Singular Integral Operators

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Abstract

Notes for the course *Singular Integral Operators* lectured at the University of Jyväskylä in Autumn 2023.

Contents

1 Introduction

This course will focus on singular integral operators, which are operators of the form

$$Tf(x) = \int K(x,y)f(y)\ dy,$$

where the kernel $K(x, y)$ has a singularity on the diagonal $x = y$. These operators appear naturally e.g. in the theory of partial differential equations, and they have been studied for over a century. The prototypical example is the Hilbert transform

$$Hf(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} \, dy.$$

The basic questions we will study concern the mapping properties of singular integral operators: for which $1 \leq p \leq \infty$ and under what hypotheses on the kernel K is the operator T bounded on L^p, in the sense that

$$\|Tf\|_{L^p} \leq C\|f\|_{L^p}.$$

The material we will cover reflects both the long tradition of this field, and the fact that it is still an active area of research. We will begin by studying the Hilbert and Riesz transforms, which date back almost 100 years back. Then, we will move on to the Calderón-Zygmund theory, which revolutionized the field in the 1950s. Finally, we will discuss singular integrals in the weighted setting, which is a much more recent topic. The grand finale will be the proof of the A_2 theorem, which was shown by Tuomas Hytönen in 2012 [Hyt12]. We will follow a short and elegant proof from [Ler16] which uses a cutting-edge technique called *sparse domination.*

The field of singular integral operators is huge, and we will only scratch the surface in this course. We refer interested readers to the textbooks [Duo01, Gra14a, Gra14b, Ste70, Ste93] for more thorough treatments of the subject.

2 Preliminaries

Before getting started in earnest, we recall briefly some useful facts and definitions. For proofs and details, see e.g. Chapters 1 and 2 of [Gra14a].

In these notes we sometimes use the notation $A \lesssim B$, which stands for "there exists a dimensional constant $C \geq 1$ such that $A \leq CB$." We write $A \sim B$ instead of $A \lesssim B \lesssim A$.

2.1 Schwartz functions and tempered distributions

Definition 2.1 (Schwartz functions). A function $f \in C^\infty(\mathbb{R}^n)$ is a *Schwartz function*, denoted by $f \in \mathcal{S}(\mathbb{R}^n)$, if for every pair of multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ we have

$$\rho_{\alpha,\beta}(f) \coloneqq \sup_{x \in \mathbb{R}^n} |x^\alpha \cdot \partial^\beta f(x)| < \infty.$$

We will say that a function decays rapidly if it decays at ∞ faster than any polynomial. Hence, Schwartz functions are precisely those $C^\infty(\mathbb{R}^n)$ functions which decay rapidly and whose all partial derivatives decay rapidly.

Example 2.2. Any smooth and compactly supported function is a Schwartz function, so that $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$. A simple example of a non-compactly supported Schwartz function is $e^{-|x|^2}$.

One of the reasons Schwartz functions are useful is the following density result.

Lemma 2.3. *The Schwartz functions are dense in $L^p(\mathbb{R}^n)$ for all $1 \leq p < \infty$.*

Note that $\mathcal{S}(\mathbb{R}^n)$ is a vector space. A topology on $\mathcal{S}(\mathbb{R}^n)$ can be defined using the family of semi-norms $\rho_{\alpha,\beta}$, and it is compatible with the following notion of convergence.

Definition 2.4 (convergence in $\mathcal{S}(\mathbb{R}^n)$). Given $f \in \mathcal{S}(\mathbb{R}^n)$ and a sequence $f_k \in \mathcal{S}(\mathbb{R}^n)$, we say that f_k coverges to f in $\mathcal{S}(\mathbb{R}^n)$ if for all multi-indices $\alpha, \beta \in \mathbb{N}^n$

$$\lim_{k\to\infty} \rho_{\alpha,\beta}(f_k - f) = 0.$$

Definition 2.5 (tempered distributions). We denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of $\mathcal{S}(\mathbb{R}^n)$, i.e., the space of all continuous linear functionals $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$. The elements of $\mathcal{S}'(\mathbb{R}^n)$ are called *tempered distributions*.

Given $T \in \mathcal{S}'(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$, instead of writing $T(f)$ we will write $\langle T, f\rangle$, and we will call it *the action of T on f*.

We have the following useful characterization of tempered distributions:

Lemma 2.6. *A linear functional $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is a tempered distribution if and only if there exist $m, k \in \mathbb{N}$ and $C > 0$ such that for all $f \in \mathcal{S}(\mathbb{R}^n)$*

$$|\langle T, f\rangle| \leq C \sum_{|\alpha|\leq m,\ |\beta|\leq k} \rho_{\alpha,\beta}(f).$$

Example 2.7. Any function $g \in L^p(\mathbb{R}^n), 1 \leq p \leq \infty$, gives rise to a tempered distribution $T_g \in \mathcal{S}'(\mathbb{R}^n)$ defined via $\langle T_g, f\rangle = \int f(x)g(x)\ dx$.

Example 2.8. Any finite Borel measure μ gives rise to a tempered distribution $T_\mu \in \mathcal{S}'(\mathbb{R}^n)$ defined via $\langle T_\mu, f\rangle = \int f\ d\mu$.

In the case of tempered distributions as above, we will often identify T_g with g, and T_μ with μ. For example, the statement "$T \in \mathcal{S}'(\mathbb{R}^n)$ is a $C^\infty(\mathbb{R}^n)$ function" should be understood as "there exists $f \in C^\infty(\mathbb{R}^n)$ such that $T = T_f$." The Hilbert transform we will define shortly will provide us with an example of a tempered distribution which is neither a locally integrable function, nor a measure.

Many common operations performed on functions can be extended by duality to tempered distributions. For example, given $h \in \mathcal{S}(\mathbb{R}^n)$ and $T \in \mathcal{S}'(\mathbb{R}^n)$, we define their convolution as a tempered distribution $T * h \in \mathcal{S}'(\mathbb{R}^n)$ given by

$$\langle T * h, f\rangle \coloneqq \langle T, \tilde{h} * f\rangle,$$

where $\tilde{h}(x) = h(-x)$. Similarly, the product of $h \in \mathcal{S}(\mathbb{R}^n)$ and $T \in \mathcal{S}'(\mathbb{R}^n)$ can be defined as a tempered distribution $hT \in \mathcal{S}'(\mathbb{R}^n)$ given by

$$\langle hT, f\rangle \coloneqq \langle T, hf\rangle.$$

Proposition 2.9. *Given $h \in \mathcal{S}(\mathbb{R}^n)$ and $T \in \mathcal{S}'(\mathbb{R}^n)$ the convolution $T * h$ belongs to $C^\infty(\mathbb{R}^n)$. Moreover,*

$$T * h(x) = \langle T, h(\cdot - x)\rangle.$$

2.2 Fourier transform

Definition 2.10. The Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$ is defined as

$$\hat{f}(\xi) \coloneqq \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx.$$

Sometimes we will denote it by $\mathcal{F}(f)$ instead of $\hat{f}$.

