Coefficients and VRF diagonals for FMP and EMP Polynomial Filters as generated using the Matlab Symbolic Toolbox

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The layout of the doc:

I. Polynomial filter coefficients up to the 8th degree in word format for EMP filters (The $\Gamma(j,i)$'s where i denotes the polynomial degree and where j is the coefficient index);
II. Polynomial filter coefficients up to the 8th degree in word format for FMP filters;
III. The EMP filter VRF diagonals; and
IV. The FMP filter VRF diagonals.
V. Finally, N_s up to the 8th degree.

I. EMP Polynomial filter coefficients up to the 8th degree

Matlab was used to generate the provided equations. Calculations were cross-validated up to the fourth degree to those in, Chapter 13, Appendix 13.6.

The $\Gamma(j,i)$ for EMP filters up to the 8th-degree, given here in reverse order, i.e. the order of normal usage:

- 0th-degree:

$$\Gamma(0,0)=\frac{1}{n+1}$$

- 1st-degree:

$$\Gamma(1,1)=\frac{6}{(n+2)^{(2)}}$$

$$\Gamma(0,1)=\frac{2(2n+1)}{(n+2)^{(2)}}$$

- 2nd-degree:

$$\Gamma(2,2)=\frac{30}{(n+3)^{(3)}}$$

$$\Gamma(1,2)=\frac{18(2n+1)}{(n+3)^{(3)}}$$

$$\Gamma(0,2)=\frac{3(3n^2+3n+2)}{(n+3)^{(3)}}$$

- 3rd-degree:

$$\Gamma(3,3)=\frac{140}{(n+4)^{(4)}}$$

$$\Gamma(2,3)=\frac{120(2n+1)}{(n+4)^{(4)}}$$

$$\Gamma(1,3)=\frac{20(6n^2+6n+5)}{(n+4)^{(4)}}$$

$$\Gamma(0,3)=\frac{8(2n^3+3n^2+7n+3)}{(n+4)^{(4)}}$$

- 4th-degree:

$$\Gamma(4,4)=\frac{630}{(n+5)^{(5)}}$$

$$\Gamma(3,4)=\frac{700(2n+1)}{(n+5)^{(5)}}$$

$$\Gamma(2,4)=\frac{1050(n^2+n+1)}{(n+5)^{(5)}}$$

$$\Gamma(1,4)=\frac{50(2n+1)(3n^2+3n+10)}{(n+5)^{(5)}}$$

$$\Gamma(0,4)=\frac{5(5n^4+10n^3+55n^2+50n+24)}{(n+5)^{(5)}}$$

- 5^{th}-degree:

$$\Gamma(5,5)=\frac{2772}{(n+6)^{(6)}}$$

$$\Gamma(4,5)=\frac{3780(2n+1)}{(n+6)^{(6)}}$$

$$\Gamma(3,5)=\frac{1260(6n^2+6n+7)}{(n+6)^{(6)}}$$

$$\Gamma(2,5)=\frac{420(8n^3+12n^2+34n+15)}{(n+6)^{(6)}}$$

$$\Gamma(1,5)=\frac{126(5n^4+10n^3+55n^2+50n+28)}{(n+6)^{(6)}}$$

$$\Gamma(0,5)=\frac{6(2n+1)(3n^4+6n^3+77n^2+74n+120)}{(n+6)^{(6)}}$$

- 6^{th}-degree:

$$\Gamma(6,6)=\frac{12012}{(n+7)^{(7)}}$$

$$\Gamma(5,6)=\frac{19404(2n+1)}{(n+7)^{(7)}}$$

$$\Gamma(4,6)=\frac{16170(3n^2+3n+4)}{(n+7)^{(7)}}$$

$$\Gamma(3,6)=\frac{2940(2n+1)(5n^2+5n+21)}{(n+7)^{(7)}}$$

$$\Gamma(2,6)=\frac{294(30n^4+60n^3+345n^2+315n+202)}{(n+7)^{(7)}}$$

$$\Gamma(1,6)=\frac{588(2n+1)(n^2+n+2)(n^2+n+21)}{(n+7)^{(7)}}$$

$$\Gamma(0,6)=\frac{7(7n^6+21n^5+385n^4+735n^3+2128n^2+1764n+720)}{(n+7)^{(7)}}$$

