1. Let A and B be two invertible matrices. Prove:

(a) $(AB)^{-1} = B^{-1}A^{-1}$ (5%)

(b) $(A^T)^{-1} = (A^{-1})^T$ (5%)

2(a). If square matrix A has an inverse, prove that the inverse is unique. (5%)

2(b). If square matrix A is nonsingular, prove that the solution of matrix equation $Ax = b$ is unique. (5%)

3. If square matrix A is row equivalent to a unit matrix I, prove that A is invertible. (10%)

4. Determine the value of a for which the following system has (a) no solution (b) only one solution. (10%)

$$\begin{aligned} x + 2y + z &= 2 \\ 2x - 2y + 3z &= 1 \\ x + 2y - az &= a \end{aligned}$$

5(a). Let

$$A = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{bmatrix}$$

Compute $\det(A)$. (5%)

5(b) Let

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix}$$

Find elementary matrices E such that $EB = A$ (5%)

6. Let

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

Find the inverse of A by row operations. (10%)

7. Solve the following system by Gauss elimination method. (10%)

$$\begin{aligned} x_1 - 2x_2 + x_3 - 4x_4 &= 1 \\ x_1 + 3x_2 + 7x_3 + 2x_4 &= 2 \\ x_1 - 12x_2 - 11x_3 - 16x_4 &= 5 \end{aligned}$$

8. For the following definition of a set S and a vector space V, determine whether S is a subspace of V. If your answer is "No", you must explain why. (10%)

(a) S is all polynominals of the form $p(t) = at^2$ where $a \in R$

$V = P_n$

(b) $S = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} | \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0\}$

$V = M_{22}$

(c) $S = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} | \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0\}$

$V = M_{22}$

(d) $S = \{(x, y, z) | y = x + z\}$

$V = R^3$

(e) $S = \{(3s, 2 + 5s) | s \in R\}$

$V = R^2$

9. Determine each of the following statements is true or false. (20%)

(a) Let A, B be matrices. Then $(A + B)^2 = A^2 + 2AB + B^2$.

(b) If matrix A is invertible and scalar $r \neq 0$, then $(rA)^{-1} = rA^{-1}$.

(c) Let A, B, C be matrices. If $AB = AC$, then $B = C$.

(d) Let A, B be matrices.

If $AB = O$(zero matrix), then either $A = O$ or $B = O$.

(e) The inverse of a symmetric matrix is also a symmetric matrix.

(f) The transpose of an elementary matrix is an elementary matrix.

(g) The row echelon form of a matrix is unique.

(h) If A is an $n \times n$ matrix, then $\det(AA^T) \geq 0$.

(i) If matrix A is invertible, then $Ax = O$(zero matrix) has only solution $x = O$.

(j) A matrix A is invertible if and only if $\det(A) \neq 0$.