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The Forced Simple Pendulum

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ABSTRACT

1 Introduction

The equation of motion for a pendulum of a mass m and length of L is:

$$mL^2\frac{d^2\theta}{dt^2} + k\frac{d\theta}{dt} + mgL\sin(\theta) = FL\cos(\Omega t) \tag{1}$$

We measure time relative to the period of free oscillations of small amplitude, so we can introduce:

$$t = \tau\sqrt{\frac{L}{g}} \tag{2}$$

We introduce:

$$\Omega = (1-\eta)\sqrt{\frac{g}{L}} \tag{3}$$

To investigate the situation where the behavior of the pendulum when the forcing frequency, Ω is slightly less than the natural frequency, $\frac{g}{L}$.

To convert equation (1) into dimensionless form, we recall that: $t = \tau\sqrt{\frac{L}{g}}$ (equation (2)) and by the chain rule, $\frac{d\tau}{dt} = \frac{d\theta}{dt}\frac{dt}{d\tau}$ give the following equations:

$$\begin{aligned}\frac{d\theta}{dt} &= \sqrt{\frac{g}{L}}\frac{d\theta}{d\tau}\\ \frac{d^2\theta}{dt^2} &= \frac{g}{L}\frac{d^2\theta}{d\tau^2}\end{aligned} \tag{4}$$

Which can be substituted into equation (1) after dividing by mgL and substituting equation (2) and (3).

$$(\frac{L}{g})(\frac{g}{L})\frac{d^2\theta}{d\tau^2} = \frac{k}{mgL}\sqrt{\frac{g}{L}}\frac{d\theta}{d\tau} + \sin(\theta) = \frac{F}{mg}\cos((1-\eta)\sqrt{\frac{g}{L}}\sqrt{\frac{L}{g}}\tau) \tag{5}$$

The dimensionless form of equation (1) is then:

$$\frac{d^2\theta}{d\tau^2} = \alpha\frac{d\theta}{d\tau} + \sin(\theta) = \beta\cos((1-\eta)\tau) \tag{6}$$

With parameters

$$\begin{aligned}\alpha &= \frac{k}{mL\sqrt{gL}}\\ \beta &= \frac{F}{mg}\end{aligned} \tag{7}$$

[1]

To solve the equation, the RK4(5) (Runge–Kutta–Fehlberg) method will be used. I will write my own version of the RK4(5) instead of using `scipy.integrate.RK45` that uses the Dormand-Prince pair of formulas [**?**]. To solve a 2nd order differential equation with any Runge-Kutta method, the ODE in question will have to be expressed as a 1st order ODE. From equation (6) and taking the following substitutions:

$$\begin{aligned} u' &= \theta'' \\ v' &= \theta \end{aligned} \tag{9}$$

The ODE will be a system of equations of the form $\mathbf{Y}' = A\mathbf{Y} - \vec{\mathbf{b}}$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} -\sin(\nu) + \beta\cos((1-\eta)\tau) \\ 0 \end{bmatrix} \tag{10}$$

[1]Note that:

$$\frac{k}{mgL}\sqrt{\frac{g}{L}} = \frac{k}{mL\sqrt{g}\cdot\sqrt{g}}\sqrt{\frac{g}{L}} = \frac{k}{mL\sqrt{gL}} \tag{8}$$