Cones, rectifiability, and SIOs

Damian Dąbrowski

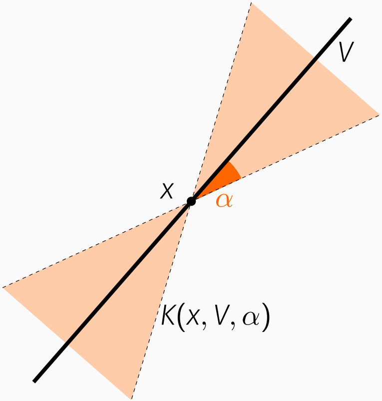
Universitat Autònoma de Barcelona, Barcelona Graduate School of Mathematics (BGS-Math)

Thanks to support from MDM-2014-0445.

Cones

Given $x \in \mathbb{R}^d$, $V \in G(d, m)$, $\alpha \in (0, 1)$, set

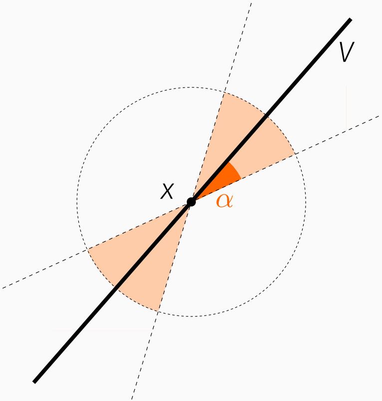
$$K(x, V, \alpha) = \{y \in \mathbb{R}^d \ : \ \mathrm{dist}(y, x + V) < \alpha|y - x|\}.$$



Cones

Given $x \in \mathbb{R}^d$, $V \in G(d, m)$, $\alpha \in (0, 1)$, set

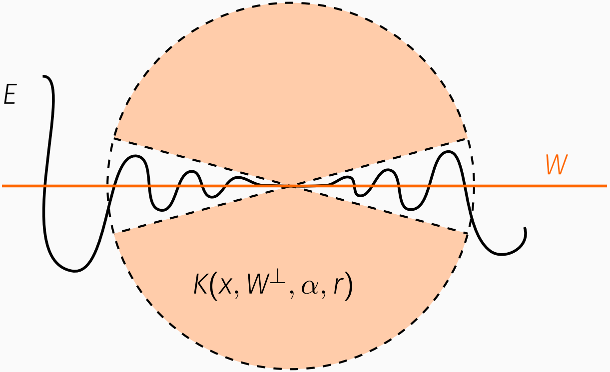
$$K(x, V, \alpha, r) = K(x, V, \alpha) \cap B(x, r).$$



Tangent planes

A plane $W \in G(d, n)$ is a **tangent plane** to E at x if for all $\alpha \in (0, 1)$ there exists $r > 0$ such that

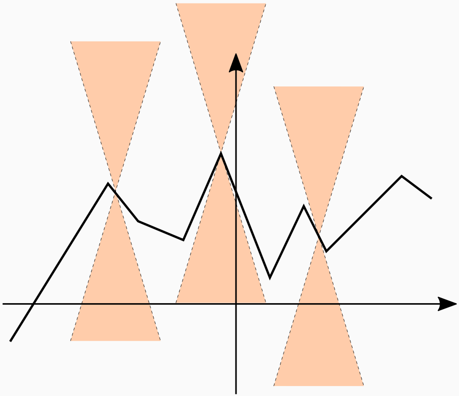
$$E \cap K(x, W^{\perp}, \alpha, r) = \varnothing.$$



Cones and Lipschitz graphs

Easy to show: $E \subset \mathbb{R}^d$ is a subset of an n-dimensional Lipschitz graph iff there exists $V \in G(d, d-n)$, $\alpha \in (0,1)$, such that

$$x \in E \quad \Rightarrow \quad E \cap K(x, V, \alpha) = \varnothing.$$



Rectifiability

Rectifiable sets and measures

A set $E \subset \mathbb{R}^d$ is *n-rectifiable* if there exists a countable number of n-dimensional Lipschitz graphs Γ_i such that

$$\mathcal{H}^n\left(E \setminus \bigcup_i \Gamma_i\right) = 0.$$

Rectifiable sets and measures

A set $E \subset \mathbb{R}^d$ is *n*-**rectifiable** if there exists a countable number of n-dimensional Lipschitz graphs Γ_i such that

$$\mathcal{H}^n\left(E \setminus \bigcup_i \Gamma_i\right) = 0.$$

A measure μ on $\mathbb{R}^d$ is *n*-**rectifiable** if it is of the form

$$\mu = f\mathcal{H}^n|_E$$

for some n-rectifiable $E \subset \mathbb{R}^d$ and $f \in L^1_{loc}(E)$.

