The Maths of

Forward LIBOR Models

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1 The Model Setup

We start by picking the set of N simple forward rates $L_1, ..., L_N$ for forward dates $T_1, ..., T_N$ on deposits maturing on dates $T_2, ..., T_{N+1}$. The tenors of the underlying money-market deposits (expressed as day-count fractions) are given by $\delta_1, ..., \delta_N$. We typically set all the tenors to be the same and equal to 1, 3 or 6 months. We choose one of the above $N+1$ dates (let's call it $T_{k_{ext}}$).

Now, we move to a world in which prices are expressed in units of one-dollar notional zero-coupon bond maturing at $T_{k_{ext}}$. This zero-coupon bond serves as the price denominator (borrowing from French, the mathematical finance literature calls a price denominator a "numeraire").

The n-th rate follows a random process of the following general form:

$$d\,L_n(t) = \mu_n(t)d\,t + \sigma_n(t)\,dW^{(n)}_{T_{k_{ext}}}(t) \tag{1}$$

Here, $\mu_n(t)$ is the drift; $\sigma_n(t)$ is the volatility; and $dW^{(n)}_{T_{k_{ext}}}(t)$ is the increment of a Brownian motion for the n-th forward rate at time t (for the sake of simplicity, we do not explicitly state the functional dependence of the drift and volatility functions on the forward rate levels). The Brownian motion increment $dW^{(n)}_{T_{k_{ext}}}$ is a normally distributed random number with zero mean and standard deviation equal to $\sqrt{d\,t}$.

The instantaneous correlation $\rho_{i,j}(t)$ between forward rates L_i and L_j is equal to the correlation between Brownian increments $dW^{(i)}_{T_{k_{ext}}}(t)$ and $dW^{(j)}_{T_{k_{ext}}}(t)$. In stochastic calculus we write:

$$dW^{(i)}_{T_{k_{ext}}}(t)dW^{(j)}_{T_{k_{ext}}}(t) = \rho_{i,j}(t)\,dt \tag{2}$$

The instantaneous covariance $\sigma_{i,j}(t)$ between rates L_i and L_j is given by the standard formula:

$$\sigma_{i,j}(t) = \rho_{i,j}(t)\sigma_i(t)\sigma_j(t) \tag{3}$$

In the Appendix we show that the drift functions are entirely determined by volatilities and correlations. We also derive the following formulas:

$$\begin{aligned}
\mu_n &= \sum_{i=k_{ext}}^{n} \frac{\delta_i}{1+\delta_i L_i}\sigma_{i,n} && \text{when} \quad k_{ext} \le n \\
\mu_n &= 0 && \text{when} \quad k_{ext} = n+1 \\
\mu_n &= -\sum_{i=n+1}^{k_{ext}-1} \frac{\delta_i}{1+\delta_i L_i}\sigma_{i,n} && \text{when} \quad k_{ext} > n
\end{aligned} \tag{4}$$

2 The Forward-Neutral Approach

Let's write $P_1(t), ..., P_{N+1}(t)$ for the current prices of one-dollar notional zero-coupon bonds maturing on $T_1, T_2, ..., T_{N+1}$.

Let's write $V(t)$ for the value of a security (plain or derivative) at time t. The forward value $V_T(t)$ of this security for time T is defined by:

$$V_T(t) = \frac{V(t)}{P(t,T)} \tag{5}$$

where $P(t,T)$ is the price of a one-dollar zero-coupon bond maturing at time T.

Assuming that the market risk of this security can be hedged, we must value it in the so called *risk-neutral World.* If, in addition, we value this security in units of a zero-coupon bond maturing at time T, the security's expected value at any future time point prior to T must be equal to the current forward value $V_T(t)$:

$$V_T(t) = E_t^{(T)}\left[V_T(t')\right] \tag{6}$$

The superscript (T) in the expectations operator means that the expectations are taken in a country with one-dollar notional zero-coupon bond maturing at time T serving as a currency unit. The subscript t means that expectations are calculated at time t. Simply, the expected forward price for a future time point must be equal to the current forward price.

The term *forward-neutral approach* refers to the fact that we first find security values denominated in zero-coupon bonds of a particular maturity. Next, we convert that value to dollars by multiplying it by the exchange rate–the zero-coupon bond dollar price. A related term *forward-neutral measure* refers to the fact that we calculate expected values and/or randomly evolve relevant market variables in a country with a one-dollar zero-coupon bond of a specific maturity serving as one currency unit.

This means that for a given choice of forward rates' volatilities and correlations we must choose drifts so that forward prices, the forward date is the maturity of our numeraire zero-coupon bond, of all zero-coupon bonds follow random processes with zero-drifts. In mathematical terms we say that these forward prices are "martingales."

