Equilibrium measures on curves

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Funded by the European Union

European Research Council
Established by the European Commission

Suomen Akatemia
Finlands Akademi
Research Council of Finland

What is an equilibrium measure?

Logarithmic energy

Given a compactly supported Radon measure μ on $\mathbb{R}^n$ its logarithmic energy is

$$\mathcal{I}(\mu) := \iint \log \frac{1}{|x-y|} \, d\mu(x) d\mu(y) \in (-\infty, +\infty].$$

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Examples:

- $\mathcal{I}(\delta_0) = +\infty$
- $\mathcal{I}(\mu) < +\infty$ whenever μ is α-Frostman for some $\alpha > 0$, i.e.

$$\mu(B(x,r)) < Cr^{\alpha} \quad x \in \mathbb{R}^n, r > 0.$$

Equilibrium measures

Suppose that $E \subset \mathbb{R}^n$ is compact, and let $\mathcal{P}(E) := \{\mu : \operatorname{supp} \mu \subset E, \mu(E) = 1\}$.

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Minimize $\mathcal{I}(\mu)$ among $\mu \in \mathcal{P}(E)$.

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Theorem (Frostman, Fuglede)

If there is at least one $\nu \in \mathcal{P}(E)$ with $\mathcal{I}(\nu) < \infty$, then there exists a unique minimizer of $\mathcal{I}(\mu)$ among $\mu \in \mathcal{P}(E)$.

This unique minimizer is called the **(logarithmic) equilibrium measure for E**.

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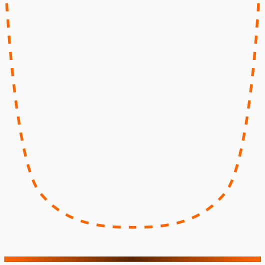

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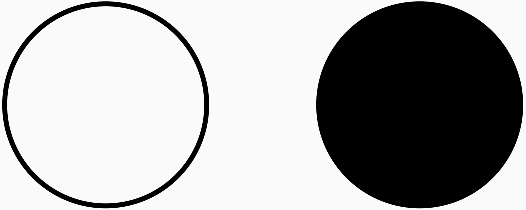

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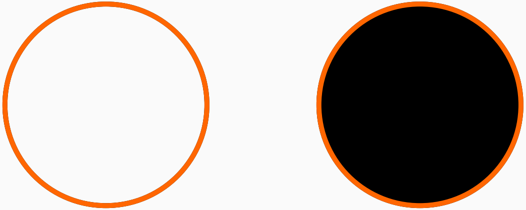

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- $E = \mathbb{S}^{n-1}$ or $E = B(0, 1) \subset \mathbb{R}^n$ $\rightsquigarrow$ $\mu = \mathcal{H}^{n-1}|_{\mathbb{S}^{n-1}}$

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- Identifying the equilibrium measure is a limit version of an important discrete problem: for a fixed $E \subset \mathbb{R}^n$ and $N \in \mathbb{N}$, what's the configuration $x_1, \ldots, x_N \subset E$ minimizing

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- For $n \geq 3$ a connection to higher order PDEs?

Main result

- If $n = 2$, a lot of results on the structure of equilibrium measures. For example:

Theorem (F. and M. Riesz, 1916)

If $\Gamma \subset \mathbb{R}^2$ is a Jordan curve of finite length, then

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Theorem (D.-Orponen '25)

If $\Gamma \subset \mathbb{R}^n$ is a $C^{1,\alpha}$ curve with $\alpha > 0$, then $\mu \ll \mathcal{H}^1|_\Gamma$.

Before nothing was known even for C^∞ curves.

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- $\mathcal{U}\nu(x) \sim -\log|x|$ as $|x| \to \infty$
- $\mathcal{I}(\nu) = \int \mathcal{U}\nu(x)\, d\nu(x)$
- $\Delta^{n/2}(\mathcal{U}\nu) = \nu$ in a weak sense

$\rightsquigarrow$ if $n = 2$ then $\mathcal{U}\nu$ is harmonic on $\mathbb{R}^2 \setminus \operatorname{supp} \nu$

$\rightsquigarrow$ if $n = 4$ then $\mathcal{U}\nu$ is biharmonic on $\mathbb{R}^4 \setminus \operatorname{supp} \nu$

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For $0 < \alpha < 1$

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$$\mathcal{U}\nu \in C^{\alpha}(\mathbb{R}^n) \quad \Leftrightarrow \quad \nu(B(x,r)) \leq Cr^{\alpha}.$$

If we show that our equilbrium measure satisfies $\mathcal{U}\mu \in \mathrm{Lip}(\mathbb{R}^n)$, we'll get $\mu \in L^{\infty}(\mathcal{H}^1|_{\Gamma})$!