The Fourier transform is a homeomorphism of $\mathcal{S}(\mathbb{R}^n)$ to itself, and its inverse is given by

$$\check{f}(x) \coloneqq \int_{\mathbb{R}^n} f(\xi)e^{2\pi i x\cdot\xi}\,dx = \hat{f}(-x),$$

sometimes denoted by $\mathcal{F}^{-1}(f)$.

The Plancherel identity asserts that for any $f \in \mathcal{S}(\mathbb{R}^n)$

$$\|f\|_{L^2(\mathbb{R}^n)} = \|\hat{f}\|_{L^2(\mathbb{R}^n)}.$$

By the density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, this allows us to extend the Fourier transform to an isometry of $L^2(\mathbb{R}^n)$.

One may further extend the definition of Fourier transform to all tempered distributions using duality: for any $T \in \mathcal{S}'(\mathbb{R}^n)$ we define $\widehat{T} \in \mathcal{S}'(\mathbb{R}^n)$ via

$$\langle \widehat{T}, f\rangle \coloneqq \langle T, \hat{f}\rangle.$$

We list a few properties of the Fourier transform we will use later on.

Lemma 2.11. *If $f \in \mathcal{S}(\mathbb{R}^n)$ and $T \in \mathcal{S}'(\mathbb{R}^n)$, then*

(i) $\mathcal{F}(\partial^\alpha f) = (2\pi i\xi)^\alpha \hat{f}$,

(ii) $\partial^\alpha \hat{f} = \mathcal{F}((-2\pi i x)^\alpha f)$,

(iii) $\widehat{T * f} = \hat{T}\hat{f}$.

2.3 Weak and strong type inequalities

In this subsection we assume that (X, μ) and (Y, ν) are two measure spaces.

Definition 2.12. Given $1 \leq p, q \leq \infty$ and an operator T mapping functions from a dense subset of $L^p(X, \mu)$ to measurable functions on (Y, ν), we say that T *is of strong type* (p, q) if there exists $C > 0$ such that

$$\|Tf\|_{L^q(Y,\nu)} \leq C\|f\|_{L^p(X,\mu)}.$$

We say that T *is of weak type* (p, q) if there exists $C > 0$ such that for all $\lambda > 0$

$$\nu(\{y \in Y \ : \ |Tf(y)| > \lambda\}) \leq C \left(\frac{\|f\|_{L^p(X,\mu)}}{\lambda} \right)^q.$$

It is easy to see that strong type (p, q) implies weak type (p, q).

Definition 2.13 (sublinear operator). An operator T defined on a linear space of measurable functions on (X, μ) and taking values in measurable functions on (Y, ν) is sub-linear if

$$|T(f + g)| \leq |Tf| + |Tg| \quad \text{and} \quad |T(\lambda f)| = |\lambda||Tf|.$$

The Marcinkiewicz interpolation theorem stated below plays a crucial role in the theory of singular integral operators.

Theorem 2.14. *Let $1 \leq p_0 < p_1 \leq \infty$. Suppose that T is a sub-linear operator mapping $L^{p_0}(X, \mu) + L^{p_1}(X, \mu)$ to the set of measurable functions on (Y, ν). If T is of weak type (p_0, p_0) and (p_1, p_1), then it is of strong type (p, p) for all $p_0 < p < p_1$.*

3 The Hilbert and Riesz transforms

In this section we will study the prototypical singular integral operator, the Hilbert transform, as well as its higher dimensional counterparts, the Riesz transforms. The Hilbert transform arises naturally e.g. in the study of boundary values of analytic functions, in questions regarding the convergence of Fourier transform, or in signal processing. While we will not study these applications, they may be chosen as a presentation topic to pass the course.

3.1 The Hilbert transform on $\mathcal{S}(\mathbb{R})$

The Hilbert transform is the singular integral operator associated with kernel $K(x, y) = \frac{1}{\pi(x-y)}$. We begin by defining it for Schwartz functions.

As a first attempt at defining it, one could try to simply integrate against the kernel:

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy.$$

However, the expression above is highly problematic. Even for a very nice function f, say, $f \in C_c^\infty(\mathbb{R})$, it is easy to see that as soon as $f(x) \neq 0$, the integral above is not well-defined! This is because $(x-y)^{-1}$ has a singularity at x which is not integrable.

To avoid this issue, we first consider the following *truncated Hilbert transform*.

Definition 3.1 (truncated Hilbert transform). For $f \in \mathcal{S}(\mathbb{R})$ and $\varepsilon > 0$, we define the truncated Hilbert transform of f as

$$H_\varepsilon f(x) \coloneqq \frac{1}{\pi}\int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\,dy = \frac{1}{\pi}\int_{|y|>\varepsilon} \frac{f(x-y)}{y}\,dy.$$

Note that, by the rapid decay of Schwartz functions, $H_\varepsilon f(x)$ is well-defined for every $x \in \mathbb{R}$.

Definition 3.2 (Hilbert transform). For $f \in \mathcal{S}(\mathbb{R})$, we define the Hilbert transform of f as

$$Hf(x) \coloneqq \lim_{\varepsilon\to 0} H_\varepsilon f(x) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|y|>\varepsilon} \frac{f(x-y)}{y}\,dy.$$

Clearly, for $x \notin \operatorname{supp} f$ this is well-defined, and in fact

$$Hf(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy \qquad \text{for } x \notin \operatorname{supp} f. \tag{3.1}$$

Let us show that $Hf(x)$ is well-defined also for $x \in \operatorname{supp} f$.

Lemma 3.3. *For any $f \in \mathcal{S}(\mathbb{R})$ and $x \in \mathbb{R}$ the limit $\lim_{\varepsilon\to 0} H_\varepsilon f(x)$ exists, and we have*

$$Hf(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{f(x-y)-f(x)}{y}\,dy + \frac{1}{\pi}\int_{|y|>1} \frac{f(x-y)}{y}\,dy. \tag{3.2}$$

Proof. Fix $\varepsilon > 0$. Note that, since the kernel $\frac{1}{y}$ is odd, it has zero mean on any symmetric pair of intervals around the origin, and in particular

$$\int_{\varepsilon<|y|<1} \frac{1}{y}\,dy = 0.$$

It follows that

$$\int_{|y|>\varepsilon} \frac{f(x-y)}{y}\,dy = \int_{\varepsilon<|y|<1} \frac{f(x-y)-f(x)}{y}\,dy + \int_{|y|>1} \frac{f(x-y)}{y}\,dy.$$

The second integral on the right hand side is just a constant that does not depend on ε. Concerning the first integral, observe that by the mean value theorem the integrand is uniformly bounded

$$\left|\frac{f(x-y)-f(x)}{y}\right| \leq \|f'\|_{L^\infty(\mathbb{R})},$$

and so the limit exists and we have

$$\lim_{\varepsilon\to 0}\int_{\varepsilon<|y|<1}\frac{f(x-y)-f(x)}{y}\,dy = \int_0^1 \frac{f(x-y)-f(x)}{y}\,dy.$$

□

We showed that the Hilbert transform is a well-defined, linear operator defined on $\mathcal{S}(\mathbb{R})$. Later on, we will be interested in extending it to the L^p spaces for $1 < p < \infty$. One way to do that is by showing that H is of strong type (p,p), i.e. that for all $f \in \mathcal{S}(\mathbb{R})$ we have

$$\|Hf\|_{L^p(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})}.$$

After establishing such inequality, we may use the density of $\mathcal{S}(\mathbb{R})$ in $L^p(\mathbb{R})$ to extend the Hilbert transform to functions in $L^p(\mathbb{R})$. The exercise below shows that we may only hope for the strong type (p,p) inequality to hold for $1 < p < \infty$.

