Problem 2: Flash Floods

1 Modelling Flash Flooding and the Flood Hydrograph

In this section we seek to formulate a simple model for the formation of a flash flood in order to help us understand why this surprising and dangerous phenomenon occurs. The conventional approach to the problem might be to try and solve the nonlinear Navier-Stokes equations in 3D along the length of the river. This is an extremely difficult numerical task complicated by the fact that nearly all river flows are in reality turbulent (and contain debris) and so involve phenomena occuring not only on the scale of the river but also down to sub-centimetre lengthscales. The vast disparity in the lengthscales involved, ranging from the sub-centimetre up to the length of the river, which may well be many tens of kilometres, make this an almost insurmountable task. In order to circumvent these difficulties we adopt an engineering type approach to the problem which is based on experimentally-derived laws for turbulent flows in rivers and channels.

As an aside, we note that turbulent flows occur at high Reynolds number Re, which is defined as

$$Re = \frac{\rho U L}{\mu}, \tag{1}$$

where ρ is the density of the fluid, μ is its viscosity, L is a typical lengthscale and U a typical velocity of the flow. Typically turbulence sets in for Reynolds numbers above a critical value which depends on the flow configuration, which is typically in the range 2,000-5,000. At smaller Reynolds numbers, the flow is *laminar* (meaning it is well-ordered). For example, laminar flow in a channel has a central fast moving region and much slower flow towards the edge of the channel. This corresponds to experience that fast flows (large U) and large-scale flows (large L) tend to be turbulent and chaotic whereas slow or small-scale flows are much more well-organised.

