C5.4 Option 2: Michaelmas Term 2022

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In network science, triadic closure is the idea that, in a network changing through time, two nodes sharing a common neighbour are more likely to become neighbours, or in other words, friends of friends are more likely to become friends. In this report we model a friendship network across two subgroups of a population, with intra- and inter-group friendships being born and dying along a Markov process with a triadic closure mechanism. We will introduce the mean-field approximation and compare it with simulation results to demonstrate multiple agreeing stable states for both simulations and the approximation.

Markov models | Triadic closure | Mean field approximation

1. Overview and Outlook

We seek to build on the work of Grindrod et al. (1) (2012), wherein the authors modelled a friendship network over discrete time steps using a triadic closure mechanism (friends of friends are more likely to become friends).

In our setting we instead have a somewhat bipartite network where n_1 and n_2 pupils attend two schools, respectively, in a town. Friendships are formed and lost randomly, independently and distinctly, and we have the triadic closure mechanism active.

We use mean field techniques presented by Grindrod et al. (1) to compare deterministic approximation with simulations, and to demonstrate multiple distinct stable states for the deterministic dynamics, for a fixed set of parameters.

We follow the same setup as in (1).

1.1 Notation and Preliminaries. In this report, an **adjacency matrix**, $A = (a_{ij}) \in \mathbf{R}^{n\times n}$, for a network has $a_{ij} = a_{ji} = 1$ if there is an edge between nodes i and j, and $a_{ij} = a_{ji} = 0$ otherwise (here we make use of the conditions $a_{ij} = a_{ji}$, implying an undirected network, and $a_{ii} = 0$ for all i, as there are no self-loops). We let S_n be the set of all such adjacency matrices, and let $\mathbf{1}_n$ be the adjacency matrix for the n-vertex clique. Denote by R_n the set of symmetric $n \times n$ real matrices with elements all in $[0, 1]$, and zeros on the diagonal. We also employ the **Hadamard product**, denoted by $A_1 \circ A_2$, the matrix obtained by elementwise multiplication. The network represented by this product is the network of common edges of the networks represented by its terms.

In a paper of Grindrod and Higham (2), the authors define the term **evolving network model** to describe a stochastic rule generating a sequence of adjacency matrices $\{A_k\}_{k=0}^{K}$, where each $A_k \in S_n$. Here A_k represents the state of our system at time t_k, where $t_0 < t_1 < \ldots < t_K$ are equally spaced time points.

2. The Model

Our model is an expansion of the model used by Grindrod et al. (1). The authors use Markovian methods to describe a simple network evolving through time, and as such consider a "first-order" evolving network model, i.e. a model characterised by the conditional probabilities $\mathbf{P}(A_{k+1} \mid A_k)$. We follow the definition of the expected value of A_{k+1} given A_k, $\langle A_{k+1} \mid A_k\rangle$, as given in (1).

By construction, $\langle A_{k+1} \mid A_k\rangle \in R_n$, and $(\langle A_{k+1} \mid A_k\rangle)_{ij} = \mathbf{P}((A_{k+1})_{ij} = 1 \mid A_k)$. We assume that the first-order evolving network model, $\mathbf{P}(A_{k+1} \mid A_k)$, is edge-independent, as in (1), so that the expected value itself defines a random graph in R_n. The importance of edge-independence will become clear when we arrive at determining a suitable approximation to the expected value of our network, in equation [4]. We also follow the conventions of (1) to define

$$\mathcal{F}(A_k) = (\mathbf{1} - \omega(A_k)) \circ A_k + \alpha(A_k) \circ (\mathbf{1} - A_k), \qquad [1]$$

where $\alpha(A_k)$ and $\omega(A_k)$ are given mappings $S_n \to R_n$, representing conditional birth and death rates respectively, as introduced in (2). The process by which we generate successive states of our system is identical to (1), and we follow the same pseudocode in our own simulation methods described in Section 5.

Now we reach the point of divergence from (1), where in this treatment, the authors study a simple network of n nodes fixed, with no other distinguishing internal structure to predetermine their process. On the contrary, we study two "copies" of these networks, of not necessarily similar size. Our copies, or schools, as we refer to them henceforth, are initially disconnected, with no links (friendships) between nodes (students) of different schools. Over time, the system will evolve by a mixture of edge-independent random friendship births and deaths, and the triadic closure mechanism, as we will demonstrate.