- 7^{th}-degree:

$$\Gamma(7,7)=\frac{51480}{(n+8)^{(8)}}$$

$$\Gamma(6,7)=\frac{96096(2n+1)}{(n+8)^{(8)}}$$

$$\Gamma(5,7)=\frac{144144\left(2n^2+2n+3\right)}{(n+8)^{(8)}}$$
$$\Gamma(4,7)=\frac{36960(2n+1)\left(3n^2+3n+14\right)}{(n+8)^{(8)}}$$
$$\Gamma(3,7)=\frac{9240\left(10n^4+20n^3+122n^2+112n+81\right)}{(n+8)^{(8)}}$$
$$\Gamma(2,7)=\frac{672(2n+1)\left(15n^4+30n^3+360n^2+345n+707\right)}{(n+8)^{(8)}}$$
$$\Gamma(1,7)=\frac{48\left(42n^6+126n^5+2065n^4+3920n^3+11837n^2+9898n+4566\right)}{(n+8)^{(8)}}$$
$$\Gamma(0,7)=\frac{32(2n+1)\left(n^6+3n^5+100n^4+195n^3+1159n^2+1062n+1260\right)}{(n+8)^{(8)}}$$

- 8th-degree:

$$\Gamma(8,8)=\frac{218790}{(n+9)^{(9)}}$$
$$\Gamma(7,8)=\frac{463320(2n+1)}{(n+9)^{(9)}}$$
$$\Gamma(6,8)=\frac{540540\left(3n^2+3n+5\right)}{(n+9)^{(9)}}$$
$$\Gamma(5,8)=\frac{108108(2n+1)\left(7n^2+7n+36\right)}{(n+9)^{(9)}}$$
$$\Gamma(4,8)=\frac{270270\left(3n^4+6n^3+39n^2+36n+29\right)}{(n+9)^{(9)}}$$
$$\Gamma(3,8)=\frac{41580(2n+1)\left(3n^4+6n^3+74n^2+71n+162\right)}{(n+9)^{(9)}}$$
$$\Gamma(2,8)=\frac{990\left(42n^6+126n^5+1995n^4+3780n^3+12075n^2+10206n+5260\right)}{(n+9)^{(9)}}$$
$$\Gamma(1,8)=\frac{108(2n+1)\left(15n^6+45n^5+1290n^4+2505n^3+14963n^2+13718n+18264\right)}{(n+9)^{(9)}}$$
$$\Gamma(0,8)=\frac{27\left(3n^8+12n^7+518n^6+1512n^5+12467n^4+22428n^3+47492n^2+36528n+13440\right)}{(n+9)^{(9)}}$$

II. FMP Polynomial filter coefficients up to the 8th degree

Matlab was used to generate the provided equations. Calculations were cross-validated up to the fourth degree to those in, Chapter 13, Appendix 13.6, where the first five iterations can be found.

The $\Gamma(j,i)$ for FMP filters up to the 8th-degree, again in reverse order, i.e. the order of normal usage:

- 0th-degree:
  $\Gamma(0,0) = 1-\theta$

- 1st-degree:
  $\Gamma(1,1) = (1-\theta)^2$
  $\Gamma(0,1) = 1-\theta^2$

- 2nd-degree:
  $\Gamma(2,2) = \tfrac{1}{2}(1-\theta)^3$
  $\Gamma(1,2) = \tfrac{3}{2}(1-\theta)^2(1+\theta)$
  $\Gamma(0,2) = 1-\theta^3$

- 3rd-degree:
  $\Gamma(3,3) = \tfrac{1}{6}(1-\theta)^4$
  $\Gamma(2,3) = (1-\theta)^3(1+\theta)$
  $\Gamma(1,3) = \tfrac{1}{6}(1-\theta)^2(11+14\theta+11\theta^2)$
  $\Gamma(0,3) = 1-\theta^4$

- 4th-degree:
  $\Gamma(4,4) = \tfrac{1}{24}(1-\theta)^5$
  $\Gamma(3,4) = \tfrac{5}{12}(1-\theta)^4(1+\theta)$
  $\Gamma(2,4) = \tfrac{5}{24}(1-\theta)^3(7+10\theta+7\theta^2)$
  $\Gamma(1,4) = \tfrac{5}{12}(1-\theta)^2(1+\theta)(5+2\theta+5\theta^2)$
  $\Gamma(0,4) = 1-\theta^5$

- 5th-degree:
  $\Gamma(5,5) = \tfrac{1}{120}(1-\theta)^6$
  $\Gamma(4,5) = \tfrac{1}{8}(1-\theta)^5(1+\theta)$
  $\Gamma(3,5) = \tfrac{1}{24}(1-\theta)^4(17+26\theta+17\theta^2)$
  $\Gamma(2,5) = \tfrac{5}{8}(1-\theta)^3(3+5\theta+5\theta^2+3\theta^3)$
  $\Gamma(1,5) = \tfrac{1}{60}(1-\theta)^2(137+202\theta+222\theta^2+202\theta^3+137\theta^4)$
  $\Gamma(0,5) = 1-\theta^6$