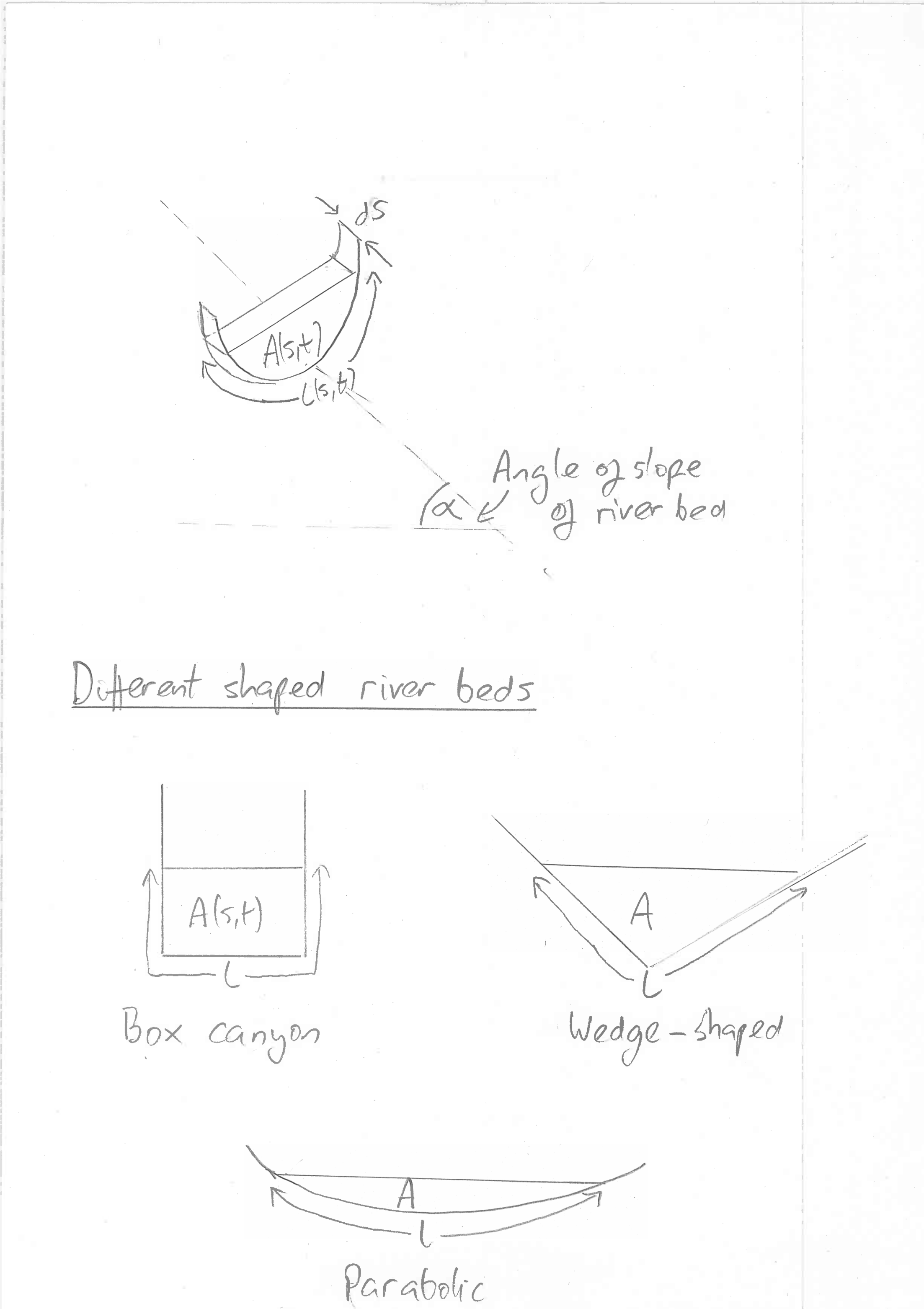

Figure 1: (a) Depiction of an infinitessimal section of the river bed illustrating the definitions of $A(s,t)$, $l(s,t)$, ds and α. (b) Typical river bed cross-sections for which model equations could be derived.

Variables and parameters in the problem. We start by considering a typical cross-section of a river bed. The key variables we will use in our investigation are:

$$\begin{array}{ccc} \text{Time} & t & \text{(Independent Variable)}, \\ \text{Distance Along River} & s & \text{(Independent Variable)}, \\ \text{Cross-sectional area of bed filled with water} & A(s,t) & \text{(Dependent Variable)}, \\ \text{Flow rate of water down river bed} & Q(s,t) & \text{(Dependent Variable)}, \\ \text{Area-averaged fluid velocity} & \bar{u}(s,t) & \text{(Dependent Variable)}, \\ \text{Perimeter of bed wetted by river} & l(A(s,t)) & \text{(Dependent Variable)}, \\ \text{Shear stress per unit area of river bed} & \tau(s,t) & \text{(Dependent Variable)}, \end{array} \tag{2}$$

while the key parameters in the problem are

$$\begin{array}{ll} \text{Density of water} & \rho, \\ \text{Angle of inclination of river bed} & \alpha, \\ \text{Friction factor} & f, \end{array} \tag{3}$$

Incompressibility and a conservation equation. We start by writing down an equation for the conservation of water in the river bed. The assumptions we shall make are that:

- water is an incompressible fluid (*i.e.* a given volume of water retains the same volume as it flows)
- the water in the river bed is not absorbed into the bed, does not evaporate and is not augmented by additional inflows (due say to rain).

The first of these assumptions is almost exact however the second is not and we may well wish to relax it to allow, say for inflow from tributaries or water loss by absorption into the river bed. A standard conservation equation for water in the river bed can be readily derived by considering a thin cross-sectional slice of the river, illustrated in Figure 1(a), between s and $s + ds$ (where ds is vanishingly small). Then the volume of water dV contained in this section is given by

$$dV = A(s,t)ds.$$

The flow rate of water into this section at s is $Q(s,t)$, and the flow rate out at $s + ds$ is $Q(s + ds, t)$. The water is incompressible and so the rate of change of the volume of water in our little section of river bed must equal the flow rate into the section minus the flow rate out. It follows that

$$\frac{d}{dt}(dV) = Q(s,t) - Q(s+ds,t) \approx -\frac{\partial Q}{\partial s}(s,t)ds,$$
$$\Longrightarrow \frac{\partial A}{\partial t}(s,t)ds \approx -\frac{\partial Q}{\partial s}(s,t)ds,$$

and on cancelling the factor ds from both sides then rearranging we obtain the following PDE relating Q and A:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial s} = 0. \tag{4}$$

The flux Q of water down the river bed can be related to the area-averaged fluid velocity by the expression

$$Q(s,t) = A(s,t)\bar{u}(s,t),$$

so that we can now rewrite the conservation equation (4) in the form

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial s}\left(A(s,t)\bar{u}(s,t)\right) = 0. \tag{5}$$