Purely unrectifiable sets

We say that $F \subset \mathbb{R}^d$ is **purely n-unrectifiable** if for every Γ - Lipschitz image of $\mathbb{R}^n$

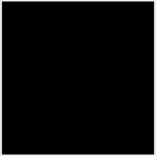
$$\mathcal{H}^n(F \cap \Gamma) = 0.$$

Purely unrectifiable sets

We say that $F \subset \mathbb{R}^d$ is **purely n-unrectifiable** if for every Γ - Lipschitz image of $\mathbb{R}^n$

$$\mathcal{H}^n(F \cap \Gamma) = 0.$$

Example

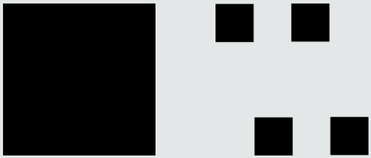


Purely unrectifiable sets

We say that $F \subset \mathbb{R}^d$ is **purely n-unrectifiable** if for every Γ - Lipschitz image of $\mathbb{R}^n$

$$\mathcal{H}^n(F \cap \Gamma) = 0.$$

Example

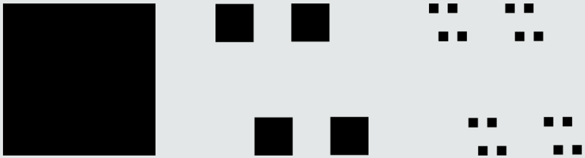


Purely unrectifiable sets

We say that $F \subset \mathbb{R}^d$ is **purely n-unrectifiable** if for every Γ - Lipschitz image of $\mathbb{R}^n$

$$\mathcal{H}^n(F \cap \Gamma) = 0.$$

Example

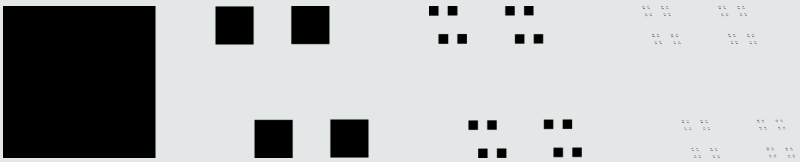


Purely unrectifiable sets

We say that $F \subset \mathbb{R}^d$ is **purely n-unrectifiable** if for every Γ - Lipschitz image of $\mathbb{R}^n$

$$\mathcal{H}^n(F \cap \Gamma) = 0.$$

Example



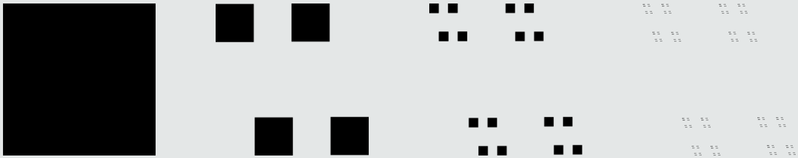
F is purely 1-unrectifiable and satisfies $1 \leq \mathcal{H}^1(F) \leq \sqrt{2}$.

Purely unrectifiable sets

We say that $F \subset \mathbb{R}^d$ is **purely n-unrectifiable** if for every Γ - Lipschitz image of $\mathbb{R}^n$

$$\mathcal{H}^n(F \cap \Gamma) = 0.$$

Example



F is purely 1-unrectifiable and satisfies $1 \leq \mathcal{H}^1(F) \leq \sqrt{2}$.

Any set of finite $\mathcal{H}^n$ measure can be decomposed into a rectifiable and purely unrectifiable part.

Why do we care?

Applications in:

Why do we care?

Applications in:

- boundedness of singular integral operators,

Why do we care?

Applications in:

- boundedness of singular integral operators,
- study of removable sets for bounded analytic functions,

Why do we care?

Applications in:

- boundedness of singular integral operators,
- study of removable sets for bounded analytic functions,
- optimal regularity of domains that ensure L^p solvability of the Dirichlet problem,

Why do we care?

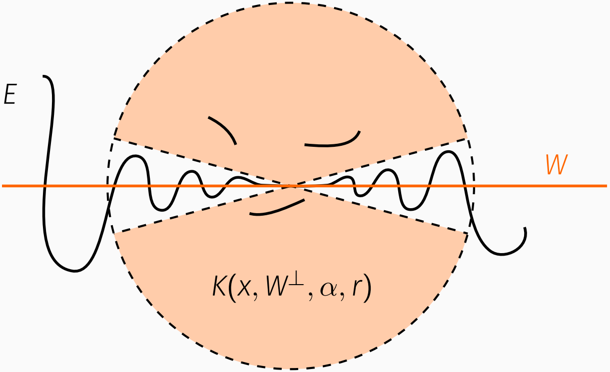
Applications in:

- boundedness of singular integral operators,
- study of removable sets for bounded analytic functions,
- optimal regularity of domains that ensure L^p solvability of the Dirichlet problem,
- study of singular sets of harmonic maps, free boundaries...