First, let A_k and B_k be the adjacency matrices for the two schools A and B, respectively. Let N_k be the adjacency matrix for the whole network (town) with inter-school relationships included. Hence, we write

$$N_k = \begin{pmatrix} A_k & * \\ * & B_k \end{pmatrix},$$

where the $*$'s are initially filled with zeros (at $k = 0$) and represent the inter-school friendships. We use a simple model for birth and death of friendships, defining constant rates. We let ω_a and α_a (resp. ω_b and α_b) be the death and birth rates for school A (resp. B), respectively, and let ω_{ab} and α_{ab} be the inter-school death and birth rates respectively.

Following the main ideas of (1), we define ω and α as

Significance Statement

We demonstrate, both analytically and via simulation, that in a network composed of two initially disconnected subgroups, it is possible, under certain fixed parameter sets, to have multiple states of equilibrium to which the network can move towards in its time-evolution.

follows:

$$\omega(N_k) = \begin{pmatrix} \omega_a \mathbf{1}_{n_1} & \omega_{ab}\mathbf{1}_{n_2,n_1} \\ \omega_{ab}\mathbf{1}_{n_1,n_2} & \omega_b \mathbf{1}_{n_2} \end{pmatrix} = \Omega, \quad [2]$$

$$\begin{aligned} \alpha(N_k) &= \begin{pmatrix} \alpha_a \mathbf{1}_{n_1} & \alpha_{ab}\mathbf{1}_{n_2,n_1} \\ \alpha_{ab}\mathbf{1}_{n_1,n_2} & \alpha_b \mathbf{1}_{n_2} \end{pmatrix} + \epsilon \circ N_k^2 \\ &= A + \epsilon \circ N_k^2, \end{aligned} \quad [3]$$

where $\mathbf{1}_{r,s}$ is an $r \times s$ matrix with 1's in every entry. We assume that $0 < \alpha_a, \alpha_b, \alpha_{ab} \ll 1$, and to ensure probabilities within $[0,1]$, $0 < \epsilon(n-1) < 1 - \alpha_i$, where $i \in \{a, b, ab\}$. We will see through experimental data and approximations that the first assumption is the one which gives rise to the fascinating behaviour with which this report is concerned. Despite this, it is perhaps not intuitive to have friendship birth rates being so low, especially in a large school, although this will be discussed in a later section evaluating our model. Both terms follow by considering the system evolution on various subsets of our network, including both intra- and inter-school friendships, all complete with the triadic closure term. The nature of this term captures the globality of triadic closure within the model, where mutual friends at different schools have the same chance of becoming friends as mutual friends at the same school do. Note that A and Ω do not depend on the time, k.

3. The Mean Field Approximation

The mean field approximation, as introduced to network science by Barabási, Albert and Jeong, (3), is an approach to provide a heuristic for how an evolving network model will move through time. When applied to our case, the approximation brings to light some counter-intuitive behaviour of the system regarding the long-run edge density of the network, and we iteratively search for the stable points arising from this mean-field approximation. We introduce some machinery, and then move on to investigating such behaviour, starting with a simplified version of our problem, building upon that exercise with a fuller treatment. Finally we come back to evaluate our model and compare the results of the approximation with simulations.

3.1 Classes of Conjugate Random Graphs.

We recap some definitions and basic ideas from **classes of conjugate random graphs**, as given in (1). The definition we give for this idea is with respect to a map $R_n \to R_n$, and we assume that $\mathcal{F}$, as in equations [1], [2] and [3], can be extended to such a map. As we will see in equation [4], such an extension in our case will be natural, as we consider applying $\mathcal{F}$ iteratively to a sequence of expected adjacency matrices, which are elements of R_n and not necessarily S_n.

Let $\lambda \in \Lambda$ denote a parameter ranging over a domain Λ within some suitable space and let $W : \Lambda \to R_n$. We say that $W(\Lambda)$ is a *class of conjugate random graphs* if for each $\lambda \in \Lambda$, there exists a unique element $g(\lambda) \in \Lambda$ such that

$$W(g(\lambda)) = \mathcal{F}(W(\lambda)).$$

Notice that from this definition, the parametrised set $\{W(\lambda) : \lambda \in \Lambda\} \subseteq R_n$ is positively invariant under $\mathcal{F}$, and that the action of $\mathcal{F}$ on R_n can be reduced to the action of g on Λ.

