Algorithms To Make Two G^1–Continuous Cubic Bézier Curves G^2–Continuous

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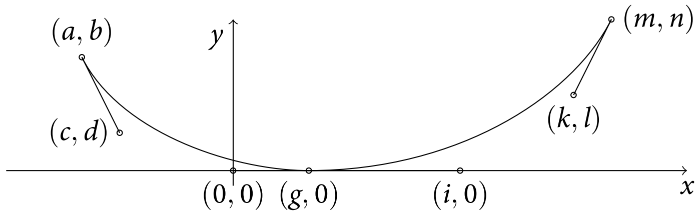
1 G^2–continuity For Cubic Bézier Curves By Moving On-Curve Nodes Tangentially

Given two G^1–continuous cubic Bézier curves $\binom{x}{y} = (1-t)^3\binom{a}{b} + 3t(1-t)^2\binom{c}{d} + t^3\binom{g}{0}$ and $\binom{x}{y} = (1-t)^3\binom{0}{0} + 3t(1-t)^2\binom{i}{0} + 3t^2(1-t)\binom{k}{l} + t^3\binom{m}{n}$ with $0 < g < i$ we want to determine g such that the curves are G^2–continuous in their joint $(g, 0)$.



For the G^2–continuity at their joint $(g, 0)$ we have to equalize the curvature on both sides:

$$\frac{2d}{3g|g|} = \frac{2l}{3(i-g)|i-g|}$$

Because of $0 < g < i$, this simplifies to

$$\frac{d}{g^2} = \frac{l}{(i-g)^2}.$$

Special case $d = l$:

$$g = i - g \quad \Rightarrow g = \frac{i}{2}$$

Solving for g we get

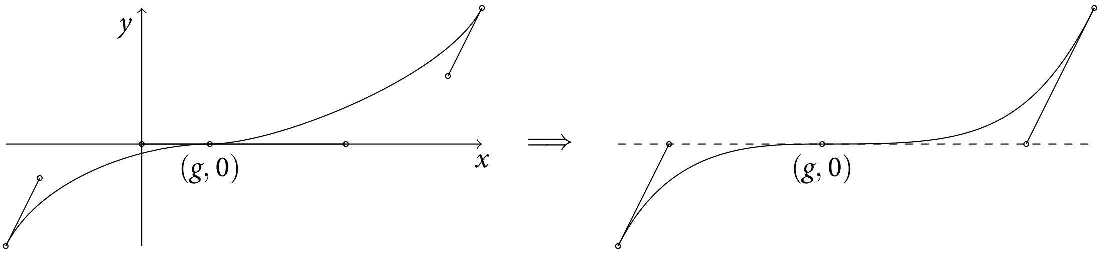
$$\boxed{g = \begin{cases} \frac{(d-\sqrt{dl})}{d-l} \cdot i & \text{if } d \neq l, \\ \frac{i}{2} & \text{else.} \end{cases}} \quad \text{or} \quad g = \begin{cases} \frac{(d+\sqrt{dl})}{d-l} \cdot i & \text{if } d \neq l, \\ \frac{i}{2} & \text{else.} \end{cases}$$

If we assume $d \geq 0$, and $l \geq 0$ (which is the generic case in typedesign) the second solution can lead to cases where $g < 0$, whereas the first solution always lead to $g \geq 0$ (remember that the geometric mean $\sqrt{dl}$ lies between d and l). Therefore, the algorithm in *Curvatura* uses the first solution.

The peculiar thing is, that the ratio $\frac{(d+\sqrt{dl})}{d-l}$ is the ratio of the geometric mean $\sqrt{dl}$ between d and l!

1.1 Special Case: Joint Node Is Inflection Point

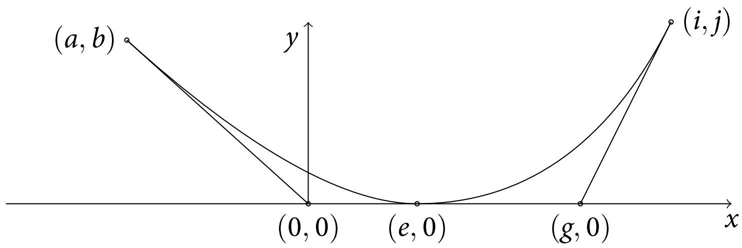
If the joint $(g, 0)$ is an inflection point (i.e. the curvature of both segments have different signs there), G^2–continuity implies that the curvature must be 0 at $(g, 0)$. This can only be satisfied if the control points lie on the tangent in $(g, 0)$:



This is critical in practice, as after rounding the handles may exceed the tangent triangles. Therefore, *Curvatura* treats inflection point by mirroring one part with regards to the tangent in $(g, 0)$, applies the generic formula and finally mirrors the part back. In this way, the curvature in $(g, 0)$ is the same on both sides but with different signs.

