# Testing Random-Number Generators


1. Chi-square test
2. Kolmogorov-Smirnov Test
3. Serial-correlation Test
4. Two-level tests
5. K-dimensional uniformity or k-distributivity
6. Serial Test
7. Spectral Test

# Testing Random-Number Generators

Goal: To ensure that the random number generator produces a random stream.

- Plot histograms
- Plot quantile-quantile plot
- Use other *t*ests
- Passing a test is necessary but not sufficient
- Pass $\neq$ Good

  Fail $\Rightarrow$ Bad
- New tests $\Rightarrow$ Old generators fail the test
- Tests can be adapted for other distributions

# Chi-Square Test

- Most commonly used test
- Can be used for any distribution
- Prepare a histogram of the observed data
- Compare observed frequencies with theoretical

$k$ = Number of cells

$o_i$ = Observed frequency for $i$th cell

$e_i$ = Expected frequency

$$D = \sum_{i=1}^{k} \frac{(o_i - e_i)^2}{e_i}$$

- $D$=0 $\Rightarrow$ Exact fit
- $D$ has a chi-square distribution with $k$-1 degrees of freedom.

$\Rightarrow$ Compare D with $\chi^2_{[1-\alpha; k-1]}$ Pass with confidence $\alpha$ if $D$ is less

# Example 27.1

- 1000 random numbers with $x_0 = 1$
- $\chi^2_{[0.9;9]} = 14.68$
- Observed difference = 10.380
- Observed is Less $\Rightarrow$ Accept IID U(0, 1)

$$x_n = (125 x_{n-1} + 1) \bmod (2^{12})$$

| Cell | Obsrvd | Exptd | $\frac{(o-e)^2}{e}$ |
|---|---|---|---|
| 1 | 100 | 100.0 | 0.000 |
| 2 | 96 | 100.0 | 0.160 |
| 3 | 98 | 100.0 | 0.040 |
| 4 | 85 | 100.0 | 2.250 |
| 5 | 105 | 100.0 | 0.250 |
| 6 | 93 | 100.0 | 0.490 |
| 7 | 97 | 100.0 | 0.090 |
| 8 | 125 | 100.0 | 6.250 |
| 9 | 107 | 100.0 | 0.490 |
| 10 | 94 | 100.0 | 0.360 |
| Total | 1000 | 1000.0 | 10.380 |

# Chi-Square for Other Distributions

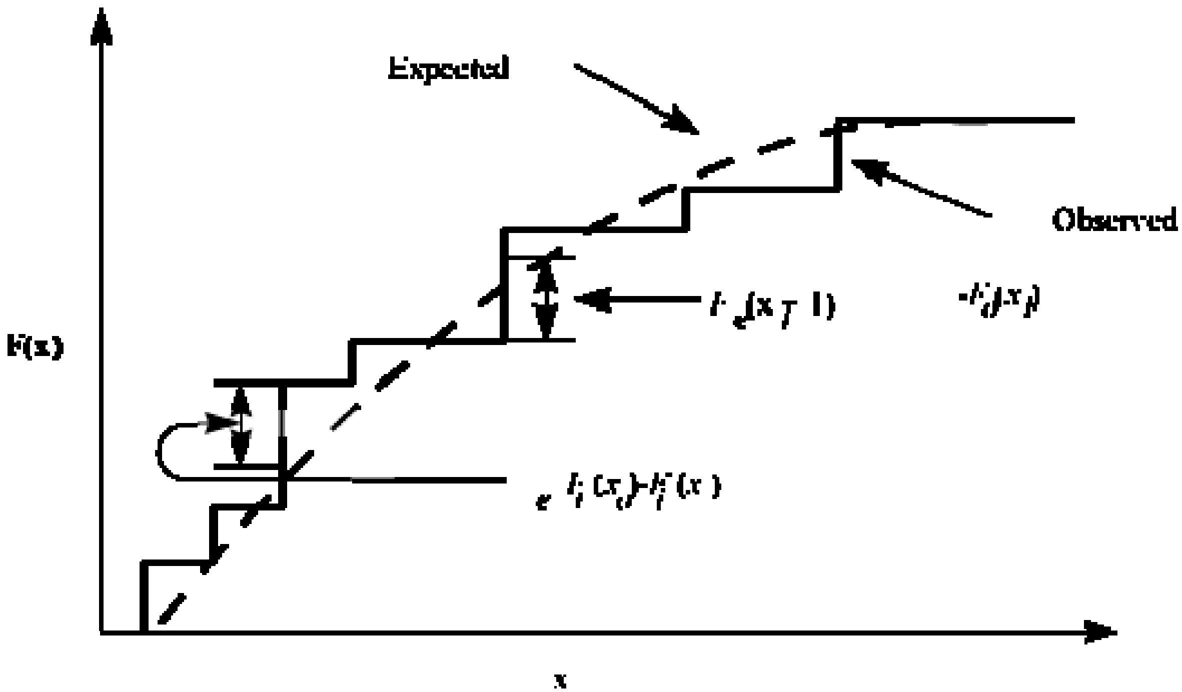
- Errors in cells with a small $e_i$ affect the chi-square statistic more
- Best when $e_i$'s are equal.

