Singular Integral Operators

Exercise 2 - 07.11.2023

Exercise 1 (1 point). Show that for every Hölder continuous $\Omega : \mathbb{S}^{n-1} \to \mathbb{C}$ the kernel $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\} \to \mathbb{C}$ defined by

$$K(x,y) = \frac{\Omega\left(\frac{x-y}{|x-y|}\right)}{|x-y|^n}$$

is a standard kernel.

Solution. We have to check that the size and smoothness conditions hold. We assume that $\Omega \in C^\alpha(\mathbb{S}^{n-1}; \mathbb{C})$.

Size condition) Trivial: for $x, y \in \mathbb{R}^n$, $x \neq y$, it holds

$$|K(x,y)| \leq \frac{\left|\Omega\left(\frac{x-y}{|x-y|}\right)\right|}{|x-y|^n} \leq \frac{\|\Omega\|_\infty}{|x-y|^n}.$$

Smoothness conditions) K is of convolution-type, so it suffices to prove that

$$(*)\quad \boxed{|K(x,y) - K(x,y')| \leq C_1 \cdot \frac{|y-y'|^\alpha}{|y-y|^{n+\alpha}}, \quad \text{for } |x-y| > 2|y-y'|,}$$

where $C_1 > 0$ is to be determined and possibly depend on Ω, n, α. Notation: we assume that $C_\alpha > 0$ is such that

$$|\Omega(\xi) - \Omega(\xi')| \leq C_\alpha |\xi - \xi'|^\alpha, \text{ for } \xi, \xi' \in \mathbb{S}^{n-1}.$$

Now, for $|x-y| > 2|y-y'|$ we write:

$$|K(x,y) - K(x,y')| \overset{\text{def}}{=} \left| \frac{\Omega\left(\frac{x-y}{|x-y|}\right)}{|x-y|^n} - \frac{\Omega\left(\frac{x-y'}{x-y'}\right)}{|x-y'|^n} \right|$$

$$\leq \left| \frac{\Omega\left(\frac{x-y}{|x-y|}\right)}{|x-y|^n} - \frac{\Omega\left(\frac{x-y'}{|x-y'|}\right)}{|x-y|^n} \right| + \left| \frac{\Omega\left(\frac{x-y'}{|x-y'|}\right)}{|x-y|^n} - \frac{\Omega\left(\frac{x-y'}{|x-y'|}\right)}{|x-y'|^n} \right|$$

$=: (1) + (2)$.

(1): We first observe that

$$\left|\frac{x-y}{|x-y|}-\frac{x-y'}{|x-y'|}\right| \le \left|\frac{x-y}{|x-y|}-\frac{x-y'}{|x-y|}\right| + \left|\frac{x-y'}{|x-y|}-\frac{x-y'}{|x-y'|}\right|$$

$$\le \frac{1}{|x-y|}|y-y'| + \frac{\cancel{|x-y'|}}{|x-y|}\frac{1}{\cancel{|x-y'|}}\Big||x-y'|-|x-y|\Big|$$

$$\le 2\,\frac{|y-y'|}{|x-y|},$$

$(\star)$

which yields

$$(1) \le \frac{1}{|x-y|^n}\left|\Omega\left(\frac{x-y}{|x-y|}\right)-\Omega\left(\frac{x-y'}{|x-y'|}\right)\right| \le \frac{C_\alpha}{|x-y|^n}\left|\frac{x-y}{|x-y|}-\frac{x-y'}{|x-y'|}\right|^\alpha$$

$$\overset{(\star)}{\le} \frac{2C_\alpha}{|x-y|^n}\frac{|y-y'|^\alpha}{|x-y|^\alpha} = 2C_\alpha\frac{|y-y'|^\alpha}{|x-y|^{n+\alpha}}.$$