Approximate tangent planes

A plane $W \in G(d,n)$ is an approximate **tangent plane** to E at x if for all $\alpha \in (0,1)$

$$\lim_{r\to 0} \frac{\mathcal{H}^n(E \cap K(x, W^\perp, \alpha, r))}{r^n} = 0.$$



Approximate tangents characterize rectifiability

Theorem (Federer '47)

Let $E \subset \mathbb{R}^d$, $\mathcal{H}^n(E) < \infty$. Then E is n-rectifiable iff for $\mathcal{H}^n$-a.e. $x \in E$ there is a unique approximate tangent plane to E at x, i.e. for all α

$$\lim_{r \to 0} \frac{\mathcal{H}^n(E \cap K(x, W^\perp, \alpha, r))}{r^n} = 0.$$

Theorem (Federer '47)

Let $E \subset \mathbb{R}^d$, $\mathcal{H}^n(E) < \infty$. Then E is n-rectifiable iff for $\mathcal{H}^n$-a.e. $x \in E$ there is a unique approximate tangent plane to E at x, i.e. for all α

$$\lim_{r \to 0} \frac{\mathcal{H}^n(E \cap K(x, W^\perp, \alpha, r))}{r^n} = 0.$$

Analogous result holds for μ satisfying $0 < \Theta^{n,*}(\mu, x) < \infty$,

$$\Theta^{n,*}(\mu, x) = \limsup_{r \to 0} \frac{\mu(B(x,r))}{r^n}, \qquad \Theta^{n}_{*}(\mu, x) = \liminf_{r \to 0} \frac{\mu(B(x,r))}{r^n}.$$

Approximate tangents characterize rectifiability

Theorem (Federer '47)

Let $E \subset \mathbb{R}^d$, $\mathcal{H}^n(E) < \infty$. Then E is n-rectifiable iff for $\mathcal{H}^n$-a.e. $x \in E$ there is a unique approximate tangent plane to E at x, i.e. for all α

$$\lim_{r \to 0} \frac{\mathcal{H}^n(E \cap K(x, W^\perp, \alpha, r))}{r^n} = 0.$$

Analogous result holds for μ satisfying $0 < \Theta^{n,*}(\mu, x) < \infty$,

$$\Theta^{n,*}(\mu, x) = \limsup_{r \to 0} \frac{\mu(B(x,r))}{r^n}, \qquad \Theta^{n}_{*}(\mu, x) = \liminf_{r \to 0} \frac{\mu(B(x,r))}{r^n}.$$

Fact

μ is rectifiable $\quad \Rightarrow \quad 0 < \Theta^{n,*}(\mu, x) = \Theta^{n}_{*}(\mu, x) < \infty$ a.e.

Conical energy

Let $V \in G(d, d-n)$, $\alpha \in (0,1)$, $1 \leq p < \infty$. The (V, α, p) conical energy of E at $x \in E$ up to scale $R > 0$ is

$$\mathcal{E}_{E,p}(x, V, \alpha, R) = \int_0^R \left(\frac{\mathcal{H}^n(E \cap K(x, V, \alpha, r))}{r^n} \right)^p \frac{dr}{r}.$$

Conical energy

Let $V \in G(d, d-n)$, $\alpha \in (0,1)$, $1 \leq p < \infty$. The (V, α, p) **conical energy** of E at $x \in E$ up to scale $R > 0$ is

$$\mathcal{E}_{E,p}(x, V, \alpha, R) = \int_0^R \left(\frac{\mathcal{H}^n(E \cap K(x, V, \alpha, r))}{r^n} \right)^p \frac{dr}{r}.$$

More generally: for a Radon measure μ on $\mathbb{R}^d$ define

$$\mathcal{E}_{\mu,p}(x, V, \alpha, R) = \int_0^R \left(\frac{\mu(K(x, V, \alpha, r))}{r^n} \right)^p \frac{dr}{r}.$$

Rectifiability implies finite energy

Theorem (D. '20)

Let $1 \leq p < \infty$. Suppose μ is n-rectifiable. Then, for μ-a.e. x there is $V_x \in G(d, d-n)$ such that for all $\alpha \in (0,1)$

$$\mathcal{E}_{\mu,p}(x, V_x, \alpha, 1) = \int_0^1 \left(\frac{\mu(K(x, V_x, \alpha, r))}{r^n} \right)^p \frac{dr}{r} < \infty.$$