$\Rightarrow$ Use an equi-probable histogram with variable cell sizes

- Combine adjoining cells so that the new cell probabilities are approximately equal.
- The number of degrees of freedom should be reduced to $k$-$r$-1 (in place of $k$-1), where r is the number of parameters estimated from the sample.
- Designed for discrete distributions and for large sample sizes only $\Rightarrow$ Lower significance for finite sample sizes and continuous distributions
- If less than 5 observations, combine neighboring cells

# Kolmogorov-Smirnov Test

- Developed by A. N. Kolmogorov and N. V. Smirnov
- Designed for continuous distributions
- Difference between the observed CDF (cumulative distribution function) $F_o(x)$ and the expected cdf $F_e(x)$ should be small.



# Kolmogorov-Smirnov Test

- $K^+$ = maximum observed deviation below the expected cdf
- $K^-$ = minimum observed deviation below the expected cdf

$$K^+ = \sqrt{n} \max_x (F_o(x) - F_e(x))$$

$$K^- = \sqrt{n} \max_x (F_e(x) - F_o(x))$$

- $K^+ < K_{[1-\alpha;n]}$ and $K^- < K_{[1-\alpha;n]}$ ⇒ Pass at α level of significance.
- Don't use max/min *of* $Fe(x_i)-F_o(x_i)$
- Use $F_e(x_{i+1})-F_o(x_i)$ for $K^-$
- For $U(0, 1)$: $F_e(x)=x$
- $F_o(x) = j/n$,
  where $x > x_1, x_2, \ldots, x_{j-1}$

$$K^+ = \sqrt{n} \max_j \left(\frac{j}{n} - x_j\right)$$

$$K^- = \sqrt{n} \max_j \left(x_j - \frac{j-1}{n}\right)$$

# Example 27.2

30 Random numbers using a seed of $x_0$=15:

$$x_n = 3x_{n-1} \bmod 31$$

- The numbers are:

14, 11, 2, 6, 18, 23, 7, 21, 1, 3, 9, 27, 19, 26, 16, 17, 20, 29, 25, 13, 8, 24, 10, 30, 28, 22, 4, 12, 5, 15.

# Example 27.2 (Cont)

The normalized numbers obtained by dividing the sequence by 31 are:

0.45161, 0.35484, 0.06452, 0.19355, 0.58065, 0.74194,
0.22581, 0.67742, 0.03226, 0.09677, 0.29032, 0.87097,
0.61290, 0.83871, 0.51613, 0.54839, 0.64516, 0.93548,
0.80645, 0.41935, 0.25806, 0.77419, 0.32258, 0.96774,
0.90323, 0.70968, 0.12903, 0.38710, 0.16129, 0.48387.

# Example 27.2 (Cont)

- $K_{[0.9;n]}$ value for $n$ = 30 and $a$ = 0.1 is 1.0424

$$
\begin{aligned}
K^- &= \sqrt{n} \max_j \left(x_j - \frac{j-1}{n}\right) \\
&= \sqrt{30} \times 0.03026 \\
&= 0.1767
\end{aligned}
$$

$$
\begin{aligned}
K^+ &= \sqrt{n} \max_j \left(\frac{j}{n} - x_j\right) \\
&= \sqrt{30} \times 0.03026 \\
&= 0.1767
\end{aligned}
$$

| $j$ | $x_j$ | $\frac{j}{n} - x_j$ | $x_j - \frac{j-1}{n}$ |
|---|---|---|---|
| 1 | 0.03226 | 0.00108 | 0.03226 |
| 2 | 0.06452 | 0.00215 | 0.03118 |
| 3 | 0.09677 | 0.00323 | 0.03011 |
| 4 | 0.12903 | 0.00430 | 0.02903 |
| 5 | 0.16129 | 0.00538 | 0.02796 |
| 6 | 0.19355 | 0.00645 | 0.02688 |
| 7 | 0.22581 | 0.00753 | 0.02581 |
| 8 | 0.25806 | 0.00860 | 0.02473 |
| ⋮ | | | |
| 29 | 0.93548 | 0.03118 | 0.00215 |
| 30 | 0.96774 | 0.03226 | 0.00108 |
| | Max | 0.03226 | 0.03226 |

- Observed<Table
  $\Rightarrow$ Pass

# Chi-square vs. K-S Test

| K-S test | Chi-Square Test |
|---|---|
| Small samples | Large Sample |
| Continuous distributions | Discrete distributions |
| Differences between observed and expected cumulative probabilities (CDFs) | Differences between observed and hypothesized probabilities (pdfs or pmfs). |
| Uses each observation in the sample without any grouping ⇒makes a better use of the data | Groups observations into a small number of cells |
| Cell size is not a problem | Cell sizes affect the conclusion but no firm guidelines |
| Exact | Approximate |