A force balance and a constitutive equation for the fluid velocity $\bar{u}$. One of the advantages of dealing with turbulent flows, rather than laminar ones, is that the flow velocity through the cross section of the river is fairly uniform (although it fluctuates markedly over small space and timescales). This is in contrast to laminar flows which are slow near the edges of the channel and fast in the middle. For turbulent flows it is therefore possible to characterise the “skin friction” (or drag) exerted by the sides of the channel in terms of the area-average fluid velocity $\bar{u}$ in the channel and this is the approach adopted by hydraulic engineers. They experimentally measure skin friction as a function of $\bar{u}$ for a variety of possible channel surfaces. Their results show that the drag is proportional to $\bar{u}^2$ and they use this as motivation for the following phenomenological law for the shear stress τ on the walls of a channel

$$\tau = f\rho\bar{u}^2. \tag{6}$$

Here τ measures the friction force per unit area, in the direction opposite to the flow, exerted by the channel walls on the fluid; and f is a friction factor that depends on the properties of the channel wall and channel geometry. If $l(A)$ is the length of river bed in contact with the flow for the cross-section at s, then for vanishingly small section of the river bed between s and $s + ds$ the frictional force exerted by the river bed on the fluid is

$$dF_{frict} = \tau \; l(A)ds. \tag{7}$$

We can also calculate the force dF_{grav} along the channel exerted on the water in this vanishingly small section of the river bed. This is given by the mass of fluid in this section $dm = \rho A ds$ multipled by the component of gravity along the channel $g \sin\alpha$ thus giving

$$dF_{grav} = g\sin\alpha \; \rho A ds. \tag{8}$$

Assuming that we can neglect the inertia of the flow (*i.e.* the force required to accelerate the flow) we can simply balance the frictional force with the gravitational force to obtain the following expression for the shear stress

$$\tau = \frac{g\sin\alpha \; \rho A}{l(A)}$$

If we now substitute for τ from the phenomenologial shear relation this allows us to obtain an expression for the area-averaged fluid velocity

$$\bar{u} = \sqrt{\frac{g \sin \alpha A}{f l(A)}}. \tag{9}$$

Finally, substitute for $\bar{u}$ in (5) to obtain the following first order hyperbolic PDE for A

$$\boxed{\frac{\partial A}{\partial t} + \sqrt{\frac{g}{f}} \frac{\partial}{\partial s} \left(A^{3/2} \sqrt{\frac{\sin \alpha}{l(A)}} \right) = 0.} \tag{10}$$

Tasks: Calculate how the perimeter $l(A)$ of the wetted bed depends on the cross-sectional area A of the bed filled by water for various possible river bed cross-sections. Obtain solutions to the hyperbolic PDE (10).

2 The method of characteristics

In this section we will consider a method, termed the method of characteristics, for solving (quasilinear) first order hyperbolic PDEs such as (10).

2.1 Advection equations

The simple linear advection equation. In order to illustrate this method we start by considering the simple linear hyperbolic first-order PDE

$$\frac{\partial c}{\partial t} + k \frac{\partial c}{\partial x} = 0. \tag{11}$$

We now define the family of *characteristic curves* to be solutions to the ODE

$$\frac{dx}{dt} = k. \tag{12}$$

This ODE has solution

$$x(t; P) = kt + P, \tag{13}$$

in which the constant of integration P parametrises the characteristic curves (in this case the characteristics are a family of straight lines, as illustrated in figure 2(a)). In particular the value of P tells us where characteristic crosses the x-axis. Along a particular characteristic $x(t; P)$, since $k = dx/dt$ it follows from (11) that

$$\frac{\partial c}{\partial t} + \frac{dx}{dt} \frac{\partial c}{\partial x} = 0, \tag{14}$$

which is an exact integral and can be rewritten, via the chain rule, as

$$\frac{d}{dt} c(x(t; P), t) = 0, \qquad \text{along a characteristic curve } x(t; P). \tag{15}$$

