Convertion of ALT-AZ Coordinates to a Tilted and Rotated Coordinate System

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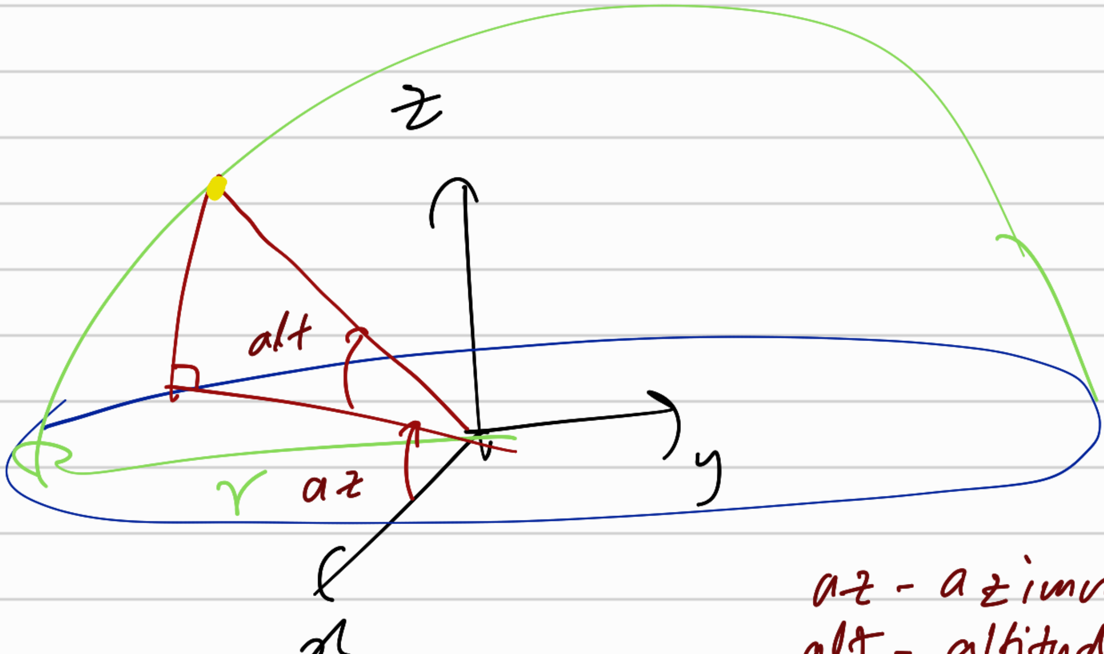
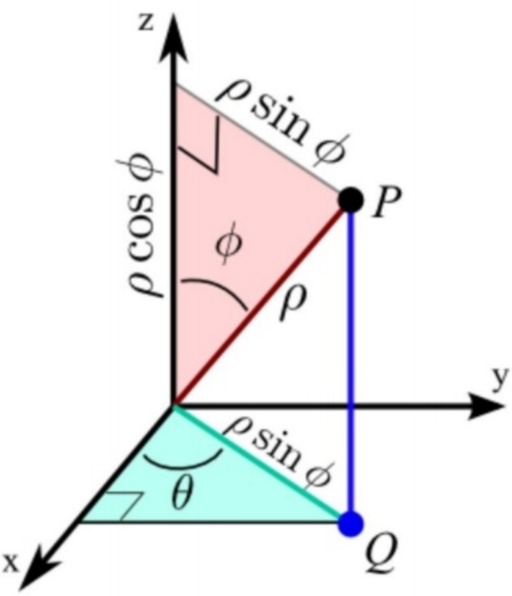
Assume the stars are on a surface of the inside of a sphere with radius r.