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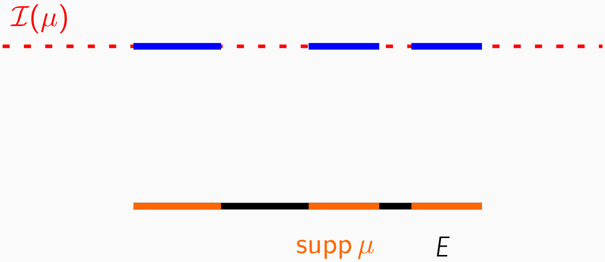

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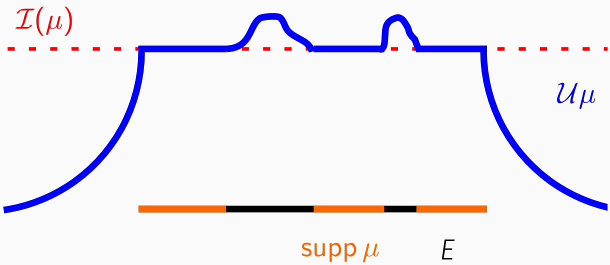

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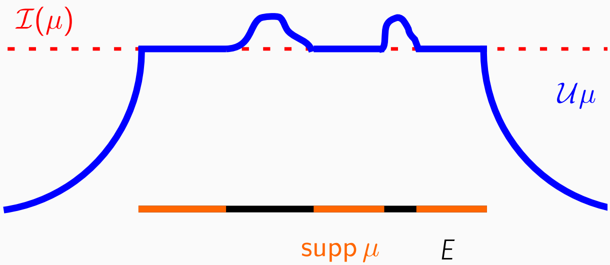

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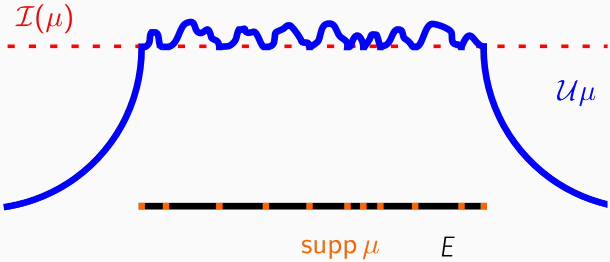

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Given the equilibrium measure μ on a $C^{1,\alpha}$-graph Γ, how do we use the information above to conclude $\mathcal{U}\mu \in \mathrm{Lip}(\mathbb{R}^n)$?

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Idea: perhaps it's enough to show that $\mathcal{U}\mu \in \mathrm{Lip}(\Gamma)$?

Plan of the proof

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Step 2. Prove Sobolev regularity of $\mathcal{U}\mu$ on Γ.

Step 1: Potentials seen “intrinsically”

Injectivity of potentials

Theorem (Frostman, Fuglede)

Given a compact set E and measures $\mu, \nu \in \mathcal{P}(E)$ we have

$$\mathcal{U}\mu|_E = \mathcal{U}\nu|_E \quad \Leftrightarrow \quad \mu = \nu.$$

Thus, the operator $\mathcal{U} : \mathcal{P}(E) \to \{\text{functions on } E\}$ is injective.

- What are its mapping properties on subspaces of $\mathcal{P}(E)$?
- When does it have a bounded inverse?

