Deriving Quanto Drift Adjustment to Foreign Interest Rate Process

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The Set-up

We denominate all prices in terms of the amount of domestic currency accumulated in a continuously compounded money market account. Let's say we put $1 into the account at time t=0 and keep reinvesting the account's balance at the prevailing short-rate — at time t we reinvest at rate $r_\$(t)$. The value of this account at time t is given by the familiar formula:

$$B_\$(t) = \exp\left(\int_0^t r_\$(s)ds\right)$$

The value of a corresponding money market account in a foreign country is given by:

$$B_F(t) = \exp\left(\int_0^t r_F(s)ds\right)$$

Following the Harrison-Pliska theorem, the stochastic process followed by our market risk factors must be adjusted so that prices of derivative products, expressed in units of $B_\$(t)$, are martingales. That is the expected value of a given price in the future is equal to the respective current value. It also means that the drift term of any given price process must be equal to zero.

Deriving Risk-Neutral Drift for Spot FX Process

Let's start by applying the above to determine the drift for the spot FX rate $X(t)$, US$ dollar price of one unit of foreign currency, while pricing in units of ("pricing under" is a frequently used term) the domestic money market account. To do that, we look at the process followed by the value of the foreign money market account in units of the domestic money market account and required that its drift be zero:

$$d\left(X(t)\frac{B_F(t)}{B_\$(t)}\right) = \frac{B_F(t)}{B_\$(t)}dX(t) + X(t)d\left(\frac{B_F(t)}{B_\$(t)}\right) + dX(t)d\left(\frac{B_F(t)}{B_\$(t)}\right)$$

where

$$d\left(\frac{B_F(t)}{B_\$(t)}\right) = (r_F - r_\$)\frac{B_F(t)}{B_\$(t)}dt,\ \frac{\mathrm{d}X(t)}{X(t)} = \mu_X(t)dt + \sigma_X(t)dW_X^{(\$)}(t), \text{ and thus } \mathrm{d}X(t)d\left(\frac{B_F(t)}{B_\$(t)}\right) = 0$$

Requiring this drift to be zero gives us the well known result:

$$\mu_X(t) = r_\$(t) - r_F(t)$$

Later, we will need the following standard solution for the stochastic equation followed by $X(t)$:

$$X(T) = X(t)\exp\left(\int_t^T (r_\$(s) - r_F(s) - \tfrac{1}{2}\sigma_X^2(s))ds + \int_t^T \sigma_X(s)dW_X^{(\$)}(s)\right)$$

Drift-Adjustment to Foreign Interest Rates: the Quick Approach

By applying Ito's lemma, we find the process followed, under the domestic money market account, by the inverse of the exchange rate:

$$\frac{dY}{Y} = \left[(r_F - r_\$) + \sigma_X^2\right]dt - \sigma_X dW_X^{(\$)} \quad \text{where} \quad Y = \frac{1}{X}$$

From the previous section we know that under the foreign money market account Y follows the following process:

$$\frac{dY}{Y} = (r_F - r_\$)dt - \sigma_X dW_X^{(F)} \quad \text{where } dW_Y^{(F)} = -dW_X^{(F)} \text{ and } \sigma_X = \sigma_Y$$

This implies the following simple relationship between increments of Brownian process driving the spot FX under foreign and domestic money market accounts:

$$dW_X^{(F)} = dW_X^{(\$)} - \sigma_X dt$$

Let's now write $dW_{z_f}^{(F)}$ for the Brownian motion, under foreign money market account, driving the state variable z_f in a one-factor term-structure model for the foreign interest rates. We write $dW_{z_f}^{(\$)}$ for the corresponding Brownian motion under domestic money market account. We express dW_{z_f} as a linear combination of dW_X and a second Brownian process $dW_\perp$ that has zero correlation with dW_X :

$$dW_{z_f}^{(\$)} = \rho_{X,z_f} dW_X^{(\$)} + \sqrt{1-\rho_{X,z_f}^2}\, dW_\perp^{(\$)} \text{ and } dW_{z_f}^{(F)} = \rho_{X,z_f} dW_X^{(F)} + \sqrt{1-\rho_{X,z_f}^2}\, dW_\perp^{(F)}$$

Since $dW_\perp$ has zero correlation with the spot FX process, i.e., with dW_X , it must be the same under both the domestic and foreign money market accounts,, i.e., $dW_\perp^{(\$)} = dW_\perp^{(F)}$. Given the simple relationship between $dW_X^{(F)}$ and $dW_X^{(\$)}$ we derived above, we obtain the following relationship:

$$dW_{z_f}^{(F)} = dW_{z_f}^{(\$)} - \rho_{X,z_f} \sigma_X dt$$

This gives us our final result. While moving from foreign to domestic money market account as a unit of value, we need to adjust the drift for the process followed by the risk factor z_f by the following amount:

$$-\rho_{X,z_f} \sigma_{z_f} \sigma_X$$

By repeating the arguments used to derive drift adjustment in a one-factor model, we extend the above result to a multi-factor foreign short-rate model driven by *N* state variables $z_f^{(1)}, \ldots, z_f^{(N)}$. The drift adjustment to the *i*-th state variable is given by:

$$-\rho_{X,z_f^{(i)}} \sigma_{z_f^{(i)}} \sigma_X$$

Drift-Adjustment to Foreign Interest Rates: Application of Girsanov's Theorem

Now let's find adjustments we must apply to drifts of risk factors driving changes in foreign interest rates while pricing under the domestic money market measure. To do that, we take advantage of the fact that we can express the domestic price of a foreign zero-coupon bond in two equivalent ways:

$$X(0)E_F\left[\exp\left(-\int_0^T r_F(s)ds\right)\right] = E_\$\left[X(T)\exp\left(-\int_0^T r_\$(s)ds\right)\right]$$

