Linear Algebra & Geometry

why is linear algebra useful in computer vision?

References:

- Any book on linear algebra!
- [HZ] - chapters 2, 4

Some of the slides in this lecture are courtesy to Prof. Octavia I. Camps, Penn State University

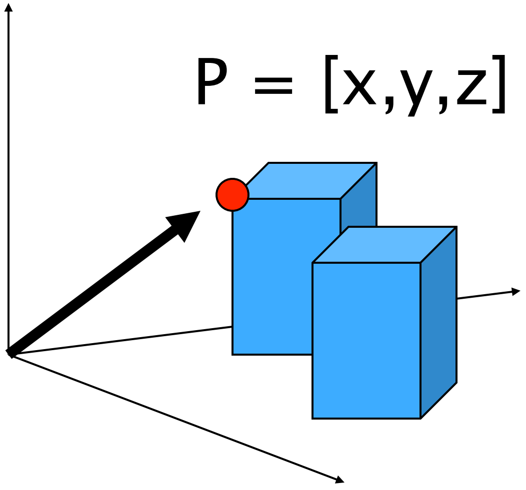
Why is linear algebra useful in computer vision?

- Representation
  - 3D points in the scene
  - 2D points in the image
- Coordinates will be used to
  - Perform geometrical transformations
  - Associate 3D with 2D points
- Images are matrices of numbers
  - Find properties of these numbers

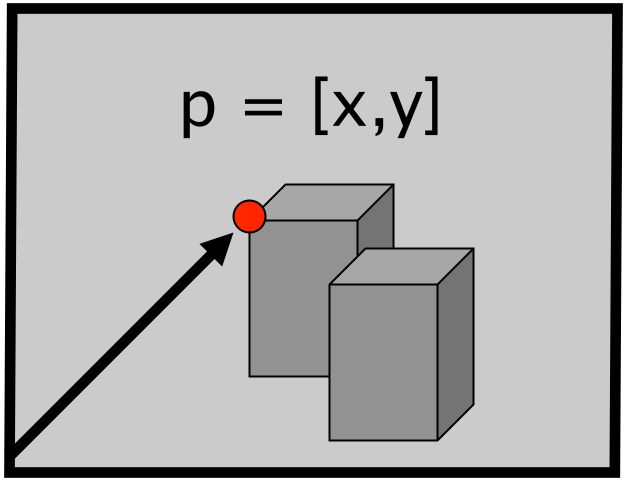
Agenda

1. Basics definitions and properties
2. Geometrical transformations
3. SVD and its applications

Vectors (i.e., 2D or 3D vectors)



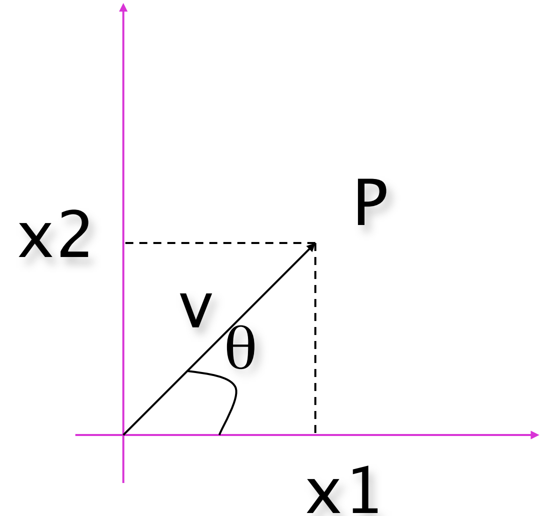
3D world



Image

Vectors (i.e., 2D vectors)

$$\mathbf{v} = (x_1, x_2)$$



Magnitude: $\|\mathbf{v}\| = \sqrt{x_1^{\,2} + x_2^{\,2}}$

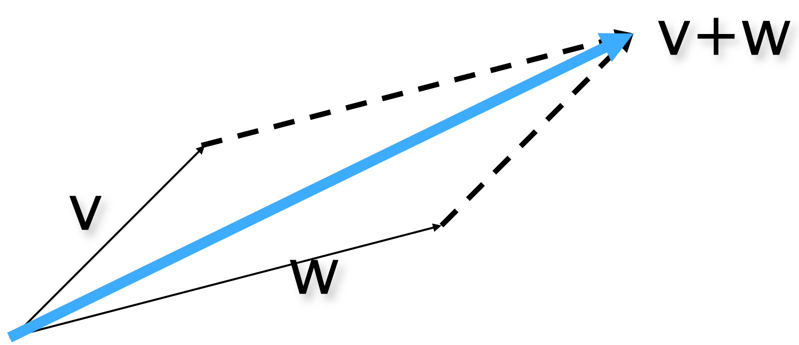
If $\|\mathbf{v}\| = 1$, $\mathbf{v}$ Is a UNIT vector

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{x_1}{\|\mathbf{v}\|}, \frac{x_2}{\|\mathbf{v}\|}\right)$$ Is a unit vector