3 Forward Rates' Volatilities and Correlations

3.1 Introduction

One way to specify the forward rates' volatilities and correlations is by introducing a covariance function Ω in the following way:

$$\sigma_i(t)\sigma_j(t)\rho_{i,j}(t) = \Omega(t, T_i - t, T_j - t) \tag{7}$$

Here, $\Omega(t, s_1, s_2)$ is the instantaneous covariance, at time t, between two fixed-tenor forward rates which reset in respectively s_1 and s_2 years from time t.

A more convenient way is through factor decomposition:

$$\sigma_n(t)\, dW_{T_{k_{ext}}}^{(n)}(t) = \sum_{i=1}^{n_f} \sigma_i^{(f)}(t, T_n - t) dW_{T_{k_{ext}}}^{(f),i}(t) \tag{8}$$

Here $\sigma_i^{(f)}(t, T_n - t)$ is the i-th factors' volatility function and $dW_{T_{k_{ext}}}^{(f),i}(t)$ is the i-th normally distributed random shock (factor) with zero mean and standard deviation $\sqrt{dt}$. The factor volatility functions $\sigma_i^{(f)}(t, T - t)$ in can depend on the forward rates' levels (for simplicity we suppressed this dependence in our notation).

Typically, we choose orthogonal factors (principal components) which means that the n_f random shocks are mutually independent, i.e., have zero correlation.

3.2 Lognormal Forward Model

In the lognormal forward model forward rate volatility is proportional to the forward rate level. The volatility and correlation structure is introduced via factor decomposition:

$$\sigma_n(t)\, dW_{T_{k_{ext}}}^{(n)}(t) = L_n(t) \sum_{i=1}^{n_f} \sigma_i^{(f)}(t, T_n - t) dW_{T_{k_{ext}}}^{(f),i}(t) \tag{9}$$

where the lognormal factor volatility functions $\sigma_i^{(f)}(t, T_n - t)$ are rate-level independent.

3.3 Normal HJM Model

By choosing the forward rate volatilities in the following way:

$$\sigma_n(t)\, dW^{(n)}_{T_{k_{ext}}}(t) = (1 + \delta_n L_n(t) \sum_{i=1}^{n_f} \sigma_i^{(f)}(t, T_n - t) dW^{(f),i}_{T_{k_{ext}}}(t) \tag{10}$$

we obtain the forward model representation of the Normal HJM model. Here the factor volatility functions $\sigma_i^{(f)}(t, T - t)$ are rate-level independent.

3.4 Separable Dependence on Rate Levels

In a separable model, the factor volatility functions are of the following form:

$$\sigma_i^{(f)}(t, T - t) = F(L_1, ..., L_n) v_i(t, T - t) \tag{11}$$

A simple example is the following model:

$$\sigma_i^{(f)}(t, T_n - t) = (\alpha + L(t, T_n)) v_i(t, T_n - t) \tag{12}$$

This model is dominated by the normal behavior when the rates are low and by lognormal behavior when the rates are high. The constant α determines the mixing between the normal and the lognormal models. By varying α we can control the level of implied Black cap/floor volatility skews ($\alpha = 0$ means is no skew).

4 Cap/Floor Valuation

We value a caplet on the n-th LIBOR rate L_n. The caplet is struck at K and resets at T_n and pays at T_{n+1}.

We pick T_{n+1} as the maturity date of our zero-coupon bond numeraire. Under this numeraire, the LIBOR rate follows the following risk-neutral process:

$$d\,L_n(t) = \sigma_n(t)\,dW^{(n)}_{T_{n+1}}(t) \tag{13}$$

We set the drift to zero because L_n is a simple linear function of the forward price for T_{n+1} of a zero-coupon bond maturing at T_n.

The time t dollar caplet value is equal to the expected value of the caplet payoff function discounted from the caplet payment date times the day-count fraction:

$$V_c(t) = \delta_n P_{\$}(t, T_{n+1}) E_t^{(T_{n+1})}\left[\max(L_n(T_n) - K, 0)\right] \tag{14}$$

The above value is equivalent to the value of a call option (with delayed payment) on the forward price P_n/P_{n+1} at T_n:

$$V_c(t) = P_{\$}(t, T_{n+1}) E_t^{(T_{n+1})}\left[\frac{P_n(T_n)}{P_{n+1}(T_n)} - (1 + \delta_n K), 0)\right] \tag{15}$$

Note that the expectation is calculated in the zero-coupon numeraire world (not in the dollar world). In this world, the expected future levels of L_n and P_n/P_{n+1} are equal to their respective current levels:

$$L_n(t) = E_t^{(T_{n+1})}\left[L_n(T_n)\right] \qquad \text{and} \qquad \frac{P_n(t)}{P_{n+1}(t)} = E_t^{(T_{n+1})}\left[\frac{P_n(T_n)}{P_{n+1}(T_n)}\right] \tag{16}$$

4.1 Lognormal Forward Model

In the **LFM**, L_n follows the multi-factor lognormal process with zero-drift:

$$\frac{d\,L_n(t)}{L_n(t)} = \sum_{i=1}^{N_f} \gamma_i(t, T_n - t)\, dW_{T_{n+1}}^{(f),i}(t) \tag{17}$$

Here N_f is the number of factors; $\gamma_i(t, T_n - t)$ is the i-th factor's volatility function; and we assume that factors are independent.