Intrinsic Carleson-Wallin

Theorem (Carleson, Wallin '60s)

For $0 < \beta < 1$

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Proposition (D.-Orponen '25) [almost true]

If Γ is a Lipschitz graph with $\mathrm{Lip}(\Gamma) \ll 1$, then for $\nu \in \mathcal{P}(\Gamma)$

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Almost-almost true: for $\beta \in [0,1]$ and $p \in (1,\infty)$ we show that

$$\mathcal{U} : \dot{H}^{\beta-1,p}(\Gamma) \to \dot{H}^{\beta,p}(\Gamma)$$

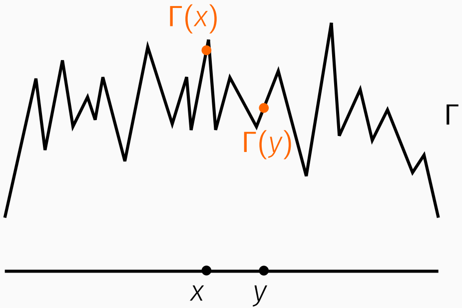
is bounded with bounded inverse, provided $\mathbf{Lip}(\Gamma) \ll_p 1$.

Graph potentials

Given $A : \mathbb{R} \to \mathbb{R}^{n-1}$ denote by $\Gamma(x) = (x, A(x)) \in \mathbb{R}^n$ the graph map.

Given a compactly supported measure ν on $\mathbb{R}$ we define its **graph potential** $\mathcal{U}^\Gamma : \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{U}^\Gamma \nu(x) = \int_{\mathbb{R}} \log \frac{1}{|\Gamma(x) - \Gamma(y)|} \, d\nu(y).$$



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Note that

$$\text{measure } \nu \text{ on } \Gamma \text{ and } \mathcal{U}\nu : \Gamma \to \mathbb{R} \quad \leftrightsquigarrow \quad \text{measure } \tilde{\nu} \text{ on } \mathbb{R} \text{ and } \mathcal{U}^\Gamma \tilde{\nu} : \mathbb{R} \to \mathbb{R}.$$

$$\tilde{\nu} = (\pi_1)_* \nu$$

$$\mathcal{U}^\Gamma \tilde{\nu}(x) = \mathcal{U}\nu(\Gamma(x))$$

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$$T^\Gamma_\beta := \Delta^{(1-\beta)/2} \mathcal{U}^\Gamma \Delta^{\beta/2}$$

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- The case $\beta \in \{0,1\}$ relies on the L^p-boundedness of nice enough SIOs on Lipschitz graphs.
- For $\beta \in (0,1)$ we use complex interpolation.

In what follows, we'll pretend we have the "almost true" result:

Proposition

If Γ is a Lipschitz graph with $\mathrm{Lip}(\Gamma) \ll 1$, then for $\nu \in \mathcal{P}([0,1])$

$$\mathcal{U}^{\Gamma}\nu \in C^{\beta}([0,1]) \quad \Leftrightarrow \quad \nu(B(x,r)) \leq Cr^{\beta}.$$

We move on to **Step 2**:

Goal

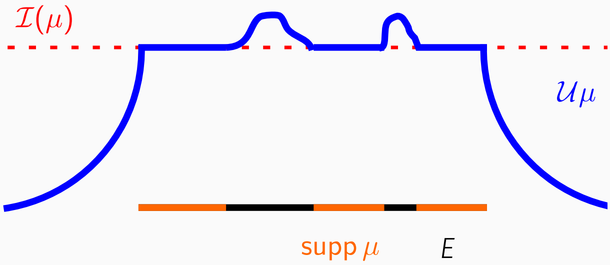
If μ is the (projection of the) equilibrium measure on a $C^{1,\alpha}$-graph Γ, then $\mathcal{U}^{\Gamma}\mu \in \mathrm{Lip}([0,1])$.

This will imply $\mu \ll \mathcal{H}^1$.

Step 2: Lipschitz regularity of $\mathcal{U}^{\Gamma}\mu$

Planar case

- Recall: for the equilibrium measure we have $\mathcal{U}^{\Gamma}\mu = \text{const}$ on $\operatorname{supp}\mu$.



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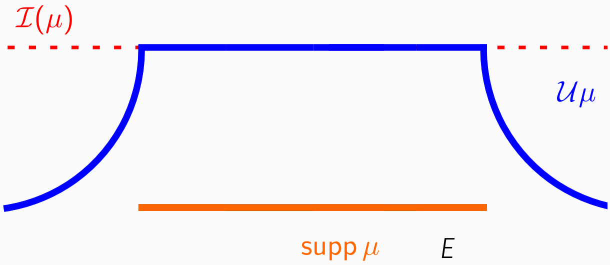

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- This is the case in $\mathbb{R}^2$!