Orientation: $\theta = \tan^{-1}\left(\frac{x_2}{x_1}\right)$

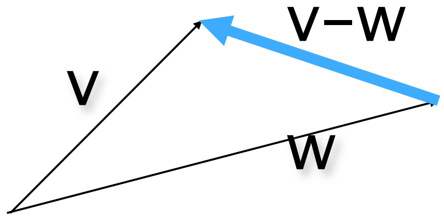
Vector Addition

$$\mathbf{v} + \mathbf{w} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$



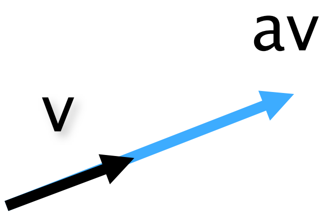
Vector Subtraction

$$\mathbf{v} - \mathbf{w} = (x_1, x_2) - (y_1, y_2) = (x_1 - y_1, x_2 - y_2)$$

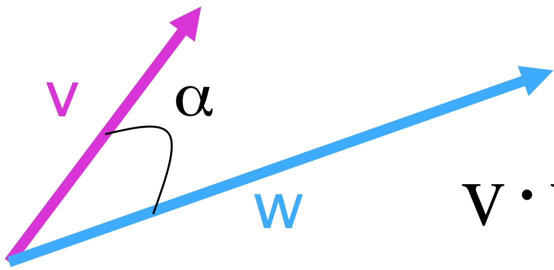


Scalar Product

$$a\mathbf{v} = a(x_1, x_2) = (ax_1, ax_2)$$



Inner (dot) Product



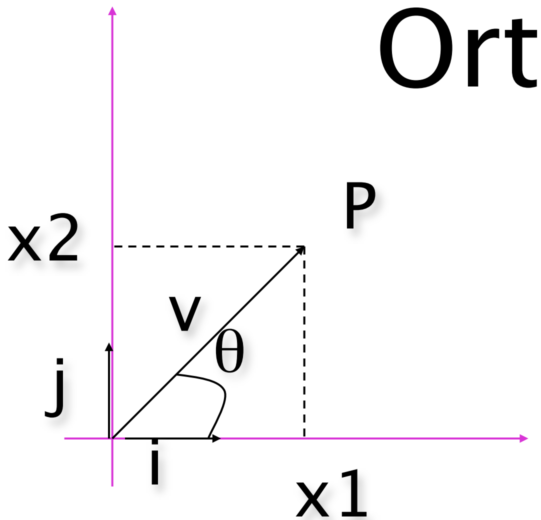
$$v \cdot w = (x_1, x_2) \cdot (y_1, y_2) = x_1 y_1 + x_2 y_2$$

The inner product is a SCALAR!

$$v \cdot w = (x_1, x_2) \cdot (y_1, y_2) = \| v \| \cdot \| w \| \cos\alpha$$

$$\text{if} \quad v \perp w, \quad v \cdot w = ? \; = 0$$

Orthonormal Basis



$$\mathbf{i} = (1,0) \qquad \|\mathbf{i}\| = 1$$

$$\mathbf{j} = (0,1) \qquad \|\mathbf{j}\| = 1$$

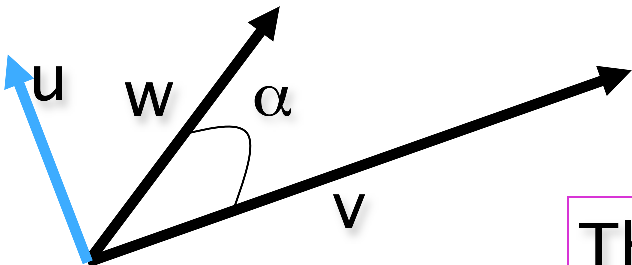
$$\mathbf{i} \cdot \mathbf{j} = 0$$

$$\mathbf{v} = (x_1, x_2) \qquad \mathbf{v} = \mathrm{x}_1\mathbf{i} + \mathrm{x}_2\mathbf{j}$$

$$\mathbf{v} \cdot \mathbf{i} = ? \;\; = (\mathrm{x}_1\mathbf{i} + \mathrm{x}_2\mathbf{j}) \cdot \mathbf{i} = \mathrm{x}_1 1 + \mathrm{x}_2 0 = \mathrm{x}_1$$

$$\mathbf{v} \cdot \mathbf{j} = (\mathrm{x}_1\mathbf{i} + \mathrm{x}_2\mathbf{j}) \cdot \mathbf{j} = \mathrm{x}_1.0 + \mathrm{x}_2.1 = \mathrm{x}_2$$

Vector (cross) Product



$$u = v \times w$$

The cross product is a VECTOR!

$$\text{Magnitude:} \|u\| = \|v \times w\| = \|v\| \|w\| \sin \alpha$$

Orientation:

$$u \perp v \Rightarrow u \cdot v = (v \times w) \cdot v = 0$$

$$u \perp w \Rightarrow u \cdot w = (v \times w) \cdot w = 0$$

if $\quad v \,//\, w\,? \qquad \rightarrow u = 0$

Vector Product Computation

$\mathbf{i} = (1,0,0)$ $\quad \|\mathbf{i}\| = 1$ $\quad \mathbf{i} = \mathbf{j} \times \mathbf{k}$

$\mathbf{j} = (0,1,0)$ $\quad \|\mathbf{j}\| = 1$ $\quad \mathbf{j} = \mathbf{k} \times \mathbf{i}$

$\mathbf{k} = (0,0,1)$ $\quad \|\mathbf{k}\| = 1$ $\quad \mathbf{k} = \mathbf{i} \times \mathbf{j}$

$$\mathbf{u} = \mathbf{v} \times \mathbf{w} = (x_1, x_2, x_3) \times (y_1, y_2, y_3)$$

$$= (x_2 y_3 - x_3 y_2)\mathbf{i} + (x_3 y_1 - x_1 y_3)\mathbf{j} + (x_1 y_2 - x_2 y_1)\mathbf{k}$$

Matrices

$$A_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & \boxed{a_{nm}} \end{bmatrix} \longleftrightarrow$$

Pixel's intensity value

Sum: $C_{n\times m} = A_{n\times m} + B_{n\times m}$ $\quad c_{ij} = a_{ij} + b_{ij}$

A and B must have the same dimensions!

Example: $\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 4 & 6 \end{bmatrix}$

Matrices

$$A_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} \quad \mathbf{a}_{\mathrm{i}}$$

$$B_{m\times p} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mp} \end{bmatrix}$$

$$\mathbf{b}_{\mathrm{j}}$$

Product:

$$C_{n\times p} = A_{n\times \boxed{m}} B_{\boxed{m}\times p}$$

$$\mathrm{c}_{\mathrm{ij}} = \mathbf{a}_{\mathrm{i}} \cdot \mathbf{b}_{\mathrm{j}} = \sum_{\mathrm{k}=1}^{\mathrm{m}} \mathrm{a}_{\mathrm{ik}} \mathrm{b}_{\mathrm{kj}}$$

A and B must have compatible dimensions!

$$A_{n\times n} B_{n\times n} \neq B_{n\times n} A_{n\times n}$$

Matrix Transpose

Definition:

$$\mathbf{C}_{m\times n} = \mathbf{A}^T_{n\times m}$$
$$c_{ij} = a_{ji}$$

Identities:

$$(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$$
$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$

If $\mathbf{A} = \mathbf{A}^T$, then $\mathbf{A}$ is *symmetric*

Matrix Determinant

Useful value computed from the elements of a *square* matrix **A**

$$\det \begin{bmatrix} a_{11} \end{bmatrix} = a_{11}$$

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21}$$

Matrix Inverse

Does not exist for all matrices, necessary (but not sufficient) that the matrix is square

