DEPARTMENT OF PHYSICS AND ASTRONOMY

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The Forced Simple Pendulum

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Contents

ABSTRACT

1 Introduction

The forced simple pendulum looks, deceivingly simple. A bob on a string where the other end is attached to a fixed point is allowed to oscillate with an initial angular velocity experiences a damping force that acts against the pendulum - if the damping force (determined partly by a damping coefficient) is large enough, it will affect the motion of the oscillating pendulum.

The equation of motion for a pendulum of a mass m and length of L is:

$$mL^2\frac{d^2\theta}{dt^2}+k\frac{d\theta}{dt}+mgL\sin(\theta)=FL\cos(\Omega t) \quad (1)$$

Where:

- m is the mass of the pendulum bob
- L is the length of the pendulum
- θ is the angular displacement of the pendulum from vertical
- $\dot{\theta}$ is the angular velocity (rate of change of θ)
- $\ddot{\theta}$ is the angular acceleration
- k is the damping coefficient (dependent on the specific damping mechanism)
- g is acceleration due to gravity
- F is the magnitude of the external force
- Ω is the frequency of the external force F
- t is the time
- $k\dot{\theta}$ is the damping force

In the following sections, I introduce the method on how to change the 2nd order differential equation Eqn. (1) into a matrix equation in the form of $\vec{\mathbf{Y}}\prime = A\vec{\mathbf{Y}} - \vec{\mathbf{b}}$, such that they can be solved numerically using the Runge-Kutta method.

I will then investigate various scenarios, which will be introduced at the end of the Methodology section. Results are then compared to known data.

The Rayleigh-Lorentz pendulum [**?**], named after Lord Rayleigh and Hendrik Lorentz, is a simple pendulum where the forcing frequency is changing due to the change of the pendulum length It has been shown that from [**?**]:

$$\frac{E(t)}{f(t)}=\frac{E(0)}{f(0)} \quad (2)$$

Stating that the ratio of average energy to frequency is constant. This is an example of an adiabatic invariant - meaning that it is a conservation law that stays constant only when changes to the parameters are done slowly. The case of a uniform and exponential change in the pendulum length have been investigated in the case of a simple pendulum in [**?**, **?**, **?**] (for a uniform change) and [**?**] (for both uniform and exponential), where the linear variation was determined by the equation $l(t)=l_0(1+\epsilon t)$, and the exponentially varying case is given by the equation $l(t)=l_0\exp\epsilon t$. Where ϵ in both equations is a small parameter of unit s^{-1}. There are many other variations that I can investigate, for example a non-linear (quadratic or higher order) variation, or even a random variation (Brownian motion or random walks). A random walk is an interesting option for a number of reasons:

- Real world systems are experience random fluctuations. For example, a skyscraper or bridge will not experience a constant force.
- Stability of the pendulum system could be assessed. What is the optimum pendulum (one that is resilient to random changes in forces)?
- Resonant frequencies can be identified easily
- Long term behaviour prediction

2 Methodology

Since time is measured relative to the period of free oscillations of small amplitude, I can write:

$$t = \tau\sqrt{\frac{L}{g}} \tag{3}$$

And:

$$\Omega = (1-\eta)\sqrt{\frac{g}{L}} \tag{4}$$

To investigate the situation where the behavior of the pendulum when the forcing frequency, Ω is slightly less than the natural frequency, $\frac{g}{L}$. To convert equation (1) into dimensionless form, we recall that: $t = \tau\sqrt{\frac{L}{g}}$ and by the chain rule, $\frac{d\tau}{dt} = \frac{d\theta}{dt}\frac{dt}{d\tau}$ give the following equations:

$$\begin{aligned} \frac{d\theta}{dt} &= \sqrt{\frac{g}{L}}\frac{d\theta}{d\tau} \\ \frac{d^2\theta}{dt^2} &= \frac{g}{L}\frac{d^2\theta}{d\tau^2} \end{aligned} \tag{5}$$

Which can be substituted into equation (1) after dividing by mgL and substituting equation (2) and (3).

$$\begin{aligned} &(\frac{L}{g})(\frac{g}{L})\frac{d^2\theta}{d\tau^2} + \frac{k}{mgL}\sqrt{\frac{g}{L}}\frac{d\theta}{d\tau} + \sin(\theta) \\ &= \frac{F}{mg}\cos((1-\eta)\sqrt{\frac{g}{L}}\sqrt{\frac{L}{g}}\tau) \end{aligned} \tag{6}$$

