1 Problem 1

(a) The joint distribution $p(X, Y)$

$X =$	0	1	2
$Y = 0$	1/16	0	0
1	1/8	1/8	0
2	1/16	1/4	1/16
3	0	1/8	1/8
4	0	0	1/16

(1)

(b) The marginals $p(X)$ and $p(Y)$

$X =$	0	1	2
$p(X)$	1/4	1/2	1/4

$Y =$	0	1	2	3	4
$p(Y)$	1/16	1/4	3/8	1/4	1/16

(2)

(c) The conditionals $p(X|Y)$ and $p(Y|X)$

$X =$	0	1	2
$Y = 0$	1	0	0
1	1/2	1/2	0
2	1/6	2/3	1/6
3	0	1/2	1/2
4	0	0	1

$X =$	0	1	2
$Y = 0$	1/4	0	0
1	1/2	1/4	0
2	1/4	1/2	1/4
3	0	1/4	1/2
4	0	0	1/4

(3)

(d) The distribution of $Z = Y - X$, $p(Z)$

$Z =$	0	1	2
$p(Z)$	1/4	1/2	1/4

(4)

2 Problem 2

The conditional probabilities implied by this situation are as follows:

- The probability of testing positive given that you have the disease is $p(t|d) = 0.99$.
- The probability of testing positive given that you *don't* have the disease is $p(t|\tilde{d}) = 0.01$.
- The marginal probability of having the disease is only $p(d) = 10^{-4}$ and the probability of not having the disease is $p(\tilde{d}) = 1 - 10^{-4}$.
- Therefore, the marginal probability of testing positive is

$$p(t) = p(t|d)\,p(d) + p(t|\tilde{d})\,p(\tilde{d}) = 0.99 \times 10^{-4} + 0.01\,(1 - 10^{-4}) = 100.98 \times 10^{-4} \tag{5}$$

The value that the patient really cares about, though is the probability that they have the disease given that they tested positive $p(d|t)$. This — by Bayes — is

$$p(d|t) = \frac{p(d)\,p(t|d)}{p(t)} = \frac{0.99\times 10^{-4}}{100.98\times 10^{-4}} \approx 0.0098 \ll 1. \tag{6}$$

3 Problem 3

4 Problem 4

We are given the three statements

1. $p(A,B|C) = p(A|C)\,p(B|C)$
2. $p(A|B,C) = p(A|C)$
3. $p(B|A,C) = p(B|C)$

To see that statement 1 implies statement 2, apply the chain rule to find

$$p(A|C)\,p(B|C) \stackrel{1}{=} p(A,B|C) = p(B|C)\,p(A|B,C). \tag{7}$$

Cancelling $p(B|C)$ on both sides, we find statement 2. Therefore, it is clear that statement 1 implies statement 2. Also, since we have only used the chain rule, the inverse also applies. Specifically, applying the chain rule to statement 2, we find

$$p(A|B,C) = \frac{p(A,B|C)}{p(B|C)} \stackrel{2}{=} p(A|C) \rightarrow [\text{Statement 1}]. \tag{8}$$

Similarly, statement 1 implies statement 3 as follows

$$p(A|C)\,p(B|C) \stackrel{1}{=} p(A,B|C) = p(A|C)\,p(B|A,C) \rightarrow [\text{Statement 3}] \tag{9}$$

and the inverse

$$p(B|A,C) = \frac{p(A,B|C)}{p(A|C)} \stackrel{3}{=} p(B|C) \rightarrow [\text{Statement 1}]. \tag{10}$$

Finally, since the equivalence holds between 1 and 2 and also between 1 and 3, it is clear that 2 and 3 are also equivalent.

5 Problem 5

(a) By Bayes' Theorem,

$$p(H|E_1,E_2) = \frac{p(E_1,E_2|H)\,p(H)}{p(E_1,E_2)}. \tag{11}$$

Therefore, set (ii) is clearly sufficient for this calculation. Without any conditional independence assumptions, Equation (11) cannot be simplified any further so the other two sets are not sufficient. In particular, $p(E_1,E_2|H) \neq p(E_1|H)\,p(E_2|H)$ unless $E_1 \perp E_2|H$.

(b) Since $E_1 \perp E_2|H$, $p(E_1, E_2|H) = p(E_1|H)\, p(E_2|H)$ and Equation (11) becomes

$$p(H|E_1, E_2) = \frac{p(E_1|H)\, p(E_2|H)\, p(H)}{p(E_1, E_2)}. \tag{12}$$

Therefore, sets (i) and (ii) are now sufficient. Set (iii) is not sufficient because it would require that $E_1 \perp E_2$ but E_1 and E_2 are only *conditionally* independent.

6 Problem 6

7 Problem 7

8 Problem 8

(a) The set A is $\{X_2, X_3, X_4, X_5, X_8\}$. Clearly, all the nodes that are directly connected to X_1 (i.e. $\{X_2, X_3, X_4, X_8\}$) must be included in A because a direct connection always constitutes an active path. The inclusion of X_5 is not immediately obvious but if we just look at the part of the graph containing X_5, we find the V-structure $X_1 \to X_3 \leftarrow X_5$. If we condition on X_3 (which we will do because it is one of the directly connected nodes), it couples its parents X_1 and X_5. Therefore, to satisfy the condition $X_1 \perp \chi - A - \{X_1\}|A$, we must also include X_5 in A. After the inclusion of X_5, there are no other active paths between X_1 and other nodes outside of A — this can be easily seen by trying them all.