Exercise 3.4 (1 point). Let $f = \mathbf{1}_{[0,1]}$. Show that for $x \in \mathbb{R}\setminus\{0,1\}$

$$\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\frac{f(y)}{x-y}\,dy = \log\left|\frac{x}{x-1}\right|.$$

Conclude that the Hilbert transform is neither of strong type (∞,∞) nor of strong type $(1,1)$.

So our goal is estimating $\|Hf\|_{L^p(\mathbb{R})}$. As a warm-up, we prove that for $f \in \mathcal{S}(\mathbb{R})$ we have $Hf \in L^p(\mathbb{R})$ for all $1 < p \leq \infty$. This is a consequence of the following asymptotic identity.

Lemma 3.5. *For $f \in \mathcal{S}(\mathbb{R})$ we have*

$$\lim_{|x|\to\infty} x \cdot Hf(x) = \frac{1}{\pi}\int_{\mathbb{R}} f(y)\,dy.$$

Proof. The proof is similar to that of (3.2). We use the oddness of kernel $\frac{1}{y}$ once again to get that for any $x \in \mathbb{R}$ with $|x| > 0$

$$\begin{aligned}\pi x \cdot Hf(x) &= \lim_{\varepsilon \to 0} x \int_{|y|>\varepsilon} \frac{f(x-y)}{y}\, dy \\ &= \lim_{\varepsilon \to 0} x \int_{\varepsilon<|y|<\frac{|x|}{2}} \frac{f(x-y)-f(x)}{y}\, dy + x \int_{\frac{|x|}{2}<|y|<2|x|} \frac{f(x-y)}{y}\, dy \\ &\qquad + x \int_{|y|>2|x|} \frac{f(x-y)}{y}\, dy = I_1 + I_2 + I_3.\end{aligned}$$

Regarding I_1, note that for $|y| < |x|/2$ we have $|x|/2 \leq |x-y| \leq 3|x|/2$, and so by the mean value theorem

$$|I_1| \leq |x|^2 \sup_{|x|/2\leq|\xi|\leq 3|x|/2} |f'(\xi)| \sim \sup_{|x|/2\leq|\xi|\leq 3|x|/2} |\xi^2 f'(\xi)| \xrightarrow{|x|\to\infty} 0,$$

where in the last step we used the rapid decay of Schwartz functions.

Concerning I_3, we have $|x-y| \geq |x|$ whenever $|y| > 2|x|$, and so

$$|I_3| \leq |x| \int_{|y|>2|x|} \frac{|f(x-y)|}{2|x|}\, dy \leq \int_{|z|>|x|} |f(z)|\, dz \xrightarrow{|x|\to\infty} 0,$$

since f is integrable.

Finally,

$$I_2 - \int f(x-y)\, dy = \int_{\frac{|x|}{2}<|y|<2|x|} \left(\frac{x}{y} - 1\right) f(x-y)\, dy - \int_{|y|<|x|/2, \text{ or } |y|>2|x|} f(x-y)\, dy,$$

which gives

$$\begin{aligned}\Big|I_2 - \int f(x-y)\, dy\Big| &\leq \int_{\frac{|x|}{2}<|y|<2|x|} \left|\frac{x-y}{y}\right| |f(x-y)|\, dy + \int_{|y|<|x|/2, \text{ or } |y|>2|x|} |f(x-y)|\, dy \\ &\lesssim \frac{1}{|x|} \int |zf(z)|\, dy + \int_{|z|>|x|/2} |f(z)|\, dy \xrightarrow{|x|\to\infty} 0.\end{aligned}$$

□

Corollary 3.6. *For every $f \in \mathcal{S}(\mathbb{R})$ we have $Hf \in L^p(\mathbb{R})$ for all $1 < p \leq \infty$.*

Proof. Note that by (3.2) and the mean value theorem we have

$$\|Hf\|_{L^\infty(\mathbb{R})} \lesssim \|f'\|_{L^\infty(\mathbb{R})} + \sup_{x\in\mathbb{R}} |x \cdot f(x)|, \tag{3.3}$$

so the Hilbert transform of a Schwartz function is bounded. Thus, whether $Hf \in L^p$ for $1 \leq p < \infty$ depends only on the decay rate of Hf at infinity. By Lemma 3.5, for $|x|$ large enough we have $|Hf(x)| \lesssim_f x^{-1}$, and it follows that $Hf \in L^p(\mathbb{R})$ for all $p > 1$. □

Exercise 3.7. Let $f \in \mathcal{S}(\mathbb{R})$. Show that $Hf \in L^1(\mathbb{R})$ if and only if $\int_{\mathbb{R}} f(y)\,dy = 0$. A hint: Modify the proof of Lemma 3.5 to estimate the asymptotics of $x^2 \cdot Hf(x)$ as $|x| \to \infty$.

3.2 The Hilbert transform on $L^2(\mathbb{R})$

In this subsection we extend the Hilbert transform to $L^2(\mathbb{R})$. We begin by computing the Fourier transform of Hf.

First, since for any $f \in \mathcal{S}(\mathbb{R})$ we have $Hf \in L^2(\mathbb{R})$ by Corollary 3.6, the Fourier transform $\widehat{Hf}$ is well-defined as a function in L^2. Below we compute its precise value.

Proposition 3.8. *For any $f \in \mathcal{S}(\mathbb{R})$ we have*

$$\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi) \quad \text{for a.e. } \xi \in \mathbb{R}. \tag{3.4}$$

To prove this, we start by taking a slightly more abstract point of view. Since the Hilbert transform is linear, and we have the estimate (3.3), we can define a tempered distribution $T_0 \in \mathcal{S}'(\mathbb{R})$ by

$$\langle T_0, f\rangle := -Hf(0) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(y)}{y}\,dy.$$

Note that

$$Hf(x) = \langle T_0, f(x - \cdot)\rangle = T_0 * f(x).$$

Taking the Fourier transform (in the sense of distributions), we see that

$$\widehat{Hf} = \widehat{T_0} \cdot \hat{f}, \tag{3.5}$$

where the product is also understood in the sense of distributions: for any $\varphi \in \mathcal{S}(\mathbb{R})$ we have $\langle \widehat{Hf}, \varphi\rangle = \langle \widehat{T_0}, \hat{f}\varphi\rangle$.

As a consequence of (3.5), to prove (3.4) it suffices to show that $\widehat{T_0}$, which *a priori* is just a tempered distribution, is in fact a function, and that $\widehat{T_0}(\xi) = -i\,\mathrm{sgn}(\xi)$.