From (1), we know that if there is some subgroup H of the group of $n \times n$ permutation matrices under which $\mathcal{F}$ is invariant, then the set of matrices in R_n which are invariant under H forms a class of conjugate random graphs for $\mathcal{F}$.

An example given by Grindrod et al. in their study of a simple network with the triadic closure mechanism active is the Erdös-Rényi graphs. In their exposition, the function $\mathcal{F}$ is almost identical to ours, but $\Omega = \omega\mathbf{1}$ and $A = \delta\mathbf{1}$. In this way, by the symmetry of the model, $\mathcal{F}$ is invariant under all permutations of the nodes, and so $\{p\mathbf{1} : p \in [0,1]\}$ is a suitable class of graphs, as these random graphs are invariant under any node permutation.

3.2 The Approximation.

We begin with the reasonable assumption that A_k can be described by its own expected graph, conditional on some information coming from prior states of the network. By using the expectation, despite losing some information regarding the true distribution of the network, we gain a certain degree of maneuverability, allowing us to make adequate simplifications to our model, as well as permitting a level of generalisation to our task.

As such we consider $\langle A_k \mid X \rangle \in R_n$ as a random graph itself, and we may wish to use our model to calculate (an estimate for) $\langle A_{k+1} \mid X \rangle \in R_n$, and so on. So we calculate

$$\langle A_{k+1} \mid X \rangle = \sum_{A_k \in S_n} \mathcal{F}(A_k)\mathbf{P}(A_k \mid X).$$

We *linearise* $\mathcal{F}$ in the above equation to obtain an iterative formula for our sequence of random graphs:

$$\langle A_{k+1} \mid X \rangle \approx \mathcal{F}\left(\sum_{A_k \in S_n} A_k \mathbf{P}(A_k \mid X) \right) = \mathcal{F}(\langle A_k \mid X \rangle). \quad [4]$$

The nonlinearities in $\mathcal{F}$ involve products of mutually independent stochastic variables, however as described in (1), edge-independence is enough to minimise the perturbing effects caused by these terms to the approximation. Hence the assumption of edge-independence across all aspects of our model is suitable to render the linearisation of $\mathcal{F}$ a valid approximation. The iterative formula in equation [4] is known as a *mean field approximation* for the evolving graph model.

The goal of this mean-field iteration problem is to find a "limiting" distribution to which an evolving network model converges. In our case, we are actually looking for a distribution for our expected adjacency network, as $\mathcal{F}(\langle A_k \mid X \rangle)$ is itself a random graph. We will see how this can be done by the application of classes of conjugate random graphs. Suppose initially that for some $k_0 \geq 0$, we may represent $\langle A_{k_0} \mid X \rangle$ by a random graph, say $W_{k_0} \in R_n$. By the mean field approximation, equation [4], we iterate with $\mathcal{F}$:

$$W_{k+1} = \mathcal{F}(W_k), \;\; k = k_0, k_0+1, k_0+2, \ldots,$$

to obtain a sequence, $\langle A_k \mid X \rangle = W_k$ for all $k \geq k_0$. This sequence of expected values gives us an idea of the limiting distribution of our network as time passes, or rather how it converges in expectation. The goal of finding this limiting distribution is achieved by searching for so-called "stable points", i.e. we want the values at which our iteration will remain stable. Hence we seek solutions to $W_{k+1} = W_k$. In our case, this convergence will allow us to say that 'X% of the networks links will be filled as time gets large', where large is dependent on the network, initial conditions and model.

We now move onto the portion of the analysis combining our structures defined earlier, classes of conjugate random graphs, with the mean-field approximation in equation [4]. Suppose $W(\Lambda)$ is one such class for $\mathcal{F}$, and if $W_{k_0} = W(\lambda_{k_0})$ for some $k_0 \in \Lambda$, then as the action of $\mathcal{F}$ on R_n is characterised by that of g on Λ, we can instead iterate with g:

$$\lambda_{k+1} = g(\lambda_k),\ k \geq k_0,$$

and so it follows that

$$\langle A_k \mid X \rangle = W_k = W(\lambda_k),\ k \geq k_0.$$

Hence using the approximation in equation [4], a class of conjugate random graphs is particularly useful, as it is when considering the mean-field iteration over R_n, this descends to one over Λ, often an easier iteration to work with, on account of functions over R_n being matrix valued, and thus difficult to effectively compute with. A stable state of equation [4] is thus a $W(\lambda_k)$ for which $\lambda_k = \lambda_{k+1}$.

