

SINGULAR INTEGRAL OPERATORS
EXERCISE I (31.10.2023)

Exercise 1 (1 point). Let $f \in \mathcal{S}(\mathbb{R})$. Show that $Hf \in L^1(\mathbb{R})$ if and only if $\int_{\mathbb{R}} f(y) dy = 0$.

Hint: Modify the proof of Lemma 3.4 from the lecture notes to estimate the asymptotics of $x^2 \cdot Hf(x)$ as $|x| \rightarrow \infty$.

Recall that in the lecture we defined a tempered distribution $T_0 \in \mathcal{S}'(\mathbb{R})$ by

$$\langle T_0, f \rangle := -Hf(0) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} \int_{|y| > \varepsilon} \frac{f(y)}{y} dy.$$

Exercise 2 (2 point). Show that the tempered distribution \widehat{T}_0 is given by a function, and that $\widehat{T}_0(\xi) = -i \operatorname{sgn}(\xi)$.

Hints:

- (i) Let $K_\varepsilon(y) = \frac{1}{y} \mathbf{1}_{|y| > \varepsilon}$, so that $\langle T_0, f \rangle = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \langle K_\varepsilon, f \rangle$ for all $f \in \mathcal{S}(\mathbb{R})$. Consider $Q_\varepsilon(y) = \frac{y}{y^2 + \varepsilon^2}$ and show that

$$\lim_{\varepsilon \rightarrow 0} (K_\varepsilon - Q_\varepsilon) = 0 \quad \text{in } \mathcal{S}'(\mathbb{R}),$$

that is, $\lim_{\varepsilon \rightarrow 0} \langle K_\varepsilon - Q_\varepsilon, f \rangle = 0$ for all $f \in \mathcal{S}(\mathbb{R})$.

- (ii) Using the above, justify rigorously that $\widehat{T}_0 = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \widehat{Q}_\varepsilon$, in the sense of distributions.
 (iii) Show that $Q_\varepsilon(x) = \mathcal{F}^{-1}(-\pi i \operatorname{sgn}(\xi) e^{-2\pi\varepsilon|\xi|})(x)$. Conclude that \widehat{T}_0 is given by a function, and that $\widehat{T}_0(\xi) = -i \operatorname{sgn}(\xi)$.

Recall that an operator $T : \mathcal{S}(\mathbb{R}) \rightarrow L^q(\mathbb{R})$ is said to be of strong type (p, q) if there exists a constant $C \in (0, \infty)$ such that $\|Tf\|_{L^q} \leq C\|f\|_{L^p}$.

Exercise 3 (1 point). Let $f = \mathbf{1}_{[0,1]}$. Show that

$$\lim_{\varepsilon \rightarrow 0} \int_{|x-y| > \varepsilon} \frac{f(y)}{x-y} dy = \log \left| \frac{x}{x-1} \right|.$$

Conclude that the Hilbert transform is neither of strong type (∞, ∞) nor of strong type $(1, 1)$.

Recall that the essential support of a locally integrable function f is the smallest closed set, denoted by $\operatorname{ess\,supp}(f)$, such that $f = 0$ a.e. on the complement of $\operatorname{ess\,supp}(f)$.

Exercise 4 (1 point). Show that if $f \in L^2(\mathbb{R})$, then for a.e. $x \notin \operatorname{ess\,supp}(f)$

$$Hf(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy.$$

Hint: You may use the fact that for $f_n(x) := (f \mathbf{1}_{B(0,n)}) * \varphi_{1/n}(x) \in C_c^\infty(\mathbb{R})$ we have $f_n \rightarrow f$ in $L^2(\mathbb{R})$. Here φ_ε is a smooth mollifier: $\varphi_\varepsilon(x) = \varepsilon^{-1} \varphi(x/\varepsilon)$, $\varphi \in C_c^\infty(\mathbb{R})$, $\mathbf{1}_{[-0.1,0.1]} \leq \varphi \leq \mathbf{1}_{[-1,1]}$, and $\|\varphi\|_{L^1} = 1$.