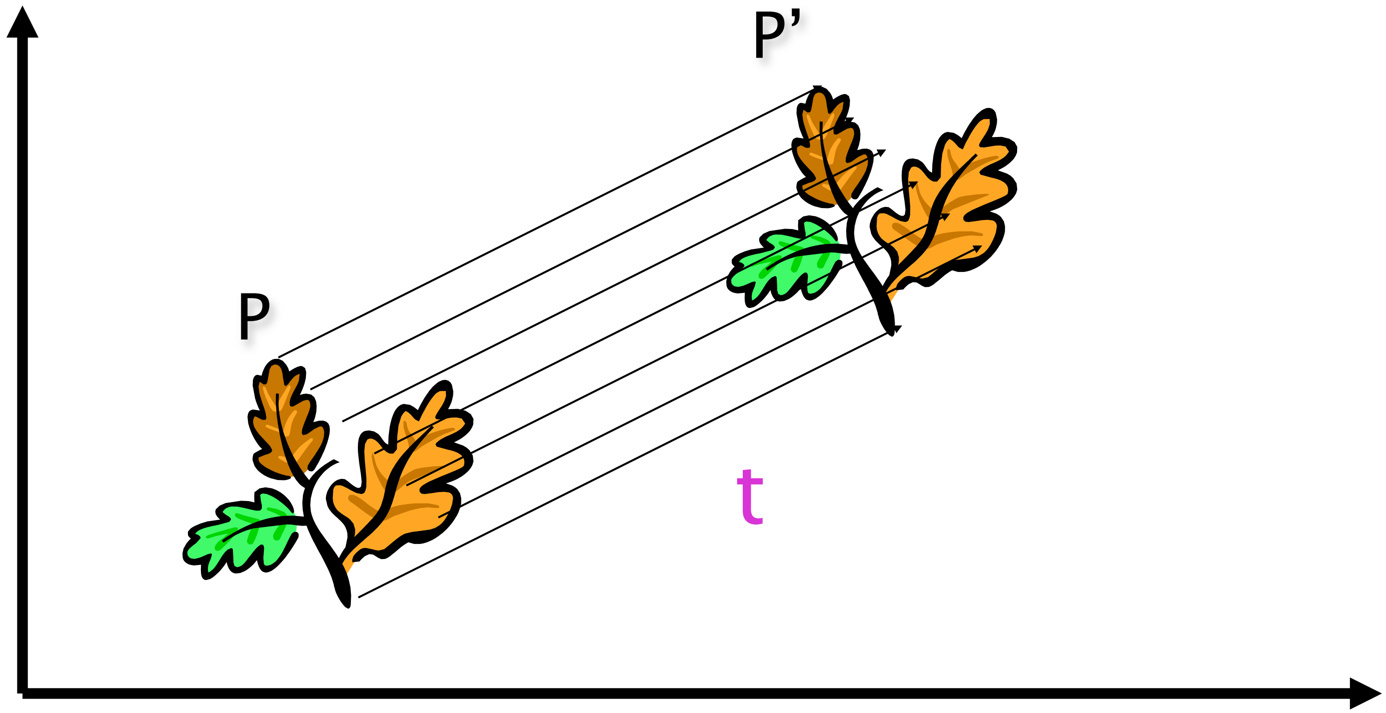
$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}, \det \mathbf{A} \neq 0$$

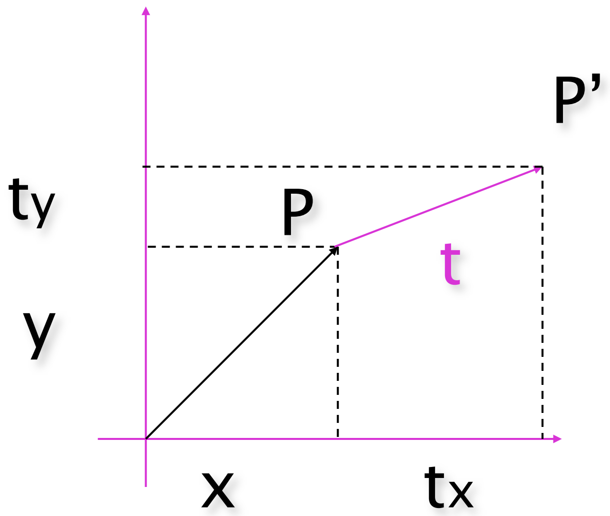
If $\det \mathbf{A} = 0$, $\mathbf{A}$ does not have an inverse.

2D Geometrical Transformations

2D Translation



2D Translation Equation

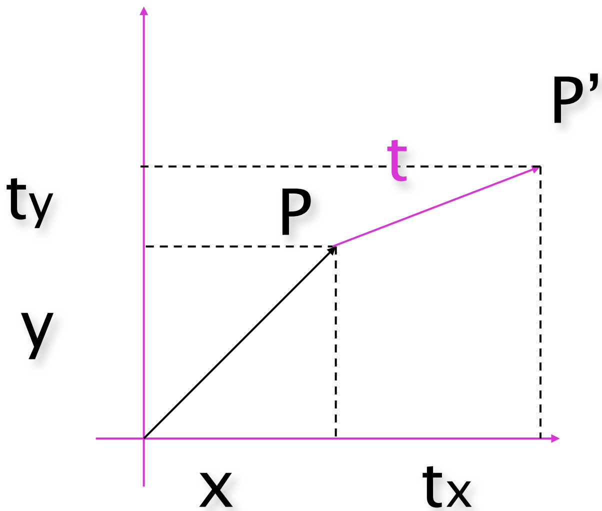


$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

$$\mathbf{P}' = \mathbf{P} + \mathbf{t} = (\mathrm{x} + \mathrm{t}_\mathrm{x}, \mathrm{y} + \mathrm{t}_\mathrm{y})$$

2D Translation using Matrices



$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

- Multiply the coordinates by a non-zero scalar and add an extra coordinate equal to that scalar. For example,

$$(x, y) \rightarrow (x \cdot z, y \cdot z, z) \quad z \neq 0$$

$$(x, y, z) \rightarrow (x \cdot w, y \cdot w, z \cdot w, w) \quad w \neq 0$$

Back to Cartesian Coordinates:

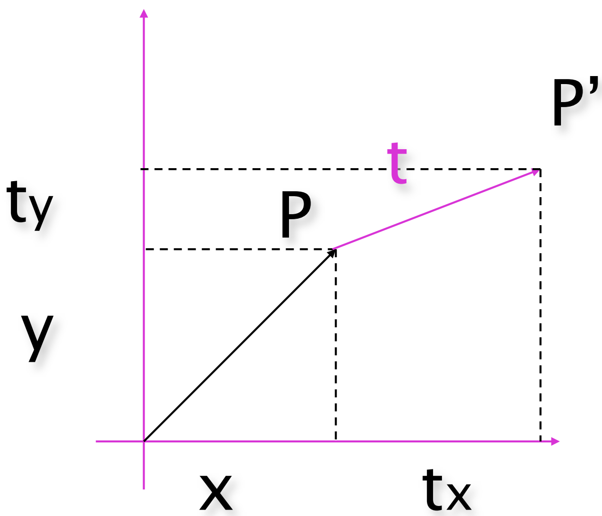
- Divide by the last coordinate and eliminate it. For example,

$$(x, y, z) \quad z \neq 0 \rightarrow (x/z, y/z)$$

$$(x, y, z, w) \quad w \neq 0 \rightarrow (x/w, y/w, z/w)$$

- NOTE: in our example the scalar was 1

2D Translation using Homogeneous Coordinates



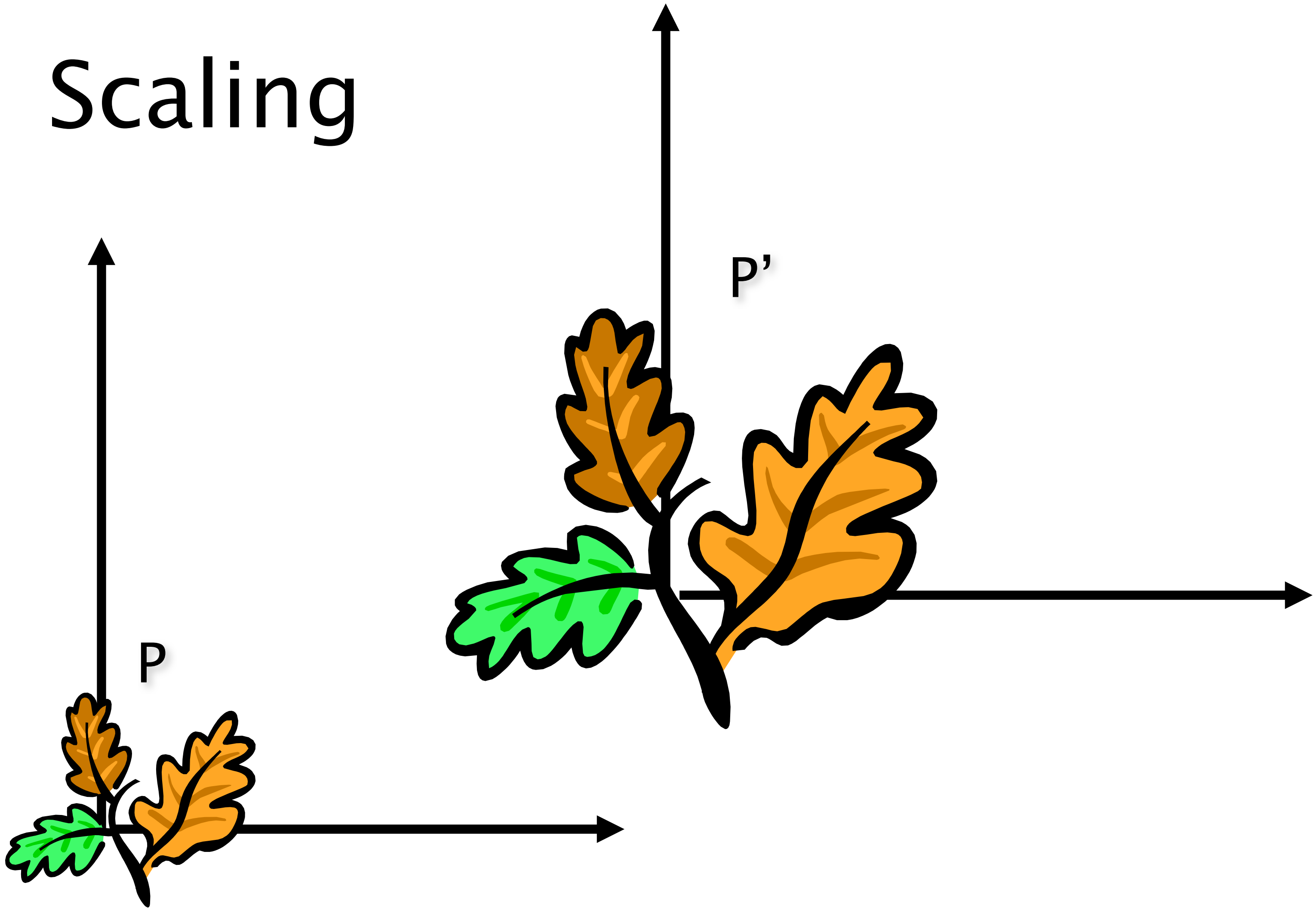
$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{t} = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

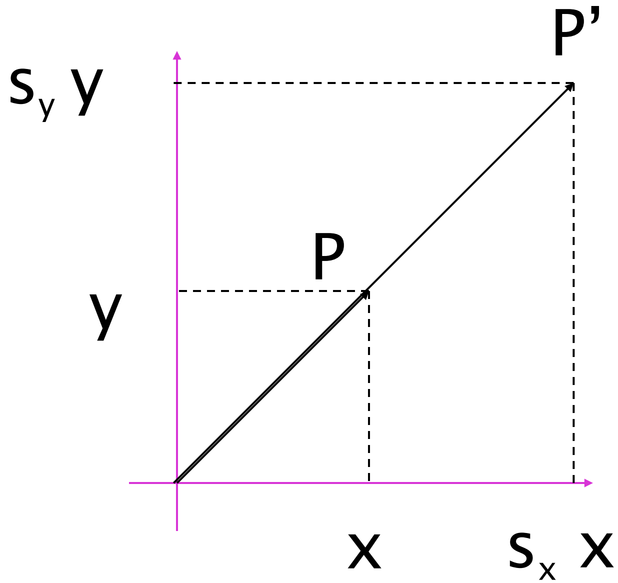
$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} & & \text{t} \end{array} \\ \begin{bmatrix} 1 & 0 & \boxed{t_x} \\ 0 & 1 & \boxed{t_y} \\ 0 & 0 & 1 \end{bmatrix} \end{array} \cdot \overset{\text{P}}{\boxed{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}}$$

$$= \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ 0 & 1 \end{bmatrix} \cdot \mathbf{P} = \mathbf{T} \cdot \mathbf{P}$$