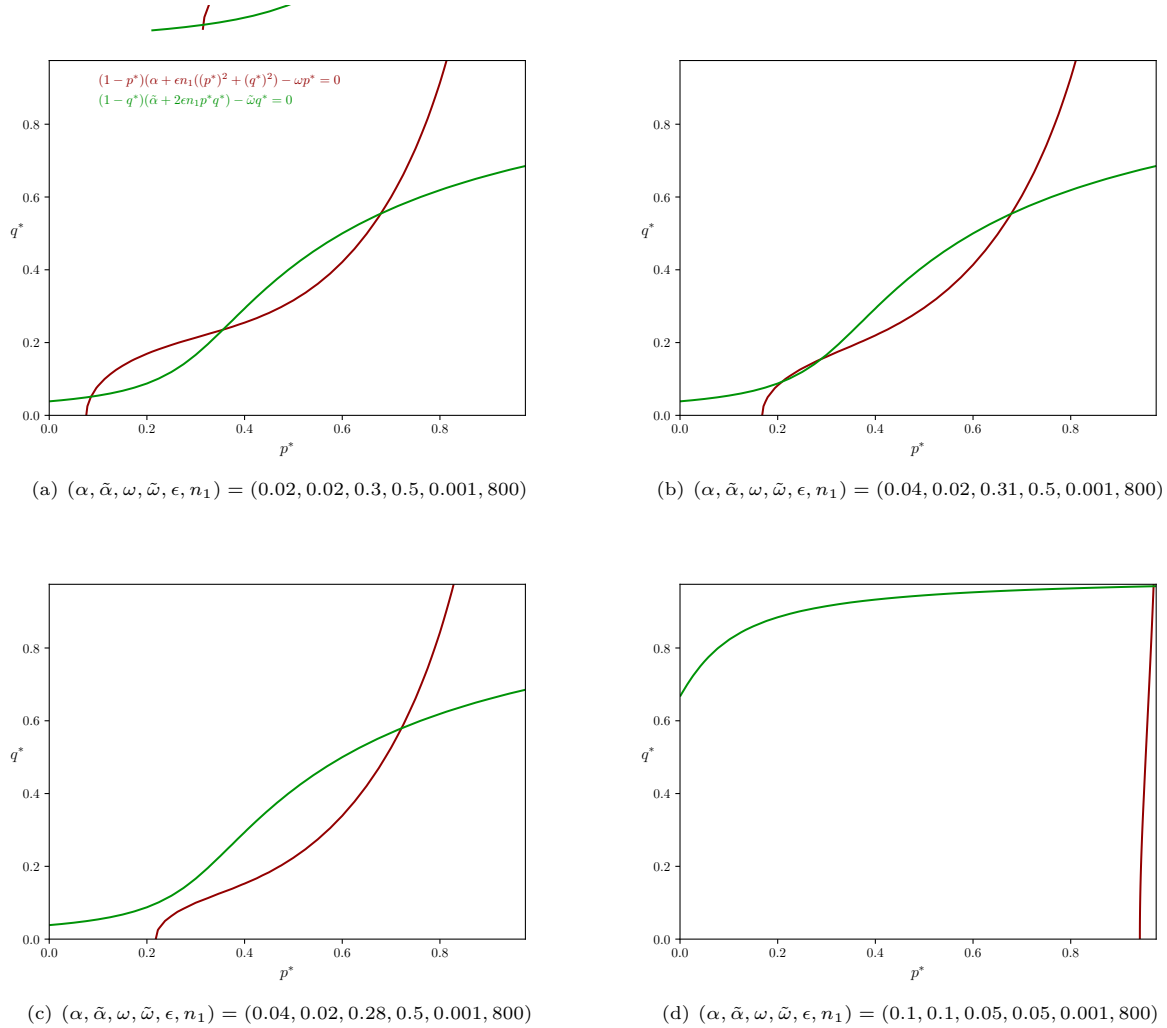
4. Stability

We apply the ideas of mean-field theory as presented to our problem, the town network with two schools. Returning to our model, and our function $\mathcal{F}$, we can see clearly that $\mathcal{F}$ is invariant under any permutation of the nodes in our town which only permute within the given blocks in A and Ω individually (which are of the same sizes and positions), and not across them. So an example of a class of conjugate random graphs would be:

$$W(p,q,r) = \begin{pmatrix} p\mathbf{1}_{n_1} & r\mathbf{1}_{n_2,n_1} \\ r\mathbf{1}_{n_1,n_2} & q\mathbf{1}_{n_2} \end{pmatrix}, \quad (p,q,r) \in [0,1]^3. \quad [5]$$

