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Distinguishability of Vector Autoregressive Symptom-Network and Common Factor models

Common Factor theoretical model A latent variable η is a causal entity which creates observable symptoms. Conditional independence of variables $P(X_i|\eta, X_j) = P(X_i|\eta) : i \neq j$ given the common factor η is the central proposition of the model. All changes in symptoms are due to changes in η and measuremenet error, meaning that all covariance is explained by η. M_1 denotes the common factor model.

Mutualistic symptom-network theoretical model Symptoms have causal effects on each others. Symptoms can change by themselves or change due to effects not within the symptom-network itself. The symptom-network is all the symptoms. M_2 denotes the symptom network model.

Empirical distinguishability The models are distinguishable up to some moment if M_1, M_2 differ in that moment.

$$
\begin{aligned}
E_{M_1}(X) - E_{M_2}(X) &= D_E \\
\mathrm{Cov}_{M_1}(X) - \mathrm{Cov}_{M_2}(X) &= D_C \\
\mathrm{Skew}_{M_1}(X) - \mathrm{Skew}_{M_2}(X) &= D_S \\
\mathrm{Kurt}_{M_1}(X) - \mathrm{Kurt}_{M_2}(X) &= D_S
\end{aligned}
$$

Change across time: Strict Longitudinal Measurement Invariance vs VAR(1) models

Regarding changes in symptoms across time, the simplest models with practical relevance that we can use for common factor and symptom-network theory are the strict longitudinal measurement invariance (s-LMI) referred to as and the vector autoregressive model of order one (VAR(1)). Here we will analytically approach differences between these models and then perform simulations to quantify how well empirically the models differ when generating from a

mixture distribution. We will first inspect what type of covariance structure VAR(1) imposes, and then similarly s-LMI.

VAR(1) symptom-network covariance structure at two subsequent time points

The VAR(1) model is defined in matrix format as $X_t = C + AX_{t-1} + \Gamma_t$, where Γ_t is independent error column vector with $E[\Gamma_t] = 0$, C is a constant assumed zero. Also assume centered X, $E[X_t] = 0$, in our case. Centering makes covariance calculations easier as the products of expected values can be mostly ignored (they become 0). A is $K \times K$ (borrowing from CT-VAR terminology) 'drift' matrix that includes all lagged effects of $K \times 1$ column vectors X_t to X_{t-1}, K being the number of observed items (symptoms). In this section the focus is on the $2K \times 2K$ covariance matrix where two subsequent measurement time points are observed. All matrices used are real-valued.

First, the covariance matrix (assumed stationary over time) is

$$\begin{aligned}
\mathrm{Cov}(X_t) &= E[X_t X_t^T] \\
&= E[(AX_{t-1} + \Gamma_t)(AX_{t-1} + \Gamma_t)^T] \\
&= E[AX_{t-1}X_{t-1}^T A^T] + \underbrace{E[\Gamma_t \Gamma_t^T]}_{=:\Psi} \\
&= AE[X_{t-1}X_{t-1}^T]A^T + \Psi \\
&= \Sigma_t = \Sigma
\end{aligned}$$

where stationarity poses that Σ is not dependent of time and so the covariance of VAR(1) is denoted as such from hereon. Γ_t is a random $K \times 1$ column vector of (serially) independent innovations at time point t, with $E[\Gamma] = 0, \mathrm{Var}(\Gamma = 1)$. Ψ is covariance of the innovations within time point t - i.e., the contemporaneous covariance. We assume that the innovations are independent and so Ψ is diagonal. The vectorized covariance matrix can also be solved to equal

$$\mathrm{vec}(\Sigma_{VAR(1)}) = (I - A \otimes A)^{-1}\mathrm{vec}(\Psi)$$

Where vec is the vectorization operator and $\otimes$ is the Kronecker product. In the above the mixed Kronecker matrix vector product is used to obtain the result. This will be handy when doing numerical demonstrations.