Lemma 3.9. *We have $\widehat{T_0}(\xi) = -i\,\mathrm{sgn}(\xi)$.*

Proof. An exercise. Some hints:

(i) Let $K_\varepsilon(y) = \frac{1}{y}\mathbf{1}_{|y|>\varepsilon}$, so that $\langle T_0, f\rangle = \frac{1}{\pi}\lim_{\varepsilon\to 0}\langle K_\varepsilon, f\rangle$, and consider $Q_\varepsilon(y) = \frac{y}{y^2+\varepsilon^2}$. Show that

$$\lim_{\varepsilon\to 0}\,(K_\varepsilon - Q_\varepsilon) = 0 \quad \text{in } \mathcal{S}'(\mathbb{R}).$$

(ii) Using the above, argue that $\widehat{T_0} = \frac{1}{\pi} \lim_{\varepsilon \to 0} \widehat{Q_\varepsilon}$, in the sense of distributions.

(iii) Show that $Q_\varepsilon(x) = \mathcal{F}^{-1}(-\pi i \operatorname{sgn}(\xi) e^{-2\pi\varepsilon|\xi|})(x)$. Conclude that $\widehat{T_0}$ is given by a function, and that $\widehat{T_0}(\xi) = -i \operatorname{sgn}(\xi)$.

□

As a corollary of Proposition 3.8 and Plancherel's identity, we can define the Hilbert transform of functions in $L^2(\mathbb{R})$.

Corollary 3.10. *For any $f \in \mathcal{S}(\mathbb{R})$ we have*

$$\|Hf\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}.$$

Consequently, the Hilbert transform extends to an isometry of $L^2(\mathbb{R})$. Moreover, for any $f \in L^2(\mathbb{R})$ its Hilbert transform satisfies

$$\widehat{Hf}(\xi) = -i \operatorname{sgn}(\xi) \hat{f}(\xi).$$

Recall that for $f \in \mathcal{S}(\mathbb{R})$ we have a nice formula for $Hf(x)$ assuming $x \notin \operatorname{supp} f$, see (3.1). It is easy to see that the same formula holds for $f \in L^2(\mathbb{R})$.

Exercise 3.11. Show that if $f \in L^2(\mathbb{R})$, then for a.e. $x \notin \operatorname{supp}(f)$

$$Hf(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy.$$

Here, $\operatorname{supp} f$ denotes the essential support of f.

3.2.1 Truncated Hilbert transform

In Definition 3.1 we introduced the truncated Hilbert transform

$$H_\varepsilon f(x) = \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy$$

for $f \in \mathcal{S}(\mathbb{R})$. However, the same definition makes sense for $f \in L^p(\mathbb{R})$ for all $1 \leq p < \infty$ To see that, we use Hölder's inequality to show that the integral defining $H_\varepsilon f$ converges absolutely:

$$\int_{|x-y|>\varepsilon} \left| \frac{f(y)}{x-y} \right| dy \leq \|f\|_{L^p} \left\| \frac{\mathbf{1}_{|x-y|>\varepsilon}}{x-y} \right\|_{L^q} < \infty,$$

where $1/p + 1/q = 1$, so that $1 < q \leq \infty$.

By the definition of Hilbert transform, we have $Hf(x) = \lim_{\varepsilon \to 0} H_\varepsilon f(x)$ for all $f \in \mathcal{S}(\mathbb{R})$ and $x \in \mathbb{R}$. It is natural to ask for a counterpart of this statement for $f \in L^2(\mathbb{R})$; for example, do we have $H_\varepsilon f \to Hf$ in L^2 sense? We are able to show this if we assume that all truncated Hilbert transforms are of strong type $(2,2)$, in a uniform way.

Proposition 3.12. *Suppose that there exists a constant $C > 0$ such that*

$$\sup_{\varepsilon>0} \|H_\varepsilon f\|_{L^2(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})} \qquad \text{for all } f \in L^2. \tag{3.6}$$

Then, for every $f \in L^2(\mathbb{R})$ we have $H_\varepsilon f \to Hf$ in L^2.

Proof. Let $f_n \in \mathcal{S}(\mathbb{R})$ be such that $f_n \to f$ in L^2. Then, $Hf_n \to Hf$ in L^2, and we have

$$\|H_\varepsilon f - Hf\|_{L^2} \leq \|H_\varepsilon f - H_\varepsilon f_n\|_{L^2} + \|H_\varepsilon f_n - Hf_n\|_{L^2} + \|Hf_n - Hf\|_{L^2} =: I_1 + I_2 + I_3.$$

The term I_3 converges to 0 because $Hf_n \to H_f$ in L^2, whereas I_1 converges to 0 because

$$\|H_\varepsilon f - H_\varepsilon f_n\|_{L^2} = \|H_\varepsilon(f - f_n)\|_{L^2} \overset{(3.6)}{\leq} C\|f - f_n\|_{L^2} \xrightarrow{n\to\infty} 0.$$

It remains to estimate $I_2 = \|H_\varepsilon f_n - Hf_n\|_{L^2}$. By (3.2) we have

$$|H_\varepsilon f_n(x) - Hf_n(x)| = \left| \frac{1}{\pi} \int_{-\varepsilon}^{\varepsilon} \frac{f_n(x-y) - f_n(x)}{y}\, dy \right| \leq \frac{2\varepsilon}{\pi} \sup_{z\in(x-\varepsilon,x+\varepsilon)} |f_n'(z)|.$$

Set $g_n(x) := \sup_{z\in(x-\varepsilon,x+\varepsilon)} |f_n'(z)|$. Since f_n' decays rapidly, we get that g_n also decays rapidly, and so

$$I_2 = \|H_\varepsilon f_n - Hf_n\|_{L^2} \leq \frac{2\varepsilon}{\pi} \|g_n\|_{L^2}.$$

Hence, for any $\delta > 0$ we may take n large enough such that $I_1 + I_3 \leq \delta$, and then $\varepsilon > 0$ small enough so that $I_2 \leq \delta$. Then, we have $\|H_\varepsilon f - Hf\|_{L^2} \leq 2\delta$, and taking $\delta \to 0$ concludes the proof. $\square$

The question remains, how to show the estimate (3.6)? We will address this later on when we prove the so-called *Cotlar's inequality* for general singular integral operators.

3.3 The Riesz transform

Before moving on to general singular integral operators and their L^p-theory, we briefly discuss another important family of operators, the Riesz transforms. They are higher dimensional counterparts of the Hilbert transform.

Definition 3.13. For $f \in \mathcal{S}(\mathbb{R}^n)$ and $1 \leq j \leq n$, we define the j-th Riesz transform of f as

$$R_j f(x) := \lim_{\varepsilon\to 0} C_n \int_{|x-y|>\varepsilon} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy,$$

where $C_n = \Gamma(\frac{n+1}{2})/\pi^{\frac{n+1}{2}}$.

As in the case of the Hilbert transform, there is a simple formula for the Fourier transform of $R_j f$.

Proposition 3.14. *For any $f \in \mathcal{S}(\mathbb{R}^n)$*

$$\widehat{R_j f}(\xi) = -i\frac{\xi_j}{|\xi|}\hat{f}(\xi). \tag{3.7}$$

The proof is similar to that of Proposition 3.4, although there are additional difficulties. The interested reader can find the full proof e.g. in [Gra14a, Proposition 5.1.14].