The mean-field approximation and subsequent iterations would, as we will see, lead to a complex system of non-linear equations in three unknowns, and proves rather tricky to characterise or understand. We will, for the moment, simplify the situation as presented, in order to demonstrate the existence of multiple stable states to the system for a fixed set of parameters. The bulk of our analysis will be concerned with this special case of the general problem, as it highlights the behaviour we wish to show without lending itself to the extra degree of complexity of the complete picture.

4.1 Simplifying the Model - 2 Parameters. We make the following initial simplifications to the model itself: first, we set



(a) $(\alpha, \tilde{\alpha}, \omega, \tilde{\omega}, \epsilon, n_1) = (0.02, 0.02, 0.3, 0.5, 0.001, 800)$

(b) $(\alpha, \tilde{\alpha}, \omega, \tilde{\omega}, \epsilon, n_1) = (0.04, 0.02, 0.31, 0.5, 0.001, 800)$

(c) $(\alpha, \tilde{\alpha}, \omega, \tilde{\omega}, \epsilon, n_1) = (0.04, 0.02, 0.28, 0.5, 0.001, 800)$

(d) $(\alpha, \tilde{\alpha}, \omega, \tilde{\omega}, \epsilon, n_1) = (0.1, 0.1, 0.05, 0.05, 0.001, 800)$

Fig. 1. Four plots of our equations [6] and [7] (p.4), varying the parameters.

$n_1 = n_2$, so that every term in the block matrix is square, and of the same size, and the schools have the same number of students. We also set $\alpha_a = \alpha_b = \alpha$, $\omega_a = \omega_b = \omega$ and rewrite $\alpha_{ab} = \tilde{\alpha}, \omega_{ab} = \tilde{\omega}$. This is equivalent to the idea that the schools share the same internal structure, in terms of how friendships are born and die independently. We thus have:

$$A = \begin{pmatrix} \alpha \mathbf{1}_{n_1} & \tilde{\alpha}\mathbf{1}_{n_1} \\ \tilde{\alpha}\mathbf{1}_{n_1} & \alpha\mathbf{1}_{n_1} \end{pmatrix},$$

$$\Omega = \begin{pmatrix} \omega \mathbf{1}_{n_1} & \tilde{\omega}\mathbf{1}_{n_1} \\ \tilde{\omega}\mathbf{1}_{n_1} & \omega\mathbf{1}_{n_1} \end{pmatrix}.$$

We accordingly simplify our class of conjugate random graphs to fit with our modified version of $\mathcal{F}$, the function determining the network's evolution through time, along the subset in which $p = q$, and rewrite them as:

$$W(p,q) = \begin{pmatrix} p\mathbf{1}_{n_1} & q\mathbf{1}_{n_1} \\ q\mathbf{1}_{n_1} & p\mathbf{1}_{n_1} \end{pmatrix}$$

This is clearly a class of conjugate random graphs for the new setting where $n_1 = n_2$, and so we restrict the problem to this two-parameter family of random graphs. We now look towards finding our function g, in order to set up our iterates and look for stable states.

We start by calculating $\mathcal{F}(W(p,q))$, and give it in terms of W, from which we can extract g.

$$\mathcal{F}(W(p,q)) = (\mathbf{1}_n - W(p,q)) \circ (A + \epsilon \mathbf{1}_n \circ W(p,q)^2) + (\mathbf{1}_n - \Omega) \circ W(p,q).$$

The main difficulty in our calculations is the value of $W(p,q)^2$:

$$W(p,q)^2 = \begin{pmatrix} p\mathbf{1}_{n_1} & q\mathbf{1}_{n_1} \\ q\mathbf{1}_{n_1} & p\mathbf{1}_{n_1} \end{pmatrix} \begin{pmatrix} p\mathbf{1}_{n_1} & q\mathbf{1}_{n_1} \\ q\mathbf{1}_{n_1} & p\mathbf{1}_{n_1} \end{pmatrix} = \begin{pmatrix} n_1(p^2+q^2)\mathbf{1}_{n_1} & 2n_1pq\mathbf{1}_{n_1} \\ 2n_1pq\mathbf{1}_{n_1} & n_1(p^2+q^2)\mathbf{1}_{n_1} \end{pmatrix}$$

