# Singular Integral Operators

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**Abstract**

Notes for the course *Singular Integral Operators* lectured at the University of Jyväskylä in Autumn 2023.

## Contents



# 1 Introduction

This course will focus on singular integral operators, which are operators of the form

$$Tf(x) = \int K(x,y)f(y)\,dy,$$

where the kernel $K(x,y)$ has a singularity on the diagonal $x=y$. These operators appear naturally e.g. in the theory of partial differential equations, and they have been studied for over a century. The prototypical example is the Hilbert transform

$$Hf(x) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-y|>\varepsilon}\frac{f(y)}{x-y}\,dy.$$

The basic questions we will study concern the mapping properties of singular integral operators: for which $1 \leq p \leq \infty$ and under what hypotheses on the kernel $K$ is the operator $T$ bounded on $L^p$, in the sense that

$$\|Tf\|_{L^p} \leq C\|f\|_{L^p}.$$

The material we will cover reflects both the long tradition of this field, and the fact that it is still an active area of research. We will begin by studying the Hilbert and Riesz transforms, which date back almost 100 years back. Then, we will move on to the Calderón-Zygmund theory, which revolutionized the field in the 1950s. Finally, we will discuss singular integrals in the weighted setting, which is a much more recent topic. The grand finale will be the proof of the $A_2$ theorem, which was shown by Tuomas Hytönen in 2012 [Hyt12]. We will follow a short and elegant proof from [Ler16] which uses a cutting-edge technique called *sparse domination.*

The field of singular integral operators is huge, and we will only scratch the surface in this course. We refer interested readers to the textbooks [Duo01, Gra14a, Gra14b, Ste70, Ste93] for more thorough treatments of the subject.

# 2 Preliminaries

Before getting started in earnest, we recall briefly some useful facts and definitions. For proofs and details, see e.g. Chapters 1 and 2 of [Gra14a].

## 2.1 Schwartz functions and tempered distributions

*Definition* 2.1 (Schwartz functions). A function $f \in C^\infty(\mathbb{R}^n)$ is a *Schwartz function*, denoted by $f \in \mathcal{S}(\mathbb{R}^n)$, if for every pair of multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ we have

$$\rho_{\alpha,\beta}(f) \coloneqq \sup_{x \in \mathbb{R}^n} |x^\alpha \cdot \partial^\beta f(x)| < \infty.$$

We will say that a function decays rapidly if it decays at $\infty$ faster than any polynomial. Hence, Schwartz functions are precisely those $C^\infty(\mathbb{R}^n)$ functions which decay rapidly and whose all partial derivatives decay rapidly.

*Example* 2.2. Any smooth and compactly supported function is a Schwartz function, so that $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$. A simple example of a non-compactly supported Schwartz function is $e^{-|x|^2}$.

One of the reasons Schwartz functions are useful is the following density result.

**Lemma 2.3.** *The Schwartz functions are dense in $L^p(\mathbb{R}^n)$ for all $1 \leq p < \infty$.*

Note that $\mathcal{S}(\mathbb{R}^n)$ is a vector space. A topology on $\mathcal{S}(\mathbb{R}^n)$ can be defined using the family of semi-norms $\rho_{\alpha,\beta}$, and it is compatible with the following notion of convergence.

*Definition* 2.4 (convergence in $\mathcal{S}(\mathbb{R}^n)$). Given $f \in \mathcal{S}(\mathbb{R}^n)$ and a sequence $f_k \in \mathcal{S}(\mathbb{R}^n)$, we say that $f_k$ coverges to $f$ in $\mathcal{S}(\mathbb{R}^n)$ if for all multi-indices $\alpha, \beta \in \mathbb{N}^n$

$$\lim_{k\to\infty} \rho_{\alpha,\beta}(f_k - f) = 0.$$

*Definition* 2.5 (tempered distributions). We denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of $\mathcal{S}(\mathbb{R}^n)$, i.e., the space of all continuous linear functionals $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$. The elements of $\mathcal{S}'(\mathbb{R}^n)$ are called *tempered distributions*.

Given $T \in \mathcal{S}'(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$, instead of writing $T(f)$ we will write $\langle T, f\rangle$, and we will call it *the action of $T$ on $f$*.