Scaling



Scaling Equation



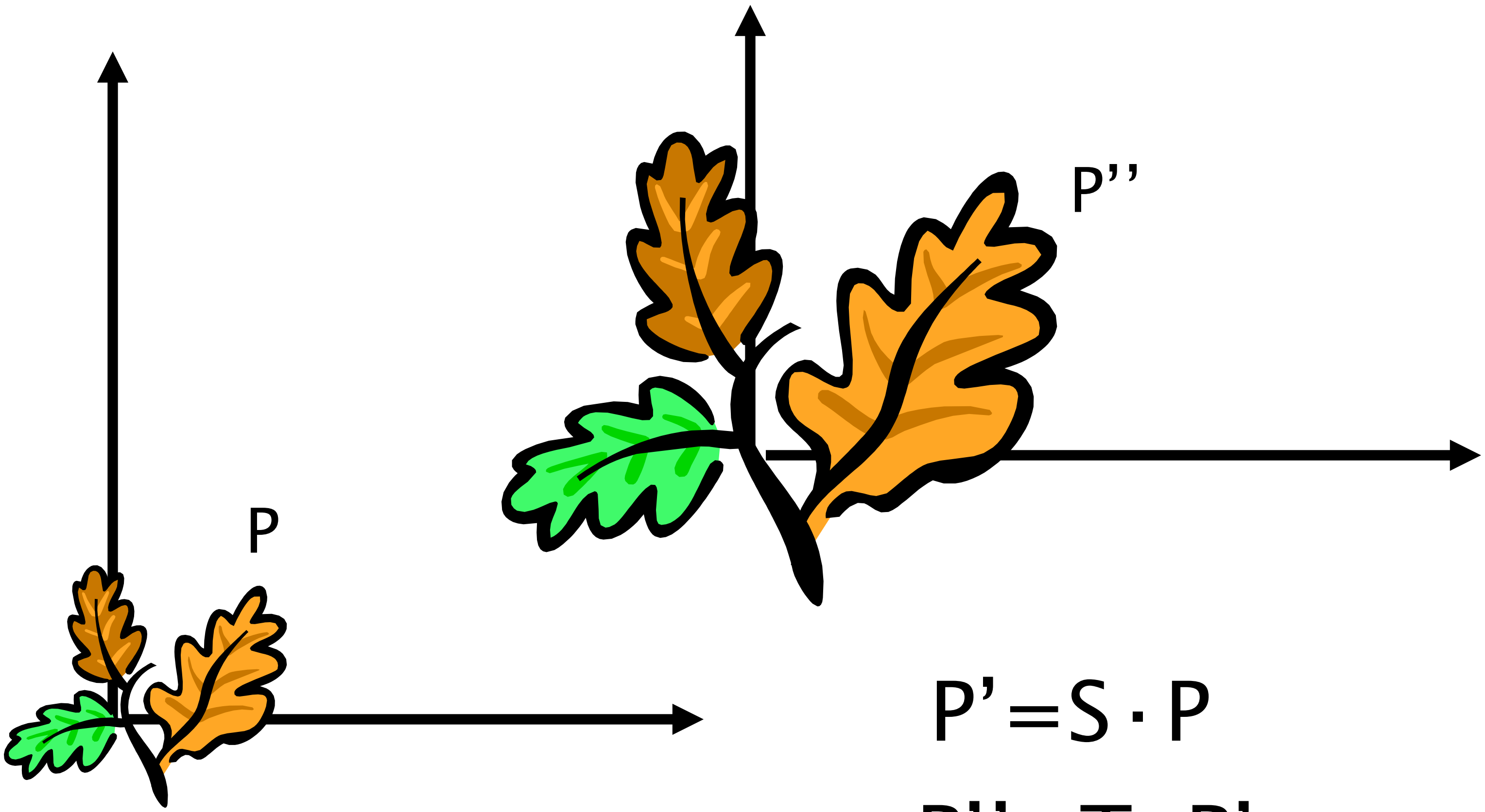
$$\mathbf{P} = (\mathrm{x}, \mathrm{y}) \rightarrow \mathbf{P'} = (\mathrm{s_x x}, \mathrm{s_y y})$$

$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{P'} = (s_x x, s_y y) \rightarrow (s_x x, s_y y, 1)$$

$$\mathbf{P'} \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S'} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \cdot \mathbf{P} = \mathbf{S} \cdot \mathbf{P}$$

Scaling & Translating



$$P' = S \cdot P$$

$$P'' = T \cdot P'$$

$$P'' = T \cdot P' = T \cdot (S \cdot P) = (T \cdot S) \cdot P = A \cdot P$$

Scaling & Translating

$$\mathbf{P''} = \mathbf{T} \cdot \mathbf{S} \cdot \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix}}_{\text{A}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} S & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}$$