Theorem (Frostman)

If $E \subset \mathbb{R}^2$ is compact and ν is the equilibrium measure on E, then the potential $\mathcal{U}\nu$ is constant on E.

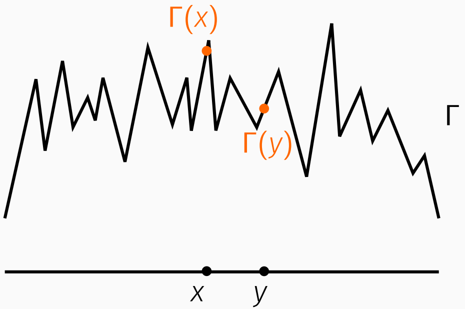
- Reason: $\mathcal{U}\nu$ is harmonic on $\mathbb{R}^2 \setminus \operatorname{supp}\nu$, and we have the maximum principle.
- This does not work in higher dimensions:

$$\Delta^{n/2}(\mathcal{U}\nu) = \nu$$

Kernel decomposition

Our graph potential has kernel

$$K^{\Gamma}(x,y) = -\log|\Gamma(x) - \Gamma(y)| = -\log|(x-y, A(x) - A(y))|.$$



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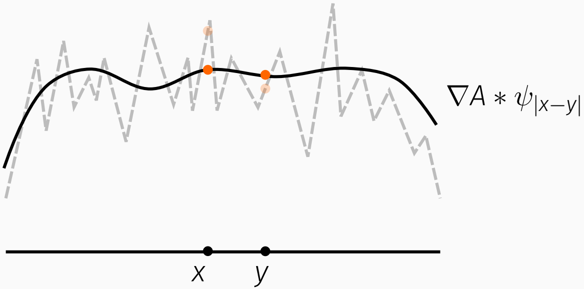
We decompose the kernel $K^{\Gamma}(x,y) = P(x,y) + R(x,y)$, where

$$\begin{aligned} P(x,y) &= -\log|(x-y, \nabla A * \psi_{|x-y|}(x)(x-y))| \\ &= -\log|x-y| - \log|(1, \nabla A * \psi_{|x-y|}(x))| \end{aligned}$$



Potential decomposition

We use the kernel decomposition to decompose the potential $\mathcal{U}^\Gamma \mu = \mathcal{P}\mu + \mathcal{R}\mu$:

$$\mathcal{P}\mu(x) := \int P(x,y)\, d\mu(y),$$

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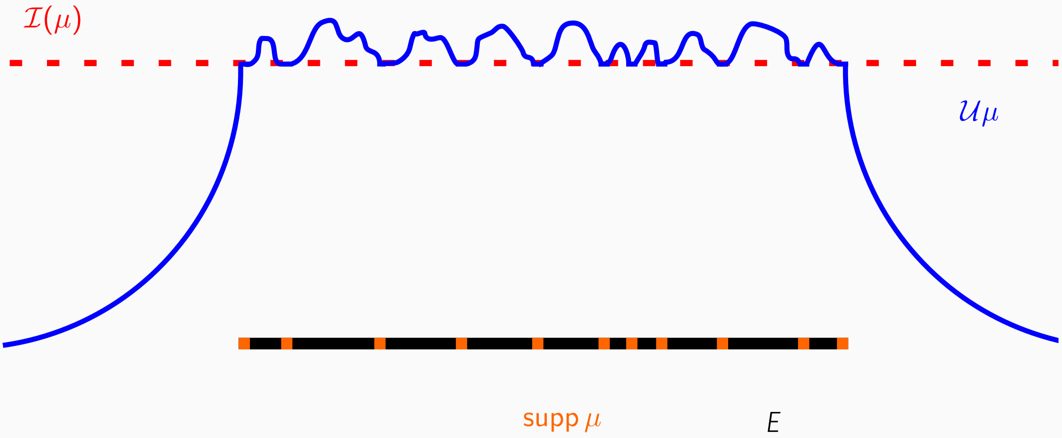
$\mathcal{R}$-Lemma

If $A \in C^{1,\alpha}$ and $\beta \in [0, 1-\alpha]$, then

$$\mu(B(x,r)) \leq Cr^\beta \quad \Rightarrow \quad \mathcal{R}\mu \in C^{\beta+\alpha}(\mathbb{R}).$$

In particular, $\mathcal{R}\mu \in C^\alpha(\mathbb{R})$.

