Algorithms To Make Two G^1–Continuous Cubic Bézier Curves G^2–Continuous and G^3–Continuous

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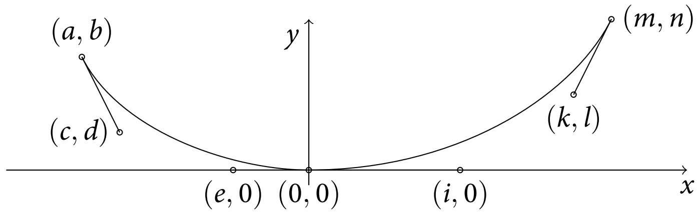
1 G^3-Continuity For Two Adjacent Cubic Bézier Curves

We consider two G^1–continuous cubic Bézier curves $\binom{x}{y} = (1-t)^3\binom{a}{b} + 3t(1-t)^2\binom{c}{d} + 3t^2(1-t)\binom{e}{0}$ and $\binom{x}{y} = 3t(1-t)^2\binom{i}{0} + 3t^2(1-t)\binom{k}{l} + t^3\binom{m}{n}$ with $e < 0 < i$:



For the G^3–continuity at their joint $(0,0)$ we have to equalize the curvature

$$\kappa(t) = \frac{x'(t)\cdot y''(t) - x''(t)\cdot y'(t)}{\left(x'(t)^2 + y'(t)^2\right)^{\frac{3}{2}}}$$

on both sides and the derivate of the curvature on both sides (note that the curvature changes the sign if the path is reversed, but the derivative of the curvature does not):

$$\frac{2l}{3i|i|} = -\frac{2d}{3e|e|} \quad \text{and} \quad \frac{18il + 2in - 12kl}{3i^2|i|} = \frac{18ed + 2eb - 12cd}{3e^2|e|}$$

Because of $e < 0 < i$ this simplifies to

$$\frac{l}{i^2} = \frac{d}{e^2} \quad \text{and} \quad \frac{9il + in - 6kl}{i^3} = -\frac{9ed + eb - 6cd}{e^3}.$$

Solving these two equations for e and i yields to two solutions

$$\boxed{e = \frac{6d(cl - k\sqrt{dl})}{dn + 18dl + bl} \quad \text{and} \quad i = \frac{6l(dk - c\sqrt{dl})}{dn + 18dl + bl}} \quad \text{or} \quad e = \frac{6d(cl + k\sqrt{dl})}{dn + 18dl + bl} \quad \text{and} \quad i = \frac{6l(dk + c\sqrt{dl})}{dn + 18dl + bl}.$$

If we assume $b > 0$, $c < 0$, $d \geq 0$, $k > 0$, $l \geq 0$ and $n > 0$ (which is the generic case in typedesign) the second solution can lead to cases where $e > 0$ or $i < 0$, whereas the first solution always lead to $e \leq 0 \leq i$. Therefore, the algorithm in *Curvatura* uses the first solution.

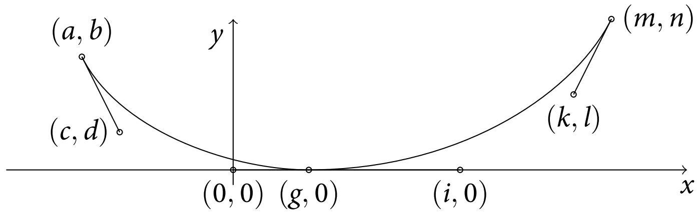
Note:

- $e = 0$ if $d = 0$
- $i = 0$ if $l = 0$

Unfortunately, the formula does not work for more than two G^1–continuous cubic Bézier curves. However, iteration seems to be stable!

2 G^2–continuity For Cubic Bézier Curves

Given two G^1–continuous cubic Bézier curves $\binom{x}{y} = (1-t)^3\binom{a}{b} + 3t(1-t)^2\binom{c}{d} + t^3\binom{g}{0}$ and $\binom{x}{y} = (1-t)^3\binom{0}{0} + 3t(1-t)^2\binom{i}{0} + 3t^2(1-t)\binom{k}{l} + t^3\binom{m}{n}$ with $0 < g < i$ we want to determine g such that the curves are G^2–continuous in their joint $(g, 0)$.



For the G^2–continuity at their joint $(0, 0)$ we have to equalize the curvature on both sides:

$$\frac{2d}{3g|g|} = \frac{2l}{3(i-g)|i-g|}$$

Because of $0 < g < i$, this simplifies to

$$\frac{d}{g^2} = \frac{l}{(i-g)^2}.$$

Special case $d = l$:

$$g = i - g \quad \Rightarrow g = \frac{i}{2}$$

Solving for g we get

$$\boxed{g = \begin{cases} \frac{(d-\sqrt{dl})}{d-l} \cdot i & \text{if } d \neq l, \\ \frac{i}{2} & \text{else.} \end{cases}} \quad \text{or} \quad g = \begin{cases} \frac{(d+\sqrt{dl})}{d-l} \cdot i & \text{if } d \neq l, \\ \frac{i}{2} & \text{else.} \end{cases}$$

If we assume $d \geq 0$, and $l \geq 0$ (which is the generic case in typedesign) the second solution can lead to cases where $g < 0$, whereas the first solution always lead to $g \geq 0$ (remember that the geometric mean $\sqrt{dl}$ lies between d and l). Therefore, the algorithm in *Curvatura* uses the first solution.