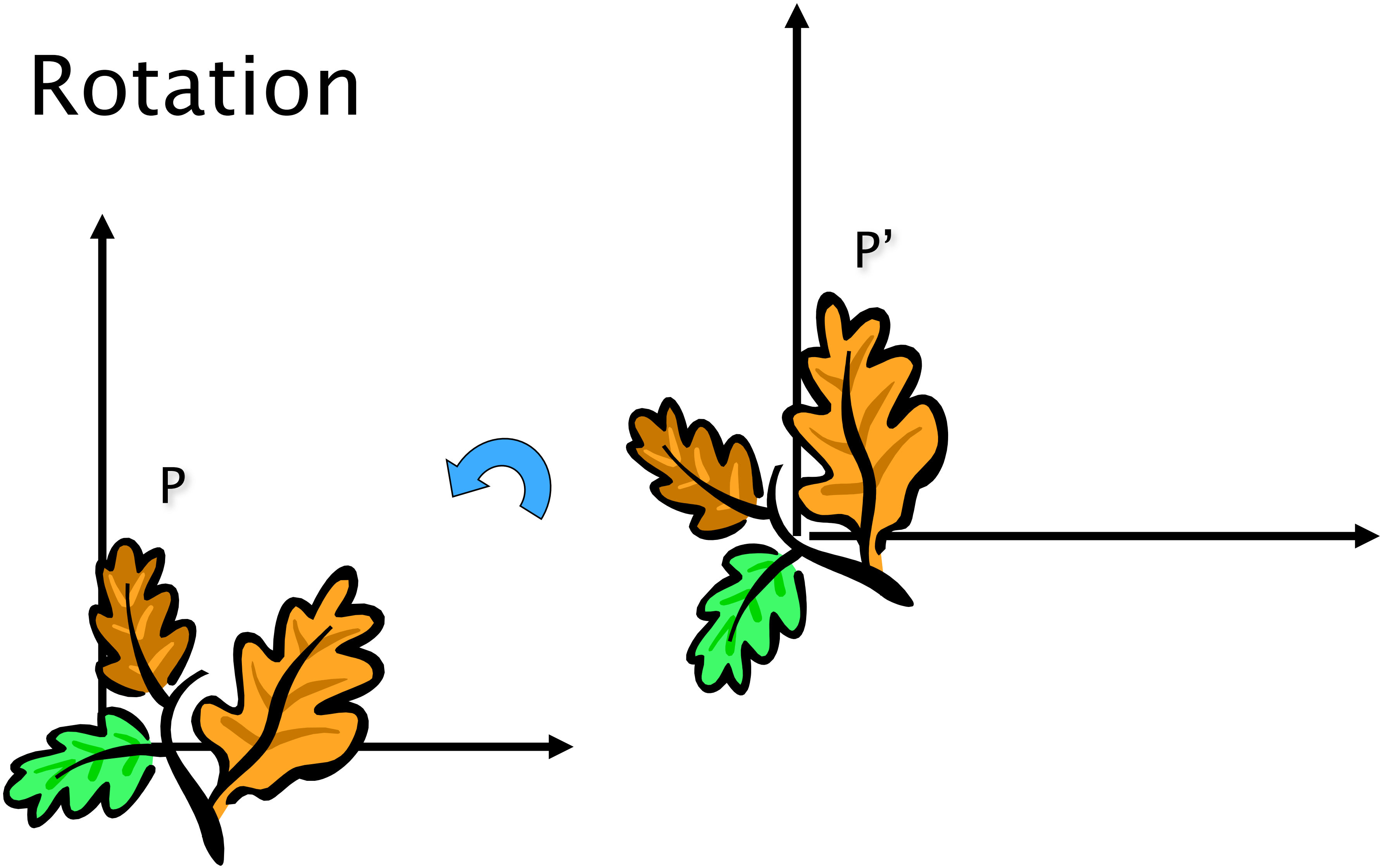
Translating & Scaling = Scaling & Translating ?

$$\mathbf{P'''} = \mathbf{T} \cdot \mathbf{S} \cdot \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}$$

$$\mathbf{P'''} = \mathbf{S} \cdot \mathbf{T} \cdot \mathbf{P} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

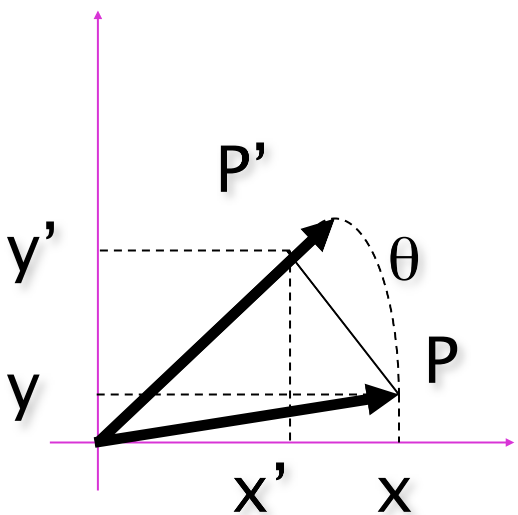
$$= \begin{bmatrix} s_x & 0 & s_x t_x \\ 0 & s_y & s_y t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + s_x t_x \\ s_y y + s_y t_y \\ 1 \end{bmatrix}$$

Rotation



Rotation Equations

Counter-clockwise rotation by an angle θ



$$x' = \cos\theta\ x - \sin\theta\ y$$

$$y' = \cos\theta\ y + \sin\theta\ x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P' = R\ P}$$

Degrees of Freedom

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

R is 2x2 → 4 elements

Note: R belongs to the category of normal matrices and satisfies many interesting properties:

$$\mathbf{R} \cdot \mathbf{R}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} = \mathbf{I}$$

$$\det(\mathbf{R}) = 1$$

Rotation+ Scaling +Translation

P'= (T R S) P

$$\mathbf{P'} = \mathbf{T}\cdot \mathrm{R}\cdot \mathbf{S}\cdot \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} R' & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} R'S & t \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

If $s_x=s_y$, this is a similarity transformation!

Transformation in 2D

- Isometries
- Similarities
- Affinity
- Projective

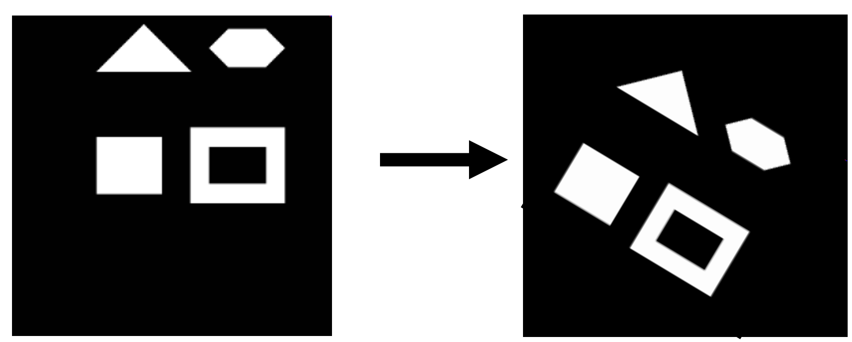
Transformation in 2D

Isometries:

[Euclideans]

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_e \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Preserve distance (areas)
- 3 DOF
- Regulate motion of rigid object

