1. Let $A$ and $B$ be two invertible matrices. Prove:

(a) $(AB)^{-1} = B^{-1}A^{-1}$ (5%)

(b) $(A^T)^{-1} = (A^{-1})^T$ (5%)

2. Let $A$ be $n \times n$ invertible matrix and $I_n$ be $n \times n$ unit matrix. Use the fact: $A\ \mathrm{adj}(A) = \det(A)\ I_n$ to prove

(a) $\det(\mathrm{adj}(A)) = [\det(A)]^{n-1}$ (5%)

(b) $\mathrm{adj}(A^{-1}) = \frac{A}{\det(A)}$ (5%)

3. Let $v_1, v_2, \cdots, v_n$ be vectors in a vector space $V$, and let $S$ be the span of $v_1, v_2, \cdots, v_n$. Prove that $S$ is a subspace of $V$. (10%)

4. Let $A$ be a $3 \times 3$ matrix. If $\det(A)=7$, find

(a) $\det(3A)$ (b) $\det(A^{-1})$ (c) $\det(2A^{-1})$

(d) $\det((2A)^{-1})$ (e) $\det(A^T)$ (10%)

5(a) Let $v_1 = (1, -3, 2)$, $v_2 = (1, 0, -4)$ and $u = (3, -9, -2)$. Determine whether $u$ is the span of $v_1$ and $v_2$ or not. (5%)

5(b) Let $v_1 = (1, 4, -2)$, $v_2 = (-2, -3, 7)$ and $u = (4, 1, h)$. Determine the value of $h$ such that $u$ is in the span of $v_1$ and $v_2$. (5%)

6. Let

$$A = \begin{bmatrix} 1 & 4 & 4 \\ 1 & 2 & 4 \\ 1 & 3 & 2 \end{bmatrix}$$

Find the inverse of $A$ by elementary row operations. (10%)

7. Solve the following system by Gauss elimination method. (10%)

$$\begin{aligned} 3x_1 - x_2 + x_3 + 7x_4 &= 13 \\ -2x_1 + x_2 - x_3 - 3x_4 &= -9 \\ -2x_1 + x_2 - 7x_4 &= -8 \end{aligned}$$

8. Determine the value of $a$ for which the following system has (a) no solution (b) infinitely many solutions. (10%)

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 2)z &= a + 4 \end{aligned}$$

9. Determine whether each of the following sets is a vector space or not. If your answer is "No", you must explain why. (10%)

(a) $\{(x, y) | x = 0 \text{ or } y = 0\}$

(b) $\{(x, y, z) | x + y + z = 1\}$

(c) $\{\begin{bmatrix} x & y \\ y & x \end{bmatrix} | x, y \in R\}$

(d) All polynomials of the form $P(t) = at^2$, where $a \in R$.

(e) $\{f | f \in C[1, 5], f(3) = 1\}$, where $C[1, 5]$ is the set of all continuous real-valued function defined on the closed interval $[1, 5]$.

10. Determine each of the following statements is true or false. (10%)

(a) Let $A, B$ be matrices. Then $(A + B)^2 = A^2 + 2AB + B^2$.

(b) Let $A, B, C$ be matrices. If $AB = AC$, then $B = C$.

(c) Let $A, B$ be matrices. If $AB = O$(zero matrix), then either $A = O$ or $B = O$.

(d) Let $A, B$ be $n \times n$ matrices. If $AB$ is invertible then both $A$ and $B$ must be invertible.

(e) The inverse of a symmetric matrix is also a symmetric matrix.