A. type check

A.1 Syntax

$$
\begin{array}{llll}
e,\ \tau & ::= & & \text{Expressions} \\
 & | & x & \quad \text{Variable} \\
 & | & \star & \quad \text{Type of Types} \\
 & | & e_1\, e_2 & \quad \text{Application} \\
 & | & \lambda x.\ e & \quad \text{Abstraction} \\
 & | & \lambda x : \sigma\ .\ e & \quad \text{Abstraction with annotation} \\
 & | & \mathsf{cast}^{\uparrow}\ e & \quad \text{Cast Up} \\
 & | & \mathsf{cast}_{\downarrow}\ e & \quad \text{Cast Down} \\
 & | & \Pi\,(x:\tau).\ \tau_2 & \quad \text{Explicit Pi type} \\
 & | & \mathbf{let}\ x = e_1\ \mathbf{in}\ e_2 & \quad \text{Let binding} \\
 & | & e :: \sigma & \quad \text{Annotation} \\
\sigma & ::= & & \text{Polytype} \\
 & | & \forall\, x:\tau.\ \rho & \quad \text{Implicit Pi Type} \\
\rho & ::= & & \text{Rhotype} \\
 & | & \tau & \quad \text{Mono} \\
 & | & \Pi\,(x:\sigma_1).\sigma_2 & \quad \text{FunPoly}
\end{array}
$$

A.2 non syntax-directed Typing

$\boxed{\Gamma \vdash e : \sigma}$ infer $\Uparrow$ check $\Downarrow$ $\delta = \Uparrow | \Downarrow$

$$\Gamma \vdash_\delta \star : \star \ \ \textsc{Ax}$$

$$\frac{x:\sigma \in \Gamma \qquad \vdash^{inst}_{\delta} \sigma \sqsubseteq \rho}{\Gamma \vdash_\delta x : \rho}\ \textsc{Var}$$

$$\frac{\Gamma, x:\tau \vdash_\Uparrow e : \rho \qquad \Gamma \vdash_\Downarrow (\Pi x:\tau.\rho) : \star}{\Gamma \vdash_\Uparrow (\lambda x.\, e) : (\Pi x : \tau.\rho)}\ \textsc{Lam-Infer}$$

$$\frac{\Gamma \vdash_\Downarrow (\Pi x:\sigma_1.\sigma_2) : \star \qquad \Gamma, x:\sigma_1 \vdash_\Downarrow e : \sigma_2}{\Gamma \vdash_\Downarrow (\lambda x.\, e) : (\Pi x : \sigma_1.\sigma_2)}\ \textsc{Lam-Check}$$

$$\frac{\sigma_2 = subst\_let(\sigma_1) \qquad \Gamma, x:\sigma_2 \vdash_\Uparrow e : \rho}{\Gamma \vdash_\Uparrow (\lambda x:\sigma_1.\, e) : (\Pi x : \sigma_2.\rho)}\ \textsc{LamAnn-Infer}$$

$$\frac{\begin{array}{c}\sigma_2 = subst\_let(\sigma_1) \\ \vdash^{dsk} \sigma_3 \sqsubseteq \sigma_2 \qquad \Gamma, x:\sigma_2 \vdash^{poly}_\Downarrow e : \sigma_4\end{array}}{\Gamma \vdash_\Downarrow (\lambda x:\sigma_1.\, e) : (\Pi x : \sigma_3.\sigma_4)}\ \textsc{LamAnn-Check}$$

$$\frac{\begin{array}{c}\Gamma \vdash_\Uparrow e_1 : (\Pi x:\sigma_1.\sigma_2) \\ \vdash^{poly}_\Downarrow e_2 : \sigma_1 \qquad \vdash^{inst}_\delta \sigma_2[x \mapsto e_2] \sqsubseteq \rho\end{array}}{\Gamma \vdash_\delta e_1\, e_2 : \rho}\ \textsc{App}$$