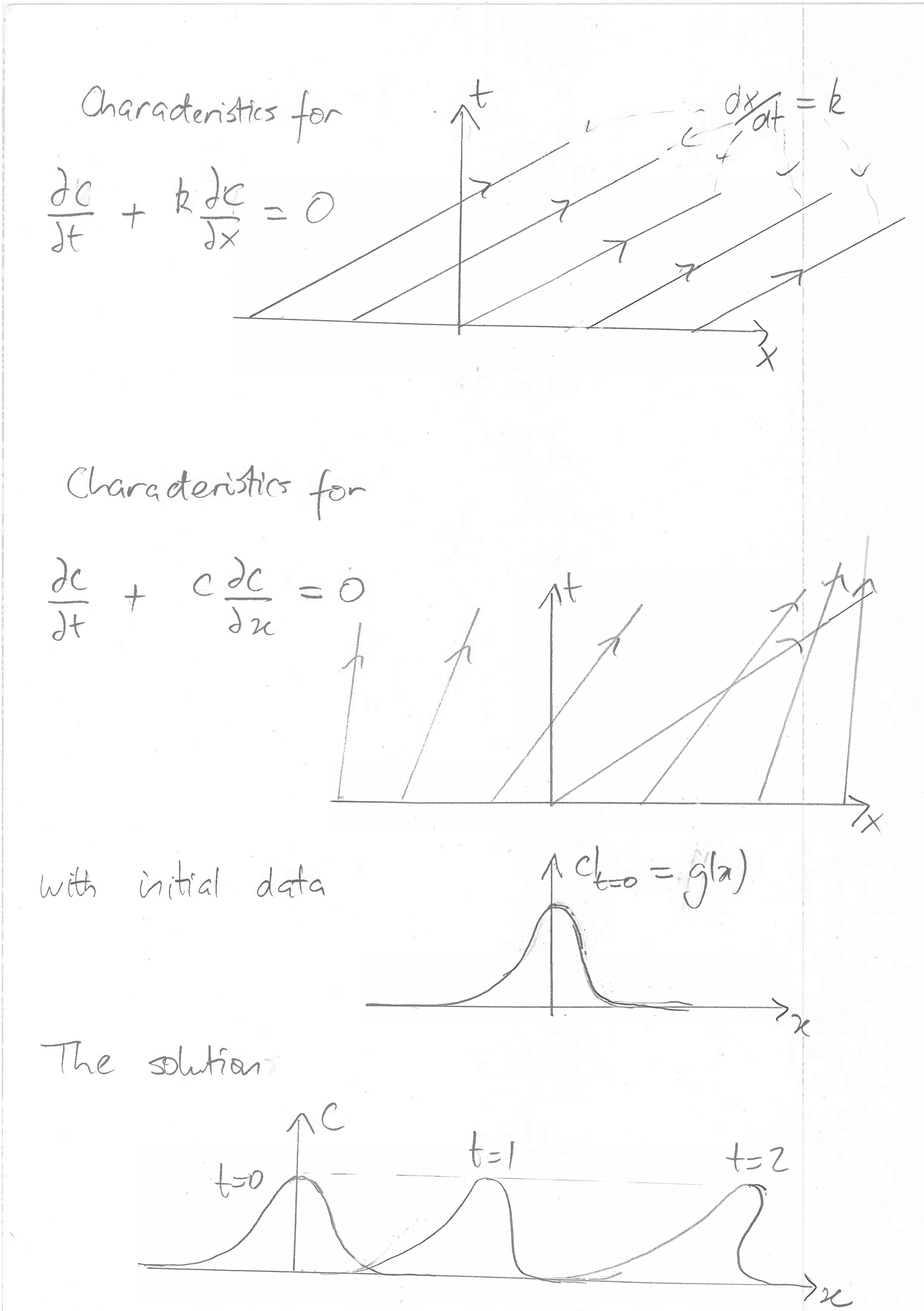

Figure 2: (a) The characteristics for the simple linear advection equation $c_t + kc_x = 0$. (b) The characteristics for the nonlinear advection equation $c_t + cc_x = 0$ with initial data illustrated in (c); the corresponding solution is illustrated in (d) and shows the development of a multi-valued solution (a "breaking wave").

In turn this implies that c has to be constant along a characteristic curve, but notice that although c is constant along any given characteristic this constant will (in general) vary between different characteristics, and the value can be related back to the initial values at $t = 0$. If the initial condition is

$$c(x,0) = g(x), \tag{16}$$

for some known function $g(x)$, then we can write c on a characteristic as $c(x(t;P),t) = c(x(0;P),0) = c(P,0) = g(P)$. Thus, given an (x,t) one only needs to determine which characteristic it lies upon, and extrapolate this back to the initial time $t = 0$. Referring back to (13) we see that $P = x - kt$ and hence

$$\boxed{c(x,t) = g(x - kt).} \tag{17}$$

The method of characteristics has considerably wider applicability than just to this equation and can, in general, be applied to any first order quasilinear PDE of the form

$$\frac{\partial c}{\partial t} + v(c,x)\frac{\partial c}{\partial x} = S(c,x),$$

where $v(c,x)$ and $S(c,x)$ are known functions of c and x.