As an immediate corollary of (3.7), we get that for all $f \in \mathcal{S}(\mathbb{R}^n)$

$$\|R_j f\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}, \tag{3.8}$$

and we may extend the Riesz transforms to $L^2(\mathbb{R}^n)$.

Finally, we give a simple application of (3.8), which also motivates the study of L^p-bounds for the Riesz transforms for $1 < p < \infty$.

Proposition 3.15. *For $f \in \mathcal{S}(\mathbb{R}^n)$ and $1 \leq j, k \leq n$ we have*

$$\partial_j \partial_k f = -R_j R_k \Delta f. \tag{3.9}$$

In consequence, for any $1 < p < \infty$ such that the bound $\|R_j f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}$ holds for all $1 \leq j \leq n$, we have

$$\|\partial_j \partial_k f\|_{L^p(\mathbb{R}^n)} \leq (C_p)^2 \|\Delta f\|_{L^p(\mathbb{R}^n)}. \tag{3.10}$$

Proof. By taking the Fourier transform of $\partial_j \partial_k f$ we get

$$\begin{aligned}
\mathcal{F}(\partial_j \partial_k f)(\xi) &= (2\pi i \xi_j)(2\pi i \xi_k)\hat{f}(\xi) \\
&= -\left(-i\frac{\xi_j}{|\xi|}\right)\left(-i\frac{\xi_k}{|\xi|}\right)(-4\pi^2|\xi|^2)\hat{f}(\xi) \\
&= -\mathcal{F}(R_j R_k \Delta f)(\xi).
\end{aligned}$$

Taking the inverse Fourier transform finishes the proof of identity (3.9). The estimate (3.10) follows immediately. $\square$

4 Calderón-Zygmund theory

In this section we begin the study of general singular integral operators.

4.1 Standard kernels and Calderón-Zygmund operators

The operators we will consider will be associated to the following kernels.

Definition 4.1 (standard kernel). We say that a Borel function $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\} \to \mathbb{C}$ is a *standard kernel* if there exists $\delta > 0$ and $C > 0$ such that

$$|K(x,y)| \leq \frac{C}{|x-y|^n}, \tag{4.1}$$

$$|K(x,y) - K(x,y')| \leq C\frac{|y-y'|^\delta}{|x-y|^{n+\delta}} \qquad \text{if } |x-y| > 2|y-y'|, \tag{4.2}$$

$$|K(x,y) - K(x',y)| \leq C\frac{|x-x'|^\delta}{|x-y|^{n+\delta}} \qquad \text{if } |x-y| > 2|x-x'|. \tag{4.3}$$

The bound (4.1) will be referred to as the *size condition*, while the other two estimates will be called the *smoothness conditions*.

Remark 4.2. The estimate $|x-y| > 2|y-y'|$ appearing in the smoothness condition can be interpreted in the following way: it is the estimate ensuring that $\frac{1}{2}|x-y| \leq |x-y'| \leq 2|x-y|$ (this follows easily from the triangle inequality).

We give a few examples.

Example 4.3. The Hilbert transform kernel $K(x,y) = \frac{1}{x-y}$ is a standard kernel on $\mathbb{R}$. More generally, the kernels $K(x,y) = \frac{x_j - y_j}{|x-y|^{n+1}}$ associated to the Riesz transforms are standard kernels on $\mathbb{R}^n$.

Example 4.4. Given $f \in C_c^\infty(\mathbb{R}^2)$ the solution to the Poisson equation $\Delta u = -2\pi f$ is given by the logarithmic potential of f

$$u(x) = \int_{\mathbb{R}^2} f(y) \log\left(\frac{1}{|x-y|}\right)\, dy.$$

It can be shown that the mixed partial derivative $\partial_{x_1}\partial_{x_2} u$ is given by the singular integral operator

$$\partial_{x_1}\partial_{x_2} u(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{\Omega_0\left(\frac{x-y}{|x-y|}\right)}{|x-y|^2} f(y)\, dy,$$

where $\Omega_0(x) = \frac{2x_1x_2}{|x|^2}$, see [CZ52, p. 130]. By the exercise below, the kernel associated to Ω_0 is a standard kernel.

Exercise 4.5. Show that for every Hölder continuous $\Omega : \mathbb{S}^{n-1} \to \mathbb{C}$ the kernel defined by

$$K(x,y) = \frac{\Omega\left(\frac{x-y}{|x-y|}\right)}{|x-y|^n}$$

is a standard kernel on $\mathbb{R}^n$.

Example 4.6. The kernel

$$K(z,w) = \frac{1}{(z-w)^2} \qquad z,w \in \mathbb{C},$$

is a standard kernel. It is associated to the *Buerling-Ahlfors transform*, which plays a fundamental role in the theory of quasiconformal mappings, see [Ast94].

The three examples above are kernels of convolution type, in the sense that $K(x,y) = K_0(x-y)$ for some $K_0 : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$. The next example shows that there are interesting kernels of non-convolution type, which justifies developing the theory in this generality.

Example 4.7 (Cauchy integral along a Lipschitz graph). Let $A : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function, and let $\Gamma = \{(t, A(t)) : t \in \mathbb{R}\} \subset \mathbb{C}$. Given $f \in \mathcal{S}(\mathbb{R})$ let $F : \Gamma \to \mathbb{C}$ be given by $F(t + iA(t)) = f(t)$. The Cauchy integral of f is defined as

$$C_\Gamma f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{F(w)}{w-z}\, dz = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(t)(1+iA'(t))}{t+iA(t)-z}\, dt,$$

and it defines an analytic function on $\mathbb{C}\setminus\Gamma$. One can compute the boundary values of $C_\Gamma f(z)$ on Γ:

$$\lim_{\varepsilon\to 0} C_\Gamma f(x + i(A(x)+\varepsilon)) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{|x-t|>\varepsilon} \frac{f(t)(1+iA'(t))}{t-x+i(A(t)-A(x))}\, dt + \frac{1}{2} f(x)$$
$$\lim_{\varepsilon\to 0} C_\Gamma f(x + i(A(x)-\varepsilon)) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{|x-t|>\varepsilon} \frac{f(t)(1+iA'(t))}{t-x+i(A(t)-A(x))}\, dt - \frac{1}{2} f(x),$$

see [Gra14b, Chapter 4.6]. This leads to the study of the Cauchy transform

$$Tf(x) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} \frac{f(y)}{x-y+i(A(x)-A(y))}\, dy,$$

whose kernel

$$K(x,y) = \frac{1}{x-y+i(A(x)-A(y))} \tag{4.4}$$

is a standard kernel of non-convolution type. For more information and the history of the Cauchy transform see [Tol14, Ver21].

Exercise 4.8. Prove that if A is Lipschitz, then the Cauchy kernel (4.4) is standard with $\delta = 1$.

We are ready to define our main object of study in this course: the Calderón-Zygmund operators.

Definition 4.9 (Calderón-Zygmund operator). We say that a linear operator $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a *Calderón-Zygmund operator* if

(i) T is of strong type $(2,2)$,

(ii) there exists a standard kernel K such that for all $f \in L^2(\mathbb{R}^n)$ with compact support
$$Tf(x) = \int K(x,y)f(y)\,dy \qquad \text{for } x \notin \operatorname{supp} f. \tag{4.5}$$

Whenever (4.5) holds, we will say that T *is associated to the kernel* K.