(2). First notice that $|x-y'| \le |x-y|+|y-y'| < \frac{3}{2}|x-y|$

and $\Big||x-y|-|x-y'|\Big| \le |y-y'|$,

so

$$\Big||x-y|^n-|x-y'|^n\Big| = \Big||x-y|-|x-y'|\Big|\,\Big||x-y|^{n-1}+\dots+|x-y'|^{n-1}\Big| \lesssim_n |y-y'|\,|x-y|^{n-1}.$$

Hence

$$|2| \le \|\Omega\|_\infty\left|\frac{1}{|x-y|^n}-\frac{1}{|x-y'|^n}\right| = \|\Omega\|_\infty\frac{\big||x-y'|^n-|x-y|^n\big|}{|x-y|^n|x-y'|^n} \lesssim \|\Omega\|_\infty\frac{|y-y'|\,|x-y|^{n-1}}{|x-y|^n|x-y'|^n}$$

$$\underset{(|x-y|>2|y-y'|)}{\lesssim} \|\Omega\|_\infty\frac{1}{|x-y|^n}\Bigg(\underbrace{\frac{|y-y'|}{|x-y|}}_{<\frac{1}{2}}\Bigg) \le \|\Omega\|_\infty\frac{1}{|x-y|^n}\frac{|y-y'|^\alpha}{|x-y|^\alpha}. \qquad \square$$

Exercise 2 (1 point). Prove that if A is Lipschitz, then the Cauchy kernel

$$K(x, y) = \frac{1}{x - y + i(A(x) - A(y))}$$

is a standard kernel with $\delta = 1$.

Solution. A is Lipschitz, so $\exists \ L > 0$ such that

$$|A(x) - A(y)| \leq L|x - y| \quad \forall \ x, y.$$

Size condition) Trivial:

$$|K(x,y)| = \frac{1}{|x-y+i(A(x)-A(y))|} \leq \frac{1}{|\operatorname{Re}(\#)|} = \frac{1}{|x-y|}.$$

Smoothness conditions) Assume $|x-y| > 2|y-y'|$. Then

$$|K(x,y) - K(x,y')| = \left| \frac{1}{x-y+i(A(x)-A(y))} - \frac{1}{x-y'+i(A(x)-A(y))} \right|$$

$$= \frac{|y-y' - i(A(y')-A(y))|}{|((x-y)+i(A(x)-A(y')))((x-y')+i(A(x)-A(y')))|} =: \boxed{\frac{|N(x,y)|}{|D(x,y)|}}.$$

We estimate N and D separately.

$$\boxed{|N(x,y)|} \leq |y-y'| + |A(y')-A(y)| \underset{\substack{\uparrow \\ A \text{ Lipschitz}}}{\leq} (L+1)|y-y'|.$$

Now, we estimate the denominator:

$$\boxed{|D(x,y)|^{-1}} = \frac{1}{|[(x-y)(x-y') - (A(x)-A(y))(A(x)-A(y'))] + i[(x-y)(A(x)-A(y')) + (x-y')(A(x)-A(y))]|}$$

$$\leq \frac{|x-y||x-y'| + |A(x)-A(y)||A(x)-A(y')| + |x-y||A(x)-A(y')| + |x-y'||A(x)-A(y)|}{|(x-y)(x-y') - (A(x)-A(y))(A(x)-A(y'))|^2 + |(x-y)(A(x)-A(y')) + (x-y')(A(x)-A(y))|^2}$$

$|x-y'| \leq \frac{3}{2}|x-y|$

$$\leq \frac{\frac{3}{2}|x-y|^2 + \frac{3}{2}L|x-y|^2 + \frac{3}{2}L|x-y|^2 + \frac{3}{2}L|x-y|^2}{|x-y|^2 |x-y'|^2} \lesssim_L \frac{1}{|x-y|^2}.$$

This proves the first smoothness condition, and the second is analogous.