A nonlinear advection equation. Rather than try to apply the method directly to general quasilinear first order PDEs of the form (18) we shall restrict our attention to the specific class of first-order nonlinear PDEs that (10) belong to, namely

$$\frac{\partial c}{\partial t} + v(c)\frac{\partial c}{\partial x} = 0. \tag{18}$$

Once more we seek to rewrite the left-hand side of this equation as an exact derivative of c along a characteristic curve $x(t)$. We do this by defining the characteristic curves by the ODEs

$$\frac{dx}{dt} = v(c), \qquad \text{with} \qquad x(0;P) = P, \tag{19}$$

such that, along a characteristic curve, (18) can be rewritten

$$\begin{aligned} \frac{\partial c}{\partial t} + \frac{dx}{dt}\frac{\partial c}{\partial x} &= 0, \\ \Longrightarrow \qquad \frac{dc}{dt} &= 0. \end{aligned}$$

In turn by integrating this last equation we see that $c = const.$ along a given characteristic curve. Given this it follows that the term on the left-hand side of equation (19) defining the characteristic is itself constant so that (19) can be straightforwardly integrated to give the following explicit expression for the characteristic curve (which is a straight line):

$$x(t;P) = v(c)t + P, \tag{20}$$

Again, c can only vary between characteristics, and not along any given characteristic, with it's value determined from the initial condition at $t = 0$. If $c(x, 0) = g(x)$ then (20) can be written

$$x(t; P) = v(g(P))t + P, \tag{21}$$

and the solution for c is

$$\boxed{c(x(t; P), t) = g(P).} \tag{22}$$

One only needs to know what characteristic (x, t) lies on to determine P (what if it lies on multiple characteristics?).

Alternatively, P can be evaluated in terms of x, t and c from (20) as $P = x - v(c)t$ giving the following *implicit* expression for c:

$$\boxed{c = g(x - v(c)t).} \tag{23}$$

Then to determine c for some (x, t), one could solve the above equation for c which in general is a nonlinear transcendental equation requiring the use of an appropriate numerical method (such as Newton's method). Furthermore, we may well expect that the solution to such an equation will be non-unique; in other words there could be multiple solutions c for certain values of (x, t).

Example of a problem which develops a multi-valued solution. In order to demonstrate what may go wrong let us consider the characteristics for the PDE

$$\frac{\partial c}{\partial t} + c\frac{\partial c}{\partial x} = 0.$$

(*i.e.* (18) with $v(c) \equiv c$) and with a "hump" initial data $g(x)$ as sketched in figure 2(c). For such initial data the characteristics are given by $x(t; P) = g(P)t + P$ and so look like the ones plotted in figure 2(b). Notably, such characteristics must intersect for sufficiently large t and thus the solution becomes multi-valued. A sketch of the resulting solution is presented in figure 2(d) and shows a wave propagating to the right and steepening as it does so until its crest turns over. This is precisely the sort of phenomena that can be observed as ocean waves approach a beach: as the depth of the water decreases, the waves steepen before eventually breaking. In the next section we see how it may be possible to generalise our treatment to study the propagation of these waves after they break, i.e.*shocks*.

2.2 Shock formation and propagation

Crossing characteristics are associated with non-physical multiple valued solutions (see, for example, figure 3(I)). In physical reality, what is usually observed is the formation of a discontinuity that then propagates (see figure 3(II)). In terms of the characteristic diagram, the discontinuity is represented by a shock, which follows a trajectory in x–t space. Any characteristics which intersect this trajectory will terminate there (*i.e.* the characteristics will only flow into the shock, from both sides), which is illustrated in figure 3(III). The