Here, we notice that in fact, it is a necessary condition to have $n_1 = n_2$, as if not, the top-left and bottom-right blocks of the matrix $\mathcal{F}(W(p,q))$ would not be equal, and so would certainly not have the form $W(p',q')$. Through some unenlightening algebra, we land on

$$\mathcal{F}(W(p,q)) = W(p(1-\omega) + (1-p)(\alpha + \epsilon n_1(p^2+q^2), q(1-\tilde{\omega}) + (1-q)(\tilde{\alpha} + 2\epsilon n_1 pq)).$$

Hence we get that

$$g(p,q) = (p(1-\omega) + (1-p)(\alpha + \epsilon n_1(p^2+q^2), q(1-\tilde{\omega}) + (1-q)(\tilde{\alpha} + 2\epsilon n_1 pq)).$$

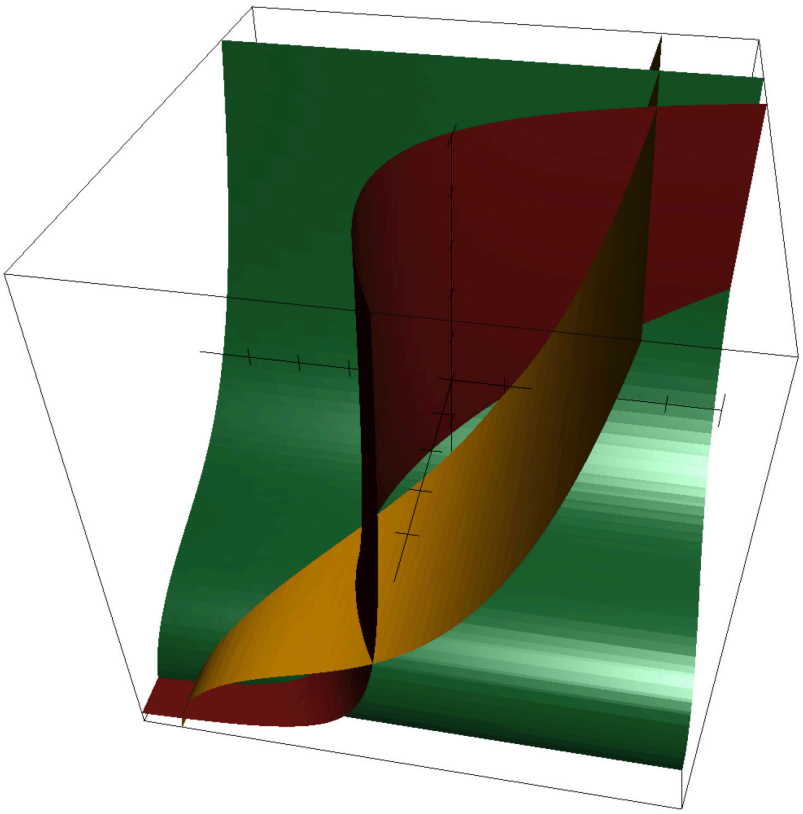


Fig. 2. The surfaces generated by equations [9-11], with three clear intersection points. The axes all lie in the range $[0, 1]$. Our parameter values are $(\alpha_a, \alpha_b, \alpha_{ab}, \omega_a, \omega_b, \omega_{ab}, \epsilon, n_1, n_2) = (0.02, 0.02, 0.01, 0.3, 0.3, 0.4, 0.001, 800, 800)$.

Thus, when iterating, we see that a stable state (p^*, q^*) satisfies:

$$(1-p^*)(\alpha + \epsilon n_1((p^*)^2 + (q^*)^2) - \omega p^* = 0, \quad [6]$$
$$(1-q^*)(\tilde{\alpha} + 2\epsilon n_1 p^* q^*) - \tilde{\omega} q^* = 0. \quad [7]$$

Thus we have a cubic and a quadratic equation in our two unknowns, which is generally a difficult system to solve (in fact, solving these proves almost as hard as the full, 3-parameter case). Through graphical visualisation of these simultaneous equations we are able to see that, for given fixed sets of parameters, we have multiple solutions to the system, i.e. multiple stable states for the distribution of our evolving network model.

