- Algebra + Topology
  ↑ enrich top Space by sequence of groups (top invariants)

"One way to improve understanding of space is applying algebraic struct. to top. space" – make use of alg. tools to underst. space.

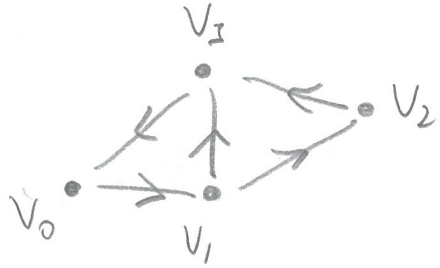
- Improve understanding of space
  - dimension (hom.)
  - existence of holes (imp. question in top. → homeomorphisms)

"Homology groups", "Cohomology groups"

$H_0, H_1, H_2, \ldots \quad H^0, H^1, H^2, \ldots$

<u>Ex</u>: Simplicial homology

Connected graph (directed)

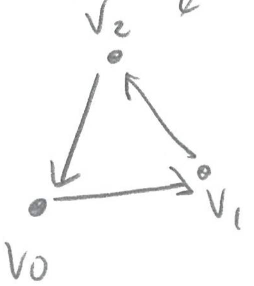


Holes ?

We count <u>two</u> holes that are each bounded by line segments → 1-simplices.

- Build large sets from simplices.
  ↳ Homology give idea about (dimension of) holes.

Ex: simplicial complex: "thick of dimension"

K 

- 1-simplex: line segment
- 2-simplex: triangle
- 3-simplex: tetrahedron
- 4-simplex: ?

⋮

can build large structures from simplices.

Complex: Closed under:

- intersection
- faces

nice closure properties.

- Can identify k-simplex by $k+1$-vertices:

In K: 1-simplices: $\langle v_0, v_1\rangle$, $\langle v_1, v_2\rangle$, $\langle v_2, v_0\rangle$

in K: 0-simplices: $\langle v_0\rangle$, $\langle v_1\rangle$, $\langle v_2\rangle$.

$K = \{\langle v_0, v_1\rangle, \langle v_1, v_2\rangle, \langle v_2, v_0\rangle, \langle v_0\rangle, \langle v_1\rangle, \langle v_2\rangle\}$ is simplicial complex.

- Orientation matters:

$\langle v_i, v_j\rangle = -\langle v_j, v_i\rangle$.

(flip direction)

Hence can "append arrows":

- A 1-cycle is a closed path.

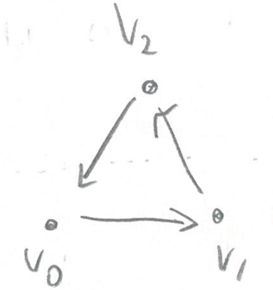
Suppose filled in:

• A 1-boundary is a 1-cycle bounding a 2-simplex.

(In general, "bounding any linear combination of 2-simplices").

Formally: Chains:

$C_r(K) =$ free group generated by r-simplices.



$C_0 = \mathbb{F}\{\langle v_0\rangle, \langle v_1\rangle, \langle v_2\rangle\}$

$C_1 = \mathbb{F}\{\langle v_0, v_1\rangle, \langle v_1, v_2\rangle, \langle v_2, v_0\rangle\}$

$C_n = 0 \quad \forall n > 1$

"free group generated by"

Boundary hom:

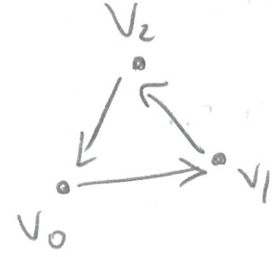
$d_K : C_K \to C_{K-1}$; $\quad 0 \xleftarrow{d^0} C_0(K) \xleftarrow{d^1} C_1(K) \leftarrow$

• linear map identifying boundary.

Ex:

$d^1(\langle v_0, v_1\rangle) = \langle v_1\rangle - \langle v_0\rangle$.

We hope $\langle v_0, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_0\rangle$ is a cycle.



Def: $Z_r(K) = \{x \in C_r(K) \mid d_r(x) = 0\} = \ker(d_r)$.

Check: $d^{\natural}_{*1}(\langle v_0, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_0\rangle)$

$= \langle v_1\rangle - \langle v_0\rangle + \langle v_2\rangle - \langle v_1\rangle$
$+ \langle v_0\rangle - \langle v_2\rangle = 0.$

So is a cycle.

Thm: every boundary is a cycle.

Converse fails!

- $\langle v_0, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_0\rangle$ is cycle, but no boundary...