The expected caplet payoff is given by the standard Black caplet formula:

$$E_t^{(T_{n+1})}\left[\max(L_n(T_n) - K, 0)\right] = \mathrm{BS}_{call}(L_n(t), K, 0, \sigma_{bs}(t, T_n), T_n - t) \tag{18}$$

$$= L_n(t)\mathbf{N}\left(\frac{\log\frac{L_n(t)}{K} + \frac{1}{2}\sigma_{bs}^2(t,T_n)\,(T_n - t)}{\sigma_{bs}(t,T_n)\sqrt{T_n - t}}\right) - K\mathbf{N}\left(\frac{\log\frac{L_n(t)}{K} - \frac{1}{2}\sigma_{bs}^2(t,T_n)\,(T_n - t)}{\sigma_{bs}(t,T_n)\sqrt{T_n - t}}\right)$$

The Black-Scholes LIBOR volatility $\sigma_{bs}(t, T_n)$ is given in terms of the factor volatility functions:

$$\sigma_{bs}^2(t, T_n) = \frac{1}{T_n - t} \sum_{i=1}^{N_f} \int_t^{T_n} \gamma_i^2(t, T_n - s) ds \tag{19}$$

4.2 Normal HJM Model

In the **NHJM**, L_n follows the following multi-factor lognormal process with zero-drift:

$$dL_n(t) = (1 + \delta_n L_n(t)) \sum_{i=1}^{N_f} \gamma_i(t, T_n - t)\, dW_{T_{n+1}}^{(f),i}(t) \tag{20}$$

Here N_f is the number of factors; $\gamma_i(t, T_n - t)$ is the i-th factor's volatility function; and factors are independent.

The forward zero-coupon price P_n/P_{n+1} (equal to $1 + \delta_n L_n$) follows the multi-factor lognormal model:

$$d\left(\frac{P_n}{P_{n+1}}\right) = \left(\frac{P_n}{P_{n+1}}\right) \delta_n \sum_{i=1}^{N_f} \gamma_i(t, T_n - t)\, dW_{T_{n+1}}^{(f),i}(t) \tag{21}$$

The expected caplet payoff is given by the standard Black-Scholes formula:

$$E_t^{(T_{n+1})}\left[\frac{P_n(T_n)}{P_{n+1}(T_n)} - (1+\delta_n K), 0)\right] = \mathrm{BS}_{call}(1+\delta_n L_n, 1+\delta_n K, 0, \sigma_{bs}(t,T_n), T_n - t) \tag{22}$$

$$= (1+\delta_n L_n)\mathbf{N}\left(\frac{\log\frac{1+\delta_n L_n}{1+\delta_n K} + \frac{1}{2}\sigma_{bs}^2(t,T_n)\,(T_n-t)}{\sigma_{bs}(t,T_n)\sqrt{T_n-t}}\right) - (1+\delta_n K)\mathbf{N}\left(\frac{\log\frac{1+\delta_n L_n}{1+\delta_n K} - \frac{1}{2}\sigma_{bs}^2(t,T_n)\,(T_n-t)}{\sigma_{bs}(t,T_n)\sqrt{T_n-t}}\right)$$

The Black-Scholes *forward-price* volatility $\sigma_{bs}(t,T_n)$ is given in terms of the factor volatility functions:

$$\sigma_{bs}^2(t,T_n) = \frac{\delta_n^2}{T_n - t}\sum_{i=1}^{N_f}\int_t^{T_n} \gamma_i^2(t, T_n - s)ds \tag{23}$$

At-the-Money Forward Black Implied Volatiltity

The at-the-money implied Black caplet volatility $\sigma_{imp}(L_n)$ can be obtained by numerically solving the following equation:

$$\frac{\delta_n\, L_n}{1+\delta_n L_n}\mathrm{BS}_{call}(1,1,0,\sigma_{imp}(L_n),T_n) = \mathrm{BS}_{call}(1,1,0,\sigma_{bs}(0,T_n),T_n) \tag{24}$$

Black Caplet Volatility Skew

For a given strike K, we find the implied Black caplet volatility $\sigma_{imp}(K)$ by numerically solving the following equation:

$$\delta_n\,\mathrm{BS}_{call}(L_n, K, 0, \sigma_{imp}(K), T_n) = \mathrm{BS}_{call}(1+\delta_n L_n, 1+\delta_n K, 0, \sigma_{bs}(0,T_n), T_n) \tag{25}$$

4.3 The Mixed Model

In the mixed model, L_n follows the following multi-factor process with zero-drift:

$$dL_n(t) = (\alpha + \delta_n L_n(t))\sum_{i=1}^{N_f}\gamma_i(t, T_n - t)\,dW_{T_{n+1}}^{(f),i}(t) \tag{26}$$

As we change α from 0 to 1, we gradually move from the lognormal forward model to the normal HJM model. The variable $x_{(\alpha)n}(t) = \alpha + \delta_n L_n(t)$ follows the multifactor lognormal model:

$$\frac{dx_{(\alpha)n}}{x_{(\alpha)n}} = \delta_n \sum_{i=1}^{N_f} \gamma_i(t, T_n - t)\, dW_{T_{n+1}}^{(f),i}(t) \tag{27}$$

Here N_f is the number of factors; $\gamma_i(t, T_n - t)$ is the i-th factor's volatility function; and factors are independent.