Finite energy implies rectifiability

Theorem (D. '20)

Let $1 \leq p < \infty$. Suppose μ is a Radon measure satisfying $0 < \Theta^{n,*}(\mu, x)$ and $\Theta^n_*(\mu, x) < \infty$. Assume that for μ-a.e. x there is $V_x \in G(d, d-n)$ and $\alpha \in (0,1)$ such that

$$\mathcal{E}_{\mu,p}(x, V_x, \alpha, 1) = \int_0^1 \left(\frac{\mu(K(x, V_x, \alpha, r))}{r^n} \right)^p \frac{dr}{r} < \infty.$$

Then, μ is n-rectifiable.

Finite energy implies rectifiability

Theorem (D. '20)

Let $1 \leq p < \infty$. Suppose μ is a Radon measure satisfying $0 < \Theta^{n,*}(\mu, x)$ and $\Theta^n_*(\mu, x) < \infty$. Assume that for μ-a.e. x there is $V_x \in G(d, d-n)$ and $\alpha \in (0,1)$ such that

$$\mathcal{E}_{\mu,p}(x, V_x, \alpha, 1) = \int_0^1 \left(\frac{\mu(K(x, V_x, \alpha, r))}{r^n} \right)^p \frac{dr}{r} < \infty.$$

Then, μ is n-rectifiable.

Finite energy implies rectifiability

Theorem (D. '20)

Let $1 \leq p < \infty$. Suppose μ is a Radon measure satisfying $0 < \Theta^{n,*}(\mu, x)$ and $\Theta^n_*(\mu, x) < \infty$. Assume that for μ-a.e. x there is $V_x \in G(d, d-n)$ and $\alpha \in (0,1)$ such that

$$\mathcal{E}_{\mu,p}(x, V_x, \alpha, 1) = \int_0^1 \left(\frac{\mu(K(x, V_x, \alpha, r))}{r^n} \right)^p \frac{dr}{r} < \infty.$$

Then, μ is n-rectifiable.

Question

$0 < \Theta^{n,*}(\mu, x),\ \Theta^n_*(\mu, x) < \infty,$ approximate tangents exist a.e. $\stackrel{?}{\Longrightarrow}$ μ is rectifiable

Remarks on the proofs

$$\mu \text{ is rectifiable} \quad \Rightarrow \quad \mathcal{E}_{\mu,p} < \infty \text{ a.e.}$$

Follows easily from a result of Tolsa:

Theorem (Tolsa '15)

$$\mu \text{ is rectifiable} \quad \Rightarrow \quad \int_0^1 \beta_{\mu,2}(x,r)^2 \, \frac{dr}{r} < \infty \text{ a.e.}$$

Remarks on the proofs

$$\mu \text{ is rectifiable} \quad \Rightarrow \quad \mathcal{E}_{\mu,p} < \infty \text{ a.e.}$$

Follows easily from a result of Tolsa:

Theorem (Tolsa '15)

$$\mu \text{ is rectifiable} \quad \Rightarrow \quad \int_0^1 \beta_{\mu,2}(x,r)^2 \, \frac{dr}{r} < \infty \text{ a.e.}$$

Not difficult:

$$\mathcal{E}_{\mu,1}(x,V,\alpha,1) = \int_0^1 \frac{\mu(K(x,V,\alpha,r))}{r^n} \frac{dr}{r} \lesssim \int_0^1 \beta_{\mu,2}(x,r)^2 \, \frac{dr}{r} < \infty,$$

Remarks on the proofs

$$\mu \text{ is rectifiable} \quad \Rightarrow \quad \mathcal{E}_{\mu,p} < \infty \text{ a.e.}$$

Follows easily from a result of Tolsa:

Theorem (Tolsa '15)

$$\mu \text{ is rectifiable} \quad \Rightarrow \quad \int_0^1 \beta_{\mu,2}(x,r)^2 \, \frac{dr}{r} < \infty \text{ a.e.}$$

Not difficult:

$$\mathcal{E}_{\mu,1}(x,V,\alpha,1) = \int_0^1 \frac{\mu(K(x,V,\alpha,r))}{r^n} \frac{dr}{r} \lesssim \int_0^1 \beta_{\mu,2}(x,r)^2 \, \frac{dr}{r} < \infty,$$

and

$$\mathcal{E}_{\mu,p}(x,V,\alpha,1) \leq \Theta^{n,*}(\mu,x)^{p-1} \, \mathcal{E}_{\mu,1}(x,V,\alpha,1).$$