Defn: $B_r(K) = \{x \in C_r(K) \mid \exists y \in C_{r+1}(K) \text{ s.t. } d^{r+1}(y) = x\}$

Note $C_2(K) = 0$, so

$B_1(K) = 0.$

Thus, in part.,

$\langle v_0, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_0\rangle$

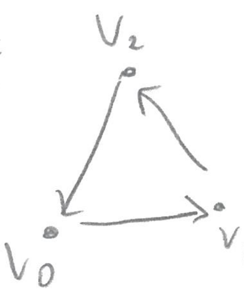
is not a boundary (agrees with intuition!

Homology groups:

$$H_r(K) = Z_r(K) / B_r(K).$$

(measures how cycles fail to bound higher-dim spaces)

Ex:



$H_0(K) \cong \mathbb{Z}$ ← connected components

$H_1(K) \cong \mathbb{Z}$ ← one hole

see next page

$H_n(K) = 0 \quad \forall n > 1.$

__Ex:__

$H_1(K) \cong \mathbb{Z}$: know $B_1(K) = 0$.

So find char. for $Z_1(K)$:

- $C_1(K) = \{ \alpha \langle v_0, v_1 \rangle + \beta \langle v_1, v_2 \rangle + \gamma \langle v_2, v_0 \rangle \mid \alpha, \beta, \gamma \in \mathbb{Z} \}$.

So need $d^1(x) = 0$

$\Leftrightarrow \alpha(\langle v_1 \rangle - \langle v_0 \rangle) + \beta(\langle v_2 \rangle - \langle v_1 \rangle) + \gamma(\langle v_0 \rangle - \langle v_2 \rangle) = 0$.

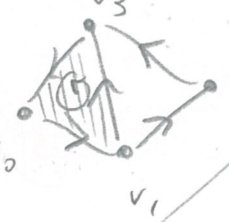
Solve system given $\alpha = \beta = \gamma$.

So $Z_1(K) = \{ \lambda(\langle v_0, v_1 \rangle + \langle v_1, v_2 \rangle + \langle v_2, v_0 \rangle) \mid \lambda \in \mathbb{Z} \}$

$\cong \mathbb{Z}$.

__More ex:__

K



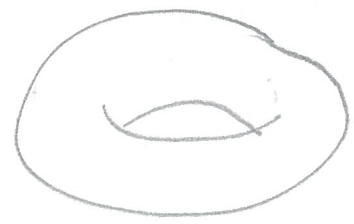
$H_0(K) \cong \mathbb{Z}$

$H_1(K) \cong \mathbb{Z}$

$H_2(K) \cong \mathbb{Z}^*$

$H_n(K) \cong 0 \quad \forall n > 2$.

Triangulation: Homeomorphism between simplicial complex (in $\mathbb{R}^n$) and top. space in question.



[SS: "nice" intuition/characterisation]

$\mathbb{T}^2$: $H_0(\mathbb{T}^2) \cong \mathbb{Z}$

$H_1(\mathbb{T}^2) \cong \mathbb{Z} \oplus \mathbb{Z}$

$H_2(\mathbb{T}^2) \cong \mathbb{Z} \leftarrow$ "empty volume" = two-dim hole

$H_n(\mathbb{T}^2) \cong 0 \quad \forall n > 2$

$\Rightarrow$ rank of group gives no. of holes! (in each dimension)

Cohomology:

- Dual notion (allows for richer structure: cochains of different dimensions can be adjoined; cup product).
- Arrows are flipped:

$$0 \to C^0(K) \xrightarrow{d^0} C^1(K) \xrightarrow{d^1} C^2(K) \xrightarrow{d^2} \cdots$$

- $Z^k(K) = \ker(d^k)$
- $B^k(K) = \operatorname{im}(d^{k-1})$.
- $H^k(K) = Z^k(K) / B^k(K)$.

"CO" - chain
- cycle
- boundary"

As space: Čech: Ordinal δ.

- What are cochain complexes?
- Take open cover:

(δ under order top.)

Basis open intervals + rays
$(\alpha, \beta) = \{\gamma \mid \alpha < \gamma < \beta\} \subseteq \delta$.

In part., singletons $\{\gamma\}$ are open iff $\gamma \in \mathrm{Succ}(\delta)$.

$U_\delta = \{\alpha \mid \alpha < \delta\}$

open rays.