We have the following useful characterization of tempered distributions:

**Lemma 2.6.** *A linear functional $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is a tempered distribution if and only if there exist $m, k \in \mathbb{N}$ and $C > 0$ such that for all $f \in \mathcal{S}(\mathbb{R}^n)$*

$$|\langle T, f\rangle| \leq C \sum_{|\alpha|\leq m,\ |\beta|\leq k} \rho_{\alpha,\beta}(f).$$

*Example* 2.7. Any function $g \in L^p(\mathbb{R}^n), 1 \leq p \leq \infty$, gives rise to a tempered distribution $T_g \in \mathcal{S}'(\mathbb{R}^n)$ defined via $\langle T_g, f\rangle = \int f(x)g(x)\ dx$.

*Example* 2.8. Any finite Borel measure $\mu$ gives rise to a tempered distribution $T_\mu \in \mathcal{S}'(\mathbb{R}^n)$ defined via $\langle T_\mu, f\rangle = \int f\ d\mu$.

In the case of tempered distributions as above, we will often identify $T_g$ with $g$, and $T_\mu$ with $\mu$. For example, the statement "$T \in \mathcal{S}'(\mathbb{R}^n)$ is a $C^\infty(\mathbb{R}^n)$ function" should be understood as "there exists $f \in C^\infty(\mathbb{R}^n)$ such that $T = T_f$." The Hilbert transform we will define shortly will provide us with an example of a tempered distribution which is neither a locally integrable function, nor a measure.

Many common operations performed on functions can be extended by duality to tempered distributions. For example, given $h \in \mathcal{S}(\mathbb{R}^n)$ and $T \in \mathcal{S}'(\mathbb{R}^n)$, we define their convolution as a tempered distribution $T * h \in \mathcal{S}'(\mathbb{R}^n)$ given by

$$\langle T * h, f\rangle \coloneqq \langle T, \tilde{h} * f\rangle,$$

where $\tilde{h}(x) = h(-x)$. Similarly, the product of $h \in \mathcal{S}(\mathbb{R}^n)$ and $T \in \mathcal{S}'(\mathbb{R}^n)$ can be defined as a tempered distribution $hT \in \mathcal{S}'(\mathbb{R}^n)$ given by

$$\langle hT, f\rangle \coloneqq \langle T, hf\rangle.$$

**Proposition 2.9.** *Given $h \in \mathcal{S}(\mathbb{R}^n)$ and $T \in \mathcal{S}'(\mathbb{R}^n)$ the convolution $T * h$ belongs to $C^\infty(\mathbb{R}^n)$. Moreover,*

$$T * h(x) = \langle T, h(\cdot - x)\rangle.$$

## 2.2 Fourier transform

*Definition* 2.10. The Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$ is defined as

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx.$$

Sometimes we will denote it by $\mathcal{F}(f)$ instead of $\hat{f}$.

The Fourier transform is a homeomorphism of $\mathcal{S}(\mathbb{R}^n)$ to itself, and its inverse is given by

$$\check{f}(x) := \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x\cdot\xi}\, dx = \hat{f}(-x),$$

sometimes denoted by $\mathcal{F}^{-1}(f)$.

The Plancherel identity asserts that for any $f \in \mathcal{S}(\mathbb{R}^n)$

$$\|f\|_{L^2(\mathbb{R}^n)} = \|\hat{f}\|_{L^2(\mathbb{R}^n)}.$$

By the density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, this allows us to extend the Fourier transform to an isometry of $L^2(\mathbb{R}^n)$.

One may further extend the definition of Fourier transform to all tempered distributions using duality: for any $T \in \mathcal{S}'(\mathbb{R}^n)$ we define $\widehat{T} \in \mathcal{S}'(\mathbb{R}^n)$ via

$$\langle \widehat{T}, f\rangle := \langle T, \hat{f}\rangle.$$

We list a few properties of the Fourier transform we will use later on.

**Lemma 2.11.** *If $f \in \mathcal{S}(\mathbb{R}^n)$ and $T \in \mathcal{S}'(\mathbb{R}^n)$, then*

*(i)* $\mathcal{F}(\partial^\alpha f) = (2\pi i\xi)^\alpha \hat{f}$,

*(ii)* $\partial^\alpha \hat{f} = \mathcal{F}((-2\pi i x)^\alpha f)$,

*(iii)* $\widehat{T * f} = \hat{T}\hat{f}$.

## 2.3 Weak and strong type inequalities

In this subsection we assume that $(X, \mu)$ and $(Y, \nu)$ are two measure spaces.