■

Remarks on the proofs

$$\begin{array}{c} 0 < \Theta^{n,*}(\mu,x),\ \Theta^{n}_{*}(\mu,x) < \infty, \\ \mathcal{E}_{\mu,p} < \infty \text{ a.e.} \end{array} \quad \Longrightarrow \quad \mu \text{ is rectifiable}$$

- a corona decomposition result,
- prove the theorem assuming additionally $\Theta^{n,*}(\mu,x) < \infty$,
- show that

$$\begin{array}{c} 0 < \Theta^{n,*}(\mu,x),\ \Theta^{n}_{*}(\mu,x) < \infty, \\ \mathcal{E}_{\mu,p} < \infty \text{ a.e.} \end{array} \quad \Longrightarrow \quad \Theta^{n,*}(\mu,x) < \infty.$$

$\blacksquare$

Big pieces of Lipschitz graphs

Big pieces of Lipschitz graphs

We say that $E \subset \mathbb{R}^d$ has big pieces of Lipschitz graphs (BPLG) if there exists $C, L, \kappa > 0$ such that

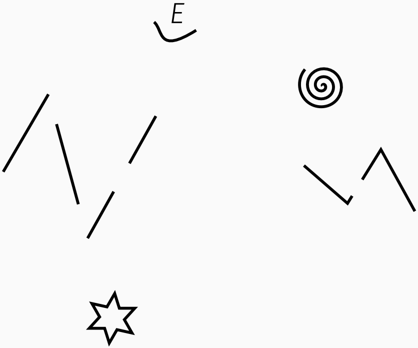
- it is AD-regular, i.e. for $x \in E,\ 0 < r < \operatorname{diam}(E)$

$$C^{-1} r^n \leq \mathcal{H}^n(E \cap B(x, r)) \leq C r^n,$$

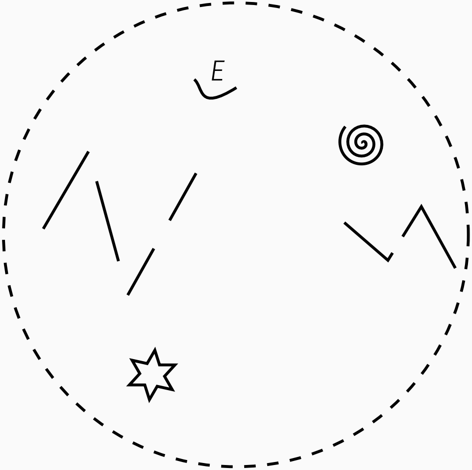
- for all balls B centered at E, $0 < r(B) < \operatorname{diam}(E)$, there exists a Lipschitz graph Γ, $\operatorname{Lip}(\Gamma) \leq L$, such that

$$\mathcal{H}^n(E \cap B \cap \Gamma) \geq \kappa r(B)^n.$$

Big pieces of Lipschitz graphs

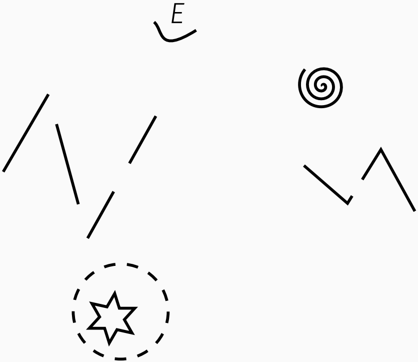


Big pieces of Lipschitz graphs

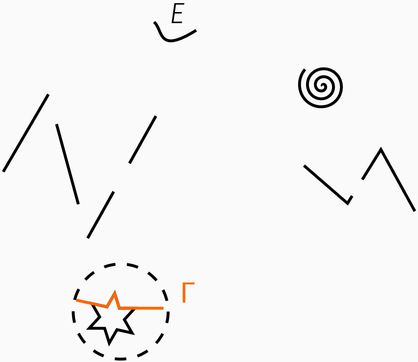


Big pieces of Lipschitz graphs

Big pieces of Lipschitz graphs



Big pieces of Lipschitz graphs



Characterizing BPLG using conical energy

Theorem (D. '20)

Suppose $E \subset \mathbb{R}^d$ is AD-regular, $1 \leq p < \infty$. Then E has BPLG iff there exist $\alpha, \kappa, M > 0$, such that the following holds.

Characterizing BPLG using conical energy

Theorem (D. '20)

Suppose $E \subset \mathbb{R}^d$ is AD-regular, $1 \leq p < \infty$. Then E has BPLG iff there exist $\alpha, \kappa, M > 0$, such that the following holds.

