1. Let A be an $m \times n$ matrix and B be an $n \times l$ matrix. If $\mathrm{CS}(A)$ denotes the column space of A, prove that $\mathrm{CS}(AB) \subseteq \mathrm{CS}(A)$. (10%)

2. Let $S = \{v_1, v_2, \cdots, v_k\}$ be a set of vectors in a vector space V. If $v_j \in S$ and v_j can be expressed as a linear combination of other vectors in S, prove that $S - \{v_j\}$ spans the same space as S. (10%)

3. Let $T : V \to W$ be a linear transformation and $v_1, v_2, \cdots, v_k \in V$. Prove
(a) If T is one to one transformation then the kernel of T contains only zero vector. (5%)
(b) If T is one to one transformation and $\{T(v_1), T(v_2), \cdots, T(v_k)\}$ is linearly dependent, then $\{v_1, v_2, \cdots, v_k\}$ is also linearly dependent. (5%)

4. Let V and W be the subspace of R^4 and are spanned by $\{(1,1,0,-1), (1,2,3,0), (2,3,3,-1)\}$ and $\{(1,2,2,-2), (2,3,2,-3), (1,3,4,-3)\}$ respectively. Compute dim $(V+W)$ and dim $(V \cap W)$. (10%)

5. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 8 \\ 2 & 4 & -3 & 10 & 9 \\ 3 & 6 & 0 & 6 & 9 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If we know that A is row equivalent to B, find bases for $NS(A)$ and $CS(A)$. (10%)

6. Let $T : P_2 \to R$ be the linear transformation defined by

$$T(p) = \int_0^1 p(x)dx$$

Find a basis for the kernel of T. (10%)

7(a) Find the coordinates of p relative to the basis $S = \{p_1, p_2, p_3\}$, where $p = 2 + x - x^2$, $p_1 = 1 - x$, $p_2 = x + x^2$ and $p_3 = 1 - x^2$. (5%)
7(b) Consider the bases $B = \{u_1, u_2\}$ and $B' = \{v_1, v_2\}$ for R^2, where $u_1 = (4,1)$, $u_2 = (3,1)$, $v_1 = (-1,-2)$ and $v_2 = (2,3)$. Find the transition matrix from B to B' (5%)

Part	Size of A	Rank$A$	Rank$(A|b)$
a	3×3	3	3
b	3×3	2	3
c	5×9	2	2
d	5×9	2	3
e	6×2	2	2

8. Determine whether the function F is a linear transformation. (10%)

(a) $F : R^2 \to R^2$ $\quad F(x, y) = (x^2, y)$

(b) $F : R^3 \to R^2$ $\quad F(x, y, z) = (3x - 4y + z, 2x - 5y + 2z)$

(c) $F : R^2 \to R^2$ $\quad F(x, y) = (x\cos\theta + y\sin\theta, x\sin\theta - y\cos\theta)$

(d) $F : P_2 \to P_2$ $\quad F(a_0 + a_1x + a_2x^2) = a_0 + a_1(x + 1) + a_2(x + 1)^2$

(e) $F : M_{22} \to R$

$$F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a^2 + b^2$$

9. Let V be a nonzero finite-dimensional vector space, and $v_1, \cdots, v_p$ are vectors in V. Mark each statement true or false. (10%)

(a) If there exists a set $\{v_1, \cdots, v_p\}$ that spans V, then dim $V \leq p$.

(b) If there exist a linearly independent set $\{v_1, \cdots, v_p\}$ in V, then dim $V \geq p$.

(c) If dim $V = p$, then there exist a spanning set of $p + 1$ vectors in V.

(d) If there exist a linearly dependent set $\{v_1, \cdots, v_p\}$ in V, then dim $V \leq p$.

(e) If $p \geq 2$ and dim $V = p$, then every set of $p - 1$ nonzero vectors is linearly independent.

10. In each part(a-e), use the information in the following table to determine whether the linear system $Ax = b$

(I) is consistent? (has at least one solution) (5%)

(II) has unique solution? (5%)