- 6th-degree:
  $\Gamma(6,6) = \tfrac{1}{720}(1-\theta)^7$

$$\Gamma(5,6) = {}^{7}\!/_{240}(1-\theta)^6(1+\theta)$$
$$\Gamma(4,6) = {}^{7}\!/_{144}(1-\theta)^5\left(5+8\theta+5\theta^2\right)$$
$$\Gamma(3,6) = {}^{7}\!/_{48}(1-\theta)^4(1+\theta)\left(7+6\theta+7\theta^2\right)$$
$$\Gamma(2,6) = {}^{7}\!/_{360}(1-\theta)^3\left(4+7\theta+4\theta^2\right)\left(29+2\theta+29\theta^2\right)$$
$$\Gamma(1,6) = {}^{7}\!/_{60}(1-\theta)^2(1+\theta)\left(21+11\theta+26\theta^2+11\theta^3+21\theta^4\right)$$
$$\Gamma(0,6) = 1-\theta^7$$

- 7th-degree:

$$\Gamma(7,7) = {}^{1}\!/_{5040}(1-\theta)^8$$
$$\Gamma(6,7) = {}^{1}\!/_{180}(1-\theta)^7(1+\theta)$$
$$\Gamma(5,7) = {}^{1}\!/_{360}(1-\theta)^6\left(23+38\theta+23\theta^2\right)$$
$$\Gamma(4,7) = {}^{7}\!/_{18}(1-\theta)^5(1+\theta)\left(1+\theta+\theta^2\right)$$
$$\Gamma(3,7) = {}^{1}\!/_{720}(1-\theta)^4\left(967+2012\theta+2442\theta^2+2012\theta^3+967\theta^4\right)$$
$$\Gamma(2,7) = {}^{7}\!/_{180}(1-\theta)^3(1+\theta)\left(67+62\theta+102\theta^2+62\theta^3+67\theta^4\right)$$
$$\Gamma(1,7) = {}^{1}\!/_{420}(1-\theta)^2\left(1089+1698\theta+2027\theta^2+2132\theta^3+2027\theta^4+1689\theta^5+1089\theta^6\right)$$
$$\Gamma(0,7) = 1-\theta^8$$

- 8th-degree:

$$\Gamma(8,8) = {}^{9}\!/_{40320}(1-\theta)^9$$
$$\Gamma(7,8) = {}^{1}\!/_{1120}(1-\theta)^8(1+\theta)$$
$$\Gamma(6,8) = {}^{1}\!/_{960}(1-\theta)^7\left(13+22\theta+13\theta^2\right)$$
$$\Gamma(5,8) = {}^{1}\!/_{80}(1-\theta)^6(1+\theta)\left(9+10\theta+9\theta^2\right)$$
$$\Gamma(4,8) = {}^{1}\!/_{1920}(1-\theta)^5\left(1069+2444\theta+3054\theta^2+2444\theta^3+1069\theta^4\right)$$
$$\Gamma(3,8) = {}^{1}\!/_{160}(1-\theta)^4(1+\theta)\left(267+332\theta+482\theta^2+332\theta^3+267\theta^4\right)$$
$$\Gamma(2,8) = {}^{1}\!/_{10080}(1-\theta)^3\left(29531+591190\theta+79581\theta^2+86756\theta^3+79581\theta^4+591190\theta^5+295531\theta^6\right)$$
$$\Gamma(1,8) = {}^{1}\!/_{280}(1-\theta)^2(1+\theta)\left(761+446\theta+1027\theta^2+572\theta^3+1027\theta^4+446\theta^5+761\theta^6\right)$$
$$\Gamma(0,8) = 1-\theta^9$$

III. EMP VRF diagonals to the 8th degree

Matlab was used to generate the provided equations. Calculations were cross-validated up to the fourth degree to those in, Chapter 13, Appendix 13.6.

The $S_X(j,i)$ denormalized VRF diagonals for EMP filters up to the 8th-degree:

- 0th-degree:

$$S_X(0,0)=\frac{1}{n+1}$$

- 1st-degree:

$$S_X(1,1)=\left(\frac{1!}{\tau^1}\right)^2\frac{12}{(n+2)^{(3)}}$$

$$S_X(0,1)=\frac{4n+6}{(n+1)^{(2)}}$$

- 2nd-degree:

$$S_X(2,2)=\left(\frac{2!}{\tau^2}\right)^2\frac{180}{(n+3)^{(5)}}$$

$$S_X(1,2)=\left(\frac{1!}{\tau^1}\right)^2\frac{192n^2+744n+684}{(n+3)^{(5)}}$$

$$S_X(0,2)=\frac{9n^2+27n+24}{(n+1)^{(3)}}$$

- 3rd-degree:

$$S_X(3,3)=\left(\frac{3!}{\tau^3}\right)^2\frac{2800}{(n+4)^{(7)}}$$

$$S_X(2,3)=\left(\frac{2!}{\tau^2}\right)^2\frac{360(9n+22)(2n+3)}{(n+4)^{(7)}}$$

