

**SINGULAR INTEGRAL OPERATORS**  
**EXERCISE II (7.11.2023)**

**Exercise 1** (1 point). Show that for every Hölder continuous  $\Omega : \mathbb{S}^{n-1} \rightarrow \mathbb{C}$  the kernel  $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) : x \in \mathbb{R}^n\} \rightarrow \mathbb{C}$  defined by

$$K(x, y) = \frac{\Omega\left(\frac{x-y}{|x-y|}\right)}{|x-y|^n}$$

is a standard kernel.

**Exercise 2** (1 point). Prove that if  $A$  is Lipschitz, then the Cauchy kernel

$$K(x, y) = \frac{1}{x - y + i(A(x) - A(y))}$$

is a standard kernel with  $\delta = 1$ .

**Exercise 3** (2 points). If  $T$  is a Calderón-Zygmund operator such that it is associated with two kernels  $K_1$  and  $K_2$ , that is,

$$Tf(x) = \int K_1(x, y)f(y) dy = \int K_2(x, y)f(y) dy \quad \text{for } x \notin \text{supp } f,$$

then  $K_1 = K_2$  a.e.

*Hint:* Assume that the claim is false. You should find a positive measure set  $E \subset \mathbb{R}^n$  and a point  $x \notin E$  such that  $K_1(x, y) - K_2(x, y)$  has a fixed sign for  $y \in E$ .

Recall that  $\mathcal{D}(\mathbb{R}^n)$  denotes the family of dyadic cubes. The notation  $A \lesssim B$  stands for “there exists a dimensional constant  $C \geq 1$  such that  $A \leq CB$ ,” and  $A \sim B$  means  $A \lesssim B \lesssim A$ . Given  $Q \in \mathcal{D}(\mathbb{R}^n)$  we write  $CQ$  to denote the cube with the same center as  $Q$  and with sidelength  $C\ell(Q)$ .

**Exercise 4** (2 point). Suppose that  $\Omega \subsetneq \mathbb{R}^n$  is an open set. Let  $\mathcal{W} \subset \mathcal{D}(\mathbb{R}^n)$  be the family of maximal<sup>1</sup> cubes contained in  $\Omega$  and satisfying  $10Q \cap \Omega^c = \emptyset$ . Prove that

- (i) the cubes in  $\mathcal{W}$  are pairwise disjoint, and  $\bigcup_{Q \in \mathcal{W}} Q = \Omega$ ,
- (ii) for every  $Q \in \mathcal{W}$  we have  $\ell(Q) \sim \text{dist}(Q, \Omega^c)$ ,
- (iii) for every  $P, Q \in \mathcal{W}$  with  $3P \cap 3Q \neq \emptyset$  we have  $\ell(P) \sim \ell(Q)$ .
- (iv) for every  $Q \in \mathcal{W}$  we have  $\#\{P \in \mathcal{W} : 3P \cap 3Q \neq \emptyset\} \lesssim 1$ .

The family  $\mathcal{W}$  is called *the Whitney decomposition* of  $\Omega$ , and it has many applications in analysis.

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<sup>1</sup>maximal with respect to inclusion