1. Let $NS(A)$ denote the null space of matrix A and $CS(A)$ denote the column space of matrix A. If A is an $m \times n$ matrix and B is an $n \times l$ matrix, prove

(a) $NS(B) \subseteq NS(AB)$ (5%)

(b) $CS(AB) \subseteq CS(A)$ (5%)

2. Let $V$ and $W$ be vector spaces, and let $T : V \to W$ be a linear transformation. If $U$ is a subspace of $V$ and $T(U) = \{T(x) | x \in U\}$ prove that $T(U)$ is a subspace of $W$. (10%)

3. If $\{v_1, \cdots, v_k\}$ is linearly independent and $v_{k+1}$ is not in the span of $\{v_1, \cdots, v_k\}$ prove that $\{v_1, \cdots, v_k, v_{k+1}\}$ is still linear independent. (10%)

4. Let $V$ be the vector space of all $2 \times 2$ matrices, and $W_1$ and $W_2$ be the subspace of $V$ defined by

$$W_1 = \{\begin{bmatrix} a & -a \\ b & c \end{bmatrix} | a, b, c \in R\} \qquad W_2 = \{\begin{bmatrix} x & y \\ -x & z \end{bmatrix} | x, y, z \in R\}$$

Find dim $(W_1 \cap W_2)$ and dim $(W_1 + W_2)$. (10%)

5. Let

$$A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If we know that $A$ is row equivalent to $B$, find bases for $NS(A)$ and $CS(A)$. (10%)

6. Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis for $R^4$. If $T : R^4 \to R^3$ is a linear transformation for which

$$T(e_1) = (1, 2, 1), \quad T(e_2) = (0, 1, 0),$$
$$T(e_3) = (1, 3, 0), \quad T(e_4) = (1, 1, 1)$$

(a) Find the rank of $T$. (5%) (b) Find the nullity of $T$. (5%)

7. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ be bases for a vector space $V$, and suppose that $a_1 = 4b_1 - b_2$, $a_2 = -b_1 + b_2 + b_3$, and $a_3 = b_2 - 2b_3$.

(a) Find the change-of-coordinates (or transition) matrix from $A$ to $B$. (5%)

(b) Find $[x]_B$ for $x = 3a_1 + 4a_2 + a_3$. (5%)

8. Determine whether the function $F$ is a linear transformation. (10%)

(a) $F : R^2 \to R^2$ $\quad F(x, y) = (x^2, y)$

(b) $F : R^3 \to R^2$ $\quad F(x, y, z) = (3x - 4y + 1, 2x - 5z + 2)$

(c) $F : R^2 \to R^2$ $\quad F(x, y) = (\cos(x) + \sin(y), \sin(x) - \cos(y))$

(d) $F : P_2 \to P_2$ $\quad F(a_0 + a_1x + a_2x^2) = (a_0 + 1) + (a_1 + 1)x + (a_2 + 1)x^2$

(e) $F : M_{22} \to R$

$$F(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = 3a - 4b + c - d$$

9. Let $V$ be a nonzero finite-dimensional vector space, and $v_1, \cdots, v_p$ are vectors in $V$. Mark each statement true or false. (10%)

(a) If there exists a set $\{v_1, \cdots, v_p\}$ that spans $V$, then dim $V \le p$.

(b) If there exist a linearly independent set $\{v_1, \cdots, v_p\}$ in $V$, then dim $V \ge p$.

(c) If dim $V = p$, then there exist a spanning set of $p + 1$ vectors in $V$.

(d) If there exist a linearly dependent set $\{v_1, \cdots, v_p\}$ in $V$, then dim $V \le p$.

(e) If $p \ge 2$ and dim $V = p$, then every set of $p - 1$ nonzero vectors is linearly independent.

10.

(a) Find a basis for the set of vectors in the plane $x + 2y + z = 0$. (5%)

(b) Show that the vector space of all continuous functions defined on the entire real line is infinite-dimensional. (5%)