(Think of as "triangulation")

Take $\alpha < \delta \to$ set of functions with domain α into $\mathbb{Z}$

Pre-sheaf $\mathcal{P}$ (gives structure to space, associates with algebra)

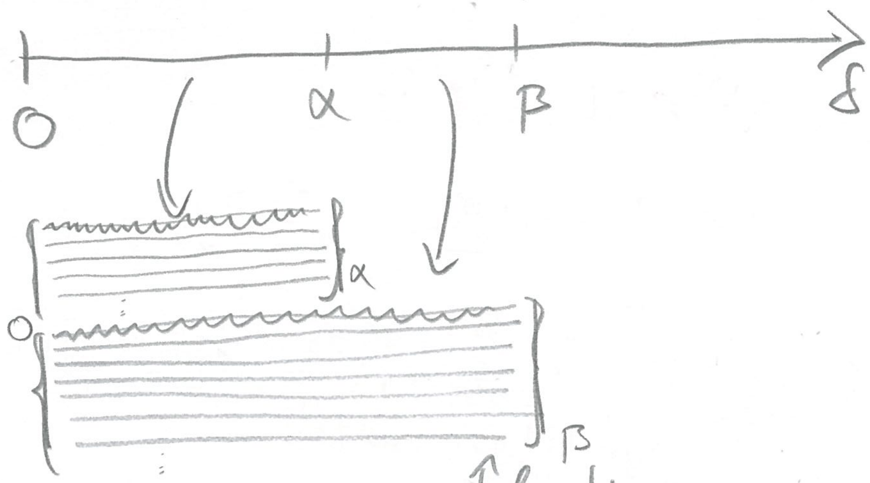
Then

$$L^j(\mathcal{U}_\delta, \mathcal{P}) = \prod_{\alpha_0 < \dots < \alpha_j < \delta} \mathcal{P}(\alpha_0)$$

cochain complexes.

$$0 \to L^0(\mathcal{U}_\delta, \mathcal{P}) \xrightarrow{d^0} L^1(\mathcal{U}_\delta, \mathcal{P}) \xrightarrow{d^1} \dots$$

Ex: For $L^0(\mathcal{U}_\delta, \mathcal{P}) = \prod_{\alpha < \delta} \mathcal{P}(\alpha)$



function from β into $\mathbb{Z}$.

Three important presheaves:

$$\mathcal{D}(\alpha) = \bigoplus_\alpha \mathbb{Z}$$

$$\mathcal{E}(\alpha) = \prod_\alpha \mathbb{Z}$$

$$\mathcal{F}(\alpha) = \prod_\alpha \mathbb{Z} / \bigoplus_\alpha \mathbb{Z}$$

$L^0(\mathcal{U}_\delta, \mathcal{E}) = \prod_{\alpha<\delta} \prod_\alpha \mathbb{Z}$.

So $\Phi \in L^0(\mathcal{U}_\delta, \mathcal{E})$ is *family of functions*, where

$$\Phi(\alpha) : \alpha \to \mathbb{Z}.$$

For $L^0(\mathcal{U}_\delta, \mathcal{D})$ we have families Φ with

$$\Phi(\alpha) : \alpha \to \mathbb{Z}, \text{ fin. supp.}$$

And $L^0(\mathcal{U}_\delta, \mathcal{F})$ yields

$$\Phi(\alpha) = [\varphi_\alpha : \alpha \to \mathbb{Z}] \text{ where}$$

$$[\varphi_\alpha] = [\psi_\alpha] \Leftrightarrow \varphi_\alpha - \psi_\alpha =^* 0.$$

Take $\Phi \in L^0(\mathcal{U}_\delta, \mathcal{E})$: if

$\Phi(\beta) \restriction \alpha =^* \Phi(\alpha)$ then

Φ is *coherent*.

Suppose there is $\varphi : \delta \to \mathbb{Z}$ with $\varphi \restriction \alpha =^* \Phi(\alpha)$ $\forall \alpha < \delta$.

Then Φ is *trivial*.

- Coherence : "similar" among initial segments
- Trivial: not "too" similar.

Thm:

$$H^{n+1}(\mathcal{U}_\delta, \mathcal{F}) = \frac{\{\Phi \mid n\text{-coherent}\}}{\{\Phi \mid n\text{-trivial}\}}$$

for $n > 1$ $\qquad = H^n(\mathcal{U}_\delta, \mathcal{D})$.

and

Thm: RECALL: fixed $\mathcal{P} = \mathcal{U}_\delta$:

$$H^n(\mathcal{U}_\delta, \mathcal{D}) = \begin{cases} \Phi = \text{0-coh functions (group of)} & \text{if } n=0 \\ \dfrac{\{\Phi \mid n\text{-coherent}\}}{\{\Phi \mid n\text{-trivial}\}} & \text{if } n>0 \end{cases}$$

These cohomology groups depend on cover!? If

- consider cover an triangulation, want

"smallest" triangulation to approximate space δ. (refinement)

$$\check{H}(\delta, \mathcal{D}) = \varinjlim_{\gamma \text{ covers } \delta} H(\gamma, \mathcal{D}).$$

BUT: $\mathcal{U}_\delta$ is nice:

Thm: $\check{H}^n(\delta, \mathcal{D}) = H^n(\mathcal{U}_\delta, \mathcal{D})$.