$$\frac{\Gamma \vdash_\Downarrow \tau_1 : \star \qquad \Gamma, x:\tau_1 \vdash_\Downarrow \tau_2 : \star}{\Gamma \vdash_\delta (\Pi x:\tau_1.\tau_2) : \star}\ \textsc{ExplicitPi}$$

$$\frac{\sigma_2 = subst\_let(\sigma) \qquad \vdash^{poly}_\Downarrow (e:\sigma_2) \qquad \vdash^{inst}_\delta e : \rho}{\Gamma \vdash_\delta (e :: \sigma) : \rho}\ \textsc{Ann}$$

$$\frac{\Gamma \vdash_\Uparrow e : \rho_2 \qquad \rho_1 \longrightarrow \rho_2}{\Gamma \vdash_\Downarrow (\mathsf{cast}^{\uparrow}\ e) : \rho_1}\ \textsc{CastUp-Check}$$

$$\frac{\Gamma \vdash_\Uparrow e : \rho_1 \qquad \rho_1 \longrightarrow \rho_2}{\Gamma \vdash_\Uparrow (\mathsf{cast}_{\downarrow}\ e) : \rho_2}\ \textsc{CastDown-Infer}$$

$$\frac{\begin{array}{c}\Gamma \vdash_\Uparrow e : \rho_1 \\ \rho_1 \longrightarrow \rho_2 \qquad \vdash^{inst}_\Downarrow \rho_2 \sqsubseteq \rho_3\end{array}}{\Gamma \vdash_\Downarrow (\mathsf{cast}_{\downarrow}\ e) : \rho_3}\ \textsc{CastDown-Check}$$

$$\frac{\vdash^{poly}_\Uparrow e_1 : \sigma \qquad \Gamma, let\ x:\sigma = subst\_let(e1) \vdash_\delta e_2 : \rho}{\Gamma \vdash_\delta (let\ x = e_1\ in\ e_2) : \rho}\ \textsc{Let}$$

$\boxed{\Gamma \vdash \sigma : \star}$

$$\frac{\Gamma \vdash_\Downarrow \tau : \star \qquad \Gamma, x:\tau \vdash_\Downarrow \rho : \star}{\Gamma \vdash_\delta (\forall x:\tau.\rho) : \star}\ \textsc{ImplicitPi}$$

$\boxed{\Gamma \vdash \rho : \star}$

$$\frac{\Gamma \vdash_\Downarrow \sigma_1 : \star \qquad \Gamma, x:\sigma_1 \vdash_\Downarrow \sigma_2 : \star}{\Gamma \vdash_\delta (\Pi x:\sigma_1.\sigma_2) : \star}\ \textsc{FunPoly}$$

$\boxed{\Gamma \vdash^{poly}_\delta \tau : \sigma}$

$$\frac{\begin{array}{c}\Gamma \vdash_\Uparrow e : \rho \qquad x = ftv(\rho) - ftv(\Gamma) \\ x:\tau \qquad \Gamma \vdash_\Downarrow (\forall x:\tau.\rho) : \star\end{array}}{\vdash^{poly}_\Uparrow e : (\forall x:\tau.\rho)}\ \textsc{Gen-Infer}$$

$$\frac{x \notin ftv(\Gamma) \qquad \Gamma \vdash_\Uparrow e : \rho \qquad pr(\sigma) = \forall x.\rho}{\vdash^{poly}_\Downarrow e : \sigma}\ \textsc{Gen-Check}$$

$\boxed{\vdash^{inst}_\delta \sigma \sqsubseteq \rho}$

$$\frac{\tau_\beta : \tau}{\vdash^{inst}_\Uparrow \forall x:\tau.\rho \sqsubseteq \rho[x \mapsto \tau_\beta]}\ \textsc{Inst-Infer}$$

$$\frac{\vdash^{dsk} \sigma \sqsubseteq \rho}{\vdash^{inst}_\Downarrow \sigma \sqsubseteq \rho}\ \textsc{Inst-Check}$$