The dimensionless form of equation (1) is then:

$$\frac{d^2\theta}{d\tau^2} = -\alpha\frac{d\theta}{d\tau} - \sin(\theta) + \beta\cos((1-\eta)\tau) \tag{7}$$

With parameters[1]:

$$\begin{aligned} \alpha &= \frac{k}{mL\sqrt{gL}} \\ \beta &= \frac{F}{mg} \end{aligned} \tag{9}$$

The RK4(5) (Runge–Kutta–Fehlberg) method will be used to solve the equation numerically. I will be using `scipy.integrate.RK45` that uses the Dormand-Prince pair of formulas [**?**]. To solve a 2nd order differential equation with any Runge-Kutta method, the ODE in question will have to be expressed as a 1st order ODE. From equation (7) and taking the substitution:

$$\omega = \frac{d\theta}{d\tau} \tag{10}$$

The ODE will be a system of equations of the form $\vec{\mathbf{Y}}' = A\vec{\mathbf{Y}} - \vec{\mathbf{b}}$.

$$\begin{bmatrix} \theta' \\ \omega' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\sin(\theta) & -\alpha \end{bmatrix}\begin{bmatrix} \theta \\ \omega \end{bmatrix} - \begin{bmatrix} 0 \\ \beta\cos((1-\eta)\tau) \end{bmatrix} \tag{11}$$

For the Rayleigh-Lorentz (R-L) pendulum , the R.H.S. term in Equation (1) can now be simplified to FL, because now L will be a random number (which emerges from the random walk). The forc-

[1]Note that:

$$\frac{k}{mgL}\sqrt{\frac{g}{L}} = \frac{k}{mL\sqrt{g}\cdot\sqrt{g}}\sqrt{\frac{g}{L}} = \frac{k}{mL\sqrt{gL}} \tag{8}$$

ing frequency is now determined by the changes in L. The matrix equation that follows is a simplified version of Eqn. (**??**).

$$\begin{bmatrix} \theta' \\ \omega' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \text{-sin}(\theta) & \text{-}\alpha \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} - \begin{bmatrix} 0 \\ \beta \end{bmatrix} \tag{12}$$

The distribution for the random walk for L will be a normal distribution using `numpy.random.Generator.normal`[2]. The reason for using a normal distribution is that normal distributions are prevalent in nature, and on an experiment regarding forces on a pendulum, I think it makes sense to have a real-world lens on how a forced pendulum behaves [**?**].

3 Setup

The code used for the entirety of the project can be found on my GitHub page.

First, I think it is useful to have a rough understanding on the effects of each parameter on the numerical solution of the pendulum. By giving the constants values and altering each one, I can (without complete certainty, of course) predict the pendulum behaviour for later experiments. The system is set up as follows:

- The pendulum has a mass m of 1kg
- The length of the string, L is 2 meters
- The damping coefficient k has a value of 0.5
- The factor η has a value of 0.1
- The magnitude of the force F is 10N
- g is $9.81ms^{-1}$
- Time interval is 0 to 50 seconds [3]

By modifying each parameter, I plot the time-domain (θ against t) and the phase portrait[4] in Figs. (1 and **??**). Each notable simulation will contain a time-domain plot and a phase portrait plot ($\frac{d\theta}{dt}$ against θ).

We notice an interesting pattern as the mass changes. As the mass of the pendulum bob increased, the motion of the pendulum went from a simple harmonic to a dampened system. This is probably due to the fact that the magnitude of the external force was not enough to keep the pendulum in a simple harmonic motion. As the force was modified, what changed was just the magnitude of oscillation of the pendulum, up to as high as 30 radians. This meant that the pendulum will have completed many full rotations (~ 9 for $\theta \sim 30$).Increasing the length allowed the pendulum to do complete rotations in some cases. An increase in length meant an increase in angular velocity. A change in length should also affect the frequency[5], as a longer pendulum length will result in a shorter frequency (longer time to complete one oscillation).

[2]Documentation can be found here

[3]As the RK4(5) algorithm goes for longer, the errors stack up. It is not wise to use a long time interval at once without taking any precautions. Therefore a relatively short time period is investigated.

[4]See Section 10.1 for an explanation on phase portraits, as this is not covered in any core module.

[5]Recall $f = \frac{1}{2\pi}\frac{g}{L}$

Projects/ForcedSimplePendulum/Plots/test_plots.png

Figure 1: Time-domain plots for the case of the simple forced pendulum, where each variable is iterated one by one.

Projects/ForcedSimplePendulum/Plots/test_plots_phase.png

Figure 2: Phase portraits for the case of the simple forced pendulum, where each variable is iterated one by one.