Now, we write the caplet payoff in the following way:

$$\delta_n \max(L_n(T_n) - K, 0) = \max(x_\alpha(T_n) - (\alpha + \delta_n K), 0) \tag{28}$$

That is, a caplet can be viewed as a delayed payment European call option, struck at $\alpha + \delta_n K$, on a portfolio that consists of one-dollar notional zero-coupon bond maturing at T_n and a short position in $(1 - \alpha)$-dollar notional zero-coupon bond maturing at T_{n+1}.

Since $x_{(\alpha)n}(T_n)$ has a lognormal distribution, the option value is given in terms of the Black-Scholes formula:

$$E_t^{(T_{n+1})}\left[\max((x_{(\alpha)n}(T_n) - (1 + \delta_n K), 0)\right] = \mathrm{BS}_{call}(\alpha + \delta_n L_n, \alpha + \delta_n K, 0, \sigma_{bs}(t, T_n), T_n - t) \tag{29}$$

$$= \frac{P_n(t)}{P_{n+1}(t)} \mathbf{N}\left(\frac{\log \frac{\alpha+\delta_n L_n}{\alpha+\delta_n K} + \frac{1}{2}\sigma_{bs}^2(t,T_n)\,(T_n - t)}{\sigma_{bs}(t,T_n)\sqrt{T_n - t}}\right) - (\alpha + \delta_n K)\mathbf{N}\left(\frac{\log \frac{\alpha+\delta_n L_n}{\alpha+\delta_n K} - \frac{1}{2}\sigma_{bs}^2(t,T_n)\,(T_n - t)}{\sigma_{bs}(t,T_n)\sqrt{T_n - t}}\right)$$

The Black-Scholes *forward-price* volatility $\sigma_{bs}(t, T_n)$ is given in terms of the factor volatility functions:

$$\sigma_{bs}^2(t, T_n) = \frac{\delta_n^2}{T_n - t} \sum_{i=1}^{N_f} \int_t^{T_n} \gamma_i^2(t, T_n - s)\, ds \tag{30}$$

At-the-Money Forward Black Implied Volatility

The at-the-money implied Black caplet volatility $\sigma_{imp}(L_n)$ can be obtained by numerically solving the following equation:

$$\frac{\delta_n L_n}{\alpha + \delta_n L_n} \mathrm{BS}_{call}(1, 1, 0, \sigma_{imp}(L_n), T_n) = \mathrm{BS}_{call}(1, 1, 0, \sigma_{bs}(0, T_n), T_n) \tag{31}$$

Caplet Volatility Skew

For a given strike K, we find the implied LIBOR Black caplet volatility $\sigma_{imp}(K)$ by numerically solving the following equation:

$$\delta_n \, \mathrm{BS}_{call}(L_n, K, 0, \sigma_{imp}(K), T_n) = \mathrm{BS}_{call}(\alpha + \delta_n L_n, \alpha + \delta_n K, 0, \sigma_{bs}(0, T_n), T_n) \tag{32}$$

By moving α from 0 to 1 we can increase the amount of Black caplet volatility skew: from no skew at $\alpha = 0$ (the Black model) to the normal model's skew at $\alpha = 1$ ($\sigma_{imp}(K) \approx L_n/K \, \sigma_{imp}(L_n)$).

5 Momentum Caps

A single momentum caplet on L_n is struck at a fixed spread Δ over the time T_{n-1} level of L_{n-1}. That is, the caplet payoff is given by:

$$\delta_n \max(L_n(T_n) - L_{n-1}(T_{n-1}) - \Delta, 0) \tag{33}$$

The payment can fall on either T_n or T_{n+1}.

We choose as our numeraire the zero-coupon bond whose maturity falls on the payment date.