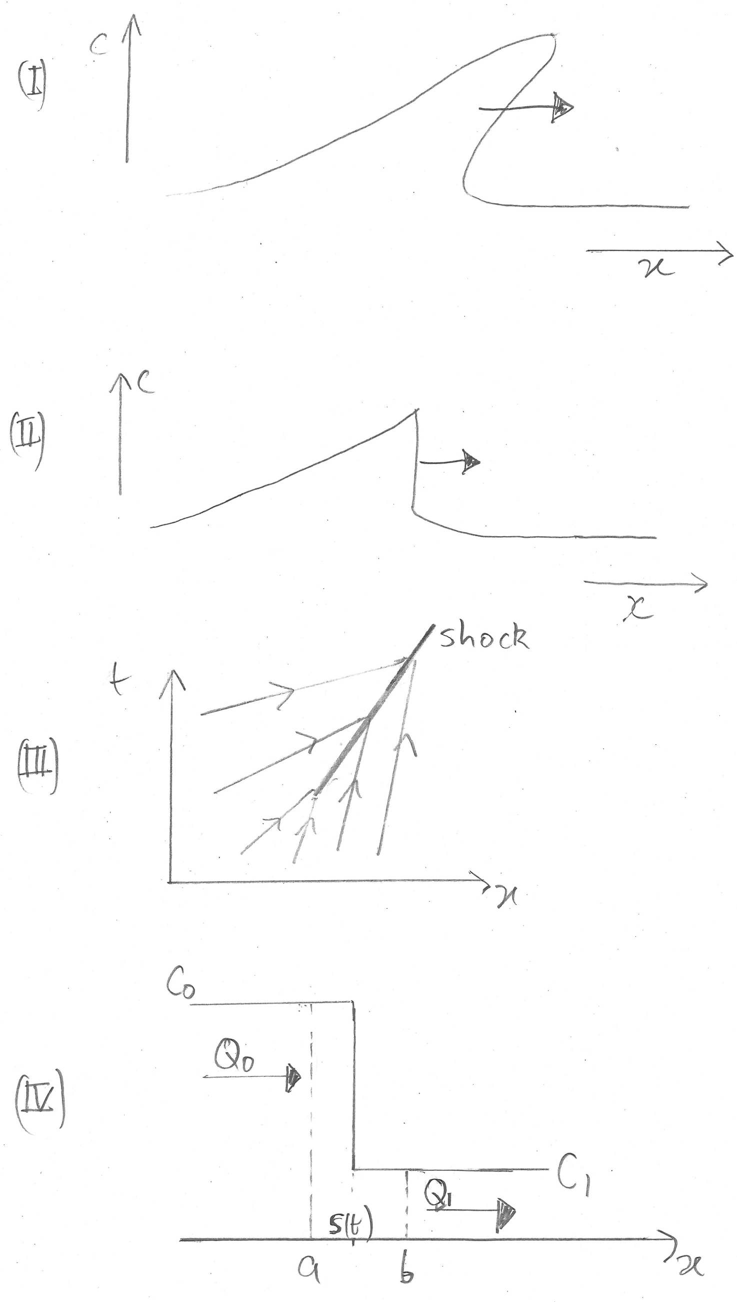

Figure 3: (I) Intersecting characteristics lead to the development of a multiple-valued solution, (II) In physics it is more usual to see a single-valued solution with a propagating discontinuity (shock), (III) a shock in the characteristic diagram, (IV) Derivation of the shock velocity from the underlying conservation law.

velocity of the shock is not determined by the PDE and can only be uniquely determined from the underying conservation law from which the PDE is derived. (*n.b.* many different conservation laws—in fact an infinite number—lead to the same PDE).

In order to make these concrete we consider the example of first order hyperbolic PDEs of the form

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x} Q(c) = 0, \tag{24}$$

In this instance we assume that the PDE arises from the conservation law

$$\frac{d}{dt} \int_a^b c(x,t)dx + [Q(c(x,t))]_a^b = 0. \tag{25}$$

which holds true for all constant values a and b. This conservation law describes the conservation of a quantity with concentration $c(x,t)$ per unit length and concentration dependent flux $Q(c)$. By focussing close to a discontinuity in the solution (*i.e.* the shock) and applying the conservation law we can determine the velocity of the discontinuity (shock). In order to do this we choose the points a and b to lie very close to, and on either side of, the discontinuity (shock). So that where we denote the position of the discontinuity by $x = s(t)$ we have $a < s(t) < b$ very close to the time t. This situation is illustrated in figure 3(IV). Assuming that a is just to the left of $x = s(t)$ and b is just to the right of $x = s(t)$ allows us to approximate the integral in (25) as follows:

$$\int_a^b c(x,t)dx \approx c_0(s(t) - a) + c_1(b - s(t)),$$

where $c_0 = c(s_-(t), t)$ and $c_1 = c(s_+(t), t)$ are the values of c just to the left of and just to the right of the shock, respectively. Thus when we differentiate with respect to time we find

$$\frac{d}{dt} \int_a^b c(x,t)dx \approx \dot{s}(t)(c_0 - c_1).$$

The second term in (25) is straightforward to evaluate and gives

$$[Q(c(x,t))]_a^b = Q(c_1) - Q(c_0).$$

If we now substitute the above two results into (25) we obtain

$$\dot{s}(t)(c_0 - c_1) + Q(c_1) - Q(c_0) = 0,$$

from which we can determine the shock velocity

$$\dot{s}(t) = \frac{Q(c_1) - Q(c_0)}{c_1 - c_0} = \frac{[Q(c)]_{s(t)}}{[c]_{s(t)}}, \tag{26}$$

where $[\cdot]_{s(t)}$ signifies the jump of the quantity in the brackets across the shock at $x = s(t)$.