4 Areas of Investigation

For the simple forced pendulum, I will investigate the case when the external force, F is almost equal to mg - $F \sim mg$. I iterate through values of k, decreasing k until something happens. With the same values of the other constants in Section 3, with the exception of $F = 9.81N$ and iterating over values of k until something happens.

For the R-L pendulum, I will implement the case where the length of the pendulum changes with a random walk, and verify if Equation (**??**) still holds for the case of a random walk. The energies in this system are simply the kinetic and potential energy, given by:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(L\omega)^2 \tag{13}$$

$$V = mgL\cos(\theta) \tag{14}$$

The angular frequency at each time step can be calculated by

$$\omega = 2\pi f = \sqrt{\frac{g}{L}} \tag{15}$$

The pendulum solver will run for a number of iterations, and then an average of the angular displacement, angular velocity, length of pendulum, frequency and energy will be taken at every time step. 4 important graphs will be plotted, being the time-domain plot, phase domain plot, plot of average values of L against t, and the difference of the average energy-frequency ratio at $t = 0$ to the values of energy-frequency at each time step.

5 Results

5.1 $F \sim mg$

By iterating through different values of k, the following time-domain and phase plots are obtained

Figure 3: Caption

(when something interesting happened - the not so interesting plots can be found in Section **??**). There are 2 points of interest I would like to bring into attention: As k decreases from 5 to 1: As k decreases from 1 to 0.1:

These will be discusses further in Section **??**.

Figure 4: Time-domain plot for the forced simple pendulum as k is increased. (Left column: raw data; Right column: data normalised by 2π

6 Analysis

6.1 $F \sim mg$

The case of k from 5 to 1 on first glance seems promising, that there may be some periodicity in terms of the angular displacement (see graph of $k = 2$ in Fig. **??**. However, the reality is that it was just the pendulum building up to reach a constant peak amplitude, as seen in Fig. **??** and also Fig **??**.

Something was happening at $k = 0.35230$.

7 Conclusion

7.1 Future Work

Future work could involve extending the Rayleigh-Lorentz pendulum to the case of a double pendulum, and using a different distribution for the random walk for L (i.e. the Pareto distribution, Gaussian distribution, etc.) for the case of a single and double pendulum.

8 Addendum

This section outlines 3 concepts studied in PHYS6017, namely random walks, Monte Carlo simulations, and random walks. A brief description of each concept is stated before some examples in and out of academia are given.

8.1 Random Walks

Random walks are a way to generate data. A simple example would be the case of a particle on a 2-D plane, where the particle is allowed to move in one of the four directions possible, and the choice of direction is random, usually according to some probability distribution.

8.1.1 In Academia

In [**?**], by knowing the sound intensity at a few detection points, sound fields (more specifically, the steady state high frequency SPL) in non-diffuse spaces can be predicted via random walks, built upon [**?**]. Random walks have been used in biophysics for an algorithm reconstructing supercoiled DNA [**?**] (supercoiled meaninng the amount of "twist" in a DNA strand, thereby giving the magnitude of strain on it) [**?**]. The DNA is found to take a random walk, which can be tied to the travelling salesman problem. Furthermore, mapping random walks onto complex network structures (where the moving probability is taken as information flow), and [**?**] discovered that information had a directional flow (when random walks were mapped), as opposed to a bidirectional flow (along the backbone of a network).

8.1.2 Outside of Academia

Random walks were used in cybersecurity, where cyber threats were detected via a "self-avoiding" (random) walk [**?**]. The patterns were extracted and compared to a threat database. Random walks were also used to predict pedestrian and vehicular movement in London [?], and proved to be very effective, compared to a segment-based centrality measures [?]. A random walk was also used in a simple risk business [?], seeing the probability of ruin as the number of contracts go up.

8.2 Monte Carlo Simulations

Prevalent in the world of finance, physics, engineering, statistics and many more, the famous casino in Monte Carlo was inspiration to the Monte Carlo simulation. It is a way to solve complex problems typically involving many variables, which may be challenging to solve analytically. They rely on randomness to generate large numbers of outcomes via simulations, where probabilities or expectation values can be calculated.

8.2.1 In Academia

The Monte Carlo (MC) method was used to do a simulation of the Ising model in in 2-D [?][?], a very important model in statistical mechanics. It is also used in robotics, where it is used to improve localisation of a moving robot in a dynamic enviroment [?] by using a vision-based MC localization. MC method is also used in machine learning for gradient estimation - the pathwise, score function and measure-valued gradient estimators were used [?].

