Almost Exact Reconstruction of Sparse Signals from Pseudo-Random Samples

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***Abstract*—This project report tries to introduce the reader to a recent topic of interest to the Signal Processing community, Compressed Sensing, with a simple yet concrete example. The example details a problem which is solved by imposing a non-linear regularization involving l_1 norm. Well maintained code, *l_1-MAGIC*, written by Justin Romberg and Emmanuel Candes, is used to solve the optimization problem at hand.**

I. Introduction

There is soon going to be another digital revolution that will enable the development and deployment of new sensors and sensing systems with ever increasing performance and resolution. The Nyquist-Shannon sampling theorem, which states that a signal's information is preserved if it is *uniformly* sampled at a rate at least two times faster than its Fourier bandwidth. Unfortunately, in many important and emerging applications, the Nyquist-Shannon rate can be so high that we end up with too many samples and must compress in order to store or transmit them. In other applications the cost of signal acquisition is prohibitive, either because of a high cost per sample, or because state-of-the-art samplers cannot achieve the high sampling rates required by Nyquist-Shannon.

Compressed Sensing that promises image/signal reconstruction was discovered just five years ago by Candes and Tao and by Donoho. It is become a buzzword in the Signal Processing community. This paradigm shift in sensing called 'Compressed sensing' and has made possible applications like the 'Single Pixel Camera' built by Richard Baraniuk and Kevin Kelly of Rice University. Such a camera manages to derive the image mathematically from largely few randomly selected measurements as compared to a conventional digital camera.

Other examples include radar imaging and novel imaging modalities outside visible wavelengths.

A. The Key Idea

Transform based compression algorithms reduce the effective dimensionality of an N-dimensional signal x by representing it in terms of a sparse set of coefficients α in a basis expansion $x = \psi\alpha$, with ψ an $N \times N$ basis matrix. A sparse signal is one that only has $K << N$ of the coefficients α are nonzero and need to be stored or transmitted. A compressible signal is one that has the coefficients α, when sorted, decay rapidly enough to zero that α can be well-approximated as K-sparse. The sparsity and compressibility properties are pervasive in many signals of interest.

B. The Key Results in Compressed Sensing

- For many $M \times N$ matrices ϕ, the unique K-sparse solution, x, to the equation $\phi\mathbf{x^*} = \mathbf{y}$, can be recovered *exactly*. (!)
- N must be much larger than K. However, M (the number of measurements) need only be a little larger than K. Specifically, M must be roughly $K\log(N/K)$. Notice that the dependence on N is logarithmic, and enormous speed-up can be achieved.
- The K-sparse solution is found by l_1-minimization, which can be proved to be equivalent to l_0-minimization under certain assumptions on the measurement matrix, ϕ.
- Random matrices ϕ almost always satisfy those assumptions.

II. A Definitive Example

Consider a signal f(t) which is a sum of three sinusoids with frequencies 697Hz, 941Hz and 1633Hz. The frequencies are chosen mutually coprime. If we sample the signal f(t) at 40000Hz for 0.125s, then we end up with a vector f of length n = 5000. The top plot of Figure 1 shows the portion of the signal. The plot also includes 500 random samples of f (10 times less samples), b, shown as black dots. The bottom plot shows the Discrete Cosine Transform (the basis) of f, c, with clearly three spikes at appropriate frequencies. There are a few dozen significant nonzero coefficients. The signal is thus sparse in the DCT basis. We now look at a simple but representative problem.

A. Problem Description

Can f be recovered from b given that f is sparse in DCT basis and how the pseudo-random sampling (the matrix, Δ, obtained when a subset of rows in the identity operator is retained) was done?

B. More precisely..

In simple mathematical terms,

$$f = [IDCT]c$$

$$b = \Delta f$$

$$\Rightarrow \quad \Delta[IDCT]c = b;\ A = \Delta[IDCT]$$

Then,

$$Ac = b.$$

We want to compute x that satisfies $Ax = b$. It is easy to see that there are more unknowns than equations. So there are a lot of solutions, x, of which one is c. If c can somehow be obtained then it is an easy matter to recover f.

C. How to go about solving the problem?

In principle picking from all of x the solution should involve counting the non-zeros (the l_0 norm). It turns that this is a combinatorial problem which is NP-hard. Surprisingly, David Donoho, Emmanuel Candès and Terence Tao showed that l_0 can be replaced with l_1. They proved that with overwhelming probability the two problems have the same solution.

So, we try to solve the following problem,

$$\min_x \|x\|_1 \ s.t.\ Ax = b$$

The l_1-MAGIC provides us with a number of functions solving variants of l_1 optimization problems of which 'l1eq_pd' can solve this particular problem.

III. MATLAB Implementation

The 'l1eq_pd.m' is placed in the current directory with the 'Compressed_Sensing.m'. Upon execution of the 'Compressed_Sensing.m' program the command window shows the iterations with the relevant parameters. A bunch of figures open up and in the command prompt the signal f and the recovered ones are sounded.