Potential decomposition



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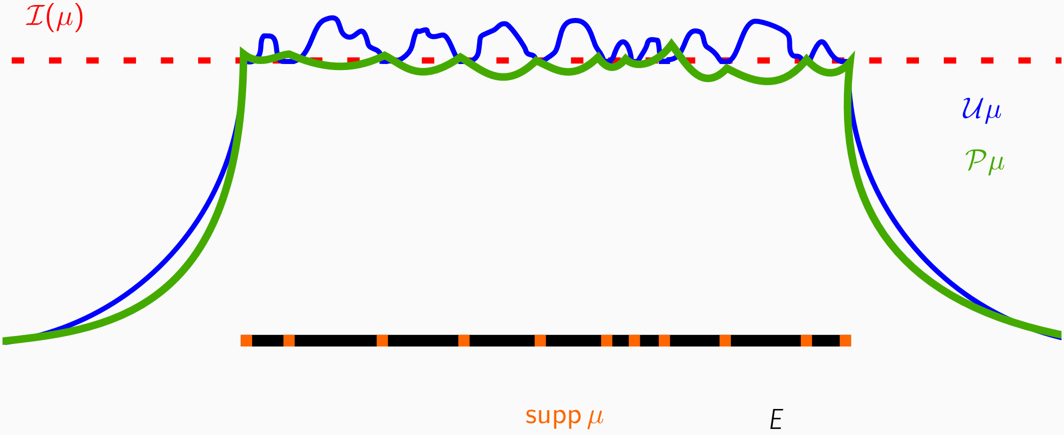

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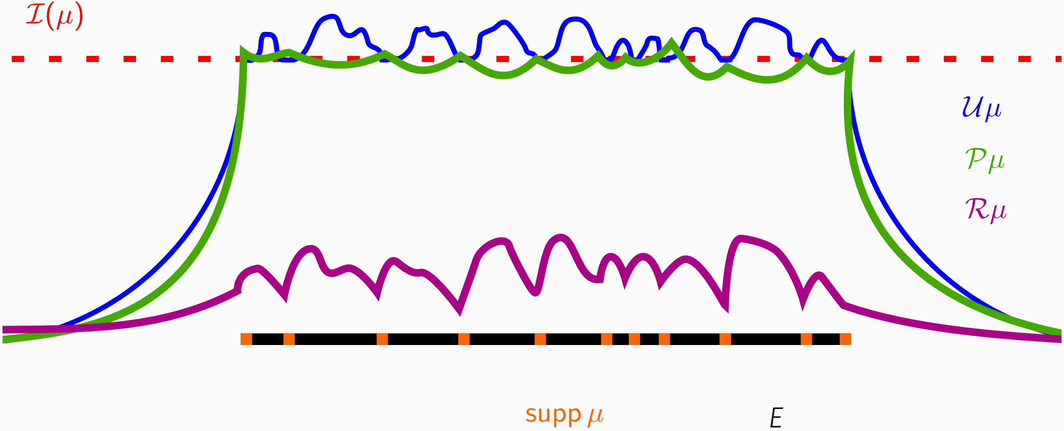

Bootstrapping scheme

- $\mathcal{U}^{\Gamma}\mu = \mathcal{P}\mu + \mathcal{R}\mu$
- $\mathcal{P}\mu$ convex on $\mathbb{R} \setminus \operatorname{supp}\mu$
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$$\begin{tikzcd}
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& \mu(B(x,r)) \leq Cr^\alpha \arrow[ul, Rightarrow, "\mathcal{R}\text{-Lemma}"] &
\end{tikzcd}$$

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\begin{tikzcd}[column sep=large, row sep=large]
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$$\mathcal{U}^\Gamma \mu \in \mathrm{Lip}([0,1]) \xRightarrow{\text{Step 1}} \mu(B(x,r)) \leq Cr \xRightarrow{\mathcal{R}\text{-Lemma}} \text{(bullets above)}$$

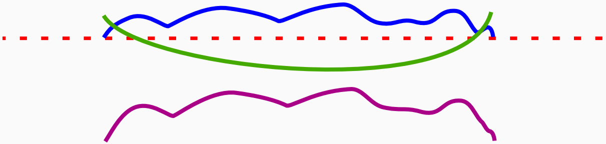
Sums of Hölder and convex functions

An elementary lemma

Suppose that $U : [0,1] \to \mathbb{R}$ satisfies $U = P + R$ where

- P is convex on $[0,1]$
- $R \in C^\beta([0,1])$
- $U(0) = U(1) = \inf_{t \in [0,1]} U(t)$.