8.2.2 Outside of Academia

Many applications were developed for proton therapy using the Monte Carlo method, such as VMCpro [?] (patient dose calculation), MCNPX [?], FLUKA [?] and Geant4 [?] [?] (all-particle code that can work with motion and magnetic fields). TOPAS (TOol for PArticle Simulation) [?][?] was then developed using the Geant4 toolkit to make it more accessible to medical physicsts, radiobiologists and clinicians. Other than proton therapy, MC method is used in risk analysis of the energy efficiency investments in buildings [?], by calculating probability distribution of energy reduction. MC method is also used in quality control [?]. The important characteristics of a product can be identified, allowing the manufacturer to prioritise certain aspects of their product to ensure customer satisfaction.

8.3 Random Numbers

Random numbers are numbers generated randomly where the current number generated has no relationship to the previous number generated, and neither will the current number have an effect on the next. They are used extensively in fields such as cryptography and statistics.

8.3.1 In Academia

An efficient stock market resembles a random number generator [?]. Therefore, random numbers were used to test the Efficient Market Hypothesis (EMH) [?] by using the Overlapping Serial Test [?], a form of test for randomness. Randomised numerical linear algebra [?] could be applied into statistics [?]. Random numbers have been used in investigating access games by having a random number of players [?].

8.3.2 Outside of Academia

True random number generators (TRNG), as outlined in [?] are used in cryptography (where the TRNG has to meet certain criteria [?]). TRNG was also used in audio encryption [?]. Following the (not so recent) cryptocurrency boom, blockchain-based random number generators [?] are explored and their uses in blockchain games [?]. Random numbers are also used in clinical trials [?], which assign participants to either of the available groups (i.e. a placebo and non-placebo group for a vaccine test).

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10 Appendix A

10.1 Phase Plots (or Phase portrait)

Phase plots (meaning plot of $\frac{d^2\theta}{dt^2}$ against $\frac{d\theta}{dt}$ in this case), are different from the usual time-domain plots (θ against t). The classic time-domain plot is a good way to visualise the change of the pendulum's position over time. The time-domain plots give the oscillation period (thereby the frequency), amplitude, and it is easy to see if any damping/external forces have been applied onto the system (the pendulum). A phase portrait, on the other hand give visualisation of the system's evolution in phase space. The intersection of trajectories at a point (or the approach of trajectories to a point) can indicate equilibrium points, stability (via attractors or repellers), limit cycles (repeated behaviour) or chaotic behaviour.

`Projects/ForcedSimplePendulum/WrittenReport/figs/The-phase-space-of-the-simpl`

Figure 5: Phase plot for a simple pendulum. Taken from [**?**].

Projects/ForcedSimplePendulum/WrittenReport/figs/Pendulum_phase_portrait_illustration.svg

Figure 6: How a phase portrait is compared to a time-domain plot for a simple pendulum. The blue dot for the illustration of the system corresponds to the blue dots on the time series graph and the phase portrait (and so do the other coloured dots). Taken from WikiMedia.

It is useful to consider both plots for this project. I will introduce some simple examples of the comparison of time-domain and phase-domain plots.

Projects/ForcedSimplePendulum/Plots/demo_simple_harmonic_motion.jpg

Figure 7: Time and Phase domain plots for simple harmonic motion.

Projects/ForcedSimplePendulum/Plots/demo_limit_cycles.jpg

Figure 8: Due to the presence of damping (seen in the time-domain plot), these plots showcase the concept of limit cycles (in the phase-domain plot), where trajectories spiral into each other.

Projects/ForcedSimplePendulum/Plots/demo_chaos.jpg

Figure 9: An example of chaotic behaviour. It definitely would not have been obvious just by looking at the time-domain plot alone.

11 Appendix B

This section contains graphs that have not been used in the main body of the text or not necessary to show, but showcase the iterative process where I select values of parameters used.

Figure 10: $F \sim mg$ for $k = 50$ to $k = 10$.

Figure 11: $F \sim mg$ for $k = 5$ to $k = 1$.

11.1 Plots for the case of $F \sim mg$

Initially, I tested for values of k of a large magnitude: From the plots it just looked to me as if the pendulum was not doing a whole lot - oscillating around the $0/2\pi$ mark, which is the lowest point of the pendulum's motion. I then reduced k from 5 to 1. Clearly something was happening in that range, so I investigate where

Figure 12: $F \sim mg$ for $k = 1$ to $k = 0.1$.