We make a few clarifying remarks regarding the definition of Calderón-Zygmund operators.

Remark 4.10. We stress that the definition of a Calderón-Zygmund operator *assumes* that the operator is bounded on L^2. We already know that this is true for the Hilbert transform and the Riesz transforms, and so they are Calderón-Zygmund operators (the property (ii) was shown in Exercise 3.11).

While the other operators mentioned in Examples 4.4, 4.6, 4.7 are also bounded on L^2, in general it is far from obvious. For example, proving the L^2-boundedness of the Cauchy transform on Lipschitz graphs was a major open problem for decades, and it was only solved in [CMM82]. We will not cover this result.

There are some sufficient conditions on kernels K that imply the L^2-boundedness of associated operators, see [Duo01, Chapter 4]. This may be a topic for a presentation.

The following exercise shows that a Calderón-Zygmund operator uniquely determines its kernel.

Exercise 4.11. If T is a Calderón-Zygmund operator such that (4.5) holds with two kernels K_1 and K_2, then $K_1 = K_2$ a.e.

The converse is not true. The trivial kernel $K = 0$ is associated both with the zero operator $T = 0$ and with the identity operator $T = I$. In general, for any $b \in L^\infty(\mathbb{R}^n)$ the pointwise multiplication operator

$$Tf(x) = b(x)f(x)$$

is a Calderón-Zygmund operator associated with the kernel $K = 0$. However, this is the only ambiguity.

Lemma 4.12. *Suppose that T_1 and T_2 are two Calderón-Zygmund operators associated with the same kernel K. Then, there exists $b \in L^\infty(\mathbb{R}^n)$ such that*

$$T_1 f = T_2 f + bf.$$

Proof. Let $T = T_1 - T_2$, so that T is a Calderón-Zygmund operator associated with the kernel $K = 0$. Our aim is to show that $Tf = bf$ for some $b \in L^\infty$. We will only prove this identity for characteristic functions, the case of general $f \in L^2$ follows by the density of simple functions in L^2.

First, we claim that for all measurable sets $E, F \subset \mathbb{R}^n$ with $0 < |E|, |F| < \infty$ we have $T(\mathbf{1}_E) = \mathbf{1}_E T(\mathbf{1}_E)$ and

$$\mathbf{1}_F T(\mathbf{1}_E) = T(\mathbf{1}_{E\cap F}). \tag{4.6}$$

Indeed, we have $T(\mathbf{1}_E)(x) = 0$ for a.e. $x \notin E$, since T is associated to $K = 0$. This gives $T(\mathbf{1}_E) = \mathbf{1}_E T(\mathbf{1}_E)$, and also it shows that $\mathbf{1}_F T(\mathbf{1}_E) = \mathbf{1}_{E\cap F} T(\mathbf{1}_E)$. By linearity of T,

$$\begin{aligned}\mathbf{1}_{E\cap F} T(\mathbf{1}_E) &= \mathbf{1}_{E\cap F} T(\mathbf{1}_{E\cap F}) + \mathbf{1}_{E\cap F} T(\mathbf{1}_{E\setminus F}) \\ &= \mathbf{1}_{E\cap F} T(\mathbf{1}_{E\cap F}) + \mathbf{1}_{E\cap F}\mathbf{1}_{E\setminus F} T(\mathbf{1}_{E\setminus F}) = \mathbf{1}_{E\cap F} T(\mathbf{1}_{E\cap F}) + 0.\end{aligned}$$

This gives (4.6).

Formally, we would like to define $b = T1$, but since $1 \notin L^2$, we have to work a bit to make this rigorous. Let $\{Q\}_{Q\in\mathcal{Q}}$ be a family of closed unit cubes tiling $\mathbb{R}^n$. Let $b_Q = T(\mathbf{1}_Q)$. Note that $\operatorname{supp} b_Q = \operatorname{supp} T(\mathbf{1}_Q) \subset Q$.

By the Lebesgue differentiation theorem, for a.e. $x \in \mathbb{R}^n$ we have

$$|b_Q(x)| = \lim_{r\to 0} \frac{\left|\int_{B(x,r)} b_Q \, dy\right|}{|B(x,r)|}. \tag{4.7}$$

We use the Cauchy-Schwarz inequality and the L^2-boundedness of T to get

$$\begin{aligned}\left|\int_{B(x,r)} b_Q \, dy\right| &= \left|\int_{B(x,r)} \mathbf{1}_{B(x,r)} T(\mathbf{1}_Q) \, dy\right| \overset{(4.6)}{=} \left|\int_{B(x,r)} T(\mathbf{1}_{Q\cap B(x,r)}) \, dy\right| \\ &\le |B(x,r)|^{1/2} \|T(\mathbf{1}_{Q\cap B(x,r)})\|_{L^2} \le C|B(x,r)|^{1/2}|Q \cap B(x,r)|^{1/2}.\end{aligned}$$

Together with (4.7) this gives $|b_Q(x)| \le C$ for a.e. $x \in \mathbb{R}^n$, so that $b_Q \in L^\infty$. Recalling that $\operatorname{supp} b_Q \subset Q$, we get that

$$b := \sum_{Q\in\mathcal{Q}} b_Q \in L^\infty.$$

We claim that for any bounded measurable $E \subset \mathbb{R}^n$ with $0 < |E| < \infty$ we have $T\mathbf{1}_E = b\mathbf{1}_E$. Indeed, such E intersects only a finite number of $Q \in \mathcal{Q}$, and then

$$T(\mathbf{1}_E) = \sum_{Q\in\mathcal{Q}} T(\mathbf{1}_{E\cap Q}) \overset{(4.6)}{=} \mathbf{1}_E \sum_{Q\in\mathcal{Q}} T(\mathbf{1}_Q) = \mathbf{1}_E b.$$

$\square$

Our goal is to prove the following fundamental result due to Calderón and Zygmund.

Theorem 4.13. *Suppose that T is a Calderón-Zygmund operator. Then, T is of weak type $(1,1)$, and of strong type (p,p) for $1 < p < \infty$.*

We will prove it over the following two subsections.

4.2 Calderón-Zygmund decomposition

Definition 4.14. The family of *dyadic cubes* in $\mathbb{R}^n$, denoted by $\mathcal{D}(\mathbb{R}^n)$, is defined as

$$\mathcal{D}(\mathbb{R}^n) = \left\{ 2^{-k}(m + [0,1)^n) = \prod_{i=1}^{n} [2^{-k}m_i, 2^{-k}m_i + 2^{-k}) \ : \ m \in \mathbb{Z}^n, k \in \mathbb{Z} \right\}.$$

Given $Q \in \mathcal{D}(\mathbb{R}^n)$, we will denote its sidelength by $\ell(Q)$. We set

$$\mathcal{D}_k(\mathbb{R}^n) = \{Q \in \mathcal{D}(\mathbb{R}^n) : \ell(Q) = 2^{-k}\}.$$

When the ambient space $\mathbb{R}^n$ is clear from context, we will write $\mathcal{D}$ instead of $\mathcal{D}(\mathbb{R}^n)$. Note that in our definition dyadic cubes are half-open, half-closed, so that for a fixed $k \in \mathbb{Z}$ the family $\mathcal{D}_k(\mathbb{R}^n)$ consists of pairwise-disjoint cubes, and it is a partition of $\mathbb{R}^n$.

