1. Consider an $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. Prove that
(a) $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$. (5%)
(b) $a_{11} + a_{22} + \cdots + a_{nn} = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. (5%)

2. Let P and Q be $n \times n$ orthogonal matrices and $x, y \in R^n$ prove
(a) $(Qx) \cdot (Qy) = x \cdot y$ (5%)
(b) PQ is also an orthogonal matrix. (5%)

3. If square matrix A is diagonalizable, prove that $A^k (k > 1)$ is also diagonalizable. (10%)

4. If u and v are vectors of a real inner product space V, prove that
$| < u, v > | \leq ||u|| \times ||v||$. (10%)

5. Let W be the space spanned by $\{u_1, u_2, u_3\}$, where $u_1 = (1, 4, 5, 2)$, $u_2 = (2, 1, 3, 0)$ and $u_3 = (-1, 3, 2, 2)$. Find a basis for the orthogonal complement of W. (10%)

6. Let W be the space spanned by orthogonal set $\{v_1, v_2, v_3\}$, where $v_1 = (1, 1, 0, -1)$, $v_2 = (1, 0, 1, 1)$, and $v_3 = (0, -1, 1, -1)$. If $y = (3, 4, 5, 6)$ can be expressed in the form $y = \overline{y} + z$, where $\overline{y}$ is in the space of W, and z is orthogonal to W. Please find $\overline{y}$ and z. (10%)

7. Let $v_1 = (1, 1, 1)$, $v_2 = (0, 1, 1)$ and $v_3 = (0, 0, 1)$ Use the Gram-Schmidt process to convert the basis $\{v_1, v_2, v_3\}$ into an orthonormal basis. (10%)

8. Consider the vectors in R^4, let $y = (3, -1, 1, 13), v_1 = (1, -2, -1, 2), v_2 = (-4, 1, 0, 3)$.Find the distance from y to the subspace spanned by $\{v_1, v_2\}$. (10%)

9. Let

$$A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$$

If we know one eigenvalue of A is 3, find a basis for the eigenspace corresponding to this eigenvalue. (10%)

10. Let

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

If we know the eigenvalues of A are 1,-2,-2. Determine whether matrix A is diagonalizable or not. You need to explain your answer. (10%)