When the payment falls on T_{n+1}, the rates L_n and L_{n-1} follow the following processes:

$$\begin{aligned} dL_n(t) &= \sigma_n(t)dW_{T_{n+1}}^{(n)} \\ dL_{n-1}(t) &= -\tfrac{\delta_n}{1+\delta_n L_n(t)}\sigma_{n-1,n}(t) + \sigma_{n-1}(t)dW_{T_{n+1}}^{(n-1)} \end{aligned}$$

The dollar caplet value is given by:

$$V_c(t) = \delta_n P_{\$}(t, T_{n+1}) E_t^{(T_{n+1})}\left[\max(L_n(T_n) - L_{n-1}(T_{n-1}) - \Delta, 0)\right] \tag{34}$$

When the payment falls on T_n, the rates L_n and L_{n-1} follow the following processes:

$$\begin{aligned} dL_n(t) &= \tfrac{\delta_n}{1+\delta_n L_n(t)}\sigma_n^2(t) + \sigma_n(t)dW_{T_n}^{(n)} \\ dL_{n-1}(t) &= \sigma_{n-1}(t)dW_{T_n}^{(n-1)} \end{aligned}$$

The dollar caplet value is given by:

$$V_c(t) = \delta_n P_{\$}(t, T_n) E_t^{(T_n)}\left[\max(L_n(T_n) - L_{n-1}(T_{n-1}) - \Delta, 0)\right] \tag{35}$$

We can find the expected value in a direct way by integrating the payoff function with the joint distribution of $L_n(T_n)$ and $L_{n-1}(T_{n-1})$. We can also use the law of iterated expectations and integrate the caplet value at time T_{n-1} with the joint distribution of $L_n(T_{n-1})$ and $L_{n-1}(T_{n-1})$:

$$E_t\left[\max(L_n(T_n) - L_{n-1}(T_{n-1}) - \Delta, 0)\right] = E_t\left[E_{T_{n-1}}\left[\max(L_n(T_n) - L_{n-1}(T_{n-1}) - \Delta, 0)\right]\right] \tag{36}$$

5.1 Lognormal Forward Model Valuation

In the **LFM**:

$$\begin{aligned}
\sigma_n(t)dW^{(n)}_{T_{k_{ext}}} &= L_n(t)\sum_{i=1}^{N_f}\gamma_i(t,T_n-t)\,dW^{(f),i}_{T_{k_{ext}}}(t)\\
\sigma_{n-1,n}(t) &= L_{n-1}(t)L_n(t)\sum_{i=1}^{N_f}\gamma_i(t,T_{n-1}-t)\gamma_i(t,T_n-t)\\
\sigma_n^2(t) &= L_n^2(t)\sum_{i=1}^{N_f}\gamma_i^2(t,T_n-t)
\end{aligned}$$

The joint distribution of $\log(L_{n-1}(T_{n-1}))$ and $\log(L_n(T_n))$ is approximated by the normal distribution with the following covariance matrix:

$$\begin{aligned}
\sigma_{(\gamma)n-1,n} &= \sum_{i=1}^{N_f}\int_t^{T_{n-1}}\gamma_i(s,T_{n-1}-s)\gamma_i(s,T_n-s)ds\\
&= \frac{1}{2}\Big[\sigma^2_{(log)n-1}+\sum_{i=1}^{N_f}\int_t^{T_{n-1}}\gamma_i^2(s,T_n-s)ds\\
&\quad -\sum_{i=1}^{N_f}\int_t^{T_{n-1}}(\gamma_i(s,T_n-s)-\gamma_i(s,T_{n-1}-s))^2ds\Big]\\
\sigma^2_{(\gamma)n-1} &= \sum_{i=1}^{N_f}\int_t^{T_{n-1}}\gamma_i^2(s,T_{n-1}-s)ds\\
\sigma^2_{(\gamma)n} &= \sum_{i=1}^{N_f}\int_t^{T_n}\gamma_i^2(s,T_n-s)ds
\end{aligned}$$

The joint distribution of $\log(L_{n-1}(T_{n-1}))$ and $\log(L_n(T_{n-1}))$ is approximated by the normal distribution whose covariance matrix is obtained by replacing $\sigma^2_{(\gamma)n}$ in the above covariance matrix with:

$$\sigma^2_{(\gamma)n}=\sum_{i=1}^{N_f}\int_t^{T_{n-1}}\gamma_i^2(s,T_n-s)ds$$

When the payment falls on T_{n+1}, the distribution means are given by:

$$\begin{aligned}
\mu_{(log)n-1} &= -\frac{\delta_nL_n}{1+\delta_nL_n}\sigma_{(\gamma)n-1,n}-\frac{1}{2}\sigma^2_{(\gamma),n-1}\\
\mu_{(log)n} &= -\frac{1}{2}\sigma^2_{(\gamma),n}
\end{aligned}$$

When the payment falls on T_n, the distribution means are given by:

$$\begin{aligned}
\mu_{(log)n-1} &= -\frac{1}{2}\sigma^2_{(\gamma),n-1}\\
\mu_{(log)n} &= \frac{\delta_nL_n}{1+\delta_nL_n}\sigma^2_{(\gamma),n}-\frac{1}{2}\sigma^2_{(\gamma),n}
\end{aligned}$$

5.2 Mixed Model Valuation

The momentum caplet payoff can be writen in the following form:

$$\max(x_{(\alpha),n}(T_n) - \frac{\delta_n}{\delta_{n-1}} x_{(\alpha),n-1}(T_{n-1}) - \tilde{\Delta}, 0) \quad \text{where} \quad \tilde{\Delta} = \Delta + \alpha(1 - \frac{\delta_n}{\delta_{n-1}}) \tag{37}$$