Why? $\mathcal{U}_\delta$ contains <u>nice</u> set of intervals, defined on some club;

- set is contained in γ, and preserve non-triviality.

Q: ① Do non-n-trivial n-coherent families exist?

② Is there a trivialisation?

[DRAW TABLE]

- For $\delta = \omega_n$, there __is__ an n-coherent non-n-trivial family. ZFC

(Todorcevic for $n=1$)

- $\check{H}^0(\delta, \mathcal{D}) = \{0\text{-coherent functions}\}$;

  - $\varphi : \delta \to \mathbb{Z}$ is 0-coherent iff $\varphi \upharpoonright \alpha =^* 0 \quad \forall \alpha < \delta$. ZFC

  - non-trivial if $\varphi \neq^* 0$.

$\Rightarrow$ no non-0-trivial 0-coherent on ω_1 (or any $\delta > \omega_0$); this is universal.

- For $\delta = \omega_n$, if $k > n$, then

$$\check{H}^k(\delta, \mathcal{D}) = 0.$$ ZFC

- If every set is constructible ($V = L$) then any group that need not vanish will not. (by using $\square$-sequences, $\diamondsuit$-seq., can diagonalise against any trivialisation). So we have

$\check{H}^3$	0	0	0	nt
$\check{H}^2$	0	0	nt	[illegible]
$\check{H}^1$	0	nt	[illegible]	[illegible]
$\check{H}^0$	nt	nt	nt	nt
	ω_0	ω_1	ω_2	ω_3

Consist

- Using comb. properties (Todorcevic) showed

$$\check{H}^1(\delta, \mathcal{D}) = 0 \Leftrightarrow cf(\delta) \neq \omega,$$

(so, second row is indep).

- That's all we know.
- Uses ω_1 ← ordinal walks etc.

(about which Tod says

"walks can be used to derive virtually any structure defined on ω_1"; so possibly not surprising then give condition.

What about $\check{H}^2(\delta, \mathcal{D})$?

- If $V=L$ then $\check{H}^2(\omega_3, \mathcal{D}) \neq 0$.

Is it consistent that $\check{H}^2(\omega_3, \mathcal{D})$ vanishes?

OPEN!

- Relating to incompactness...

Baire Space:

[Translate into Baire!]

Similar setup can be considered on Baire Space:

$\mathcal{D}(\alpha)$ becomes $A_f = \bigoplus_{i<\omega} \bigoplus_{j=1}^{f(i)} \mathbb{Z}$

$\mathcal{E}(\alpha)$ becomes $B_f = \prod_{i<\omega} \bigoplus_{j=1}^{f(i)} \mathbb{Z}.$

$ON \to \omega^\omega$

$\alpha \to f$

$\mathcal{D}(\alpha) \to A_f$

$\mathcal{E}(\alpha) \to B_f$

$\mathcal{F}(\alpha) \to B_f / A_f.$

So far mostly set theory

Here, consider <u>algebraic approach</u>.

$$0 \to A_f \xrightarrow{\varphi} B_f \xrightarrow{\psi} B_f / A_f \to 0$$

is <u>short exact</u>:

- φ is injective
- ψ is surjective
- $\ker(\varphi) = \mathrm{im}(\psi)$ ← "looks" like cohomology group vanishes...

Apply $\varprojlim$ ← based on coherence:

- picks one element from each B~~A~~$_f$, $\langle \Phi_f \rangle$ s.t. $\Phi_f(i) : f(i) \to \mathbb{Z}$.
  or $\Phi_f \in \prod_{i<\omega} \bigoplus_{j=1}^{f(i)} \mathbb{Z}$.
- $P_{fg}(x_g) = x_f$ for all $f \leq g \in \omega^\omega$.

$\underline{\Phi} = \{\varphi_f \mid f \in \omega^\omega\} \in \prod_{f \in \omega^\omega} B_f.$

Can adapt triviality + coherence to this space.