# Serial-Correlation Test

- Nonzero covariance ⇒ Dependence. The inverse is not true
- $R_k$ = Autocovariance at lag $k$ = Cov[$x_n$, $x_{n+k}$]

$$R_k = \frac{1}{n-k}\sum_{i=1}^{n-k}(U_i - \frac{1}{2})(U_{i+k} - \frac{1}{2})$$

- For large $n$, $R_k$ is normally distributed with a mean of zero and a variance of $1/[144(n-k)]$
- $100(1-\alpha)\%$ confidence interval for the autocovariance is:

$$R_k \mp z_{1-\alpha/2}/(12\sqrt{n-k})$$

For $k \geq 1$ Check if CI includes zero

- For $k = 0$, $R_0$= variance of the sequence Expected to be 1/12 for IID $U(0,1)$

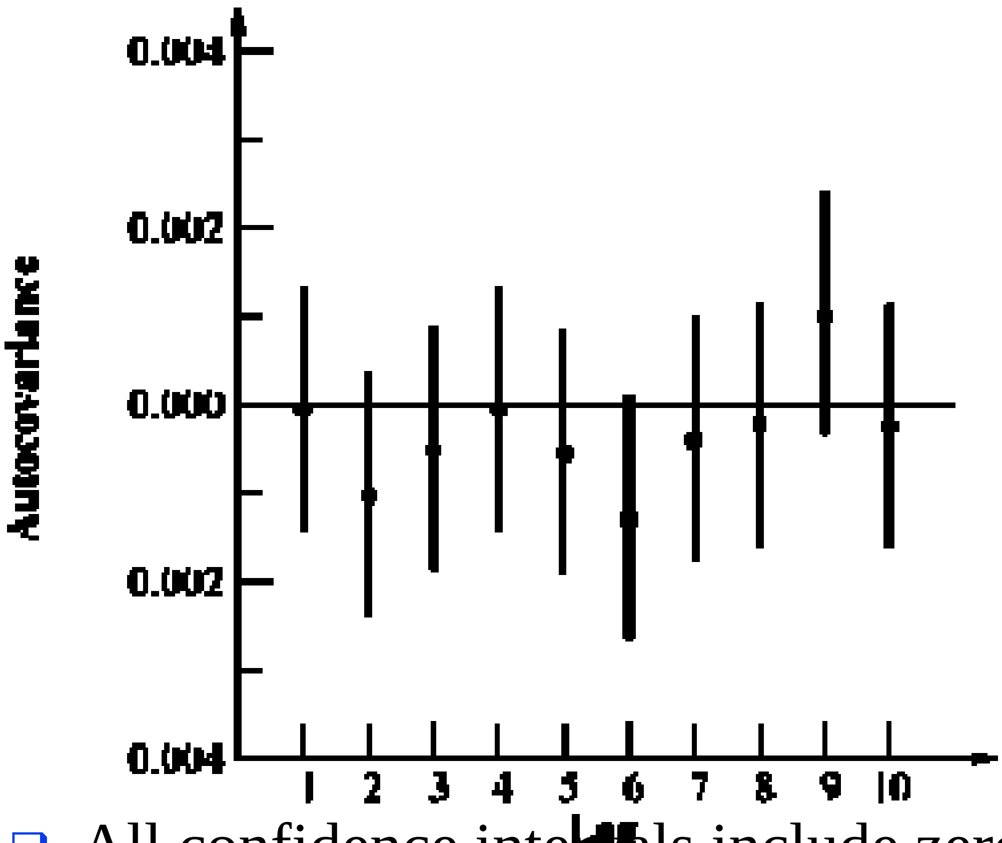
# Example 27.3: Serial Correlation Test

$$x_n = 7^5 x_{n-1} \bmod (2^{31} - 1)$$

10,000 random numbers with $x_0$=1:

| Lag | Autocovariance | St. Dev. | 90% Confidence Interval | |
|---|---|---|---|---|
| $k$ | $R_k$ | of $R_k$ | Lower Limit | Upper Limit |
| 1 | -0.000038 | 0.000833 | -0.001409 | 0.001333 |
| 2 | -0.001017 | 0.000833 | -0.002388 | 0.000354 |
| 3 | -0.000489 | 0.000833 | -0.001860 | 0.000882 |
| 4 | -0.000033 | 0.000834 | -0.001404 | 0.001339 |
| 5 | -0.000531 | 0.000834 | -0.001902 | 0.000840 |
| 6 | -0.001277 | 0.000834 | -0.002648 | 0.000095 |
| 7 | -0.000385 | 0.000834 | -0.001757 | 0.000986 |
| 8 | -0.000207 | 0.000834 | -0.001579 | 0.001164 |
| 9 | 0.001031 | 0.000834 | -0.000340 | 0.002403 |
| 10 | -0.000224 | 0.000834 | -0.001595 | 0.001148 |

# Example 27.3 (Cont)




- All confidence intervals include zero ⇒ All covariances are statistically insignificant at 90% confidence.

# Two-Level Tests

- If the sample size is too small, the test results may apply locally, but not globally to the complete cycle.
- Similarly, global test may not apply locally
- Use two-level tests

  ⇒ Use Chi-square test on $n$ samples of size $k$ each and then use a Chi-square test on the set of $n$ Chi-square statistics so obtained

⇒ Chi-square on Chi-square test.