Then $U \in C^\beta([0,1])$.



$U = P + R$

Corollary: the Missing Lemma

Missing Lemma

If μ is the equilibrium measure on Γ, and $\mathcal{U}^\Gamma\mu = \mathcal{P}\mu + \mathcal{R}\mu$ with

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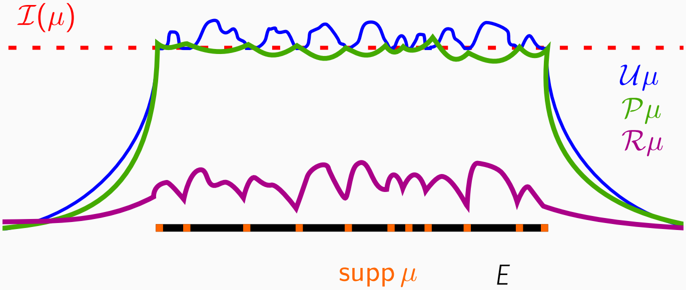

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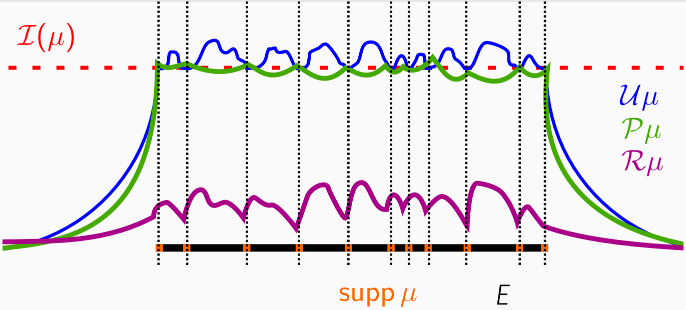

Open questions

Question 1

Suppose that $\Gamma \subset \mathbb{R}^n$ is a 1-dimensional Lipschitz graph. Is the equilibrium measure on Γ absolutely continuous with respect to $\mathcal{H}^1$? What about $\mathcal{H}^1 \ll \mu$?

1-dimensional Lipschitz graphs

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In our proof:

Step 1. Regularity of $\mathcal{U}^\Gamma \mu \quad \Rightarrow \quad$ regularity of μ ✓

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Step 2. Prove regularity of $\mathcal{U}^\Gamma \mu$. ✗

Bootstrapping falls apart ☹

Riesz equilibrium measures

Equilibrium measures can be considered for many different energies. One of the classical is the s-Riesz energy:

$$\mathcal{I}_s(\mu) = \iint \frac{1}{|x-y|^s}\, d\mu(x)d\mu(y).$$

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Suppose that $\Gamma \subset \mathbb{R}^n$ is a 1-dimensional $C^{1,\alpha}$ graph. If $0 < s < 1$, is the s-equilibrium measure on Γ absolutely continuous with respect to $\mathcal{H}^1$?

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Step 1. Regularity of $\mathcal{U}^\Gamma \mu \quad \Rightarrow \quad$ regularity of μ ✗

Unclear how to treat corresponding operators T^Γ_β ☹

Step 2. Prove regularity of $\mathcal{U}^\Gamma \mu$. ✓(probably)

Higher dimensional surfaces

Just as log-equilibrium measures on curves in $\mathbb{R}^2$ are classical, $(n-2)$-equilibrium measures on $(n-1)$-dimensional surfaces in $\mathbb{R}^n$ are classical.

Question 3

Suppose that $\Sigma \subset \mathbb{R}^n$ is a k-dimensional $C^{1,\alpha}$ graph. Is the $(k-1)$-equilibrium measure on Σ absolutely continuous with respect to $\mathcal{H}^k$? What about s-equilibrium measures for $0 < s < k$?

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Unclear how to get the corresponding elementary lemma on subharmonic functions ☹

Thank you!