We see in Figure 1(a) and Figure 1(b) that in the examples where the death rate sufficiently outweighs the birth rate we have multiple stable states, and for any situation where the death rate is not large enough, we only have one, for example Figure 1(c) or Figure 1(d). Note also that as equation [6] is a cubic equation (and outweighs equation [7] in degree), we have either three or one real intersection point(s) between the two curves, dependent upon the parameters. We note that our stable points in the case where we use the parameters in Figure 1(a) are

$$(p^*, q^*) \in \{(0.085, 0.051), (0.356, 0.236), (0.679, 0.555)\}, \quad [8]$$

accurate to three decimal places.

The stable points we arrive at from the iteration are the long-run "limits" of the network distribution, as an instance of the class of conjugate random graphs. These limits do not necessarily have to be unique due to the stochastic nature of the whole system. So if a specific system had stable states $(0.5, 0.5)$ and $(0.3, 0.1)$, say, the network would evolve in such a way that eventually it would have *either* (i) 50% of friendships within schools achieved and 50% of friendships between schools achieved, *or* (ii) 30% of friendships within schools achieved and 10% of friendships between schools achieved. Which state the network moves toward is not deterministic, and further, depends upon the initial state of our system.

4.2 The Full Story - 3 Parameters. Returning to the full version of the problem, we use identical methods to visualise stable points of the iteration arising from the mean-field approximation. We will refrain from a detailed analysis of the stable point equations, but instead focus on graphically displaying existence of multiple stable states. We switch back to the initially suggested class of conjugate random graphs:

$$W(p,q,r) = \begin{pmatrix} p\mathbf{1}_{n_1} & r\mathbf{1}_{n_2,n_1} \\ r\mathbf{1}_{n_1,n_2} & q\mathbf{1}_{n_2} \end{pmatrix},$$

as well as our initial, unaltered form for the matrix-valued function $\mathcal{F}$, determining the iteration:

$$\begin{aligned} \mathcal{F}(N_k) &= (\mathbf{1} - \omega(N_k)) \circ N_k + \alpha(N_k) \circ (\mathbf{1} - N_k), \\ \omega(N_k) &= \begin{pmatrix} \omega_a \mathbf{1}_{n_1} & \omega_{ab}\mathbf{1}_{n_2,n_1} \\ \omega_{ab}\mathbf{1}_{n_1,n_2} & \omega_b \mathbf{1}_{n_2} \end{pmatrix} = \Omega, \\ \alpha(N_k) &= \begin{pmatrix} \alpha_a \mathbf{1}_{n_1} & \alpha_{ab}\mathbf{1}_{n_2,n_1} \\ \alpha_{ab}\mathbf{1}_{n_1,n_2} & \alpha_b \mathbf{1}_{n_2} \end{pmatrix} + \epsilon \circ N_k^2, \\ &= A + \epsilon \circ N_k^2. \end{aligned}$$

Following the same process as in our restricted problem, we arrive at three equations, describing stable states for our three parameters:

$$(1-p^*)(\alpha_a + \epsilon(n_1(p^*)^2 + n_2(r^*)^2)) - \omega_a p^* = 0, \quad [9]$$
$$(1-q^*)(\alpha_b + \epsilon(n_2(q^*)^2 + n_1(r^*)^2)) - \omega_b q^* = 0, \quad [10]$$
$$(1-r^*)(\alpha_{ab} + \epsilon r^*(n_1 p^* + n_2 q^*)) - \omega_{ab} r^* = 0. \quad [11]$$

In Figure 2, the existence of multiple stable states is once again illustrated through a 3D-plot of the surfaces defined by our equations [9-11].

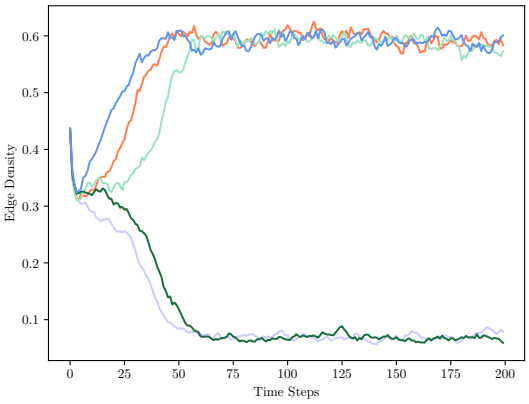


Fig. 3. Edge density $\hat{p}_k = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} N_{ij}$ of multiple paths of our network over time, along the Markov evolution as detailed in Section 2.

