Space containing VAR networks indistinguishable from S-LMI

Proof for $cJ_k + dI$ spanning (matrix) vector space

- Refinements are necessary, but central ideas should be visible.
- Some proofs might be good to show here and use this as a supplement.

Let $0 < c, d \in \mathbb{R}$. Let J_k be a k dimensional matrix of ones, the subindex of which will be omitted in the following. Let I_k be k dimensional identity matrix, similarly omitting the subindex in the following. We will first show that the space which these two matrices span contains a set of drift matrices $A \in span(J, I)$ that produce covariance matrices perfectly also explained by S-LMI model.

Let $A = cJ + dI$. Let Σ be the covariance matrix of random variables X observed at different time points $t \in \mathbb{N}$. We will omit any special cases that might occur at small $k \leq 3$ due to the factor model being just identifiable at $k = 3$. The cross-covariance is $A\Sigma = (cJ + dI)\Sigma = cJ\Sigma + d\Sigma$. Note that $J\Sigma$ is effectively summing over the covariances for each variable producing then K different values such that

$$cJ\Sigma = \begin{pmatrix} c\sum_{i=1}^{k} \sigma_{i1} & \dots & c\sum_{i=1}^{k} \sigma_{ik} \\ \vdots & \ddots & \vdots \\ c\sum_{i=1}^{k} \sigma_{i1} & \dots & c\sum_{i=1}^{k} \sigma_{ik} \end{pmatrix} = \begin{pmatrix} a & \dots & b \\ \vdots & \ddots & \vdots \\ a & \dots & b \end{pmatrix}$$

where $a \in \mathbb{R}$ and clearly $d\Sigma$ is just a scalar multiple of the covariance. Define innovations contained in $\Gamma \in \mathbb{R}^{K\times 1}$ to have mean of 0 $E[\Gamma] = 0$ and are independent $E[\Gamma\Gamma^T] = 0$. We also have that, since A completely now defines covariances of X, that all variables have identical covariance $\sigma_{12} = \sigma_{13} = \sigma_{23} ... = \sigma_{ij}$ for all $i \neq j$. This is because A defines identical cross-lagged autoregression coefficients for all $x \in X$. Note that A also has identical autoregression coefficients and hence the variances are also identical for all x. This means that

$$cJ\Sigma = \begin{pmatrix} a & \dots & a \\ \vdots & \ddots & \vdots \\ a & \dots & a \end{pmatrix} \in span(J)$$

and $d\Sigma, \Sigma \in span(J, I)$ hence also

$$A\Sigma = (cJ + dI)\Sigma \in span(J, I)$$

Also see that

$$AA = (cJ + dI)(cJ + dI) = c^2kJ + dcJ + dcJ + d^2I = J\underbrace{(c^2k + 2dc)}_{\in\mathbb{R}} + \underbrace{d^2}_{\in\mathbb{R}}I \in span(J, I)$$

where we can see that AA, A have coordinates in the space $span(J, I)$, derivable from the equation above. Using induction this can be proven for $A \times A \times ... \times A\Sigma \in span(J, I)$ which means that all cross covariances for any change in time $\Delta = t_T - t_0$ are also in this space.

Now, since we have that all covariances are the same, then for the S-LMI model with $\mathrm{Var}(\eta) = 1, E[\eta] = 0$ all factor loadings must be the same so that$\lambda_1 = \lambda_2 = ... = \lambda_k = \lambda : \mathrm{Cov}(X_i, X_k) = \lambda^2, i \neq j$. The residual cross covariances must also be the same given some Δ, as seen above. (By this point the practical constraint is evident, but will be discussed later, elsewhere. - Sakari) Now we have that the S-LMI imposed cross covariance is

$$\Lambda\Lambda^T \prod_{i=0}^{\Delta} \delta_i + \Omega_\Delta = cJ + dI \in span(J, I)$$

where $\delta \in \mathbb{R}$ is the regression coefficient for subsequent η_1, η_2 and the respective product is for multiple subsequent regressions.

Another, more general, criterion for indistinguishability that the subspace of positive semi-definite symmetric matrices $(\Sigma = \Lambda\Lambda^T + \Omega) \in \mathbb{R}^{K\times K}$ must be invariant under A. This is because $\Lambda\Lambda^T\delta + \Omega_\Delta$ is always positive semi-definite symmetric.