For all balls B centered at E, $0 < r(B) < \operatorname{diam}(E)$, there exists a set $G_B \subset E \cap B$ with $\mathcal{H}^n(G_B) \geq \kappa\, r(B)^n$, and a direction $V \in G(d, d-n)$, such that for all $x \in G_B$

$$\mathcal{E}_{E,p}(x, V, \alpha, r(B)) = \int_0^{r(B)} \left(\frac{\mathcal{H}^n(E \cap K(x, V, \alpha, r))}{r^n} \right)^p \frac{dr}{r} \leq M.$$

We will call the condition above **big pieces with bounded energy** (BPBE).

Proof of "E has BPBE $\Rightarrow$ E has BPLG"

E has BPBE $\quad\Rightarrow\quad$ E has BPLG

Can be reduced to

Theorem (Martikainen-Orponen '18)

Suppose $E \subset \mathbb{R}^d$ is AD-regular. Then E has BPLG iff there exist $\kappa, M > 0$, such that the following holds.

For all balls B centered at E, $0 < r(B) < \mathrm{diam}(E)$, there exists a set $G_B \subset E \cap B$ with $\mathcal{H}^n(G_B) \geq \kappa\, r(B)^n$, and a direction $V_B \in G(d, n)$, such that for a.e. $W \in \mathbf{B}(V_B, \kappa)$ we have $(\pi_W)_*(\mathcal{H}^n|_{G_B}) \in L^2(W)$, and

$$\int_{\mathbf{B}(V_B,\kappa)} \|(\pi_W)_*(\mathcal{H}^n|_{G_B})\|_{L^2(W)}^2 \, d\gamma_{d,n}(W) \leq M r(B)^n.$$

Bounded mean energy condition

Definition

We will say that an AD-regular set E satisfies the **bounded mean energy** condition if there exist $\alpha > 0, M > 1$, and for a.e. $x \in E$ there exists $V_x \in G(d, d-n)$, such that:

for all balls B centered at E, $0 < r(B) < \operatorname{diam}(E)$,

$$\int_{E\cap B} \mathcal{E}_{E,p}(x, V_x, \alpha, r(B))\, d\mathcal{H}^n(x) = \int_{E\cap B} \int_0^{r(B)} \left(\frac{\mathcal{H}^n(E \cap K(x, V_x, \alpha, r))}{r^n} \right)^p \frac{dr}{r} d\mathcal{H}^n(x) \leq M r(B)^n.$$

Bounded mean energy condition

Definition

We will say that an AD-regular set E satisfies the **bounded mean energy** condition if there exist $\alpha > 0, M > 1$, and for a.e. $x \in E$ there exists $V_x \in G(d, d-n)$, such that:

for all balls B centered at E, $0 < r(B) < \operatorname{diam}(E)$,

$$\int_{E\cap B} \mathcal{E}_{E,p}(x, V_x, \alpha, r(B))\, d\mathcal{H}^n(x)$$
$$= \int_{E\cap B} \int_0^{r(B)} \left(\frac{\mathcal{H}^n(E \cap K(x, V_x, \alpha, r))}{r^n} \right)^p \frac{dr}{r} d\mathcal{H}^n(x) \le Mr(B)^n.$$

Easy: BME $\Rightarrow$ BPBE. In particular, BME $\Rightarrow$ BPLG.

Bounded mean energy condition

Definition

We will say that an AD-regular set E satisfies the **bounded mean energy** condition if there exist $\alpha > 0, M > 1$, and for a.e. $x \in E$ there exists $V_x \in G(d, d-n)$, such that:

for all balls B centered at E, $0 < r(B) < \operatorname{diam}(E)$,

$$\int_{E\cap B} \mathcal{E}_{E,p}(x, V_x, \alpha, r(B))\, d\mathcal{H}^n(x)$$
$$= \int_{E\cap B}\int_0^{r(B)} \left(\frac{\mathcal{H}^n(E\cap K(x, V_x, \alpha, r))}{r^n}\right)^p \frac{dr}{r} d\mathcal{H}^n(x) \leq Mr(B)^n.$$

Easy: BME $\Rightarrow$ BPBE. In particular, BME $\Rightarrow$ BPLG. But the converse is not true!

Question

How to modify BME to get a characterization of BPLG or UR?
Replace V_x by $V_{x,r}$?