We can define cochain complexes:

$K^n(B) = \prod_{f_0 \neq \dots \neq f_n \in \omega^\omega} B_{f_0}$, with bound. hom's and resh.

$\Rightarrow 0 \to K^0(B) \to K^1(B) \to \dots$

~~Thm~~ for all $f \in \omega^\omega$

$$0 \to A_f \to B_f \to B_f/A_f \to 0$$

<u>inverse system</u>

$\Rightarrow$ inverse <u>limit</u>: high degree of coherence amongst elements...

Inverse limit:

$$0 \to \varprojlim A \to \varprojlim B \xrightarrow{q} \varprojlim B/A$$

$\leftarrow$ surjective! Then short exact!

$$\to \varprojlim{}^1 A \to \dots$$

not necessarily short exact!

first derived limit

<u>Thm</u>:

- $\varprojlim{}^n A \cong H^n(K(A))$ (NR)
- $\boxed{\varprojlim{}^n A = 0 \Leftrightarrow \text{every } n\text{-coherent is } n\text{-trivial.}}$ [Bergfalk]

SAME SITUATION AS BEFORE

LOCAL example of global ZFC case:

only properties affecting Baire space affect cohomology groups now:

- $\mathfrak{d} = cf(\omega^\omega, \leq) = cf(\omega^\omega, \leq^*)$:
  if $\mathfrak{d} = \aleph_2$ and PFA (strengths of MA ⇒ existence of filters.)
  then $\varprojlim^2 A \neq 0$ but all others vanish.

- What if we consider ω^κ? If $\kappa > \omega$ then if $H^n(\omega^\kappa) = 0$ then $H^n(\omega^\omega)$
  (can be trivialised... by restricting) ~~Thus~~ ~~con~~

Converse open!

Connection: Algebraic: Mittag-Leffler.

Suppose (A_i), (B_i), (C_i) is inverse system, (have inv. limits!)

and

$$0 \to A_i \to B_i \to C_i \to 0$$

is short exact.

There is a sufficient condition for vanishing first derived limit: ML

If <u>I is ctbl</u> and <u>system stabilises</u> then <u>first derived limit vanishes.</u>

↑ purely algebraic:

uses fact that I is countable to show $\varprojlim q$ must be surjective; thus

$$0 \to \varprojlim A \xrightarrow{\varprojlim i} \varprojlim \mathcal{B} \xrightarrow{\varprojlim q} \varprojlim{}^{1} A \to \dots$$

is short exact, and

$\varprojlim{}^{1} A$ vanishes.

This applies tons!

$\boxed{\delta < \omega_1}$: $\mathcal{L}(\mathcal{U}_\delta, \mathcal{D})$ is cochain complex,

$\mathcal{D}(\alpha) = \bigoplus_{\alpha} A \quad P_{\alpha\beta}$

$A = (\cancel{}, \cancel{P}, \delta)$ is inverse ~~limit~~ system, countable. $P_{\alpha\beta}$ are surjective, thus system is ML, δ is countable. So

$\varprojlim{}^{1} A = 0$. By Nöbeling Roos,

$$\varprojlim{}^{1} A \cong H^1(K(A)) = H^1(\mathcal{L}(\mathcal{U}_\delta, \mathcal{D}))$$
$$= H^1(\mathcal{U}_\delta, \mathcal{D}) = \check{H}^1(\delta, \mathcal{D}).$$

So: ALGEBRAIC TOOL to show Čech groups vanishes.

Uncountable ML?

- Open question!

Answering these questions might allow us to extrapolate combinatorial principles of the ordinals in question – and hence improve our understanding of the ordinals themselves beyond topological considerations.

- Two questions:
  - Do non-trivial coherent families exist? (*)
  - If they exist, can they be trivialized?

<u>Thm</u> (ZFC)

For any $\delta = \omega_n$, (*) holds.

$$\Rightarrow \check{H}^n(S, D) \neq 0.$$

<u>Thm</u> (ZFC): $\delta = \omega_n$ and $k > n$,

any k-coherent family can be trivialized.

$$\Rightarrow \check{H}^k(S, D) = 0.$$

<u>Thm</u> (ZFC + V=L).

Every ~~form~~ coh. group that can be non-trivial will be non-trivial.

$\check{H}^3$	0	0	0	nt	← open.
$\check{H}^2$	0	0	nt	cons-nt.	
$\check{H}^1$	0	nt	indep cons ~~nt~~	indep cons ~~nt~~	
$\check{H}^0$	nt	nt	nt	nt	
	ω_0	ω_1	ω_2	ω_3