*Definition* 2.12. Given $1 \le p, q \le \infty$ and an operator $T$ mapping functions from a dense subset of $L^p(X, \mu)$ to measurable functions on $(Y, \nu)$, we say that $T$ *is of strong type* $(p, q)$ if there exists $C > 0$ such that

$$\|Tf\|_{L^q(Y,\nu)} \le C\|f\|_{L^p(X,\mu)}.$$

We say that $T$ *is of weak type* $(p, q)$ if there exists $C > 0$ such that for all $\lambda > 0$

$$\nu(\{y \in Y \ :\ |Tf(y)| > \lambda\}) \le C \left(\frac{\|f\|_{L^p(X,\mu)}}{\lambda}\right)^q.$$

It is easy to see that strong type $(p, q)$ implies weak type $(p, q)$.

*Definition* 2.13 (sublinear operator). An operator $T$ defined on a linear space of measurable functions on $(X, \mu)$ and taking values in measurable functions on $(Y, \nu)$ is sub-linear if

$$|T(f+g)| \leq |Tf| + |Tg| \quad \text{and} \quad |T(\lambda f)| = |\lambda||Tf|.$$

The Marcinkiewicz interpolation theorem stated below plays a crucial role in the theory of singular integral operators.

**Theorem 2.14.** *Let $1 \leq p_0 < p_1 \leq \infty$. Suppose that $T$ is a sub-linear operator mapping $L^{p_0}(X, \mu) + L^{p_1}(X, \mu)$ to the set of measurable functions on $(Y, \nu)$. If $T$ is of weak type $(p_0, p_0)$ and $(p_1, p_1)$, then it is of strong type $(p, p)$ for all $p_0 < p < p_1$.*

# 3 The Hilbert and Riesz transforms

In this section we will study the prototypical singular integral operator, the Hilbert transform, as well as its higher dimensional counterparts, the Riesz transforms.

## 3.1 The Hilbert transform on $\mathcal{S}(\mathbb{R})$

The Hilbert transform is the singular integral operator associated with kernel $K(x, y) = \frac{1}{\pi(x-y)}$. We begin by defining it for Schwartz functions.

As a first attempt at defining it, one could try to simply integrate against the kernel:

$$\frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y} \, dy.$$

However, the expression above is highly problematic. Even for a very nice function $f$, say, $f \in C_c^\infty(\mathbb{R})$, it is easy to see that as soon as $f(x) \neq 0$, the integral above is not well-defined! This is because $(x-y)^{-1}$ has a singularity at $x$ which is not integrable.

To avoid this issue, we first consider the following *truncated Hilbert transform.*

*Definition* 3.1 (truncated Hilbert transform). For $f \in \mathcal{S}(\mathbb{R})$ and $\varepsilon > 0$, we define the truncated Hilbert transform of $f$ as

$$H_\varepsilon f(x) := \frac{1}{\pi} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} \, dy = \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(x-y)}{y} \, dy.$$

Note that, by the rapid decay of Schwartz functions, $H_\varepsilon f(x)$ is well-defined for every $x \in \mathbb{R}$.

*Definition* 3.2 (Hilbert transform). For $f \in \mathcal{S}(\mathbb{R})$, we define the Hilbert transform of $f$ as

$$Hf(x) := \lim_{\varepsilon \to 0} H_\varepsilon f(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(x-y)}{y}\, dy.$$

Clearly, for $x \notin \operatorname{supp} f$ this is well-defined, and in fact

$$Hf(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy \qquad \text{for } x \notin \operatorname{supp} f. \tag{3.1}$$

Let us show that $Hf(x)$ is well-defined also for $x \in \operatorname{supp} f$.

**Lemma 3.3.** *For any $f \in \mathcal{S}(\mathbb{R})$ and $x \in \mathbb{R}$ the limit $\lim_{\varepsilon \to 0} H_\varepsilon f(x)$ exists, and we have*

$$Hf(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x-y) - f(x)}{y}\, dy + \frac{1}{\pi} \int_{|y|>1} \frac{f(x-y)}{y}\, dy. \tag{3.2}$$

*Proof.* Fix $\varepsilon > 0$. Note that, since the kernel $\frac{1}{y}$ is odd, it has zero mean on any symmetric pair of intervals around the origin, and in particular

$$\int_{\varepsilon<|y|<1} \frac{1}{y}\, dy = 0.$$

It follows that

$$\int_{|y|>\varepsilon} \frac{f(x-y)}{y}\, dy = \int_{\varepsilon<|y|<1} \frac{f(x-y) - f(x)}{y}\, dy + \int_{|y|>1} \frac{f(x-y)}{y}\, dy.$$