- Similarly, *K-S* on *K-S*
- Can also use this to find a \`\`nonrandom'' segment of an otherwise random sequence.

# k-Distributivity

- k-Dimensional Uniformity
- Chi-square $\Rightarrow$ uniformity in one dimension
  $\Rightarrow$ Given two real numbers $a_1$ and $b_1$ between 0 and 1 such that $b_1 > a_1$

$$P(a_1 \leq u_n < b_1) = b_1 - a_1 \quad \forall b_1 > a_1$$

- This is known as 1-distributivity property of $u_n$.
- The 2-distributivity is a generalization of this property in two dimensions:

$$P(a_1 \leq u_{n-1} < b_1 \text{ and } a_2 \leq u_n < b_2)$$

$$= (b_1 - a_1)(b_2 - a_2)$$

For all choices of $a_1$, $b_1$, $a_2$, $b_2$ in [0, 1], $b_1 > a_1$ and $b_2 > a_2$

# k-Distributivity (Cont)

- k-distributed if:

$$P(a_1 \leq u_n < b_1, \ldots, a_k \leq u_{n+k-1} < b_k)$$

$$(b_1 - a_1) \cdots (b_k - a_k)$$

- For all choices of $a_i$, $b_i$ in [0, 1], with $b_i > a_i$, $i$=1, 2, ..., $k$.
- k-distributed sequence is always ($k$-1)-distributed. The inverse is not true.
- Two tests:

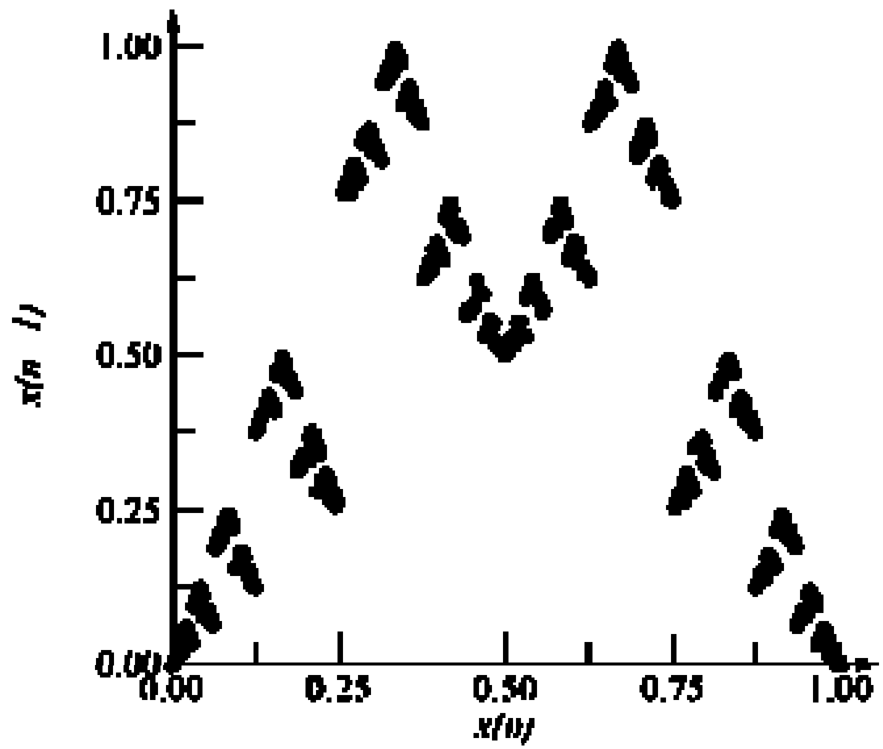
1. Serial test
2. Spectral test
3. Visual test for 2-dimensions: Plot successive overlapping pairs of numbers

# Example 27.4

- *T*ausworthe sequence generated by:

  $$x^{15} + x + 1$$

- The sequence is *k*-distributed for *k* up to $\lceil\ /l\ \rceil$, that is, *k*=1.
- In two dimensions: Successive overlapping pairs $(x_n, x_{n+1})$