$$S_X(1,3)=\left(\frac{1!}{\tau^1}\right)^2\frac{200\left(6\,n^4+51\,n^3+159\,n^2+219\,n+116\right)}{(n+4)^{(7)}}$$

$$S_X(0,3)=\frac{8\,(2\,n+3)(n^2+3\,n+5)}{(n+1)^{(4)}}$$

- 4th-degree:

$$S_X(4,4)=\left(\frac{4!}{\tau^4}\right)^2\frac{44100}{(n+5)^{(9)}}$$

$$S_X(3,4)=\left(\frac{3!}{\tau^3}\right)^2\frac{2800(32n+79)(2n+3)}{(n+5)^{(9)}}$$

$$S_X(2,4)=\left(\frac{2!}{\tau^2}\right)^2\frac{8820\left(9n^4+76n^3+239n^2+336n+185\right)}{(n+5)^{(9)}}$$

$$S_X(1,4)=\left(\frac{1!}{\tau^1}\right)^2 \frac{100\,(2n+3)(24n^5+297n^4+1476n^3+3777n^2+5198n+3172)}{(n+5)^{(9)}}$$

$$S_X(0,4)=\frac{25n^4+150n^3+575n^2+1050n+720}{(n+1)^{(5)}}$$

- 5th-degree:

$$S_X(5,5)=\left(\frac{5!}{\tau^5}\right)^2 \frac{698544}{(n+6)^{(11)}}$$

$$S_X(4,5)=\left(\frac{4!}{\tau^4}\right)^2 \frac{88200(25n+62)(2n+3)}{(n+6)^{(11)}}$$

$$S_X(3,5)=\left(\frac{3!}{\tau^3}\right)^2 \frac{75600\left(48n^4+402n^3+1274n^2+1828n+1047\right)}{(n+6)^{(11)}}$$

$$S_X(2,5)=\left(\frac{2!}{\tau^2}\right)^2 \frac{17640(2n+3)(16n^5+192n^4+952n^3+2472n^2+3501n+2230)}{(n+6)^{(11)}}$$

$$S_X(1,5)=\left(\frac{1!}{\tau^1}\right)^2 \frac{588\,(84528+226920n+267180n^2+181760n^3+79585n^4+23300n^5+4450n^6+500n^7+25n^8)}{(n+6)^{(11)}}$$

$$S_X(0,5)=\frac{6(2n+3)\left(3n^4+18n^3+113n^2+258n+280\right)}{(n+1)^{(6)}}$$

- 6th-degree:

$$S_X(6,6)=\left(\frac{6!}{\tau^6}\right)^2 \frac{11099088}{(n+7)^{(13)}}$$

$$S_X(5,6)=\left(\frac{5!}{\tau^5}\right)^2 \frac{698544(72n+179)(2n+3)}{(n+7)^{(13)}}$$

$$S_X(4,6)=\left(\frac{4!}{\tau^4}\right)^2 \frac{5336100\,(25n^4+208n^3+665n^2+974n+580)}{(n+7)^{(13)}}$$

$$S_X(3,6)=\left(\frac{3!}{\tau^3}\right)^2 \frac{25200(2n+3)(800n^5+9375n^4+46640n^3+123525n^2+180944n+120603)}{(n+7)^{(13)}}$$

$$S_X(2,6)=\left(\frac{2!}{\tau^2}\right)^2 \frac{\begin{array}{c}1764\,(5527192+14416770n+16629525n^2+11146740n^3+\\4834785n^4+1410300n^5+270540n^6+30960n^7+1620n^8)\end{array}}{(n+7)^{(13)}}$$

$$S_X(1,6)=\left(\frac{1!}{\tau^1}\right)^2 \frac{\begin{array}{c}2352(2n+3)(8n^9+203n^8+2388n^7+17268n^6+85506n^5+\\298584n^4+720634n^3+1140499n^2+1074102n+465288)\end{array}}{(n+7)^{(13)}}$$

$$S_X(0,6)=\frac{7\left(7n^6+63n^5+595n^4+2625n^3+6958n^2+9912n+5760\right)}{(n+1)^{(7)}}$$

- 7^{th}-degree:

$$S_X(7,7)=\left(\frac{7!}{\tau^7}\right)^2\frac{176679360}{(n+8)^{(15)}}$$

$$S_X(6,7)=\left(\frac{6!}{\tau^6}\right)^2\frac{22198176(2n+3)(49n+122)}{(n+8)^{(15)}}$$

$$S_X(5,7)=\left(\frac{5!}{\tau^5}\right)^2\frac{78702624\,(54n^4+447n^3+1443n^2+2157n+1334)}{(n+8)^{(15)}}$$

$$S_X(4,7)=\left(\frac{4!}{\tau^4}\right)^2\frac{42688800(2n+3)(25n^5+288n^4+1444n^3+3909n^2+5927n+4130)}{(n+8)^{(15)}}$$