The peculiar thing is, that the ratio $\frac{(d+\sqrt{dl})}{d-l}$ is the ratio of the geometric mean $\sqrt{dl}$ between d and l!

2.1 Connection To The Harmonize Algorithm Described By Simon Cozens

Simon Cozens describes at `gist.github.com/simoncozens/3c5d304ae2c14894393c6284df91be5b` an algorithm to «harmonize» Bézier curves:

- Given two adjacent cubic bezier curves (a, b), (c, d), (e, f), (g, h) and (g, h), (i, j), (k, l), (m, n) that are smooth at (g, h) we calculate the corner point (u, v) which is the intersection of the lines (c, d)—(e, f) and (i, j)—(k, l).
- Determine the ratio $p = \sqrt[4]{\frac{\left((d-f)^2+(c-e)^2\right)\left((v-j)^2+(u-i)^2\right)}{\left((j-l)^2+(i-k)^2\right)\left((v-f)^2+(u-e)^2\right)}}$.
- Set (g, h) such that it is situated at $t = \frac{p}{p+1}$ of the line (e, f)—(i, j).

We now compare this algorithm to our situation described in the section above. If the handles are not vertical and not parallel, $v = \frac{d}{c}u$ and $v = \frac{l}{k-i}(u-i)$ yields to the intersection point (u, v) with

$$u = \frac{cil}{cl - dk + di} \quad \text{and} \quad v = \frac{dil}{cl - dk + di}.$$

Substituting $u = \frac{cil}{cl-dk+di}$, $v = \frac{dil}{cl-dk+di}$ and $e = f = j = 0$ in $p = \sqrt[4]{\frac{\left((d-f)^2+(c-e)^2\right)\left((v-j)^2+(u-i)^2\right)}{\left((j-l)^2+(i-k)^2\right)\left((v-f)^2+(u-e)^2\right)}}$ yields to

$$p = \sqrt{\frac{d}{l}}.$$

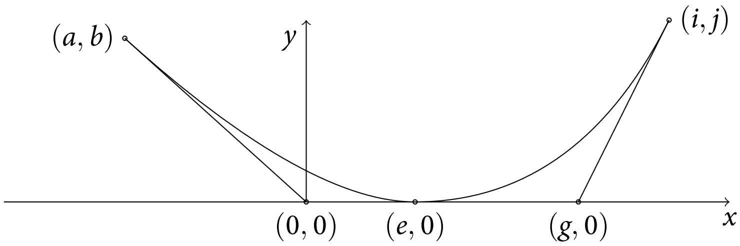
Since $\frac{p}{p+1} = t = \frac{g}{i}$, we get

$$g = \frac{p}{p+1} \cdot i = \frac{\sqrt{d/l}}{\sqrt{d/l}+1} \cdot i = \frac{(d - \sqrt{dl})}{d-l} \cdot i,$$

which is the solution for $d \neq l$ that we have determined before.

3 G^2–Continuity For Quadratic Bézier Curves

Given two G^1–continuous quadratic Bézier curves $\binom{x}{y} = (1-t)^2\binom{a}{b} + t^2\binom{e}{0}$ and $\binom{x}{y} = (1-t)^2\binom{e}{0} + 2t(1-t)\binom{g}{0} + t^2\binom{i}{j}$ with $0 < e < g$ we want to determine e such that the curves are G^2–continuous in their joint $(e, 0)$.



For the G^2–continuity at their joint $(0, 0)$ we have to equalize the curvature on both sides:

$$\frac{b}{2e^2} = \frac{e}{2(g-e)^2}$$

which yields to

$$\boxed{e = \frac{(b - \sqrt{bj})}{b-j} \cdot g} \quad \text{or} \quad e = \frac{(b + \sqrt{bj})}{b-j} \cdot g.$$

Rewriting the two solutions as

$$e = \frac{\sqrt{b}}{\sqrt{b}+\sqrt{j}} \cdot g \quad \text{or} \quad e = \frac{\sqrt{b}}{\sqrt{b}-\sqrt{j}} \cdot g$$

shows that the first solution in contrast to the second solution lies in the interval between 0 and g, and therefore is the desired solution.

Unfortunately, the formula does not work for more than two G^1–continuous quadratic ézier curves. However, iteration seems to be stable!