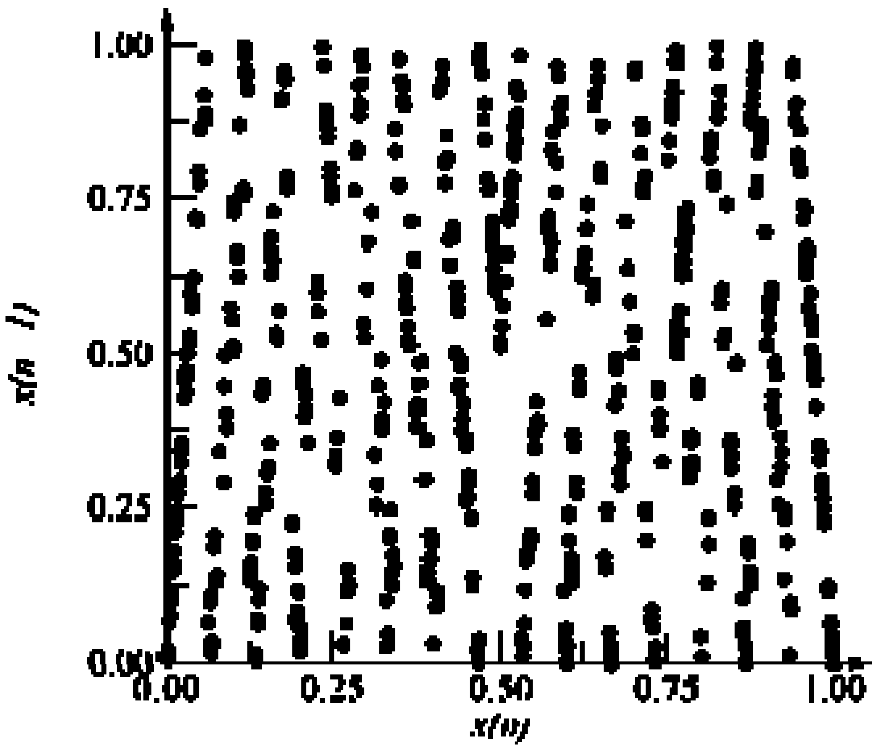


# Example 27.5

- Consider the polynomial:

  $$x^{15} + x^4 + 1$$

- Better 2-distributivity than Example 27.4

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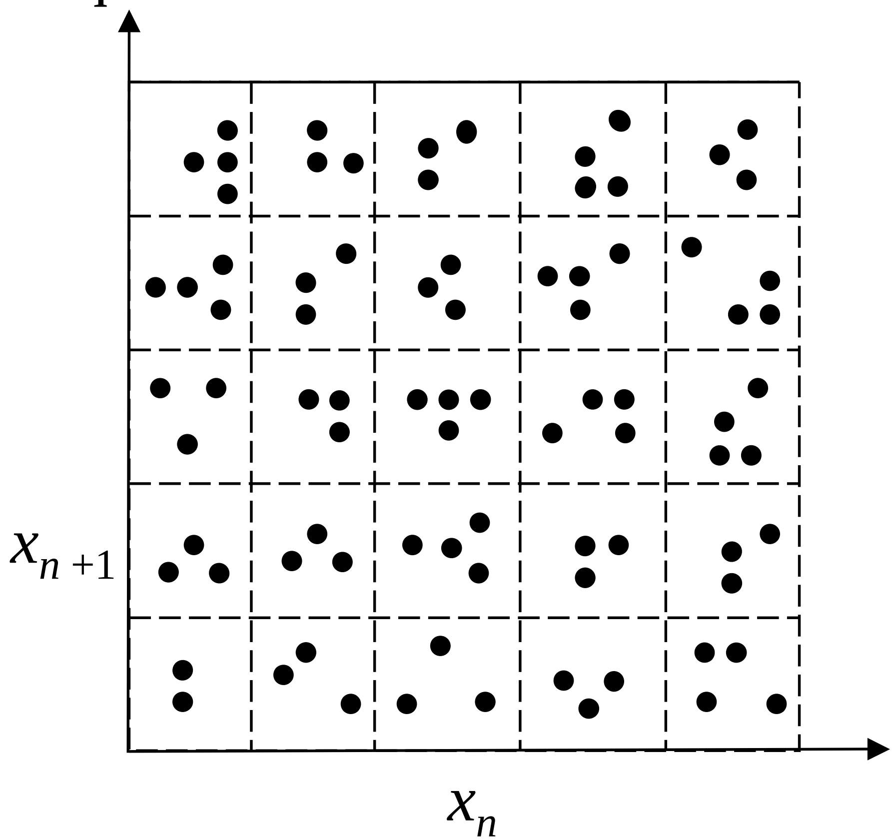
# Serial Test

- Goal: To test for uniformity in two dimensions or higher.
- In two dimensions, divide the space between 0 and 1 into $K^2$ cells of equal area



# Serial Test (Cont)

- Given $\{x_1, x_2, \ldots, x_n\}$, use $n/2$ non-overlapping pairs $(x_1, x_2)$, $(x_3, x_4)$, … and count the points in each of the $K^2$ cells.
- Expected= $n/(2K^2)$ points in each cell.
- Use chi-square test to find the deviation of the actual counts from the expected counts.
- The degrees of freedom in this case are $K^2$-1.
- For $k$-dimensions: use $k$-tuples of non-overlapping values.
- $k$-tuples must be non-overlapping.
- Overlapping $\Rightarrow$ number of points in the cells are not independent chi-square test cannot be used
- In visual check one can use overlapping or non-overlapping.
- In the spectral test overlapping tuples are used.
- Given n numbers, there are $n$-1 overlapping pairs, $n$/2 non-overlapping pairs.