In the mixed model

$$\begin{aligned}
\sigma_n(t) dW^{(n)}_{T_{k_{ext}}} &= (\alpha + \delta_n L_n(t)) \textstyle\sum_{i=1}^{N_f} \gamma_i(t, T_n - t)\, dW^{(f),i}_{T_{k_{ext}}}(t) \\
\sigma_{n-1,n}(t) &= (\alpha + \delta_{n-1} L_{n-1}(t))(\alpha + \delta_n L_n(t)) \textstyle\sum_{i=1}^{N_f} \gamma_i(t, T_{n-1} - t)\gamma_i(t, T_n - t) \\
\sigma^2_n(t) &= (\alpha + \delta_n L_n(t))^2 \textstyle\sum_{i=1}^{N_f} \gamma^2_i(t, T_n - t)
\end{aligned}$$

The joint distribution of $\log(x_{(\alpha)n-1}(T_{n-1}))$ and $\log(x_{(\alpha)n}(T_n))$ is approximated by the normal distribution with the following covariance matrix:

$$\sigma_{x_{(\alpha)n-1}, x_{(\alpha)n}} = \delta_{n-1}\delta_n \sigma_{(\gamma)n-1,n}, \qquad \sigma^2_{x_{(\alpha)n-1}} = \delta^2_{n-1}\sigma^2_{(\gamma)n-1} \qquad \sigma^2_{x_{(\alpha)n}} = \delta^2_n \sigma^2_{(\gamma)n} \tag{38}$$

Here $\sigma_{(\gamma)n-1,n}$, $\sigma^2_{(\gamma)n-1}$ and $\sigma^2_{(\gamma)n}$ are expressed in terms of the $\gamma(t, T-t)$ factor volatility functions. The formulas are the same as in the lognormal case.

The joint distribution of $\log(x_{(\alpha)n-1}(T_{n-1}))$ and $\log(x_{(\alpha)n}(T_{n-1}))$ is also approximated by the normal distribution. The covariance matrix for this distribution is given by the same formula as above.

When the payment falls on T_{n+1}, the distribution means are given by:

$$\begin{aligned}
\mu_{x_{(\alpha)n-1}} &= -\tfrac{\delta_n \delta_{n-1}}{1+\delta_n L_n} x_{(\alpha)n} \sigma_{(\gamma)n-1,n} - \tfrac{1}{2}\delta^2_{n-1}\sigma^2_{(\gamma),n-1} \\
\mu_{x_{(\alpha)n}} &= -\tfrac{1}{2}\delta^2_n \sigma^2_{(\gamma),n}
\end{aligned}$$

When the payment falls on T_n, the distribution means are given by:

$$\begin{aligned}
\mu_{x_{(\alpha)n-1}} &= -\tfrac{1}{2}\delta^2_{n-1}\sigma^2_{(\gamma),n-1} \\
\mu_{x_{(\alpha)n}} &= -\tfrac{\delta^2_n}{1+\delta_n L_n} x_{(\alpha)n} \sigma^2_{(\gamma),n} - \tfrac{1}{2}\delta^2_n \sigma^2_{(\gamma),n}
\end{aligned}$$

6 European Swaption Valuation

6.1 Closed-Form Approximate Valuation

In order to derive a closed-form approximation for the values of European swaptions we write the values of receiver and payer European swaptions at time t in the following form: [1]

$$V_{rec}(t) = \sum_{j=1}^{n} d_{j-1,j}\, P(t,T_j) E_t^{(T_j)}\Big[(\kappa - L_{j-1}(T_0))\, I_{A_{rec}}\Big] \tag{39}$$

$$V_{pay}(t) = \sum_{j=1}^{n} d_{j-1,j}\, P(t,T_j) E_t^{(T_j)}\Big[(L_{j-1}(T_0) - \kappa)\, I_{A_{pay}}\Big]$$

Here κ is the swap strike rate; T_0 is the swaption expiration date; T_n is the swaption maturity date; swap resets occur at times $T_0, \ldots, T_{n-1}$; $d_{j-1,j}$ is the day-count fraction for the time interval from T_{j-1} to T_j; $I_{A_{rec}}$ and $I_{A_{pay}}$ are equal to 1 when the swaption expires in-the-money and 0 otherwise; in mathematics, $I_{A_{rec}}$ and $I_{A_{pay}}$ are called the indicator functions of a set of events A_{rec} and A_{pay} corresponding to the swaption ending up in the money:

$$A_{pay} = \{\, w_{T_0}(T_0, n) > \kappa \,\} = \Big\{ \sum_{j=1}^{n} C_j\, P(T_0, T_j) < 1 \Big\} \tag{40}$$

$$A_{rec} = \{\, w_{T_0}(T_0, n) < \kappa \,\} = \Big\{ \sum_{j=1}^{n} C_j\, P(T_0, T_j) > 1 \Big\}$$

where $w_{T_0}(T_0, n)$ is the swap rate at time T_0; $C_j = d_{j-1,j}\,\kappa$ for $j = 1, \ldots, n-1$ and $C_n = 1 + d_{n-1,n}\,\kappa$.