az - azimuth
alt - altitude

$$0 \leq az < 2\pi$$
$$-\frac{\pi}{2} \leq alt \leq \frac{\pi}{2}$$

Spherical Coordinate System



$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

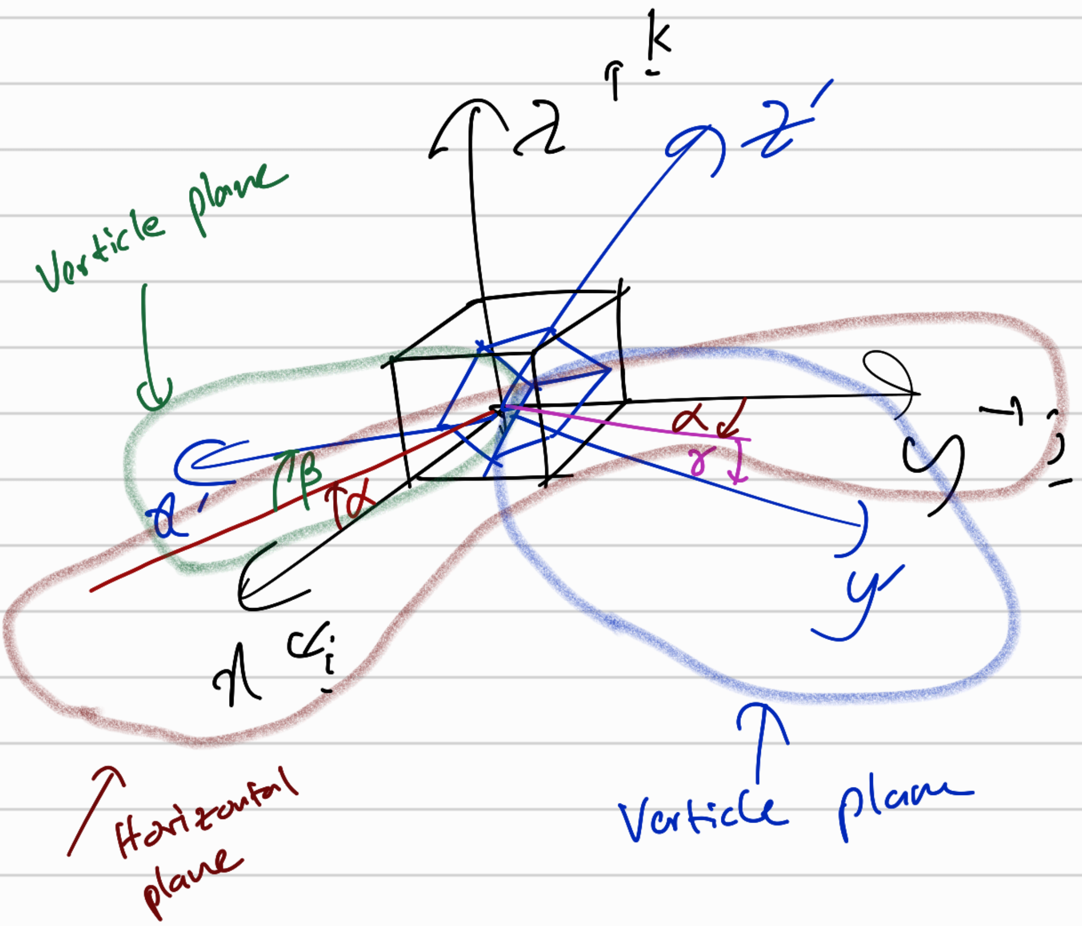
$$\theta = 2\pi - az \quad ; \quad 0 \le \theta < 2\pi$$

$$\phi = \frac{\pi}{2} - alt \quad ; \quad 0 \le \phi \le \pi$$

$$\rho = r$$

$$\text{Say, } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Suppose the coordinate axes are tilted and rotated as following.



α - the angle at which the xy plane is rotated horizontally from the desired direction (North)

β - the angle at which x-axis is tilted vertically from the α-rotated x-axis.

γ - the angle at which y-axis is tilted vertically from the α-rotated y-axis.

$$\underline{x}' = [\cos\alpha, -\sin\alpha, \sin\beta]$$

$$\underline{y}' = (\sin\alpha, \cos\alpha, -\sin\gamma)$$

$$\underline{z}' = [p, q, r]$$

$$\underline{x}' \perp \underline{z}' \Longrightarrow \langle \underline{x}', \underline{z}' \rangle = 0$$

$$p\cos\alpha - q\sin\alpha + r\sin\beta = 0 \quad \text{--- } (1)$$

$$\underline{y}' \perp \underline{z}' \Longrightarrow \langle \underline{y}', \underline{z}' \rangle = 0$$

$$p\sin\alpha + q\cos\alpha - r\sin\gamma = 0 \quad \text{--- } (2)$$

$$\langle \underline{z}', \underline{z}' \rangle = 1$$

$$p^2 + q^2 + r^2 = 1 \quad \text{--- } (3)$$

$① \times \cos\alpha + ② \times \sin\alpha$,

$$p\cos^2\alpha + p\sin^2\alpha = -r\sin\beta\cos\alpha + r\sin\gamma\sin\alpha$$