The second integral on the right hand side is just a constant that does not depend on $\varepsilon$. Concerning the first integral, observe that by the mean value theorem the integrand is uniformly bounded

$$\left| \frac{f(x-y) - f(x)}{y} \right| \le \|f'\|_{L^\infty(\mathbb{R})},$$

and so the limit exists and we have

$$\lim_{\varepsilon \to 0} \int_{\varepsilon<|y|<1} \frac{f(x-y) - f(x)}{y}\, dy = \int_{0}^{1} \frac{f(x-y) - f(x)}{y}\, dy.$$

□

We showed that the Hilbert transform is a well-defined, linear operator defined on $\mathcal{S}(\mathbb{R})$. Later on, we will be interested in extending it to the $L^p$ spaces for $1 < p < \infty$. One way to do that is by showing that $H$ is of strong type $(p,p)$, i.e. that for all $f \in \mathcal{S}(\mathbb{R})$ we have

$$\|Hf\|_{L^p(\mathbb{R})} \le C_p \|f\|_{L^p(\mathbb{R})}.$$

After establishing such inequality, we may use the density of $\mathcal{S}(\mathbb{R})$ in $L^p(\mathbb{R})$ to extend the Hilbert transform to functions in $L^p(\mathbb{R})$. The exercise below shows that we may only hope for the strong type $(p,p)$ inequality to hold for $1 < p < \infty$.

*Exercise* 3.4. Let $f = \mathbf{1}_{[0,1]}$. Show that

$$\lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{f(y)}{y}\, dy = \log \left| \frac{x}{x-1} \right|.$$

Conclude that the Hilbert transform is neither of strong type $(\infty, \infty)$ nor of strong type $(1,1)$.

So our goal is estimating $\|Hf\|_{L^p(\mathbb{R})}$. As a warm-up, we prove that for $f \in \mathcal{S}(\mathbb{R})$ we have $Hf \in L^p(\mathbb{R})$ for all $1 < p \leq \infty$. This is a consequence of the following asymptotic identity.

**Lemma 3.5.** *For $f \in \mathcal{S}(\mathbb{R})$ we have*

$$\lim_{|x| \to \infty} x \cdot Hf(x) = \frac{1}{\pi} \int_{\mathbb{R}} f(y)\, dy.$$

*Proof.* The proof is similar to that of (3.2). We use the oddness of kernel $\frac{1}{y}$ once again to get that for any $x \in \mathbb{R}$ with $|x| > 0$

$$\begin{aligned} \pi x \cdot Hf(x) &= \lim_{\varepsilon \to 0} x \int_{|y|>\varepsilon} \frac{f(x-y)}{y}\, dy \\ &= \lim_{\varepsilon \to 0} x \int_{\varepsilon < |y| < \frac{|x|}{2}} \frac{f(x-y) - f(x)}{y}\, dy + x \int_{\frac{|x|}{2} < |y| < 2|x|} \frac{f(x-y)}{y}\, dy \\ &\qquad + x \int_{|y|>2|x|} \frac{f(x-y)}{y}\, dy = I_1 + I_2 + I_3. \end{aligned}$$

Regarding $I_1$, note that for $|y| < |x|/2$ we have $|x|/2 \leq |x-y| \leq 3|x|/2$, and so by the mean value theorem

$$|I_1| \leq |x|^2 \sup_{|x|/2 \leq |\xi| \leq 3|x|/2} |f'(\xi)| \sim \sup_{|x|/2 \leq |\xi| \leq 3|x|/2} |\xi^2 f'(\xi)| \xrightarrow{|x| \to \infty} 0,$$

where in the last step we used the rapid decay of Schwartz functions.

Concerning $I_3$, we have $|x-y| \geq |x|$ whenever $|y| > 2|x|$, and so

$$|I_3| \leq |x| \int_{|y|>2|x|} \frac{|f(x-y)|}{2|x|}\, dy \leq \int_{|z|>|x|} |f(z)|\, dz \xrightarrow{|x| \to \infty} 0,$$

since $f$ is integrable.