2 G^2–Continuity For Quadratic Bézier Curves By Moving On-Curve Nodes Tangentially

Given two G^1–continuous quadratic Bézier curves $\binom{x}{y} = (1-t)^2\binom{a}{b} + t^2\binom{e}{0}$ and $\binom{x}{y} = (1-t)^2\binom{e}{0} + 2t(1-t)\binom{g}{0} + t^2\binom{i}{j}$ with $0 < e < g$ we want to determine e such that the curves are G^2–continuous in their joint $(e, 0)$.



For the G^2–continuity at their joint $(0, 0)$ we have to equalize the curvature on both sides:

$$\frac{b}{2e^2} = \frac{e}{2(g-e)^2}$$

which yields to

$$\boxed{e = \frac{(b - \sqrt{bj})}{b - j} \cdot g} \quad \text{or} \quad e = \frac{(b + \sqrt{bj})}{b - j} \cdot g.$$

Rewriting the two solutions as

$$e = \frac{\sqrt{b}}{\sqrt{b} + \sqrt{j}} \cdot g \quad \text{or} \quad e = \frac{\sqrt{b}}{\sqrt{b} - \sqrt{j}} \cdot g$$

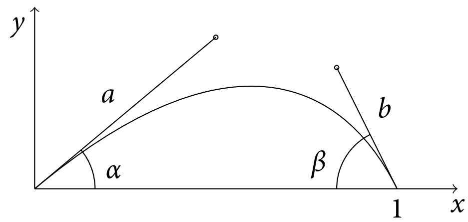
shows that the first solution in contrast to the second solution lies in the interval between 0 and g, and therefore is the desired solution.

Unfortunately, the formula does not work for more than two G^1–continuous quadratic Bézier curves. However, iteration seems to be stable (not proved yet).

3 G^2–continuity For Cubic Bézier Curves By Moving Handles

3.1 Curvature Depending On The Angles To The Chord

Given a cubic Bézier curve $\binom{x}{y} = 3t(1-t)^2\binom{a\cos(\alpha)}{a\sin(\alpha)} + 3t^2(1-t)\binom{1-b\cos(\alpha)}{b\sin(\alpha)} + t^3\binom{1}{0}$ with $a, b \geq 0$



we calculate the curvature at $t = 0$

$$\kappa_\alpha = \kappa(0) = \begin{cases} \frac{2}{3a^2}\big(b\sin(\alpha+\beta) - \sin(\alpha)\big) & \text{if } a \neq 0, \\ \pm\infty & \text{else.} \end{cases}$$

and $t = 1$:

$$\kappa_\beta = \kappa(1) = \begin{cases} \frac{2}{3b^2}\big(a\sin(\alpha+\beta) - \sin(\beta)\big) & \text{if } b \neq 0, \\ \pm\infty & \text{else.} \end{cases}$$

Solving the equation system

$$\kappa_\alpha = \frac{2}{3a^2}\big(b\sin(\alpha+\beta) - \sin(\alpha)\big) \quad \text{and} \quad \kappa_\beta = \frac{2}{3b^2}\big(a\sin(\alpha+\beta) - \sin(\beta)\big)$$

for a and b is done by solving the (depressed) quartic equation

$$0 = 27\kappa_\alpha\kappa_\beta^2 \cdot x^4 + 36\kappa_\alpha\sin(\beta)\kappa_\beta \cdot x^2 - 8\sin(\beta)a^3 \cdot x + 8\sin(\alpha)\sin(\beta)a^2 + 12\kappa_\alpha\sin(\beta)^2$$

for b and then $a = \frac{2\sin(\beta)+3\kappa_\beta b^2}{2\sin(\alpha+\beta)}$ if $\alpha + \beta \neq 0$. If $\alpha + \beta = 0$, the upper equation system simplifies to

$$\kappa_\alpha = -\frac{2\sin(\alpha)}{3a^2} \quad \text{and} \quad \kappa_\beta = \frac{2\sin(\alpha)}{3b^2}$$

with the non–negative solutions

$$a = \sqrt{-\frac{2\sin(\alpha)}{3\kappa_\alpha}} \quad \text{and} \quad b = \sqrt{\frac{2\sin(\alpha)}{3\kappa_\beta}}.$$

Now we can take the average curvature between segments and adjust the handles as described above and the curves will join G^2–continuous.

In fact, not every average curvature will produce useable handles on both sides. Therefore, iteratively taking the average curvature is recommended.

If we have a sharp end at α (i.e. a non–smooth node) it is additionally convenient to choose a such that $\max|\kappa(t)|$ is minimal. In practice this can be done by varying b and calculating $a = \frac{2\sin(\beta)+3\kappa_\beta b^2}{2\sin(\alpha+\beta)}$ and $\max|\kappa(t)|$ for each «sensible» $b \geq 0$ (b should not exceed the tangent triangle). If $\alpha + \beta = 0$, b is fixed and a can be varyied independently of b.