3 A numerical scheme for accurate solution of hyperbolic PDEs with shocks: The Godunov method

As we have seen accounting for the correct form of the conservation law is vital if we want to properly capture the propagation of a shock once it has formed. We thus make use of a numerical scheme that is partially based on the conservation law (25) rather than trying to base it entirely on the PDE (24). This scheme is an example of a Godunov method which itself is a type of Finite Volume Method (see Leveque [1] for further details). As is common when solving a PDE we divide space up into finite intervals by denoting the set of discrete points x_i by

$$x_i = \left(x_L - \frac{\Delta}{2}\right) + i\Delta, \qquad \text{for} \quad i = 1, 2, 3, \ldots \tag{27}$$

where Δ is the grid spacing and we denote the "half points" by

$$x_{i+1/2} = x_L + i\Delta, \qquad \text{for} \quad i = 1, 2, 3, \ldots. \tag{28}$$

Here x_L is the left hand edge of the domain (see figure 4(I) for an illustration of the spatial-discretization). We approximate the solution $c(x,t)$ to the PDE and its conservation law by the piecewise uniform function

$$c(x,t) = \begin{cases} \ldots & \\ c_{i-1}(t) & \text{for} \quad x_{i-3/2} < x < x_{i-1/2} \\ c_i(t) & \text{for} \quad x_{i-1/2} < x < x_{i+1/2} \\ c_{i+1}(t) & \text{for} \quad x_{i+/2} < x < x_{i+3/2} \\ \ldots & \end{cases}, \tag{29}$$

Here the finite volumes are the regions between the half points, for example the i-th finite volume lies between $x_{i-1/2} < x < x_{i+1/2}$, is centred on x_i, and within this region we approximate the solution by the uniform function $c(x,t) \approx c_i(t)$.

In order to determine how $c_i(t)$ evolves, Godunov suggested a procedure in which the solution is calculated exactly from piecewise uniform initial data at $t = t_n$, *i.e.* $c(x, t_n) = c^{(n)}(x)$ which is of the form (29), forward a short timestep to the $(n+1)$-th timestep at $t = t_{n+1}$. We saw how to do this exactly in §2. However there are several things to note about this procedure, (i) the solution we obtain at $t = t_{n+1}$ no longer has the piecewise uniform form (29), (ii) the solution may contain shocks, as might be expected from the discontinuous nature of the initial data, and (iii) it may also contain rarefactions (which we have not so far discussed).

In order to deal with (i) and ensure that the solution remains piecewise uniform within the finite volumes (*i.e.* is of the form (29)) Godunov suggested averaging the exact solution we obtain using the method of characteristics at $t = t_{n+1}$ over each finite volume. After this averaging procedure, the approximate solution $c(x, t_{n+1}) = c^{(n+1)}(x)$ now has the correct piecewise uniform form (29), furthermore this solution also clearly satisfies the conservation law because both the exact solution technique and the averaging procedure ensure conservation of c. This two step procedure (which is illustrated in figure 4(II)-(IV)) is then repeated