Finally,

$$I_2 - \int f(x-y)\, dy = \int_{\frac{|x|}{2} < |y| < 2|x|} \left( \frac{x}{y} - 1 \right) f(x-y)\, dy - \int_{|y|<|x|/2, \text{ or } |y|>2|x|} f(x-y)\, dy,$$

which gives

$$\Big|I_2 - \int f(x-y)\,dy\Big| \le \int_{\frac{|x|}{2}<|y|<2|x|} \left|\frac{x-y}{y}\right| |f(x-y)|\,dy + \int_{|y|<|x|/2, \text{ or } |y|>2|x|} |f(x-y)|\,dy$$
$$\lesssim \frac{1}{|x|}\int |zf(z)|\,dy + \int_{|z|>|x|/2} |f(z)|\,dy \xrightarrow{|x|\to\infty} 0.$$

□

**Corollary 3.6.** *For every $f \in \mathcal{S}(\mathbb{R})$ we have $Hf \in L^p(\mathbb{R})$ for all $1 < p \le \infty$.*

*Proof.* Note that by (3.2) and the mean value theorem we have

$$\|Hf\|_{L^\infty(\mathbb{R})} \lesssim \|f'\|_{L^\infty(\mathbb{R})} + \sup_{x\in\mathbb{R}} |x \cdot f(x)|, \tag{3.3}$$

so the Hilbert transform of a Schwartz function is bounded. Thus, whether $Hf \in L^p$ for $1 \le p < \infty$ depends only on the decay rate of $Hf$ at infinity. By Lemma 3.5, for $|x|$ large enough we have $|Hf(x)| \lesssim_f x^{-1}$, and it follows that $Hf \in L^p(\mathbb{R})$ for all $p > 1$. □

*Exercise* 3.7. Let $f \in \mathcal{S}(\mathbb{R})$. Show that $Hf \in L^1(\mathbb{R})$ if and only if $\int_{\mathbb{R}} f(y)\,dy = 0$. A hint: Modify the proof of Lemma 3.5 to estimate the asymptotics of $x^2 \cdot Hf(x)$ as $|x| \to \infty$.

## 3.2 The Hilbert transform on $L^2(\mathbb{R})$

In this subsection we extend the Hilbert transform to $L^2(\mathbb{R})$. We begin by computing the Fourier transform of $Hf$.

First, since for any $f \in \mathcal{S}(\mathbb{R})$ we have $Hf \in L^2(\mathbb{R})$ by Corollary 3.6, the Fourier transform $\widehat{Hf}$ is well-defined as a function in $L^2$. Below we compute its precise value.

**Proposition 3.8.** *For any $f \in \mathcal{S}(\mathbb{R})$ we have*

$$\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi) \quad \textit{for a.e. } \xi \in \mathbb{R}. \tag{3.4}$$

To prove this, we start by taking a slightly more abstract point of view. Since the Hilbert transform is linear, and we have the estimate (3.3), we can define a tempered distribution $T_0 \in \mathcal{S}'(\mathbb{R})$ by

$$\langle T_0, f\rangle := -Hf(0) = \lim_{\varepsilon\to 0} \frac{1}{\pi}\int_{|y|>\varepsilon} \frac{f(y)}{y}\,dy.$$

Note that

$$Hf(x) = \langle T_0, f(x - \cdot)\rangle = T_0 * f(x).$$

Taking the Fourier transform (in the sense of distributions), we see that

$$\widehat{Hf} = \widehat{T_0} \cdot \hat{f}, \tag{3.5}$$

where the product is also understood in the sense of distributions: for any $\varphi \in \mathcal{S}(\mathbb{R})$ we have $\langle \widehat{Hf}, \varphi\rangle = \langle \widehat{T_0}, \hat{f}\varphi\rangle$.

As a consequence of (3.5), to prove (3.4) it suffices to show that $\widehat{T_0}$, which *a priori* is just a tempered distribution, is in fact a function, and that $\widehat{T_0}(\xi) = -i\,\mathrm{sgn}(\xi)$.