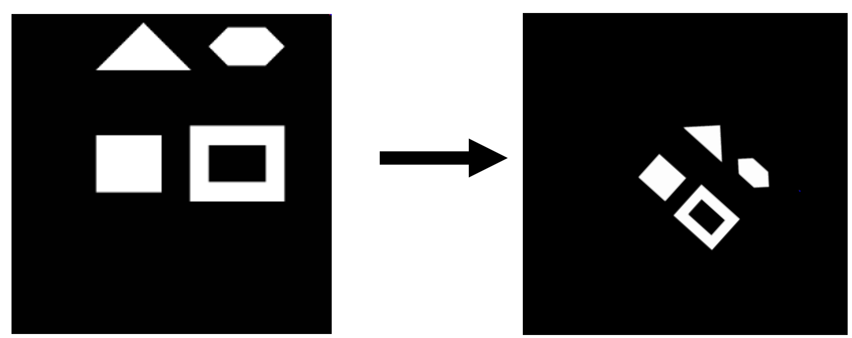


Transformation in 2D

Similarities:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s\,R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_s \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Preserve
  - ratio of lengths
  - angles
- 4 DOF

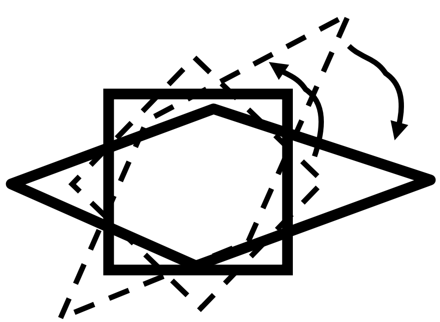


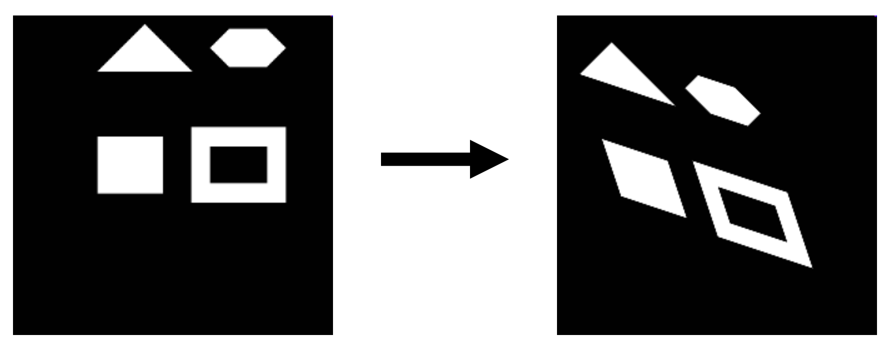
Transformation in 2D

Affinities:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_a \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = R(\theta) \cdot R(-\phi) \cdot D \cdot R(\phi) \qquad D = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$



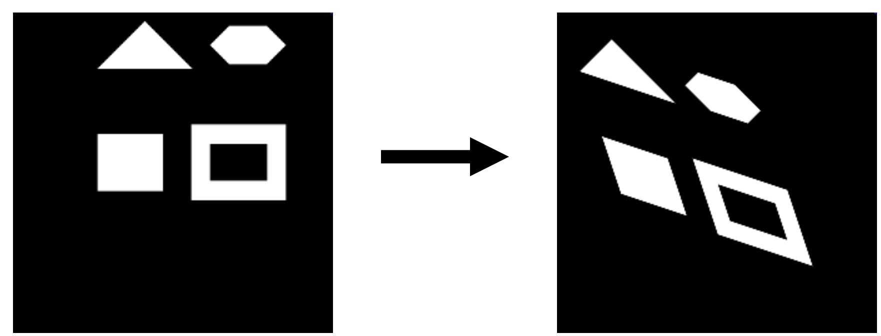


Transformation in 2D

Affinities: $$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_a \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = R(\theta) \cdot R(-\phi) \cdot D \cdot R(\phi) \quad D = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

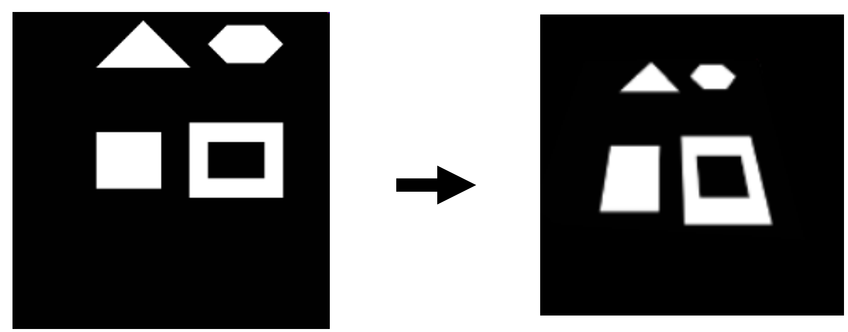
- Preserve:
  - Parallel lines
  - Ratio of areas
  - Ratio of lengths on collinear lines
  - others…
- 6 DOF



Transformation in 2D

Projective: $$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & t \\ \boxed{v} & b \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_p \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- 8 DOF
- Preserve:
  - cross ratio of 4 collinear points
  - collinearity
  - and a few others…



Eigenvalues and Eigenvectors

A eigenvalue λ and eigenvector $\mathbf{u}$ satisfies

$$\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$$

where $\mathbf{A}$ is a square matrix.

- Multiplying $\mathbf{u}$ by $\mathbf{A}$ scales $\mathbf{u}$ by λ

Eigenvalues and Eigenvectors

Rearranging the previous equation gives the system

$$\mathbf{Au} - \lambda\mathbf{u} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = 0$$

which has a solution if and only if $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.

- The eigenvalues are the roots of this determinant which is polynomial in λ.
- Substitute the resulting eigenvalues back into $\mathbf{Au} = \lambda\mathbf{u}$ and solve to obtain the corresponding eigenvector.

Singular Value Decomposition

$$\mathbf{U\Sigma V^T = A}$$

- Where **U** and **V** are orthogonal matrices, and **Σ** is a diagonal matrix. For example:

$$\overset{U}{\begin{bmatrix} -.40 & .916 \\ .916 & .40 \end{bmatrix}} \times \overset{\Sigma}{\begin{bmatrix} 5.39 & 0 \\ 0 & 3.154 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.05 & .999 \\ .999 & .05 \end{bmatrix}} = \overset{A}{\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}}$$

Singular Value decomposition

- **Singular values**: Non negative square roots of the eigenvalues of $\mathbf{A^tA}$. Denoted σ_i, $i=1,\dots,n$

- SVD: If $\mathbf{A}$ is a real m by n matrix then there exist orthogonal matrices $\mathbf{U}$ ($\in\mathbb{R}^{m\times m}$) and $\mathbf{V}$ ($\in\mathbb{R}^{n\times n}$) such that

$$A = U\,\Sigma\,V^{-1} \qquad \mathrm{U}^{-1}\mathrm{AV} = \Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \cdot & \\ & & & \sigma_N \end{bmatrix}$$