We point out several important properties of the dyadic cubes:

(i) For any $Q, P \in \mathcal{D}(\mathbb{R}^n)$ we have either $Q \cap P = \varnothing$, or $Q \subset P$, or $P \subset Q$.

(ii) For every $Q \in \mathcal{D}_k(\mathbb{R}^n)$ there is a unique $\widehat{Q} \in \mathcal{D}_{k-1}(\mathbb{R}^n)$ such that $Q \subset \widehat{Q}$. We will call $\widehat{Q}$ *the parent of* Q.

(iii) Every $Q \in \mathcal{D}_k(\mathbb{R}^n)$ contains exactly 2^n cubes from $\mathcal{D}_{k+1}(\mathbb{R}^n)$. We will call these cubes *the children of* Q.

These properties endow $\mathcal{D}(\mathbb{R}^n)$ with a natural tree structure based on the parent-child relation.

The following is the main result of this subsection, and it is crucial for the proof of Theorem 4.13.

Proposition 4.15. *Let $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$. There exists a decomposition of f of the form*

$$f = g + \sum_{Q \in \mathcal{B}} b_Q,$$

where $\mathcal{B}$ is a collection of disjoint dyadic cubes, and which satisfies the following:

(i) the "good part" g satisfies $\|g\|_{L^1} \leq \|f\|_{L^1}$ and $\|g\|_{L^\infty} \leq 2^n\alpha$,

(ii) each "bad function" b_Q is supported on $\overline{Q}$, satisfies $\int_Q b_Q = 0$, and

$$\|b_Q\|_{L^1} \leq 2^{n+1}\alpha|Q|, \tag{4.8}$$

(iii) for each $Q \in \mathcal{B}$ we have

$$\alpha \leq \frac{1}{|Q|}\int_Q |f| \leq 2^n\alpha, \tag{4.9}$$

(iv) we can estimate the total measure of cubes in $\mathcal{B}$ by

$$\sum_{Q\in\mathcal{B}} |Q| \leq \frac{\|f\|_{L^1}}{\alpha}.$$

Proof. We will say that a cube $Q \in \mathcal{D}$ is *bad* if

$$\frac{1}{|Q|}\int_Q |f| > \alpha.$$

A bad cube Q is called maximal if there in no other bad cube Q' such that $Q \subsetneq Q'$.

We claim that every bad cube is contained in some maximal bad cube. If that was not true, then there would be a sequence of bad cubes $Q_1, Q_2, \ldots$ such that $\ell(Q_k) \to \infty$. At the same time,

$$\alpha < \frac{1}{|Q_k|}\int_{Q_k} |f| \leq \frac{\|f\|_{L^1}}{|Q_k|} \xrightarrow{k\to\infty} 0,$$

which is a contradiction.

Let $\mathcal{B}$ be the family of maximal bad cubes. Since they are dyadic and maximal, they are disjoint. For every $Q \in \mathcal{B}$ we define

$$b_Q := \left(f - \frac{1}{|Q|}\int_Q f\right)\mathbf{1}_Q,$$

and

$$g := f - \sum_{Q\in\mathcal{B}} b_Q.$$

We begin by proving (iii). The lower bound in (4.9) is just the definition of bad cubes. The upper bound follows from maximality: for every $Q \in \mathcal{B}$ its parent $\widehat{Q}$ is not a bad cube, and so

$$\frac{1}{|Q|}\int_Q |f| \leq \frac{|\widehat{Q}|}{|Q|}\frac{1}{|\widehat{Q}|}\int_{\widehat{Q}} |f| \leq 2^n\alpha.$$

Concerning (ii), the first two properties follow immediately from the definition, and

$$\|b_Q\|_{L^1} \leq \int_Q |f|dx + \int_Q \left| \frac{1}{|Q|} \int_Q f dx \right| dy \leq 2 \int_Q |f|dx \overset{(4.9)}{\leq} 2^{n+1}\alpha|Q|.$$

We move on to (i). Note that

$$g(x) = \begin{cases} f(x) & \text{for } x \notin \bigcup_{Q\in\mathcal{B}} Q \\ \frac{1}{|Q|}\int_Q f & \text{for } x \in Q \in \mathcal{B}. \end{cases}$$

Hence,

$$\begin{aligned} \|g\|_{L^1} &= \int_{\mathbb{R}^n \setminus \bigcup_{Q\in\mathcal{B}} Q} |f|dx + \sum_{Q\in\mathcal{B}} \int_Q \left| \frac{1}{|Q|} \int_Q f dy \right| dx \\ &= \int_{\mathbb{R}^n \setminus \bigcup_{Q\in\mathcal{B}} Q} |f|dx + \sum_{Q\in\mathcal{B}} \left| \int_Q f dy \right| \leq \|f\|_{L^1}. \end{aligned}$$

To see $\|g\|_{L^\infty} \leq 2^n\alpha$, note that $|g(x)| \leq 2^n\alpha$ for $x \in Q \in \mathcal{B}$ by (4.9). Let $x \notin \bigcup_{Q\in\mathcal{B}} Q$, so that $g(x) = f(x)$. Then, for all dyadic cubes containing x we have $\frac{1}{|Q|}\int_Q |f|dx \leq \alpha$. By the (dyadic version of) Lebesgue differentiation theorem for a.e. $y \in \mathbb{R}^n$ we have

$$|f(y)| = \lim_{\ell(Q)\to 0,\, y\in Q\in\mathcal{D}} \frac{1}{|Q|} \int_Q |f|dz$$

Together with the estimate $\frac{1}{|Q|}\int_Q |f| \leq \alpha$, this shows that for a.e. $x \notin \bigcup_{Q\in\mathcal{B}} Q$ we have $|g(x)| = |f(x)| \leq \alpha < 2^n\alpha$.

Finally, we show (iv). By the definition of bad cubes,

$$\sum_{Q\in\mathcal{B}} |Q| \leq \sum_{Q\in\mathcal{B}} \frac{\int_Q |f|}{\alpha} \leq \frac{\|f\|_{L^1}}{\alpha},$$

where in the last inequality we also used that the cubes in $\mathcal{B}$ are disjoint. □

4.3 The L^p theory for Calderón-Zygmund operators

In this subsection we prove Theorem 4.13, whose statement we repeat below.

Theorem. *Suppose that T is a Calderón-Zygmund operator. Then, T is of weak type $(1,1)$, and of strong type (p,p) for $1 < p < \infty$.*

We begin by reducing matters to the weak type (1,1) estimate.

Weak type (1,1) implies strong type (p,p). Fix a Calderón-Zygmund operator T. By definition, it is of strong type (2,2). Hence, as soon as we know that it is of weak type (1,1), it follows from the Marcinkiewicz interpolation theorem (Theorem 2.14) that T is of strong type (p,p) for all $1<p<2$. To get the same for $2<p<\infty$ we argue by duality as follows.