4.3 Evaluation of the Model. As in the restricted 2-parameter case, we require the death rate to sufficiently outweigh the birth rate in order for there to be multiple stable states for the 3-parameter problem. This behaviour does seem to make intuitive sense. The more natural of the parameter sets, where birth rate actually outweighs death rate, give rise to the more natural idea of a single stable state. On the other hand, when the parameters are more unnatural, with the death rate sufficiently outweighing the birth rate, we arrive at the somewhat infeasible position where we have three stable states.

In the context of model parameters, one would likely expect that within schools, the birth rate of friendships would be higher than the death rate, while between schools you might expect similar birth and death rates. As such, in the majority of these cases, where $\alpha_a > \omega_a$, and the same for school B, and $\alpha_{ab} = \omega_{ab}$, the stable states are very close to $(1, 1, r^*)$, where r^* tends to be slightly more variable than the stable state intra-school friendships. What this implies is that, in the majority of expected instances of our model, we would have that, in the long-run, almost everyone becomes friends, between schools and within them. From my own observational experience over seven years in one school, I would argue that this does seem to be the case in reality, with most students becoming at least well-acquainted, if not good friends, with their peers by the time at which they reached the end of their secondary education, although this is but speculation.

We should note that, in the 3-parameter version of the problem, graphical data would suggest that there are parameter sets for which there are no stable states, and a preliminary

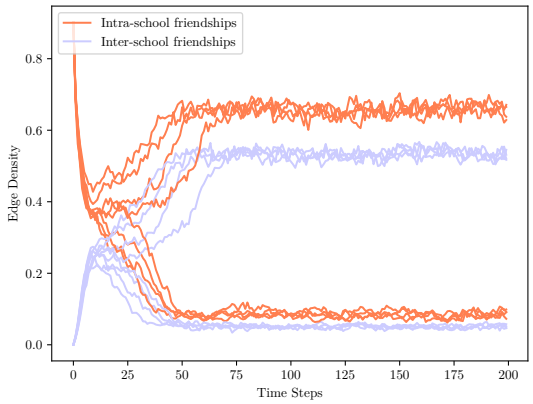


Fig. 4. Edge density of a number of simulations, highlighting the difference between the intra- and the inter-school friendships. We use the same parameters and starting state as in Figure 3.

analysis would indicate that suitably small death rates are a cause for this behaviour.

5. Simulations

We check the empirical data, from our simplified scenario, against numerical simulations. Using the same pseudocode for our evolving network model as in (1), we analyse the edge density of our network over time, both as a whole town and as individual schools.

In a loose sense, our simulations follow this structure - at a given stage, to find the next stage we move through the entries of the expected next adjacency matrix, and to generate the next instance in the sequence we simply select an edge with probability equal to the corresponding entry.

We see in Figure 3 that the existence of multiple stable states within the problem is backed up by experimental data. In this figure we have the parameters set as $(\alpha, \tilde{\alpha}, \omega, \tilde{\omega}, \epsilon, n_1) = (0.02, 0.02, 0.3, 0.5, 0.016, 50)$, and our starting states are $(p, q) = (0.885, 0)$. The only differences between the parameters used in Figure 3 and Figure 1 are the values of ϵ and n_1, however the multiple $n_1\epsilon$ is kept constant, and we notice from the form of our equations [6] and [7], this doesn't actually change the stable points. Hence we expect stable points identical to those of Figure 1, which the data supports, with the (total) edge density at stable points being roughly 0.05 and 0.6. These correspond with the two extremal stable points of Figure 1.

Figure 4 is yet more evidence towards the accuracy of the approximation given in equation 4. We have stable states for p being around 0.1 and 0.65, and those for q around 0.05 and 0.55. This lines up with the actual intersection points as given in [8].

6. Conclusions

In summary of our findings, we have been able to observe the existence of multiple stable states of our system, both through the mean-field approximation as a method of deterministically approximating the behaviour of the system and through automated simulations according to the model.

We might try to improve the effectiveness of such approximations by modelling the two birth and death rates differently. For example, performing our analysis with them viewed as random variables may lead to a greater accuracy of both our model and predictions, or alternatively we could instead have them vary as deterministic functions of time/network size/node degree. If there are fewer friendships within a school, we would expect a greater birth rate and lower death rate, while if there are more we would expect a lower birth rate and greater death rate. Such modifications would inevitably introduce a far greater level of complexity to the model, and would alter our methods of analysis.

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