[1] Alan Brace, Gatarek D., and Musiela M., *The Market Model of Interest Rate Dynamics*, UNSW Working Paper, May 1995.

A Deriving the Drift Formula

A.1 Case 1: Terminal Measure Approach

We take the zero-coupon bond maturing at time T_{N+1} as our numeraire and derive drift formulas for all our N forward LIBOR rates starting with L_N. Taking the last date as the maturity of our zero-coupon bond numeraire is frequently referred to as the *terminal measure approach.*

Drift for $\mathbf{L_N}$

The random process followed by the forward rate L_N has a simple relationship to the process followed by the forward price $P_N(t)/P_{N+1}(t)$:

$$d\left(\frac{P_N(t)}{P_{N+1}(t)}\right) = d(1+\delta_N L_N) = \delta_N dL_N \tag{41}$$

When $P_{N+1}(t)$ is our numeraire, the forward price $P_N(t)/P_{N+1}(t)$ follows a process with zero drift. As a result, the drift for L_N must also be zero ($\mu_{L_N} = 0$).

Drift for $\mathbf{L_{N-1}}$

We find the drift for L_{N-1} so that the drift of the processes followed by the forward price P_{N-1}/P_{N+1} is zero. Using the basic theorems of stochastic calculus we write:

$$\begin{aligned} d\left(\frac{P_{N-1}(t)}{P_{N+1}(t)}\right) &= d\left(\frac{P_{N-1}(t)}{P_N(t)}\frac{P_N(t)}{P_{N+1}(t)}\right) \\ &= \frac{P_{N-1}(t)}{P_N(t)}d\left(\frac{P_N(t)}{P_{N+1}(t)}\right) + \frac{P_N(t)}{P_{N+1}(t)}d\left(\frac{P_{N-1}(t)}{P_N(t)}\right) + d\left(\frac{P_{N-1}(t)}{P_N(t)}\right)d\left(\frac{P_N(t)}{P_{N+1}(t)}\right) \\ &= (1+\delta_{N-1}L_{N-1})\delta_N dL_N + (1+\delta_N L_N)\delta_{N-1}dL_{N-1} + \delta_{N-1}\delta_N\sigma_{N-1,N}\, dt \end{aligned} \tag{42}$$

Since the drift of the left hand side is zero, we obtain the following equation:

$$0 = (1+\delta_{N-1}L_{N-1})\delta_N\mu_{L_N} + (1+\delta_N L_N)\delta_{N-1}\mu_{L_{N-1}} + \delta_{N-1}\delta_N\sigma_{N-1,N} \tag{43}$$

After setting μ_N to zero in the above equation, we obtain a formula for $\mu_{L_{N-1}}$:

$$\mu_{L_{N-1}} = -\frac{\delta_N}{1+\delta_N L_N}\sigma_{N-1,N} \tag{44}$$

Drift for $\mathbf{L_1, ..., L_{N-2}}$

Now, we find the drift for L_k $(k < N-1)$. Using stochastic calculus we write:

$$d\left(\frac{P_k(t)}{P_{N+1}(t)}\right) = d\left(\frac{P_k(t)}{P_{k+1}(t)}\frac{P_{k+1}(t)}{P_{N+1}(t)}\right) \tag{45}$$

$$= \frac{P_k(t)}{P_{k+1}(t)} d\left(\frac{P_{k+1}(t)}{P_{N+1}(t)}\right) + \frac{P_{k+1}(t)}{P_{N+1}(t)} d\left(\frac{P_k(t)}{P_{k+1}(t)}\right) + d\left(\frac{P_k(t)}{P_{k+1}(t)}\right) d\left(\frac{P_{k+1}(t)}{P_{N+1}(t)}\right)$$

Since the drifts of the processes for time T_{N+1} forward prices have zero drifts, we obtain the following equation for μ_{L_k}:

$$\frac{P_{k+1}(t)}{P_{N+1}(t)} \delta_k \mu_k dt = -d\left(\frac{P_k(t)}{P_{k+1}(t)}\right) d\left(\frac{P_{k+1}(t)}{P_{N+1}(t)}\right) \tag{46}$$

Using the chain formula for stochastic calculus we obtain:

$$d\left(\frac{P_{k+1}(t)}{P_{N+1}(t)}\right) = d\left(\frac{P_{k+1}(t)}{P_{k+2}(t)} \cdots \frac{P_N(t)}{P_{N+1}(t)}\right) = \frac{P_{k+1}(t)}{P_{N+1}(t)} \sum_{n=k+1}^{N} \frac{P_{n+1}(t)}{P_n(t)} d\left(\frac{P_n(t)}{P_{n+1}(t)}\right) + (...)dt \tag{47}$$