On the left-hand side we convert the price of a zero-coupon bond in foreign currency units using the current spot FX. This zero-coupon bond price is expressed as the expected present value, under foreign money market account, of one unit of foreign currency to be received at time *T*.

On the right-hand side we take the expected present US$ value, under domestic money market account, of the amount of US$ obtained at time *T* by converting one unit of foreign currency at the prevailing spot FX .

Using the solution for $X(T)$, we rewrite the above equation in the following form:

$$E_F\left[\exp\left(-\int_0^T r_F(s)ds\right)\right] = E_\$\left[\exp\left(-\tfrac{1}{2}\int_0^T \sigma_X^2(s)ds + \int_0^T \sigma_X(s)dW_X^{(\$)}(s)\right)\exp\left(-\int_0^T r_F(s)ds\right)\right]$$

The above equation is ready-made for the application of the multi-dimensional Girsanov theorem. This theorem relates expected values, and underlying Brownian processes, taken under the domestic money market account to those taken under the foreign money market account.

$$dW_j^{(F)} = \theta_j(t)dt + dW_j^{(\$)}, \quad j = 1,\ldots,N \quad \text{where } N \text{ is the number of market risk factors}$$

$$\text{Market risk factors are uncorrelated, i.e.,} \quad dW_i^{(\$)}dW_j^{(\$)} = \delta_{ij}\,dt$$

$$Z(T) = \exp\left\{-\tfrac{1}{2}\int_0^T \theta(s)\cdot\theta(s)ds - \int_0^T \theta(s)\cdot dW^{(\$)}\right\}$$

$$E_F\left[Y(T)\,|\,t=0\right] = E_\$\left[Z(T)Y(T)\,|\,t=0\right] \text{ for any given random variable } Y(T)$$

The above theorem says that the expected value under foreign money market account, of a random variable $Y(T)$, is equal to the expected value under domestic money market account, of a product of $Y(T)$ and the so called Radon-Nikodym derivative $Z(T)$ (note that the corresponding expected value of $Z(T)$ is by definition equal to one). The set of functions $\theta_j,\, j=1,\dots,N$ is unique, i.e., independent of the choice for the random variable $Y(T)$.

In our case:

$$Y(T)=\exp\left(-\int_0^T r_F(s)ds\right) \text{ and } Z(T)=\exp\left(-\tfrac{1}{2}\int_0^T \sigma_X^2(s)ds+\int_0^T \sigma_X(s)dW_X^{(\$)}(s)\right)$$

$$\text{thus}\quad dW_1^{(\$)}=dW_X^{(\$)},\ \ \theta_1(t)=-\sigma_X(t)\ \ \text{and}\quad \theta_2=\ldots=\theta_N=0$$

As before, $dW_{z_f}^{(F)}$ is the Brownian motion, under foreign measure, driving the state variable z_f in a one-factor term-structure model for the foreign interest rates. We express dW_{z_f} as a linear combination of dW_X and a second Brownian process $dW_\perp$ that has zero correlation with dW_X:

$$dW_{z_f}^{(\$)}=\rho_{X,z_f}dW_X^{(\$)}+\sqrt{1-\rho_{X,z_f}^2}\,dW_\perp^{(\$)} \text{ and } dW_{z_f}^{(F)}=\rho_{X,z_f}dW_X^{(F)}+\sqrt{1-\rho_{X,z_f}^2}\,dW_\perp^{(F)}$$

Girsanov's theorem gives us the following relationships:

$$dW_X^{(F)}=dW_X^{(\$)}-\sigma_X dt,\quad dW_\perp^{(F)}=dW_\perp^{(\$)},\quad \text{and}\quad \text{thus } dW_{z_f}^{(F)}=dW_{z_f}^{(\$)}-\rho_{X,z_f}\sigma_X dt$$

This gives us our final result. While moving from foreign to domestic money market account as a unit of value, we need to adjust the drift for the process followed by the risk factor z_f by the following amount:

$$-\rho_{X,z_f}\sigma_{z_f}\sigma_X$$

By repeating the arguments used to derive drift adjustment in a one-factor model, we extend the above result to a multi-factor foreign short-rate model driven by *N* state variables $z_f^{(1)},\dots,z_f^{(N)}$. The drift adjustment to the *i*-th state variable is given by:

$$-\rho_{X,z_f^{(i)}}\sigma_{z_f^{(i)}}\sigma_X$$

Notice that the above drift adjustment formulae are independent of the formulation of our term-structure model.