$\square$

Exercise 3 (2 points). If T is a Calderón-Zygmund operator such that it is associated with two kernels K_1 and K_2, that is, for all $f \in L^2(\mathbb{R}^n)$ with compact support

$$Tf(x) = \int K_1(x,y)f(y)\,dy = \int K_2(x,y)f(y)\,dy \qquad \text{for } x \notin \operatorname{supp} f,$$

then $K_1 = K_2$ a.e. $(*)$

Hint: Assume that the claim is false. You should find a positive measure set $E \subset \mathbb{R}^n$ and a point $x \notin E$ such that $K_1(x,y) - K_2(x,y)$ has a fixed sign for $y \in E$.

Solution. We assume without loss of generality that K_1, K_2 are real-valued (otherwise we argue with the real or imaginary parts).

Let $\tilde{E} := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y,\ K_1(x,y) - K_2(x,y) \neq 0\}$.

We argue by contradiction, and assume that $(*)$ does not hold. This implies that $|\tilde{E}| > 0$. In particular, at least one between

$$\tilde{E}^{\pm} := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y,\ \operatorname{sgn}(K_1(x,y) - K_2(x,y)) = \pm 1\}$$

has positive measure. Without loss of generality, we assume that $\mathcal{L}^{2n}(\tilde{E}^+) > 0$ and, for $x \in \mathbb{R}^n$, we denote $\tilde{E}^+_x := \{y \in \mathbb{R}^n : (x,y) \in \tilde{E}^+\}$. (Lebesgue measure)

Fubini's theorem implies that $\exists\ x$ s.t. $\mathcal{L}^n(\tilde{E}^+_x) > 0$.

Furthermore, $\exists\ 0 < r < R$ s.t. $(\tilde{E}^+_x \cap B(x,R)) \setminus B(x,r) =: E$ has positive measure. Observe that $\bar{E}$ is compact and $x \notin \bar{E}$.

Hence, for $f := \chi_E$ we have that

$$\int K_1(x,y) f(y)\,dy > \int K_2(x,y) f(y)\,dy,$$

which yields a contradiction. $\square$

Recall that $\mathcal{D}(\mathbb{R}^n)$ denotes the family of dyadic cubes. The notation $A \lesssim B$ stands for "there exists a dimensional constant $C \geqslant 1$ such that $A \leqslant CB$," and $A \sim B$ means $A \lesssim B \lesssim A$. Given $Q \in \mathcal{D}(\mathbb{R}^n)$ we write CQ to denote the cube with the same center as Q and with sidelength $C\ell(Q)$.

Exercise 4 (2 point). Suppose that $\Omega \subsetneq \mathbb{R}^n$ is an open set. Let $\mathcal{W} \subset \mathcal{D}(\mathbb{R}^n)$ be the family of maximal[1] cubes contained in Ω and satisfying $10Q \cap \Omega^c = \varnothing$. Prove that

(i) the cubes in $\mathcal{W}$ are pairwise disjoint, and $\bigcup_{Q\in\mathcal{W}} Q = \Omega$,
(ii) for every $Q \in \mathcal{W}$ we have $\ell(Q) \sim \operatorname{dist}(Q, \Omega^c)$,
(iii) for every $P, Q \in \mathcal{W}$ with $3P \cap 3Q \neq \varnothing$ we have $\ell(P) \sim \ell(Q)$.
(iv) for every $Q \in \mathcal{W}$ we have $\#\{P \in \mathcal{W} : 3P \cap 3Q \neq \varnothing\} \lesssim 1$.

The family $\mathcal{W}$ is called *the Whitney decomposition* of Ω, and it has many applications in analysis.

Solution. For $Q \in \mathcal{D}(\mathbb{R}^n)$, we denote $\mathcal{D}(Q) := \{Q' \in \mathcal{D}(\mathbb{R}^n) : Q' \subseteq Q\}$.

(i) We assume that $Q, Q' \in \mathcal{W}$ are such that $Q \cap Q' \neq \varnothing$. Then, by the properties of $\mathcal{D}(\mathbb{R}^n)$, either $Q \subseteq Q'$ or $Q' \subseteq Q$. By maximality of cubes in $\mathcal{W}$, this implies $Q = Q'$.