**Lemma 3.9.** *We have* $\widehat{T_0}(\xi) = -i\,\mathrm{sgn}(\xi)$.

*Proof.* An exercise. Some hints:

(i) Let $K_\varepsilon(y) = \frac{1}{y}\mathbf{1}_{|y|>\varepsilon}$, so that $\langle T_0, f\rangle = \frac{1}{\pi}\lim_{\varepsilon\to 0}\langle K_\varepsilon, f\rangle$, and consider $Q_\varepsilon(y) = \frac{y}{y^2+\varepsilon^2}$. Show that

$$\lim_{\varepsilon\to 0}\,(K_\varepsilon - Q_\varepsilon) = 0 \quad \text{in } \mathcal{S}'(\mathbb{R}).$$

(ii) Using the above, argue that $\widehat{T_0} = \frac{1}{\pi}\lim_{\varepsilon\to 0}\widehat{Q_\varepsilon}$, in the sense of distributions.

(iii) Show that $Q_\varepsilon(x) = \mathcal{F}^{-1}(-\pi i\,\mathrm{sgn}(\xi)e^{-2\pi\varepsilon|\xi|})(x)$. Conclude that $\widehat{T_0}$ is given by a function, and that $\widehat{T_0}(\xi) = -i\,\mathrm{sgn}(\xi)$.

□

As a corollary of Proposition 3.8 and Plancherel's identity, we can define the Hilbert transform of functions in $L^2(\mathbb{R})$.

**Corollary 3.10.** *For any* $f \in \mathcal{S}(\mathbb{R})$ *we have*

$$\|Hf\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}.$$

*Consequently, the Hilbert transform extends to an isometry of* $L^2(\mathbb{R})$. *Moreover, for any* $f \in L^2(\mathbb{R})$ *its Hilbert transform satisfies*

$$\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi).$$

Recall that for $f \in \mathcal{S}(\mathbb{R})$ we have a nice formula for $Hf(x)$ assuming $x \notin \mathrm{supp}\, f$, see (3.1). It is easy to see that essentially the same formula holds for $f \in L^2(\mathbb{R})$.

*Exercise* 3.11. Show that if $f \in L^2(\mathbb{R})$, then for a.e. $x \notin \mathrm{ess\,supp}(f)$

$$Hf(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy.$$

### 3.2.1 Truncated Hilbert transform

In Definition 3.1 we introduced the truncated Hilbert transform

$$H_\varepsilon f(x) = \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy$$

for $f \in \mathcal{S}(\mathbb{R})$. However, the same definition makes sense for $f \in L^p(\mathbb{R})$ for all $1 \le p < \infty$ To see that, we use Hölder's inequality to show that the integral defining $H_\varepsilon f$ converges absolutely:

$$\int_{|x-y|>\varepsilon} \left| \frac{f(y)}{x-y} \right| \, dy \le \|f\|_{L^p} \left\| \frac{\mathbf{1}_{|x-y|>\varepsilon}}{x-y} \right\|_{L^q} < \infty,$$

where $1/p + 1/q = 1$, so that $1 < q \le \infty$.

By the definition of Hilbert transform, we have $Hf(x) = \lim_{\varepsilon\to 0} H_\varepsilon f(x)$ for all $f \in \mathcal{S}(\mathbb{R})$ and $x \in \mathbb{R}$. It is natural to ask for a counterpart of this statement for $f \in L^2(\mathbb{R})$; for example, do we have $H_\varepsilon f \to Hf$ in $L^2$ sense? We are able to show this if we assume that all truncated Hilbert transforms are of strong type $(2,2)$, in a uniform way.

**Proposition 3.12.** *Suppose that there exists a constant $C > 0$ such that*

$$\sup_{\varepsilon>0} \|H_\varepsilon f\|_{L^2(\mathbb{R})} \le C\|f\|_{L^2(\mathbb{R})} \qquad \text{for all } f \in L^2. \tag{3.6}$$

*Then, for every $f \in L^2(\mathbb{R})$ we have $H_\varepsilon f \to Hf$ in $L^2$.*

*Proof.* Let $f_n \in \mathcal{S}(\mathbb{R})$ be such that $f_n \to f$ in $L^2$. Then, $Hf_n \to H_f$ in $L^2$, and we have

$$\|H_\varepsilon f - Hf\|_{L^2} \le \|H_\varepsilon f - H_\varepsilon f_n\|_{L^2} + \|H_\varepsilon f_n - Hf_n\|_{L^2} + \|Hf_n - Hf\|_{L^2} =: I_1 + I_2 + I_3.$$

The term $I_3$ converges to 0 because $Hf_n \to H_f$ in $L^2$, whereas $I_1$ converges to 0 because

$$\|H_\varepsilon f - H_\varepsilon f_n\|_{L^2} = \|H_\varepsilon(f - f_n)\|_{L^2} \overset{(3.6)}{\le} C\|f - f_n\|_{L^2} \xrightarrow{n\to\infty} 0.$$

