov chain. Then, there exists a vector $\pi = [\pi_1 \;\; \pi_2 \;\; \ldots \pi_s]^T$ such that

$$\lim_{n\to\infty} P^n = \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_{s-1} & \pi_s \\ \pi_1 & \pi_2 & & \pi_{s-1} & \pi_s \\ \vdots & & \ddots & \vdots & \\ \pi_1 & \pi_2 & \cdots & \pi_{s-1} & \pi_s \\ \pi_1 & \pi_2 & & \pi_{s-1} & \pi_s \end{bmatrix}$$

- Steady-state probabilities (Equilibrium)

$$\pi_j = \sum_{i=1}^{s} \pi_i p_{ij} \;\text{ for all } j$$

Calculation of π_j's

- Steady-state probabilities

$$\pi_j = \sum_{i=1}^{s} \pi_i p_{ij} \text{ for all } j \qquad (1)$$

$$\sum_{j=1}^{s} \pi_j = 1 \qquad (2)$$

- Calculation π_j's : Replace one row in (1) by (2), and solve for π_j's.

$$\pi_1 = \sum_{i=1}^{s} \pi_i p_{i1}$$

$$\vdots$$

$$\pi_{s-1} = \sum_{i=1}^{s} \pi_i p_{i,s-1}$$

$$\sum_{j=1}^{s} \pi_j = 1$$

Example

$$\begin{bmatrix} p_{11}-1 & p_{21} & p_{31} \\ p_{12} & p_{22}-1 & p_{32} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Let Q be the matrix obtained by replacing the last row of $(P^T - I)$ by a vector of 1's. Then,

$$Q\pi = [0,0,\dots,0,1]^T$$

Calculation of π_j's with R

- Consider the example from Web Analytics

- Let Q be the matrix obtained by replacing the last row of $(P^T - I)$ by a vector of 1's. Then,

$$Q\pi = [0,0, \ldots , 0,1]^T$$

$$\pi = Q^{-1}[0,0, \ldots , 0,1]^T$$

- Calculation of π_j using R
  hint1 diag(n) creates identity matrix with size n
  hint2 solve(M) returns inverse of matrix M

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- Calculation of π_j using R

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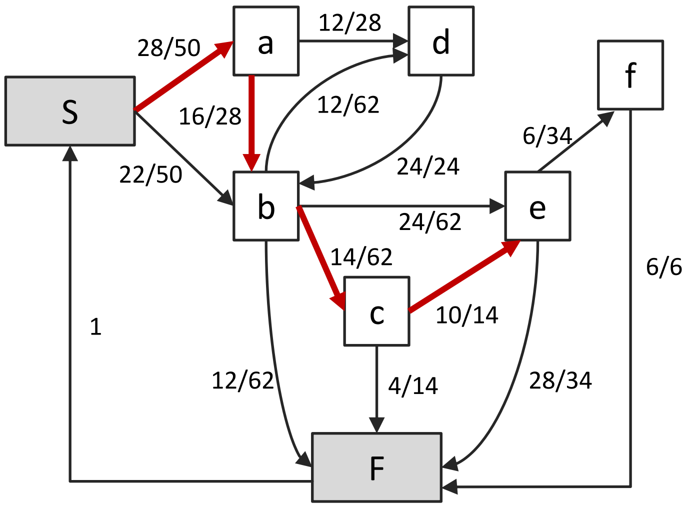
  hint2 solve(M) returns inverse of matrix M

```
> Q = t(P) – diag(8);
> Q[8,] = c(1,1,1,1,1,1,1,1);
> rhs = c(0,0,0,0,0,0,0,1);
> Pi = solve(Q) %*% rhs;
```

First Passage Time

- Consider a path: Start – A – B – C – E
- How many steps to get to state E? What is the probability that this path happens?

$$P_{S-a-b-c-e} = P_{S,a}P_{a,b}P_{b,c}P_{c,e} = \left(\frac{28}{50}\right)\left(\frac{16}{28}\right)\left(\frac{14}{62}\right)\left(\frac{10}{14}\right)$$



	S	a	b	c	d	e	f	F
S	0	28/50	22/50	0	0	0	0	0
a	0	0	16/28	0	12/28	0	0	0
b	0	0	0	14/62	12/62	24/62	0	12/62
c	0	0	0	0	0	10/14	0	4/14
d	0	0	1	0	0	0	0	0
e	0	0	0	0	0	0	6/34	28/34
f	0	0	0	0	0	0	0	1
F	1	0	0	0	0	0	0	0

- What is the expected number of steps to get to state E?

E[# steps to state E] = $\sum_{all\ path\ to\ E} length(path) Prob(path)$??

Mean First Passage Time

- Mean first passage time
  Mean first passage time m_{ij} is the expected number of transitions before we first reach state j, given we are currently in state i.

- Derivation

$$\boxed{m_{ij} = \underbrace{p_{ij}(1)}_{\substack{\text{Move to state } j \text{ in} \\ \text{the next period}}} + \underbrace{\textstyle\sum_{k \neq j} p_{ik}(1 + m_{kj})}_{\substack{\text{Move to state } \mathrm{k} \neq j \text{ in} \\ \text{the next period}}}}$$

$$\Big\downarrow \quad p_{ij} + \textstyle\sum_{k \neq j} p_{ik} = 1$$

$$\boxed{m_{ij} = 1 + \textstyle\sum_{k \neq j} p_{ik} m_{kj}}$$

Mean First Passage Time in Matrix Form

- Consider the example from Web Analytics
- Mean first passage time to state j

  $m_{ij} = 1 + \sum_{k \neq j} p_{ik} m_{kj}$ for each i

- Matrix form: first passage time to state Start
  - Let us define
    - B: sub matrix of P obtained by deleting the row and column corresponding to the state Start
    - m: vector of m_{ij}'s, $i, \neq Start, j = Start$
    - e: vector of 1's
  - Then, we obtain $m = e + Bm$ in matrix form
  - Hence, the mean first passage time can be calculated by

$$m = (I - B)^{-1} e$$

Mean First Passage Time with R

- Mean first passage time

$$m = (I - B)^{-1}e$$

- Calculation of $m_{i,Start}$ using R

```
> B =P[2:8,2:8];
> Q = diag(7) - B;
> e = c(1,1,1,1,1,1,1);
> m = solve(Q) %*% e;
```

Note

- m is a vector containing the mean first passage time to Start from i=A,B,…
- Hence, the output right contains the mean first passage time to Start from all other pages

Output

```
        [,1]
A 4.988571
B 3.560000
C 2.840336
D 4.560000
E 2.176471
F 2.000000
Final 1.000000
```

- What if we want to calculate $m_{i,A}$ or to other states?

```
> B =P[-2,-2];                    // deleting second row and column of P
```