We are left with the proof of the identity $\boxed{\bigcup_{Q\in\mathcal{W}} Q = \Omega.}$ $(*)$

The inclusion "$\subseteq$" is trivial by definition of Q. Conversely, take $x \in \Omega$.

The set Ω is open, so $\exists\ r > 0$ s.t. $B(x,r) \subset \Omega$. In particular, there exists $\tilde{Q}_x \in \mathcal{D}(\mathbb{R}^n)$ such that $x \in \tilde{Q}_x$ and $10\tilde{Q}_x \subset 100\tilde{Q}_x \subset \Omega$.

In particular, $10\tilde{Q}_x \cap \Omega^c = \varnothing$. Hence, $\exists\ Q_x \in \mathcal{W}$ (maximal) such that $x \in Q_x$, which proves the inclusion "$\supseteq$" in $(*)$.

(ii) Let $Q \in \mathcal{W}$.

On the one hand, since $10Q \cap \Omega^c = \varnothing$, it holds:

$\operatorname{dist}(Q, \Omega^c) \geq \operatorname{dist}(Q, \partial(10Q)) \approx \ell(Q)$.

Conversely, we argue by contradiction and assume that for every j $\exists\ Q_j$ s.t.

$$\ell(Q_j) < \frac{1}{j} \operatorname{dist}(Q_j, \Omega^c). \qquad (**)$$

Let $\tilde{Q}_j$ be the <u>dyadic parent</u> of Q_j (i.e. $\tilde{Q}_j \in \mathcal{D}(\mathbb{R}^n)$ is the unique cube s.t. $\ell(\tilde{Q}_j) = 2\ell(Q_j)$ and $Q_j \subset \tilde{Q}_j$). Observe that

$$\mathrm{dist}(\partial(10\tilde{Q}_j), Q_j) \leq \hat{c}\,\ell(Q_j)$$

for some $\hat{c} > 0$. Hence, for $j > \hat{c}$, we have

$$\mathrm{dist}(10\tilde{Q}_j, \Omega^c) \geq \mathrm{dist}(Q_j, \Omega^c) - \mathrm{dist}(\partial(10\tilde{Q}_j), Q_j)$$

$$\underset{(**)}{\geq} (j - \hat{c})\,\ell(Q_j) > 0$$

In particular, $10\tilde{Q}_j \cap \Omega^c = \emptyset$, which contradicts the maximality of $Q_j \in W$.

(iii) Let $P, Q \in W$ be such that $3P \cap 3Q \neq \emptyset$.

If $\ell(Q) = \ell(P)$, there is nothing to prove. Assume w.l.o.g. that $\ell(P) > \ell(Q)$.

All we have to prove is that $\ell(P) \lesssim \ell(Q)$. It is enough to observe that $3P \cap 3Q \neq \emptyset$ & $\ell(P) > \ell(Q) \Rightarrow 9P \subset Q$. Hence

$$\ell(Q) \underset{(ii)}{\sim} \mathrm{dist}(Q, \Omega^c) \geq \mathrm{dist}(9P, \Omega^c) \sim \ell(P).$$

(iv) Let $Q \in W$. If $P \in W$ is such that $3P \cap 3Q \neq 0$, by (iii) $\exists\ \hat{c} = \hat{c}(n) > 1$ such that

$$\hat{c}^{-1}\ell(P) \leq \ell(Q) \leq \hat{c}\,\ell(P).$$

Hence, $P \subseteq \tilde{c}\,Q$ for some $\tilde{c} = \tilde{c}(n) > 1$.

This finishes the proof, because:

$$\{P \in W : 3P \cap 3Q \neq \emptyset\}$$
$$\subseteq \{P \in \mathcal{D}(\mathbb{R}^n) : P \subseteq \tilde{c}\,Q \text{ and } \hat{c}^{-1}\ell(P) \leq \ell(Q) \leq \hat{c}\,\ell(P)\} =: \mathcal{A}_Q,$$

and it is easy to see that $\#\mathcal{A}_Q \lesssim_n 1$. $\square$