Singular integral operators

Boundedness of singular integral operators

Given a Radon measure μ, $f \in L^2(\mu)$, and a kernel $K(x, y)$ set

$$T_\mu f(x) = \int K(x, y) f(y) \, d\mu(y).$$

Given a Radon measure μ, $f \in L^2(\mu)$, a kernel $K(x, y)$, and $\varepsilon > 0$ set

$$T_{\mu,\varepsilon}f(x) = \int_{|x-y|>\varepsilon} K(x, y)f(y)\, d\mu(y).$$

Given a Radon measure μ, $f \in L^2(\mu)$, a kernel $K(x, y)$, and $\varepsilon > 0$ set

$$T_{\mu,\varepsilon}f(x) = \int_{|x-y|>\varepsilon} K(x, y)f(y)\, d\mu(y).$$

We say that T_μ is bounded on $L^2(\mu)$ if $\|T_{\mu,\varepsilon}\|_{L^2(\mu)\to L^2(\mu)}$ are bounded uniformly in ε.

Boundedness of singular integral operators

Given a Radon measure μ, $f \in L^2(\mu)$, a kernel $K(x,y)$, and $\varepsilon > 0$ set

$$T_{\mu,\varepsilon}f(x) = \int_{|x-y|>\varepsilon} K(x,y)f(y)\, d\mu(y).$$

We say that T_μ is bounded on $L^2(\mu)$ if $\|T_{\mu,\varepsilon}\|_{L^2(\mu)\to L^2(\mu)}$ are bounded uniformly in ε.

Examples

- Cauchy transform $\mathcal{C}_\mu f(z) = \int_{\mathbb{C}} \frac{f(w)}{z-w}\, d\mu(w)$,
- n-dimensional Riesz transform $\mathcal{R}_\mu f(x) = \int_{\mathbb{R}^d} \frac{x-y}{|x-y|^{n+1}} f(y)\, d\mu(y)$.

Question

Given a "nice" kernel K, what are the measures μ such that T_μ is bounded on $L^2(\mu)$?

Question

Given a "nice" kernel K, what are the measures μ such that T_μ is bounded on $L^2(\mu)$?

Denote by $\mathcal{K}^n(\mathbb{R}^d)$ the class of kernels of the form $K(x, y) = k(x - y)$, where $k : \mathbb{R}^d \to \mathbb{R}$ are smooth, odd, and satisfy

$$|\nabla^j k(x)| \leq C_j |x|^{-n-j}, \quad j = 0, 1, 2, \dots$$

Question

Given a “nice” kernel K, what are the measures μ such that T_μ is bounded on $L^2(\mu)$?

Denote by $\mathcal{K}^n(\mathbb{R}^d)$ the class of kernels of the form $K(x,y) = k(x-y)$, where $k : \mathbb{R}^d \to \mathbb{R}$ are smooth, odd, and satisfy

$$|\nabla^j k(x)| \leq C_j |x|^{-n-j}, \quad j = 0, 1, 2, \dots$$

Theorem (David-Semmes ’91)

Suppose μ is n-AD-regular measure on $\mathbb{R}^d$. Then,

$$\begin{array}{c} \text{for all } K \in \mathcal{K}^n(\mathbb{R}^d) \\ T_\mu \text{ is bounded on } L^2(\mu) \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \mu \text{ is uniformly} \\ \text{rectifiable.} \end{array}$$

David-Semmes conjecture

Suppose μ is n-AD-regular measure on $\mathbb{R}^d$. Then,

$$\mathcal{R}_\mu \text{ is bounded on } L^2(\mu) \quad \Leftrightarrow \quad \mu \text{ is uniformly rectifiable.}$$

David-Semmes conjecture

Suppose μ is n-AD-regular measure on $\mathbb{R}^d$. Then,

$$\mathcal{R}_\mu \text{ is bounded on } L^2(\mu) \quad \Leftrightarrow \quad \mu \text{ is uniformly rectifiable.}$$

True for $n = 1$ (Mattila-Melnikov-Verdera 1996) and $n = d - 1$ (Nazarov-Tolsa-Volberg 2012).

David-Semmes conjecture

David-Semmes conjecture

Suppose μ is n-AD-regular measure on $\mathbb{R}^d$. Then,

$$\mathcal{R}_\mu \text{ is bounded on } L^2(\mu) \quad \Leftrightarrow \quad \mu \text{ is uniformly rectifiable.}$$

True for $n = 1$ (Mattila-Melnikov-Verdera 1996) and $n = d - 1$ (Nazarov-Tolsa-Volberg 2012).

Question

If we only assume that $\mu(B(x, r)) \leq Cr^n$, what are the necessary/sufficient conditions for boundedness of $\mathcal{R}_\mu$?