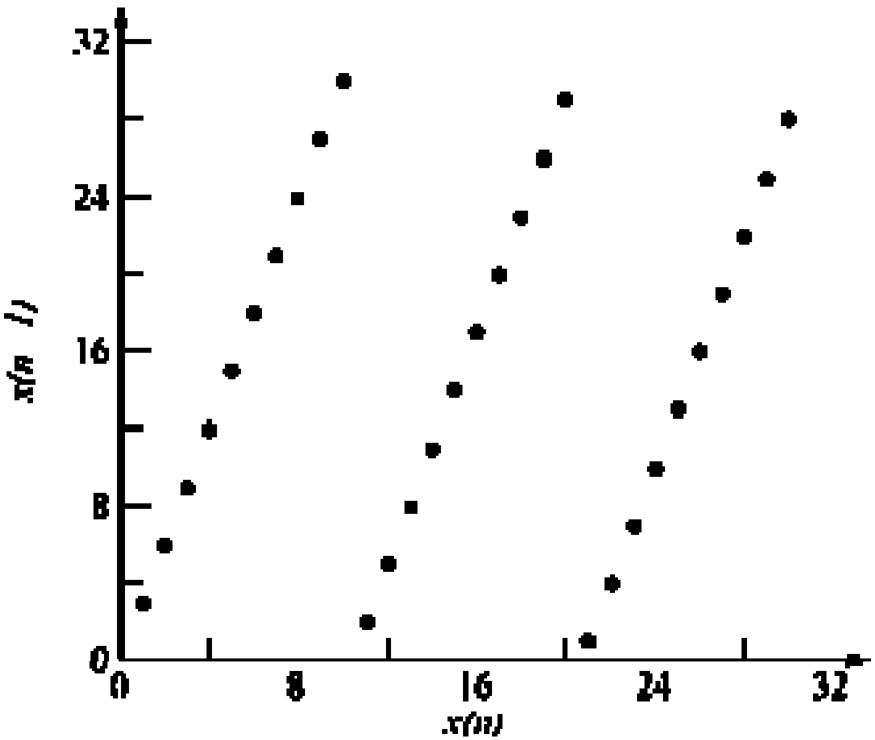
# Spectral Test

- Goal: To determine how densely the k-tuples $\{x_1, x_2, \ldots, x_k\}$can fill up the $k$-dimensional hyperspace.
- The $k$-tuples from an LCG fall on a finite number of parallel hyper-planes.
- Successive pairs would lie on a finite number of lines
- In three dimensions, successive triplets lie on a finite number of planes.

# Example 27.6: Spectral Test

$$x_n = 3x_{n-1} \bmod 31$$

Plot of overlapping pairs




- All points lie on three straight lines.

$$\begin{aligned} x_n &= 3x_{n-1} \\ x_n &= 3x_{n-1} - 31 \\ x_n &= 3x_{n-1} - 62 \end{aligned}$$

- Or:

$$x_n = 3x_{n-1} - 31k \quad k = 0, 1, 2$$

# Example 27.6 (Cont)

- In three dimensions, the points ($x_n$, $x_{n-1}$, $x_{n-2}$) for the above generator would lie on five planes given by:

$$x_n = 2x_{n-1} + 3x_{n-2} - 31k \quad k = 0, 1, \ldots, 4$$

Obtained by adding the following to equation

$$x_{n-1} = 3x_{n-2} - 31k_1 \quad k_1 = 0, 1, 2$$

Note that $k+k_1$ will be an integer between 0 and 4.

# Spectral Test (More)

- Marsaglia (1968): Successive k-tuples obtained from an LCG fall on, at most, $(k!m)^{1/k}$ parallel hyper-planes, where $m$ is the modulus used in the LCG.
- Example: $m = 2^{32}$, fewer than 2,953 hyper-planes will contain all 3-tuples, fewer than 566 hyper-planes will contain all 4-tuples, and fewer than 41 hyper-planes will contain all 10-tuples. Thus, this is a weakness of LCGs.
- Spectral Test: Determine the max distance between adjacent hyper-planes.
- Larger distance $\Rightarrow$ worse generator
- In some cases, it can be done by complete enumeration

# Example 27.7

- Compare the following two generators:

$$x_n = 3x_{n-1} \bmod 31$$

$$x_n = 13x_{n-1} \bmod 31$$

- Using a seed of $x_0$=15, first generator:

14, 11, 2, 6, 18, 23, 7, 21, 1, 3, 9, 27, 19, 26, 16, 17, 20, 29, 25, 13, 8, 24, 10, 30, 28, 22, 4, 12, 5, 15, 14.

- Using the same seed in the second generator:

9, 24, 2, 26, 28, 23, 20, 12, 1, 13, 14, 27, 10, 6, 16, 22, 7, 29, 5, 3, 8, 11, 19, 30, 18, 17, 4, 21, 25, 15, 9.