Given a Calderón-Zygmund operator $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ we consider its adjoint operator $T^t : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined by

$$\langle T^t(g), f\rangle_{L^2} = \int T^t(g)\cdot \bar{f}\, dx = \int g\cdot \overline{T(f)}\, dx = \langle g, T(f)\rangle_{L^2}.$$

This is well-defined by the Riesz representation theorem.

Exercise 4.16. Prove that if a Calderón-Zygmund operator T is associated to a standard kernel K, then its adjoint is also a Calderón-Zygmund operator, and it is associated to the standard kernel

$$K^t(x,y) = \overline{K(y,x)}.$$

Since T^t is a Calderón-Zygmund operator, it follows by the argument above that T^t is of strong type (q,q) for all $1<q<2$. Fix $f\in\mathcal{S}(\mathbb{R}^n)$, $2<p<\infty$, and let $1<q<2$ be such that $1/p+1/q=1$. Then, using that the dual of $L^p(\mathbb{R}^n)$ is $L^q(\mathbb{R}^n)$, we get

$$\begin{aligned}\|Tf\|_{L^p} &= \sup_{g\in\mathcal{S},\, \|g\|_{L^q}\le 1}\left|\int T(f)\cdot\overline{g}\right| = \sup_{g\in\mathcal{S},\, \|g\|_{L^q}\le 1}\left|\int f\cdot\overline{T^t(g)}\right| \\ &\le \|f\|_{L^p}\sup_{g\in\mathcal{S},\, \|g\|_{L^q}\le 1}\|T^t(g)\|_{L^q} \le C_q\|f\|_{L^p}\sup_{g\in\mathcal{S},\, \|g\|_{L^q}\le 1}\|g\|_{L^q} = C_q\|f\|_{L^p}.\end{aligned}$$

Hence, T is of strong type (p,p). □

Proof of the weak type (1,1) estimate. Let[1] $f\in L^1\cap L^2$. Our goal is to show that there exists a dimensional constant C such that for any $\alpha>0$

$$|\{x\in\mathbb{R}^n \ : \ |Tf(x)|>\alpha\}| \le C\frac{\|f\|_{L^1}}{\alpha}.$$

We apply the Calderón-Zygmund decomposition (Proposition 4.15) to f at level α, so that $f = g+b = g+\sum_{Q\in\mathcal{B}} b_Q$. By the linearity of T, we have $Tf = Tg+Tb$, and so

$$\begin{aligned}&|\{x\in\mathbb{R}^n \ : \ |Tf(x)|>\alpha\}| \\ &\qquad \le |\{x\in\mathbb{R}^n \ : \ |Tg(x)|>\alpha/2\}| + |\{x\in\mathbb{R}^n \ : \ |Tb(x)|>\alpha/2\}|. \qquad (4.10)\end{aligned}$$

[1]We only assume $f\in L^2$ so that Tf is well-defined, our estimates will be independent of $\|f\|_{L^2}$.

To estimate the term corresponding to g, we use Chebyshev's inequality and the fact that $\|g\|_{L^1} \leq \|f\|_{L^1}$, $\|g\|_{L^\infty} \leq 2^n\alpha$:

$$|\{x \in \mathbb{R}^n \ : \ |Tg(x)| > \alpha/2\}| \leq \frac{\|Tg\|_{L^2}^2}{(\alpha/2)^2} \lesssim \frac{\|g\|_{L^2}^2}{\alpha^2}$$
$$\leq \frac{\|g\|_{L^1}\|g\|_{L^\infty}}{\alpha^2} \leq \frac{2^n\alpha\|f\|_{L^1}}{\alpha^2} \sim \frac{\|f\|_{L^1}}{\alpha}. \quad (4.11)$$

So the first term from the RHS of (4.10) satisfies the desired inequality. We move on to the second term, which is more difficult to estimate.

For every $Q \in \mathcal{B}$ let Q^* be the cube with the same center as Q, and with sidelength $\ell(Q^*) = 2\sqrt{n}\,\ell(Q)$. We have

$$|\{x \in \mathbb{R}^n \ : \ |Tb(x)| > \alpha/2\}| \leq \sum_{Q\in\mathcal{B}} |Q^*| + |\{x \in \mathbb{R}^n \setminus \bigcup_{Q\in\mathcal{B}} Q^* \ : \ |Tb(x)| > \alpha/2\}|.$$

The first term satisfies

$$\sum_{Q\in\mathcal{B}} |Q^*| \lesssim \sum_{Q\in\mathcal{B}} |Q| \leq \frac{\|f\|_{L^1}}{\alpha}$$

by Proposition 4.15 (iv). Concerning the second term, by Chebyshev's inequality

$$|\{x \in \mathbb{R}^n \setminus \bigcup_{Q\in\mathcal{B}} Q^* \ : \ |Tb(x)| > \alpha/2\}| \leq \frac{2}{\alpha}\int_{(\bigcup_{Q\in\mathcal{B}} Q^*)^c} |Tb(x)|\, dx$$
$$\leq \frac{2}{\alpha}\sum_{Q'\in\mathcal{B}}\int_{(\bigcup_{Q\in\mathcal{B}} Q^*)^c} |Tb_{Q'}(x)|\, dx \leq \frac{2}{\alpha}\sum_{Q\in\mathcal{B}}\int_{(Q^*)^c} |Tb_Q(x)|\, dx.$$

It remains to show that the sum above is bounded by $C\|f\|_{L^1}$.

Fix $Q \in \mathcal{B}$ and let y_Q denote the center of Q. Since $\operatorname{supp} b_Q \subset Q$ and $\int_Q b_Q = 0$, we get that for $x \in (Q^*)^c$

$$Tb_Q(x) = \int_Q K(x,y)b_Q(y)\, dy = \int_Q (K(x,y) - K(x,y_Q))b_Q(y)\, dy.$$

Observe that for $x \in (Q^*)^c$ and $y \in Q$ we have $|x-y| \geq \ell(Q^*)/2 = \sqrt{n}\ell(Q)$ and $|y - y_Q| \leq \operatorname{diam}(Q)/2 = \sqrt{n}\ell(Q)/2$, so that $|x-y| \geq 2|y-y_Q|$. It follows that we may use the smoothness condition on K (4.2) to estimate

$$|Tb_Q(x)| \lesssim \int_Q \frac{|y-y_Q|^\delta}{|x-y_Q|^{n+\delta}}|b_Q(y)|\, dy \lesssim \frac{\ell(Q)^\delta}{|x-y_Q|^{n+\delta}}\|b_Q\|_{L^1}.$$

Hence,

$$\int_{(Q^*)^c} |Tb_Q(x)|\, dx \lesssim \ell(Q)^\delta \|b_Q\|_{L^1}\int_{(Q^*)^c} \frac{1}{|x-y_Q|^{n+\delta}}\, dy \leq C(\delta)\|b_Q\|_{L^1},$$

which gives

$$\sum_{Q\in\mathcal{B}} \int_{(Q^*)^c} |Tb_Q(x)| \, dx \lesssim_\delta \sum_{Q\in\mathcal{B}} \|b_Q\|_{L^1} = \|b\|_{L^1} \leq \|f\|_{L^1} + \|g\|_{L^1} \leq 2\|f\|_{L^1}.$$

This finishes the proof. $\square$

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