Pseudo-random samples of f are held in a vector b and matrix A which is rectangular are input to the l1eq_pd function. y (traditional least-squares solution) as the initial point, 5e-3 as the tolerance for primal-dual algorithm and 32 as the maximum number of primal-dual iterations are chosen for l1eq_pd function.

To improve the recovery performance we can increase either the number of iterations or the number of measurements or find a basis which gives a much sparser representation.

IV. Discussion Of The Example

It is easy to see that the l_1 solution, Fig. 2, closely resembles f, see Fig. 4, while the traditional least squares solution, Fig. 3, poorly resembles f. Also when the signals are sounded one hears similar notes and a noisy buzz respectively. Addition of a small amount of noise doesn't degrade the recovery. There is code segment where this can be checked out. It is known in the literature that the recovery process does not degrade gracefully, but rather, abruptly with increasing amounts of noise.

A. Relevant Plots

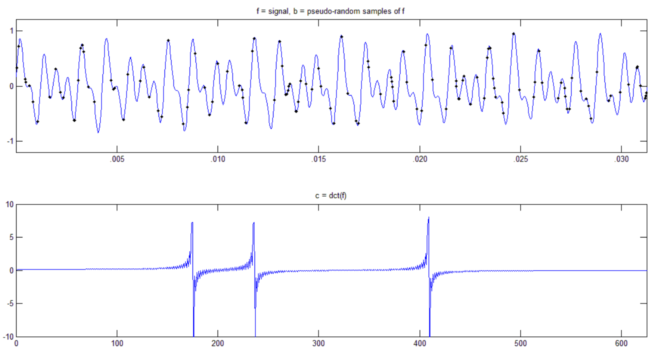

Fig. 1. The original signal and its discrete cosine transform

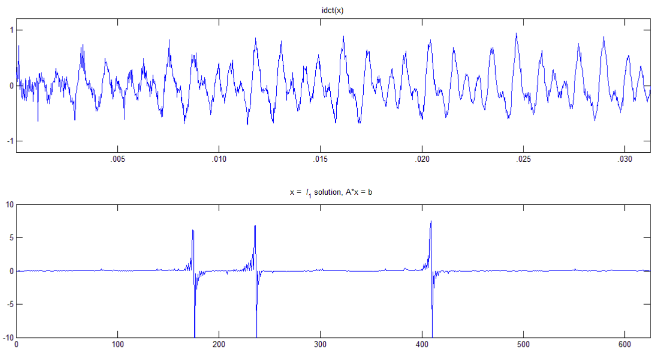

Fig. 2. The l_1 solution, x, and its inverse discrete cosine transform

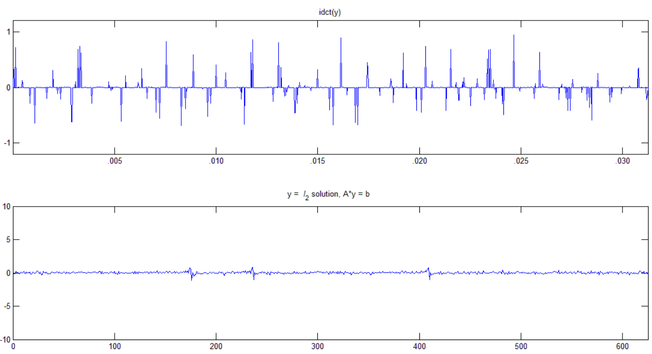

Fig. 3. The l_2 solution, y, and its inverse discrete cosine transform

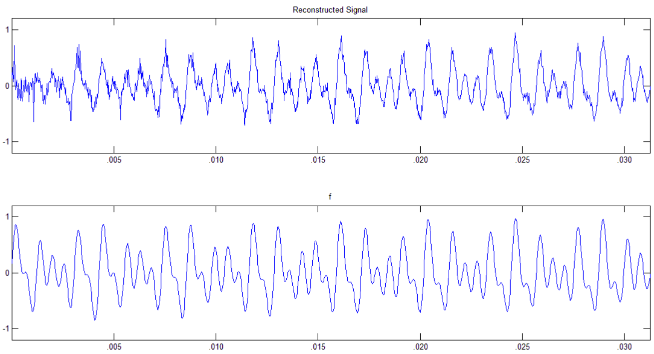

Fig. 4. The reconstructed signal and the original signal

V. CONCLUSION

In summary, most real signals, when represented in an appropriate basis, are sparse. It is interesting and a powerful notion that one can design nonadaptive measurement techniques to compress a signal without any knowledge about what the signal is made of. The number of measurements is comparable to its sparsity measure rather than its nominal length. Compressed Sensing which lets us think outside Nyquist-Shannon box has paved way for efficient data acquisition methods. The idea of compressed sensing was established by a simple yet concrete example.

VI. ACKNOWLEDGMENT

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