BPBE with $p = 1$ and SIOs

Theorem (Chang-Tolsa '17)

Let μ be a Radon measure on $\mathbb{R}^d$ satisfying $\mu(B(x,r)) \leq Cr^n$.
Suppose that μ satisfies the BPBE conditions with $p = 1$, i.e. there exist constants $\alpha, \kappa, M > 0$, such that:
for all balls B there exists a set $G_B \subset B$ with $\mu(G_B) \geq \kappa\, \mu(B)$, and a direction $V_B \in G(d, d-n)$, such that for all $x \in G_B$

$$\mathcal{E}_{\mu,1}(x, V_B, \alpha, r(B)) \leq M.$$

Then, for all $K \in \mathcal{K}^n(\mathbb{R}^d)$ we have T_μ bounded on $L^2(\mu)$.

BPBE with $p = 2$ and SIOs

Theorem (D. '20)

Let μ be a Radon measure on $\mathbb{R}^d$ satisfying $\mu(B(x,r)) \leq Cr^n$.
Suppose that μ satisfies the BPBE conditions with $p = 2$, i.e.
there exist constants $\alpha, \kappa, M > 0$, such that:
for all balls B there exists a set $G_B \subset B$ with $\mu(G_B) \geq \kappa\, \mu(B)$,
and a direction $V_B \in G(d, d-n)$, such that for all $x \in G_B$

$$\mathcal{E}_{\mu,2}(x, V_B, \alpha, r(B)) \leq M.$$

Then, for all $K \in \mathcal{K}^n(\mathbb{R}^d)$ we have T_μ bounded on $L^2(\mu)$.

BPBE with $p = 2$ and SIOs

Theorem (D. '20)

Let μ be a Radon measure on $\mathbb{R}^d$ satisfying $\mu(B(x,r)) \leq Cr^n$. Suppose that μ satisfies the BPBE conditions with $p = 2$, i.e. there exist constants $\alpha, \kappa, M > 0$, such that:
for all balls B there exists a set $G_B \subset B$ with $\mu(G_B) \geq \kappa\, \mu(B)$, and a direction $V_B \in G(d, d-n)$, such that for all $x \in G_B$

$$\mathcal{E}_{\mu,2}(x, V_B, \alpha, r(B)) \leq M.$$

Then, for all $K \in \mathcal{K}^n(\mathbb{R}^d)$ we have T_μ bounded on $L^2(\mu)$.

This is strictly stronger than the result of Chang and Tolsa:

$$\int_0^R \left(\frac{\mu(K(x,V,\alpha,r))}{r^n} \right)^2 \frac{dr}{r} \leq \int_0^R \frac{\mu(K(x,V,\alpha,r))}{r^n} \frac{\mu(B(x,r))}{r^n} \frac{dr}{r}$$
$$\leq C \int_0^R \frac{\mu(K(x,V,\alpha,r))}{r^n} \frac{dr}{r}.$$

Corona decomposition

Main lemma

Let μ be a compactly supported Radon measure on $\mathbb{R}^d$ satisfying $\mu(B(x,r)) \leq Cr^n$. Assume further that for some $V \in G(d, d-n)$, $\alpha \in (0,1)$, we have

$$\mathcal{E}_{\mu,p}(\mathbb{R}^d) = \int \mathcal{E}_{\mu,p}(x, V, \alpha, \infty)\, d\mu(x) < \infty.$$

Then, there exists a decomposition $\mathcal{D}_\mu = \bigcup_{R \in \mathrm{Top}} \mathrm{Tree}(R)$, and a corresponding family of Lipschitz graphs $\{\Gamma_R\}_{R \in \mathrm{Top}}$, satisfying:

Corona decomposition

Main lemma

Let μ be a compactly supported Radon measure on $\mathbb{R}^d$ satisfying $\mu(B(x,r)) \leq Cr^n$. Assume further that for some $V \in G(d, d-n)$, $\alpha \in (0,1)$, we have

$$\mathcal{E}_{\mu,p}(\mathbb{R}^d) = \int \mathcal{E}_{\mu,p}(x, V, \alpha, \infty)\, d\mu(x) < \infty.$$

Then, there exists a decomposition $\mathcal{D}_\mu = \bigcup_{R \in \mathrm{Top}} \mathrm{Tree}(R)$, and a corresponding family of Lipschitz graphs $\{\Gamma_R\}_{R \in \mathrm{Top}}$, satisfying:

(i) Lipschitz constants of Γ_R are uniformly bounded,

(ii) μ-almost all of $R \setminus \bigcup_{Q \in \mathrm{Stop}(R)} Q$ is contained in Γ_R,

(iii) for all $Q \in \mathrm{Tree}(R)$ we have $\Theta_\mu(2B_Q) \lesssim \Theta_\mu(2B_R)$

(iv) we have the packing condition

$$\sum_{R \in \mathrm{Top}} \Theta_\mu(2B_R)^p \mu(R) \lesssim \mu(\mathbb{R}^d) + \mathcal{E}_{\mu,p}(\mathbb{R}^d).$$