$$S_X(3,7)=\left(\frac{3!}{\tau^3}\right)^2\frac{\begin{array}{l}1219680\,(915432+2313285n+2602875n^2+1708385n^3+\\728245n^4+209825n^5+40075n^6+4625n^7+250n^8\,)\end{array}}{(n+8)^{(15)}}$$

$$S_X(2,7)=\left(\frac{2!}{\tau^2}\right)^2\frac{\begin{array}{l}14112\,(2\,n+3)\,(405n^9+9630n^8+109980n^7+782730n^6+3840660n^5+\\13364040n^4+32384190n^3+51942915n^2+50108782n+22507856)\end{array}}{(n+8)^{(15)}}$$

$$S_X(1,7)=\left(\frac{1!}{\tau^1}\right)^2\frac{\begin{array}{l}288(936292032+3034723608n+4476875200n^2+3990642796n^3+2405780559n^4+\\1036263508n^5+327831336n^6+77288092n^7+13667864n^8+1798545n^9+\\168903n^{10}+10143n^{11}+294n^{12}\,)\end{array}}{(n+8)^{(15)}}$$

$$S_X(0,7)=\frac{32(2n+3)(n^6+9n^5+130n^4+645n^3+2389n^2+4386n+3780)}{(n+1)^{(8)}}$$

- 8^{th}-degree:

$$S_X(8,8)=\left(\frac{8!}{\tau^8}\right)^2\frac{2815827300}{(n+9)^{(17)}}$$

$$S_X(7,8)=\left(\frac{7!}{\tau^7}\right)^2\frac{176679360(128n+319)(2n+3)}{(n+9)^{(17)}}$$

$$S_X(6,8)=\left(\frac{6!}{\tau^6}\right)^2\frac{\begin{array}{l}122367445200n^4+1008907099200n^3+3290602114800n^2\\+5017897684800n+3219012997200\end{array}}{(n+9)^{(17)}}$$

$$S_X(5,8)=\left(\frac{5!}{\tau^5}\right)^2\frac{39351312(2n+3)(1176n^5+13377n^4+67788n^3+187677n^2+294362n+214000)}{(n+9)^{(17)}}$$

$$S_X(4,8)=\left(\frac{4!}{\tau^4}\right)^2\frac{\begin{array}{l}91494013911300+223855841008800n+245374614084600n^2+157342613828400n^3+\\65715133383900n^4+18631206594000n^5+3526041519000n^6+407614006800n^7+\\22545022500n^8\end{array}}{(n+9)^{(17)}}$$

$$S_X(3,8)=\left(\frac{3!}{\tau^3}\right)^2\frac{1829520(2n+3)\left(\begin{array}{l}480n^9+10905n^8+122660n^7+870280n^6+4286390n^5\\+15038680n^4+36937370n^3+60457345n^2+60018014n+28027132\end{array}\right)}{(n+9)^{(17)}}$$

$$S_X(2,8)=\left(\frac{2!}{\tau^2}\right)^2\frac{\begin{array}{l}109661946292800+345931715606400n+499378967897040n^2+437715142658640n^3+\\260903295811620n^4+111823912292400n^5+35453881136520n^6+8438717715120\,n^7+\\1516469460660n^8+203817675600n^9+19670999040n^{10}+1229437440n^{11}+38419920n^{12}\end{array}}{(n+9)^{(17)}}$$

$$S_X(1,8)=\left(\frac{1!}{\tau^1}\right)^2\frac{144(2n+3)\left(\begin{array}{l}600n^{13}+24825n^{12}+510600n^{11}+6878625n^{10}+67545180n^9\\+502535340n^8+2860262740n^7+12450968165n^6+41247209428n^5\\+102524465081n^4+185319213484n^3+229832768156n^2\\+175331045616n+62633044800\end{array}\right)}{(n+9)^{(17)}}$$

$$S_X(0,8)=\frac{\begin{array}{l}81n^8+972n^7+18522n^6+136080n^5+767529n^4+\\2655828n^3+5745708n^2+7004880n+3628800\end{array}}{(n+1)^{(9)}}$$

IV. FMP VRF diagonals to the 8th degree

Matlab was used to generate the provided equations. Calculations were cross-validated up to the third degree to those in, Chapter 13, Appendix 13.6, where the first four iterations were provided.