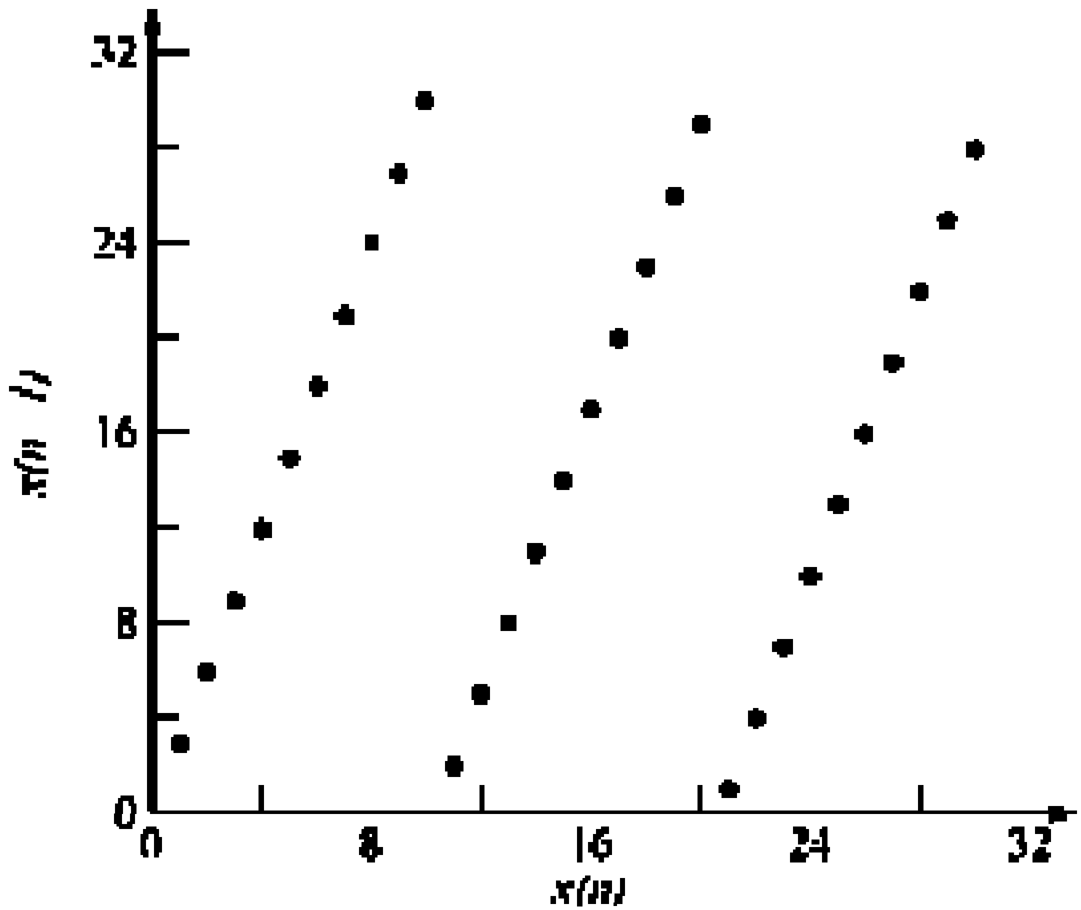
# Example 27.7 (Cont)

- Every number between 1 and 30 occurs once and only once

$\Rightarrow$ Both sequences will pass the chi-square test for uniformity

# Example 27.7 (Cont)

- First Generator:



# Example 27.7 (Cont)

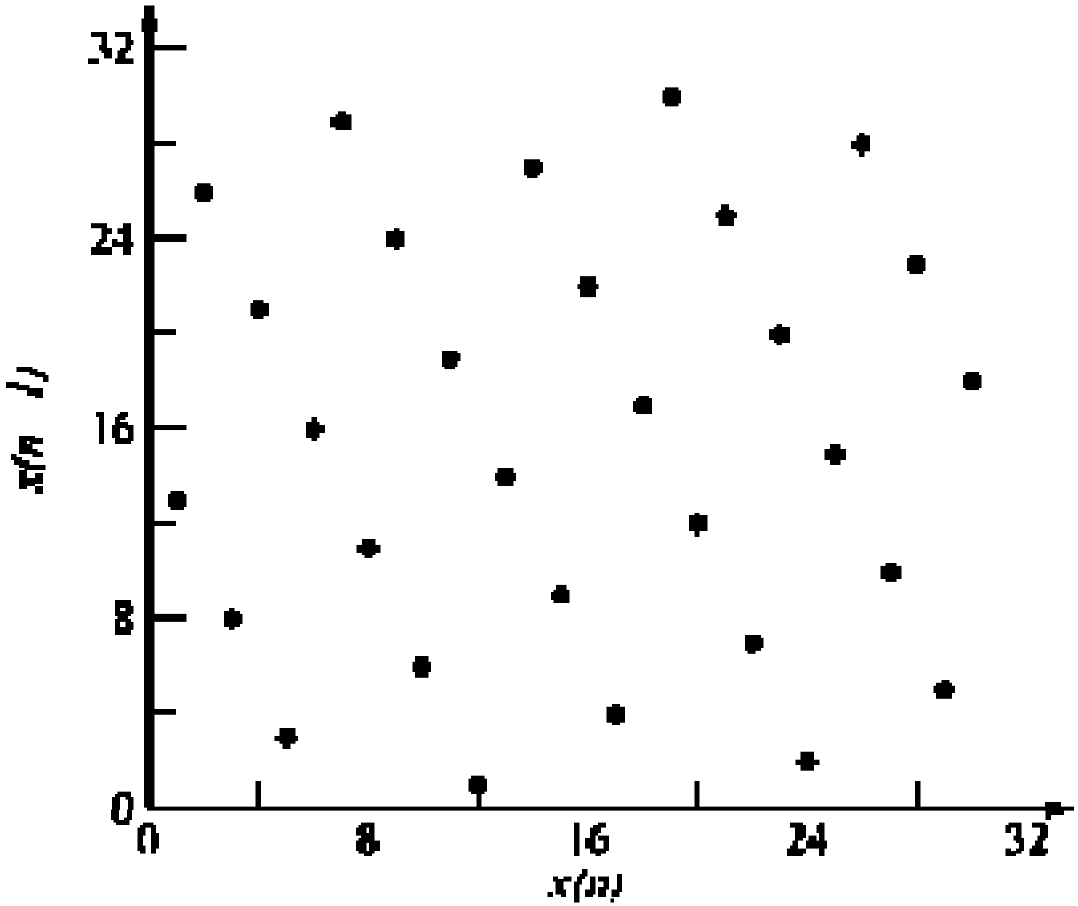
- Three straight lines of positive slope or ten lines of negative slope
- Since the distance between the lines of positive slope is more, consider only the lines with positive slope.