An Numerical Example

$$\overset{U}{\begin{bmatrix} -.39 & -.92 \\ -.92 & .39 \end{bmatrix}} \times \overset{\Sigma}{\begin{bmatrix} 9.51 & 0 & 0 \\ 0 & .77 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} = \overset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$$

- Look at how the multiplication works out, left to right:
- Column 1 of **U** gets scaled by the first value from **Σ**.

$$\overset{U\Sigma}{\begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} \qquad \overset{A_{partial}}{\begin{bmatrix} 1.6 & 2.1 & 2.6 \\ 3.8 & 5.0 & 6.2 \end{bmatrix}}$$

- The resulting vector gets scaled by row 1 of **V^T** to produce a contribution to the columns of **A**

An Numerical Example

$$\begin{array}{c}U\Sigma \\ \begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}\end{array} \times \begin{array}{c}V^T \\ \begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}\end{array} \qquad \begin{array}{c}A_{partial} \\ \begin{bmatrix} 1.6 & 2.1 & 2.6 \\ 3.8 & 5.0 & 6.2 \end{bmatrix}\end{array}$$

$$+\begin{array}{c}U\Sigma \\ \begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}\end{array} \times \begin{array}{c}V^T \\ \begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}\end{array} \qquad \begin{array}{c}A_{partial} \\ \begin{bmatrix} -.6 & -.1 & .4 \\ .2 & 0 & -.2 \end{bmatrix}\end{array}$$

$$= \qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

- Each product of (column i of **U**)·(value i from **Σ**)·(row i of $\mathbf{V}^T$) produces a

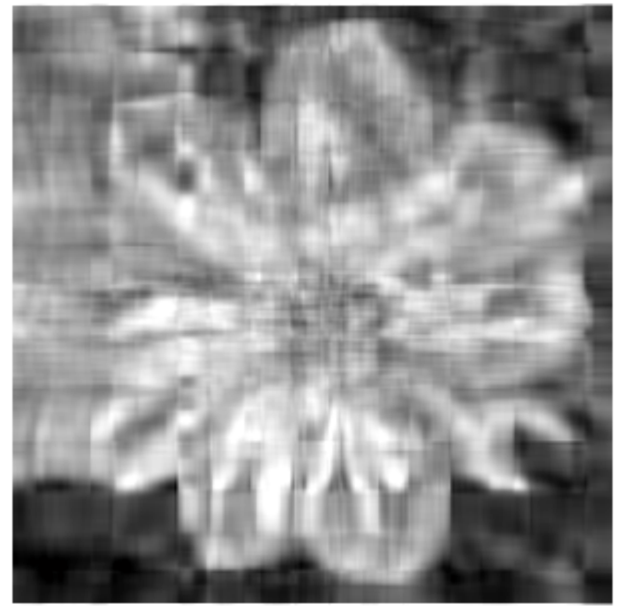
An Numerical Example

$$\overset{U}{\begin{bmatrix} -.39 & -.92 \\ -.92 & .39 \end{bmatrix}} \times \overset{\Sigma}{\begin{bmatrix} 9.51 & 0 & 0 \\ 0 & .77 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} = \overset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$$

We can look at **Σ** to see that the first column has a large effect

while the second column has a much smaller effect in this example

SVD Applications



- For this image, using **only the first 10** of 300 singular values produces a recognizable reconstruction
- So, SVD can be used for image compression

Principal Component Analysis

$$\overset{U\Sigma}{\begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} \qquad \overset{A_{partial}}{\begin{bmatrix} 1.6 & 2.1 & 2.6 \\ 3.8 & 5.0 & 6.2 \end{bmatrix}}$$

- Remember, columns of **U** are the Principal Components of the data: the major patterns that can be added to produce the columns of the original matrix
- One use of this is to construct a matrix where each column is a separate data sample
- Run SVD on that matrix, and look at the first few columns of **U** to see patterns that are common among the columns
- This is called Principal Component Analysis (or PCA) of the data samples

Principal Component Analysis

$$\overset{U\Sigma}{\begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} \qquad \overset{A_{partial}}{\begin{bmatrix} 1.6 & 2.1 & 2.6 \\ 3.8 & 5.0 & 6.2 \end{bmatrix}}$$

- Often, raw data samples have a lot of redundancy and patterns
- PCA can allow you to represent data samples as weights on the principal components, rather than using the original raw form of the data
- By representing each sample as just those weights, you can represent just the "meat" of what's different between samples.
- This minimal representation makes machine learning and other algorithms much more efficient

Properties of the SVD

- Suppose we know the singular values of **A** and we know *r* are non zero

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq \sigma_{r+1} = \ldots = \sigma_p = 0$$

  - Rank(**A**) = *r*.
  - Null(**A**) = span$\{\mathbf{v}_{r+1},\ldots,\mathbf{v}_n\}$
  - Range(**A**)=span$\{\mathbf{u}_1,\ldots,\mathbf{u}_r\}$
- $||\boldsymbol{A}||_F^2 = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_p^2$ $\qquad ||A||_2 = \sigma_1$
- *Numerical rank:* If *k* singular values of *A* are larger than a given number ε. Then the ε rank of A is *k*.
- Distance of a matrix of rank *n* from being a matrix of rank $k = \sigma_{k+1}$

Why is it useful?

- Square matrix may be singular due to round-off errors. Can compute a "regularized" solution
  - $\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}=(\mathbf{U}\,\Sigma\,\mathbf{V}^{\mathbf{t}})^{-1}\mathbf{b} = \sum_{i=1}^{n}\frac{\mathbf{u}_i^t\mathbf{b}}{\sigma_i}\mathbf{v}_i$
- If σ_i is small (vanishes) the solution "blows up"
- Given a tolerance ε we can determine a solution that is "closest" to the solution of the original equation, but that does not "blow up" $\mathbf{x}_r = \sum_{i=1}^{k}\frac{\mathbf{u}_i^t\mathbf{b}}{\sigma_i}\mathbf{v}_i$ $\sigma_k > \varepsilon,\ \ \sigma_{k+1} \le \varepsilon$

- Least squares solution is the x that satisfies
  $\mathbf{A}^t\mathbf{A}\mathbf{x}=\mathbf{A}^t\mathbf{b}$
- can be effectively solved using SVD