The $S_X(j,i)$ denormalized VRF diagonals for FMP filters up to the 8th-degree:

- 0th-degree:

$$S_X(0,0)=\frac{1-\theta}{1+\theta}$$

- 1st-degree:

$$S_X(1,1)=\left(\frac{1!}{\tau^1}\right)^2\frac{2(1-\theta)^3}{(1+\theta)^3}$$

$$S_X(0,1)=\frac{(1-\theta)(5+4\theta+\theta^2)}{(1+\theta)^3}$$

- 2nd-degree:

$$S_X(2,2)=\left(\frac{2!}{\tau^2}\right)^2\frac{3/2(1-\theta)^5}{(1+\theta)^5}$$

$$S_X(1,2)=\left(\frac{1!}{\tau^1}\right)^2\frac{1/2(1-\theta)^3(49+50\theta+13\theta^2)}{(1+\theta)^5}$$

$$S_X(0,2)=\frac{(1-\theta)(19+24\theta+16\theta^2+6\theta^3+\theta^4)}{(1+\theta)^5}$$

- 3rd-degree:

$$S_X(3,3)=\left(\frac{3!}{\tau^3}\right)^2\frac{5/9(1-\theta)^7}{(1+\theta)^7}$$

$$S_X(2,3)=\left(\frac{2!}{\tau^2}\right)^2\frac{1/2(1-\theta)^5(63+76\theta+23\theta^2)}{(1+\theta)^7}$$

$$S_X(1,3)=\left(\frac{1!}{\tau^1}\right)^2\frac{5/18(1-\theta)^3(581+970\theta+762\theta^2+298\theta^3+53\theta^4)}{(1+\theta)^7}$$

$$S_X(0,3)=\frac{(1-\theta)(69+104\theta+97\theta^2+64\theta^3+29\theta^4+8\theta^5+\theta^6)}{(1+\theta)^7}$$

- 4th-degree:

$$S_X(4,4)=\left(\frac{4!}{\tau^4}\right)^2\frac{35/288(1-\theta)^9}{(1+\theta)^9}$$

$$S_X(3,4)=\left(\frac{3!}{\tau^3}\right)^2\frac{5/72(1-\theta)^7(253+338\theta+113\theta^2)}{(1+\theta)^9}$$

$$S_X(2,4)=\left(\frac{2!}{\tau^2}\right)^2\frac{\tfrac{7}{288}(1-\theta)^5\left(12521+251144\theta+227460\theta^2+101440\theta^3+20210\theta^4\right)}{(1+\theta)^9}$$

$$S_X(1,4)=\left(\frac{1!}{\tau^1}\right)^2\frac{\tfrac{5}{72}(1-\theta)^3\left(12199+255888\theta+289230\theta^2+212160\theta^3+100130\theta^4+29880\theta^5+4490\theta^6\right)}{(1+\theta)^9}$$

$$S_X(0,4)=\frac{(1-\theta)\left(251+4100\theta+4460\theta^2+3800\theta^3+2560\theta^4+1300\theta^5+460\theta^6+100\theta^7+\theta^8\right)}{(1+\theta)^9}$$

- 5th-degree:

$$S_X(5,5)=\left(\frac{5!}{\tau^5}\right)^2\frac{\tfrac{7}{400}(1-\theta)^{11}}{(1+\theta)^{11}}$$

$$S_X(4,5)=\left(\frac{4!}{\tau^4}\right)^2\frac{\tfrac{7}{288}(1-\theta)^9\left(221+316\theta+113\theta^2\right)}{(1+\theta)^{11}}$$

$$S_X(3,5)=\left(\frac{3!}{\tau^3}\right)^2\frac{\tfrac{1}{48}(1-\theta)^7\left(11117+25176\theta+24926\theta^2+120720\theta^3+25490\theta^4\right)}{(1+\theta)^{11}}$$

$$S_X(2,5)=\left(\frac{2!}{\tau^2}\right)^2\frac{\tfrac{7}{288}(1-\theta)^5\left(87581+225176\theta+2958550\theta^2+2448800\theta^3+1297150\theta^4+430016\theta^5+71210\theta^6\right)}{(1+\theta)^{11}}$$

$$S_X(1,5)=\left(\frac{1!}{\tau^1}\right)^2\frac{\tfrac{7}{1800}(1-\theta)^3\begin{pmatrix}1028527+24540740\theta+33526360\theta^2+32509180\theta^3+23455100\theta^4+12399580\theta^5\\+478036\theta^6+1248740\theta^7+17467\theta^8\end{pmatrix}}{(1+\theta)^{11}}$$

$$S_X(0,5)=\frac{(1-\theta)\begin{pmatrix}923+15720\theta+18470\theta^2+17920\theta^3+14840\theta^4+\\10240\theta^5+5620\theta^6+2320\theta^7+670\theta^8+120\theta^9+\theta^{10}\end{pmatrix}}{(1+\theta)^{11}}$$