$$p = r(\sin\gamma\sin\alpha - \sin\beta\cos\alpha) \quad ④$$

$① \times \sin\alpha - ② \times \cos\alpha$,

$$-q\sin^2\alpha - q\cos^2\alpha = -r\sin\alpha\sin\beta - r\sin\gamma\cos\alpha$$

$$q = r(\sin\alpha\sin\beta + \sin\gamma\cos\alpha) \quad ⑤$$

$③$,

$$r^2(\sin\gamma\sin\alpha - \sin\beta\cos\alpha)^2 + r^2(\sin\alpha\sin\beta + \sin\gamma\cos\alpha)^2 + r^2 = 1$$

$$r^2\left\{\sin^2\gamma\sin^2\alpha + \sin^2\beta\cos^2\alpha + \sin^2\alpha\sin^2\beta + \sin^2\gamma\cos^2\alpha + 1\right\} = 1$$

Since $r > 0$,

$$r = +\sqrt{\frac{1}{\sin^2\gamma\sin^2\alpha + \sin^2\beta\cos^2\alpha + \sin^2\alpha\sin^2\beta + \sin^2\gamma\cos^2\alpha + 1}}$$

Define the transformation matrix A,

$$A = \begin{bmatrix} \begin{pmatrix} \underline{x}' \end{pmatrix} & \begin{pmatrix} \underline{y}' \end{pmatrix} & \begin{pmatrix} \underline{z}' \end{pmatrix} \end{bmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{bmatrix} \begin{pmatrix} \underline{x}' \end{pmatrix} & \begin{pmatrix} \underline{y}' \end{pmatrix} & \begin{pmatrix} \underline{z}' \end{pmatrix} \end{bmatrix}}_{\text{Transformation matrix}} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

coordinates to be converted

Converted coordinates in x', y', z'

$$X = A\bar{X}$$

$$A^{-1}X = A^{-1}A\bar{X} \qquad ; \bar{X} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

$$A^{-1}X = \bar{X}$$

$$\boxed{\bar{X} = A^{-1}X}$$

With the previous derivation,

$$x_1 = \rho \sin(\phi_1) \cos(\theta_1)$$

$$y_1 = \rho \sin(\phi_1) \sin(\theta_1)$$

$$z_1 = \rho \cos(\phi_1)$$

$$\frac{y_1}{x_1} = \frac{\rho \sin(\phi_1) \sin(\theta_1)}{\rho \sin(\phi_1) \cos(\theta_1)} = \tan(\theta_1)$$

$$\Rightarrow \theta_1 = \tan^{-1}\left(\frac{y_1}{x_1}\right)$$

$$\frac{x_1}{z_1} = \frac{\rho \sin(\phi_1) \cos(\theta_1)}{\rho \cos(\phi_1)}$$

$$\frac{x_1}{z_1} = \tan(\phi_1) \cos(\theta_1)$$

$$\Rightarrow \phi_1 = \tan^{-1}\left(\frac{x_1}{z_1 \cdot \cos(\theta_1)}\right)$$

From,

$$\theta = 2\pi - az \quad ; \quad 0 \leq \theta < 2\pi$$

$$\phi = \frac{\pi}{2} - alt \quad ; \quad 0 \leq \phi \leq \pi$$

the new altitude-azimuth values can be obtained.

$$\theta_1 = 2\pi - az_1$$

$$\Rightarrow az_1 = 2\pi - \theta_1$$

$$\phi_1 = \frac{\pi}{2} - alt_1$$

$$\Rightarrow alt_1 = \frac{\pi}{2} - \phi_1$$

$$\boxed{\begin{aligned} az_1 &= 2\pi - \theta_1 \\ alt_1 &= \frac{\pi}{2} - \phi_1 \end{aligned}}$$

Notes

- The local altitude and azimuth values should be first found considering the local time and position.

(β)

- The angle at which the x-axis is tilted vertically is considered positive upward, while the similar is considered positive downwards for the y-axis.

(γ)

- The newly found "az_1" and "alt_1" gives the azimuth and altitude values for the new tilted and rotated coordinates.