It remains to estimate $I_2 = \|H_\varepsilon f_n - Hf_n\|_{L^2}$. By (3.2) we have

$$|H_\varepsilon f_n(x) - Hf_n(x)| = \left| \frac{1}{\pi} \int_{-\varepsilon}^{\varepsilon} \frac{f_n(x-y) - f_n(x)}{y}\, dy \right| \le \frac{2\varepsilon}{\pi} \sup_{z\in(x-\varepsilon, x+\varepsilon)} |f_n'(z)|.$$

Set $g_n(x) \coloneqq \sup_{z\in(x-\varepsilon,x+\varepsilon)} |f_n'(z)|$. Since $f_n'$ decays rapidly, we get that $g_n$ also decays rapidly, and so

$$I_2 = \|H_\varepsilon f_n - Hf_n\|_{L^2} \le \frac{2\varepsilon}{\pi} \|g_n\|_{L^2}.$$

Hence, for any $\delta > 0$ we may take $n$ large enough such that $I_1 + I_3 \leq \delta$, and then $\varepsilon > 0$ small enough so that $I_2 \leq \delta$. Then, we have $\|H_\varepsilon f - Hf\|_{L^2} \leq 2\delta$, and taking $\delta \to 0$ concludes the proof. $\square$

The question remains, how to show the estimate (3.6)? We will address this later on when we prove the so-called *Cotlar's inequality* for general singular integral operators.

## 3.3 The Riesz transform

Before moving on to general singular integral operators and their $L^p$-theory, we briefly discuss another important family of operators, the Riesz transforms. They are higher dimensional counterparts of the Hilbert transform.

*Definition* 3.13. For $f \in \mathcal{S}(\mathbb{R}^n)$ and $1 \leq j \leq n$, we define the $j$-th Riesz transform of $f$ as

$$R_j f(x) := \lim_{\varepsilon \to 0} C_n \int_{|x-y|>\varepsilon} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy,$$

where $C_n = \Gamma(\frac{n+1}{2})/\pi^{\frac{n+1}{2}}$.

As in the case of the Hilbert transform, there is a simple formula for the Fourier transform of $R_j f$.

**Proposition 3.14.** *For any $f \in \mathcal{S}(\mathbb{R}^n)$*

$$\widehat{R_j f}(\xi) = -i \frac{\xi_j}{|\xi|} \hat{f}(\xi). \tag{3.7}$$

The proof is similar to that of Proposition 3.4, although there are additional difficulties. The interested reader can find the full proof e.g. in [Gra14a, Proposition 5.1.14].

As an immediate corollary of (3.7), we get that for all $f \in \mathcal{S}(\mathbb{R}^n)$

$$\|R_j f\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}, \tag{3.8}$$

and we may extend the Riesz transforms to $L^2(\mathbb{R}^n)$.

Finally, we give a simple application of (3.8), which also motivates the study of $L^p$-bounds for the Riesz transforms for $1 < p < \infty$.

**Proposition 3.15.** *For $f \in \mathcal{S}(\mathbb{R}^n)$ and $1 \leq j, k \leq n$ we have*

$$\partial_j \partial_k f = -R_j R_k \Delta f. \tag{3.9}$$

*In consequence, for any $1 < p < \infty$ such that the bound $\|R_j f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}$ holds for all $1 \leq j \leq n$, we have*

$$\|\partial_j \partial_k f\|_{L^p(\mathbb{R}^n)} \leq (C_p)^2 \|\Delta f\|_{L^p(\mathbb{R}^n)}. \tag{3.10}$$

*Proof.* By taking the Fourier transform of $\partial_j \partial_k f$ we get

$$\begin{aligned}\mathcal{F}(\partial_j \partial_k f)(\xi) &= (2\pi i \xi_j)(2\pi i \xi_k)\hat{f}(\xi) \\ &= -\left(-i\frac{\xi_j}{|\xi|}\right)\left(-i\frac{\xi_k}{|\xi|}\right)(-4\pi^2|\xi|^2)\hat{f}(\xi) \\ &= -\mathcal{F}(R_j R_k \Delta f)(\xi).\end{aligned}$$

Taking the inverse Fourier transform finishes the proof of identity (3.9). The estimate (3.10) follows immediately. □

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