- 6th-degree:

$$S_X(6,6)=\left(\frac{6!}{\tau^6}\right)^2\frac{\tfrac{77}{43200}(1-\theta)^{13}}{(1+\theta)^{13}}$$

$$S_X(5,6)=\left(\frac{5!}{\tau^5}\right)^2\frac{\tfrac{7}{4800}(1-\theta)^{11}\left(705+10580\theta+3970\theta^2\right)}{(1+\theta)^{13}}$$

$$S_X(4,6)=\left(\frac{4!}{\tau^4}\right)^2\frac{\tfrac{77}{1728}(1-\theta)^9\left(2071+51020\theta+5376\theta^2+27500\theta^3+6010\theta^4\right)}{(1+\theta)^{13}}$$

$$S_X(3,6)=\left(\frac{3!}{\tau^3}\right)^2\frac{\tfrac{1}{192}(1-\theta)^7\left(397415+11699340\theta+17018650\theta^2+15253800\theta^3+8669050\theta^4+3028300\theta^5+516710\theta^6\right)}{(1+\theta)^{13}}$$

$$S_X(2,6)=\left(\frac{2!}{\tau^2}\right)^2\frac{\tfrac{7}{21600}(1-\theta)^5\begin{pmatrix}38759503+1154881166\theta+18565860 4\theta^2+2041387120\theta^3+\\1636484500\theta^4+959364220\theta^5+408106840\theta^6+116899960\theta^7+175666630\theta^8\end{pmatrix}}{(1+\theta)^{13}}$$

$$S_X(1,6)=\left(\frac{1!}{\tau^1}\right)^2 \frac{{}^{7}\!/_{1800}(1-\theta)^3\begin{pmatrix}4590925+118726778\theta+181729223\theta^2+206525578\theta^3+\\186020018\theta^4+134417758\theta^5+76829942\theta^6+34696222\theta^7\\+12010017\theta^8+2952252\theta^9+40687\theta^{10}\end{pmatrix}}{(1+\theta)^{13}}$$

$$S_X(0,6)=\frac{(1-\theta)\begin{pmatrix}3431+59992\theta+73344\theta^2+76308\theta^3+\\70071\theta^4+58810\theta^5+40968\theta^6+23808\theta^7\\+10938\theta^8+3788\theta^9+928\theta^{10}+148\theta^{11}+\theta^{12}\end{pmatrix}}{(1+\theta)^{13}}$$

- 7^{th}-degree:

$$S_X(7,7)=\left(\frac{7!}{\tau^7}\right)^2 \frac{{}^{143}\!/_{1058400}(1-\theta)^{15}}{(1+\theta)^{15}}$$

$$S_X(6,7)=\left(\frac{6!}{\tau^6}\right)^2 \frac{{}^{11}\!/_{43200}(1-\theta)^{13}\left(527+820\theta+319\theta^2\right)}{(1+\theta)^{15}}$$

$$S_X(5,7)=\left(\frac{5!}{\tau^5}\right)^2 \frac{{}^{13}\!/_{43200}(1-\theta)^{11}\left(74023+1940778\theta+2140622\theta^2+1138222\theta^3+254447\theta^4\right)}{(1+\theta)^{15}}$$

$$S_X(4,7)=\left(\frac{4!}{\tau^4}\right)^2 \frac{{}^{77}\!/_{1728}(1-\theta)^9\left(22967+745504\theta+1166113\theta^2+1105544\theta^3+657977\theta^4+236888\theta^5+40877\theta^6\right)}{(1+\theta)^{15}}$$

$$S_X(3,7)=\left(\frac{3!}{\tau^3}\right)^2 \frac{{}^{11}\!/_{43200}(1-\theta)^7\begin{pmatrix}58500149+2022613558\theta+3640937712\theta^2+4370396666\theta^3+\\3759734430\theta^4+2343875066\theta^5+1045058722\theta^6+308207188\theta^7+46575589\theta^8\end{pmatrix}}{(1+\theta)^{15}}$$

$$S_X(2,7)=\left(\frac{2!}{\tau^2}\right)^2 \frac{{}^{7}\!/_{21600}(1-\theta)^5\begin{pmatrix}206729843+6789812776\theta+12392432265\theta^2+16104997784\theta^3+\\16147374428\theta^4+12813152256\theta^5+8046129964\theta^6+3984754000\theta^7+\\1506142881\theta^8+397820012\theta^9+56724918\theta^{10}\end{pmatrix}}{(1+\theta)^{15}}$$

$$S_X(1,7)=\left(\frac{1!}{\tau^1}\right)^2 \frac{{}^{1}\!/_{88200}(1-\theta)^3\begin{pmatrix}6803200609+185727608882\theta+306084540036\theta^2+384434536866\theta^3+\\396268656687\theta^4+343379221200\theta^5+250222378000\theta^6+151559468888\theta^7+\\76324843559\theta^8+31531055558\theta^9+10361801116\theta^{10}+2508307388\theta^{11}+350743218\theta^{12}\end{pmatrix}}{(1+\theta)^{15}}$$

$$S_X(0,7)=\frac{(1-\theta)\begin{pmatrix}12869+228640\theta+287650\theta^2+310400\theta^3+\\305850\theta^4+277920\theta^5+228170\theta^6+163840\theta^7+\\99490\theta^8+49440\theta^9+19410\theta^{10}+5760\theta^{11}+1210\theta^{12}+160\theta^{13}+\theta^{14}\end{pmatrix}}{(1+\theta)^{15}}$$

