SINGULAR INTEGRAL OPERATORS
EXERCISE V (28.11.2023)

Exercise 1 (1 point)**.** Prove that in $\mathbb{R}^n$ the family of all dyadic cubes containing 0 is $(1-2^{-n})$-sparse.

Exercise 2 (1 point)**.** Suppose that $\mathcal{F}_1, \dots, \mathcal{F}_k$ are sparse families of dyadic cubes, and that each $\mathcal{F}_j$ is η_j-sparse for some $\eta_j \in (0,1]$. Show that $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_k$ is $1/(\sum_{j=1}^k \eta_j^{-1})$-sparse.

Hint: Use Proposition 7.6.

Exercise 3 (3 points)**.** For any $s \in (0,1)$ let $w = |x|^{1-s}$ be a weight on $\mathbb{R}$.

(i) Show that $w \in A_2$, and $[w]_{A_2} \leqslant s^{-1}$.

(ii) Given $f_s(x) = x^{s-1}\mathbf{1}_{(0,1)}(x)$, show that $\|f_s\|_{L^2(w_s)} \leqslant s^{-1/2}$.

(iii) Prove that $\|Hf_s\|_{L^2(w_s)} \geqslant 2s^{-3/2}$, and conclude that in the estimate

$$\|Tf\|_{L^2(w)} \leqslant C[w]_{A_2}\|f\|_{L^2(w)}$$

the factor $[w]_{A_2}$ cannot be replaced by $[w]_{A_2}^s$ for any $s < 1$.