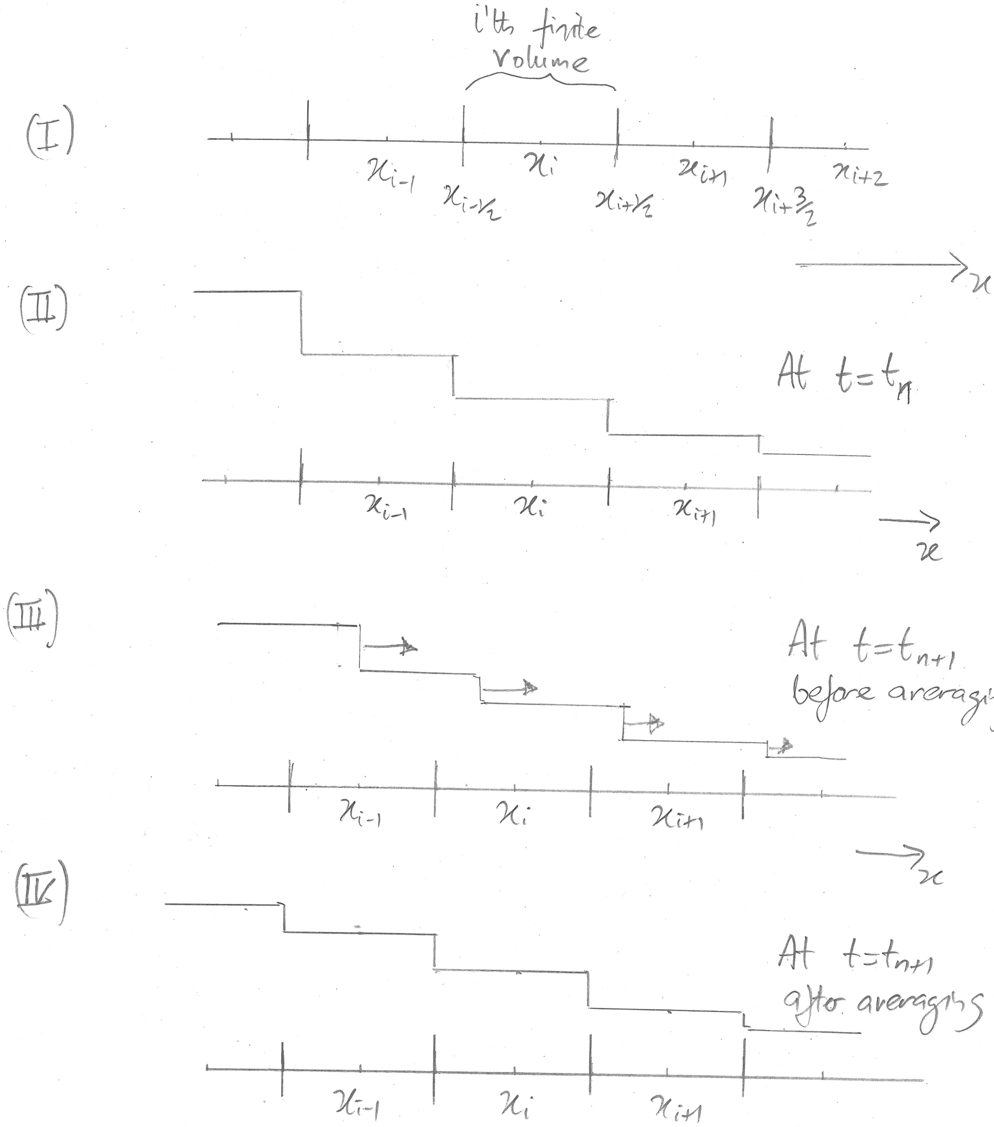

Figure 4: The Godunov method (I) The finite volumes, (II) The initial data for the problem for $t \in [t_n, t_{n+1}]$, (III) The solution at $t = t_{n+1}$ before the averaging step, (IV) The (approximate) solution at $t = t_{n+1}$ after the averaging step.

at each time step; this ensures the approximate solution that we obtain always has the piecewise uniform form (29).

The problem with the procedure outlined above is that it is rather complicated to implement. In order to simplify the implementation, Godunov then noted that the approximate solution must necessarily satisfy the conservation law (25) on each finite volume. In the case of the i-th finite volume this is

$$\frac{d}{dt}\int_{x_{i-1/2}}^{x_{i+1/2}} c(x,t)dx + [Q]_{x_{i-1/2}}^{x_{i+1/2}} = 0,$$

which, on substituting the piecewise constant appoximation (29), gives the following ODE for $c_i(t)$:

$$\frac{dc_i}{dt} + \frac{Q|_{x=x_{i+1/2}} - Q|_{x=x_{i-1/2}}}{x_{i+1/2} - x_{i-1/2}} = 0. \tag{30}$$

The question that then arises is what value of $Q(c)$ should we take at the half-points $x = x_{i+1/2}$ and $x = x_{i-1/2}$ since at the half-points c is discontinuous. In our case it is straightforward because we will only consider conservation laws for which $Q'(c) > 0$. This means that the characteristics, which satisfy $dx/dt = Q'(c)$, are always directed in the direction of increasing x (*i.e.* from left to right). Between $t = t_n$ and $t = t_{n+1}$, and before the averaging step, the solution in the $(i-1)$-th finite volume thus propagates into the i-th finite volume while that in i-th finite volume propagates into the $(i+1)$-th finite volume (see figure 4(III)). It follows that, in our case, $Q|_{x=x_{i-1/2}} = Q(c_{i-1})$ and $Q|_{x=x_{i+1/2}} = Q(c_i)$ and that we can thus rewrite (30) as

$$\frac{dc_i}{dt} + \frac{Q(c_i) - Q(c_{i-1})}{\Delta} = 0, \qquad \text{for} \quad i = 1, 2, \dots, N \tag{31}$$

where we consider a problem divided into N finite volumes. Of course for this to be a closed system of equations for the functions $c_i(t)$, with $i = 1, 2, \dots, N$ we need to specify $c|_{x=x_L} = c_0$, *i.e.* the value of c on the left-hand end of our domain. This boundary condition is called an inflow boundary condition.

The numerical integration of the above system with respect to time can then be performed with any suitable nonlinear ODE scheme or solver.

References

[1] R.J. Leveque, “Numerical Methods for Conservation Laws”. Lectures in Mathematics, ETH Zurich (1992).