- 8th-degree:

$$S_X(8,8)=\left(\frac{8!}{\tau^8}\right)^2 \frac{{}^{143}\!/\!{}_{18063360}\,(1-\theta)^{17}}{(1+\theta)^{17}}$$

$$S_X(7,8)=\left(\frac{7!}{\tau^7}\right)^2 \frac{{}^{143}\!/\!{}_{16934400}\,(1-\theta)^{15}\left(1501+2402\theta+961\theta^2\right)}{(1+\theta)^{17}}$$

$$S_X(6,8)=\left(\frac{6!}{\tau^6}\right)^2 \frac{{}^{11}\!/\!{}_{276480}\,(1-\theta)^{13}\left(89867+2472000\theta+2823740\theta^2+154536\theta^3+351111\theta^4\right)}{(1+\theta)^{17}}$$

$$S_X(5,8)=\left(\frac{5!}{\tau^5}\right)^2 \frac{{}^{13}\!/\!{}_{86400}\,(1-\theta)^{11}\begin{pmatrix}1992416+69546110\theta+115048833\theta^2+\\ 113709260\theta^3+6993106\theta^4+2567327\theta^5+444749\theta^6\end{pmatrix}}{(1+\theta)^{17}}$$

$$S_X(4,8)=\left(\frac{4!}{\tau^4}\right)^2 \frac{{}^{143}\!/\!{}_{1105920}\,(1-\theta)^{9}\begin{pmatrix}68169131+2625353920\theta+5136728860\theta^2+6575953280\theta^3\\ +5948359300\theta^4+3860392640\theta^5+1767945080\theta^6+527104000\theta^7+79031870\theta^8\end{pmatrix}}{(1+\theta)^{17}}$$

$$S_X(3,8)=\left(\frac{3!}{\tau^3}\right)^2 \frac{{}^{11}\!/\!{}_{345600}\,(1-\theta)^{7}\begin{pmatrix}2941805587+113433355680\theta+234258166681\theta^2+335281517440\theta^3\\ +362952740620\theta^4+306749181760\theta^5+203460862420\theta^6+\\ 105179655040\theta^7+40885605910\theta^8+10890984480\theta^9+1525713970\theta^{10}\end{pmatrix}}{(1+\theta)^{17}}$$

$$S_X(2,8)=\left(\frac{2!}{\tau^2}\right)^2 \frac{{}^{11}\!/\!{}_{3386880}\,(1-\theta)^{5}\begin{pmatrix}103101310615+3626811106000\theta+7218544972580\theta^2+10471463420120\theta^3\\ +12083915437050\theta^4+11493864672480\theta^5+9114681558360\theta^6+\\ 6020854458480\theta^7+3307390235130\theta^8+1486236669200\theta^9\\ +524445327620\theta^{10}+132286299640\theta^{11}+18331329190\theta^{12}\end{pmatrix}}{(1+\theta)^{17}}$$

$$S_X(1,8)=\left(\frac{1!}{\tau^1}\right)^2 \frac{{}^{1}\!/\!{}_{352800}\,(1-\theta)^{3}\begin{pmatrix}115199131591+3268696242980\theta+5668657705930\theta^2+7613750908740\theta^3\\ +8573570130810\theta^4+8346017873880\theta^5+7081756480270\theta^6+\\ 5217420325400\theta^7+3306576099610\theta^8+1804387450500\theta^9+843050135910\theta^{10}\\ +332965121170\theta^{11}+107780200870\theta^{12}+26349737600\theta^{13}+3777149890\theta^{14}\end{pmatrix}}{(1+\theta)^{17}}$$

$$S_X(0,8)=\frac{(1-\theta)\begin{pmatrix}48619+874980\theta+1121140\theta^2+1238100\theta^3+1261900\theta^4+1213620\theta^5+1092580\theta^6+898440\theta^7\\ +655360\theta^8+412260\theta^9+217780\theta^{10}+94020\theta^{11}+32140\theta^{12}+8340\theta^{13}+1540\theta^{14}+180\theta^{15}+\theta^{16}\end{pmatrix}}{(1+\theta)^{17}}$$

V. N_s

V.1. N_s up to the 8th degree

Degree	0	1	2	3	4
N_s	$2/(1-\theta)$	$3.2/(1-\theta)$	$4.3636/(1-\theta)$	$5.5054/(1-\theta)$	$6.6321/(1-\theta)$

Degree	5	6	7	8	
N_s	$7.7478/(1-\theta)$	$8.8548/(1-\theta)$	$9.955/(1-\theta)$	$11.0493/(1-\theta)$	