Now, we rewrite the formula for μ_k in the following form:

$$\delta_k \mu_{L_k} dt = -\sum_{n=k+1}^{N} \frac{P_{n+1}(t)}{P_n(t)} d\left(\frac{P_n(t)}{P_{n+1}(t)}\right) d\left(\frac{P_k(t)}{P_{k+1}(t)}\right) = -\sum_{n=k+1}^{N} \frac{\delta_n \delta_k}{1+\delta_n L_n} \sigma_{n,k} dt \tag{48}$$

This simply means that:

$$\mu_{L_k} = -\sum_{n=k+1}^{N} \frac{\delta_n}{1+\delta_n L_n} \sigma_{n,k} \tag{49}$$

A.2 Case 2: First-Reset Measure Approach

We take the zero-coupon bond maturing at time T_1 as our numeraire and derive drift formulas for all our N forward LIBOR rates starting with L_1. We call this the *first-reset measure* approach. This is because this approach is convenient for valuation of European swaptions whose underlying swap's first reset falls on the swaption expiration date.

Drift for $\mathbf{L_1}$

We find the drift for L_1 so that the drift of the processes followed by the forward price P_2/P_1 is zero. Using the basic theorems of stochastic calculus we write:

$$
\begin{aligned}
d\left(\frac{P_2(t)}{P_1(t)}\right) &= d\left(\frac{1}{1+\delta_1 L_1}\right) \\
&= -\left(\frac{P_2(t)}{P_1(t)}\right)^2 d\left(\frac{P_1(t)}{P_2(t)}\right) + \left(\frac{P_2(t)}{P_1(t)}\right)^3 d\left(\frac{P_1(t)}{P_2(t)}\right) d\left(\frac{P_1(t)}{P_2(t)}\right)
\end{aligned} \tag{50}
$$

We express the right-hand side of the above equation in terms of L_1. After setting the drift of both sides to zero we find the formula for μ_1:

$$
\mu_{L_1} = \frac{\delta_1}{1+\delta_1 L_1}\sigma_1^2 \tag{51}
$$

Drift for $\mathbf{L_1, \ldots, L_{N-2}}$

Now, we find the drift for L_k $(k > 1)$. Using stochastic calculus we write:

$$
d\left(\frac{P_{k+1}(t)}{P_1(t)}\right) = \frac{P_{k+1}}{P_1}\sum_{i=1} k \frac{P_i}{P_{i+1}} d\left(\frac{P_{i+1}(t)}{P_i(t)}\right) + \frac{P_{k+1}}{P_1}\sum_{i=1}^{k}\sum_{i=j+1}^{k}\frac{P_i}{P_{i+1}}\frac{P_j}{P_{j+1}} d\left(\frac{P_{i+1}(t)}{P_i(t)}\right) d\left(\frac{P_{j+1}(t)}{P_j(t)}\right) \tag{52}
$$

Next, we take an advantage of the fact that:

$$
d\left(\frac{P_{i+1}(t)}{P_i(t)}\right) d\left(\frac{P_{j+1}(t)}{P_j(t)}\right) = \left(\frac{P_{i+1}}{P_i}\right)^2 \left(\frac{P_{j+1}}{P_j}\right)^2 d\left(\frac{P_i(t)}{P_{i+1}(t)}\right) d\left(\frac{P_j(t)}{P_{j+1}(t)}\right) \tag{53}
$$

and obtain the following formula:

$$
\frac{P_1}{P_{k+1}} d\left(\frac{P_{k+1}(t)}{P_1(t)}\right) = \frac{P_1}{P_k} d\left(\frac{P_k(t)}{P_1(t)}\right) + \frac{P_k}{P_{k+1}} d\left(\frac{P_{k+1}(t)}{P_k(t)}\right) + \sum_{i=1}^{k-1}\frac{P_{i+1}}{P_i}\frac{P_{k+1}}{P_k} d\left(\frac{P_i(t)}{P_{i+1}(t)}\right) d\left(\frac{P_k(t)}{P_{k+1}(t)}\right) \tag{54}
$$

After taking into account that the drifts of processes for P_{k+1}/P_1 and P_k/P_1 are zero we obtain the following formula for μ_k:

$$
\mu_{L_k} = \sum_{i=1}^{k}\frac{\delta_i}{1+\delta_i L_i}\sigma_{i,k} \tag{55}
$$

A.3 General Case

By collecting the results from the two previous cases we obtain the following drift formula for the n-th forward LIBOR rate when the numeraire is a zero coupon bond maturing at time T_n:

$$
\begin{aligned}
\mu_n &= \sum_{i=k_{ext}}^{n} \frac{\delta_i}{1+\delta_i L_i}\sigma_{i,n} && \text{when} \quad k_{ext} \leq n \\
\mu_n &= 0 && \text{when} \quad k_{ext} = n+1 \\
\mu_n &= -\sum_{i=n+1}^{k_{ext}-1} \frac{\delta_i}{1+\delta_i L_i}\sigma_{i,n} && \text{when} \quad k_{ext} > n
\end{aligned}
\tag{56}
$$