PHYS6017 Project E2

COMPUTER TECHNIQUES IN PHYSICS

The forced simple pendulum

The motion of a forced simple pendulum is a widely used illustration of forced simple harmonic motion. If however we relax the approximation that the amplitude is small very complicated behaviour can occur. The period varies with amplitude so that difference between the forcing and resonant frequencies changes as the amplitude builds up. If the damping is very small the pendulum can also make complete circuits.

The equation of motion for a pendulum of mass m and length L is

$$mL^2\ddot{\theta} + k\dot{\theta} + mgl\sin\theta = FL\cos\Omega t$$

where F represents the external periodic force of frequency Ω. It is convenient to measure time relative to the period of free oscillations of small amplitude, that so write $t = \tau(L/g)^{1/2}$. The interesting phenomena occur when Ω is slightly less than the natural frequency for small oscillations. This is because the frequency of a pendulum decreases as the amplitude increases, so the pendulum comes into resonance with the forcing frequency as its amplitude builds up. It is therefore convenient to write $\Omega = (1-\eta)(g/L)^{1/2}$ where η is small.

Convert the equation to dimensionless form; it then contains three parameters $k/mL\sqrt{gL}$, F/mg and η. There are also the starting position and velocity that could the varied.

You cannot in the time available explore all values of these parameters, therefore I suggest you always start with the pendulum at rest, (you could start it inverted!) and with zero velocity and investigate *one* of the following cases.

1. The breakdown of the simple periodic behaviour. This implies a forcing term with $F \sim mg$ and starting with sufficient damping to prevent the amplitude becoming large and then reducing the damping until something interesting happens.

2. The case of extreme forcing with $F \gg mg$. Here the pendulum will settle to steady periodic motion in a circle in the vertical plane with gravity having only a small effect. Damping can be taken as zero. Find an example of this and try reducing the forcing term until this motion no longer persists.

3. More general behaviour in the case of zero damping. If F/mg and η are small the motion shows 'beats' between the natural period and the forcing period. For some values of the parameters the 'beat' frequency may be such that the pendulum motion is periodic but with a period which is a *multiple* of the forcing period, and much greater than the natural period. Several things could be investigated here: the beat frequency and the conditions for strictly periodic motion.

As F/mg is increased a critical value is reached at which complete loops start to occur. For still larger values there are ranges of F/mg where the motion is bounded and ranges where continual looping occurs. For large values of F/mg periodic solutions have the period of the forcing term.

4. Statistical behaviour in the chaotic regime. Find an example where the motion appears to be chaotic, involving occasional complete circuits. Determine the times between successive flips over the top and plot a histogram of them. See how the histogram changes as you vary one of the parameters.

In each case the aim is determine the ranges of the parameters for which different types of qualitative behaviour occur, for example, steady periodic motion, complete oscillations, or chaotic behaviour. The difficulty in this project is not in collecting some results, it is easy to get too many. You must try to present a coherent set of results so that the readers of your report can feel that they have gained some understanding of this confusing phenomenon.

Convert the equation into two first order equations. The Runge-Kutta routine is suitable for a numerical solution.

If you run the integration for a long time the errors will accumulate. An inaccurate numerical integration may show that same qualitative features as an accurate one, so make sure that you are not deceived by this. If the pendulum goes over the top θ becomes greater than 2π, and this makes it a bit difficult to display what is happening. Be careful not to get confused by this.

There is a minor typo in the equation of motion at the top of page 1, i.e., l should be L. The equation should read

$$mL^2\ddot{\theta} + k\dot{\theta} + mgL\sin\theta = FL\cos\Omega t.$$