$$\begin{aligned} x_n &= 3x_{n-1} \\ x_n &= 3x_{n-1} - 31 \\ x_n &= 3x_{n-1} - 62 \end{aligned}$$

- Distance between two parallel lines $y=ax+c_1$ and $y=ax+c_2$ is given by $|c_2 - c_1|/\sqrt{1+a^2}$
- The distance between the above lines is $31/\sqrt{10}$ or 9.80.

# Example 27.7 (Cont)

- Second Generator:



# Example 27.7 (Cont)

- All points fall on seven straight lines of positive slope or six straight lines of negative slope.
- Considering lines with negative slopes:

$$x_n = -\frac{5}{2}x_{n-1} + k\frac{31}{2} \quad k = 0, 1, \ldots, 5$$

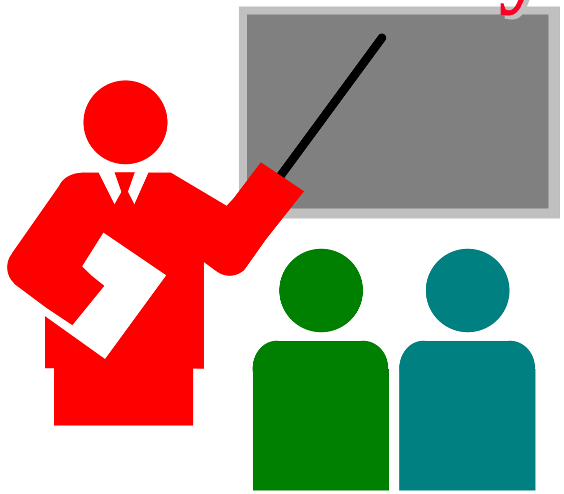
- The distance between lines is: $(31/2)/\sqrt{(1+(5/2)^2)}$ or 5.76.
- The second generator has a smaller maximum distance and, hence, the second generator has a better 2-distributivity.
- The set with a larger distance may **not** always be the set with fewer lines.

# Example 27.7 (Cont)

- Either overlapping or non-overlapping *k*-tuples can be used.
  - With overlapping *k*-tuples, we have k times as many points, which makes the graph visually more complete.The number of hyper-planes and the distance between them are the same with either choice.
- With serial test, only non-overlapping *k*-tuples should be used.
- For generators with a large *m* and for higher dimensions, finding the maximum distance becomes quite complex.

  See Knuth (1981)

# Summary



1. Chi-square test is a one-dimensional test
   Designed for discrete distributions and large sample sizes
2. K-S test is designed for continuous variables
3. Serial correlation test for independence
4. Two level tests find local non-uniformity
5. k-dimensional uniformity = k-distributivity
   tested by spectral test or serial test

# Homework

- Submit detailed answer to Exercise 27.3. Print 10,000th number also.

# Exercise 27.1

Generate 10,000 numbers using a seed of $x_0$=1 in the following generator:

$$x_n = 7^5 x_{n-1} \bmod (2^{31}-1)$$

Classify the numbers into ten equal size cells and test for uniformity using the chi-square test at 90% confidence.

# Exercise 27.2

Generate 15 numbers using a seed of $x_0$=1 in the following generator:

$$x_n = (5x_{n-1} + 1) \bmod 16$$

Perform a *K-S* test and check whether the sequence passes the test at a 95% confidence level.

# Exercise 27.3

Generate 10,000 numbers using a seed of $x_0=1$ in the following LCG:

$$x_n = 48271 x_{n-1} \bmod (2^{31} - 1)$$

Perform the serial correlation test of randomness at 90% confidence and report the result.

# Exercise 27.4

Using the spectral test, compare the following two generators

$$x_n = 7x_{n-1} \bmod 13$$

$$x_n = 11x_{n-1} \bmod 13$$

Which generator has a better 2-distributivity?