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J. Math. Anal. Appl. 292 (2004) 381–400

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Theta-function identities and the explicit formulas for theta-function and their applications

Jinhee Yi

Department of Mathematics, Pusan National University, 30 Jangjeon-dong, Geumjeong-gu, Busan 609-735, Republic of Korea

Received 1 November 2003

Submitted by B.C. Berndt

Abstract

We define two quotients of theta-functions depending on two positive real parameters. We then show how they are connected with two parameters of Dedekind eta-function and the Ramanujan–Weber class invariants. Explicit formulas for determining values of the theta-function is derived, and several examples will be given and using them, we give some complete explicit results for the complete elliptic integral of the first kind and the Gaussian hypergeometric function. Also several new modular equations for the theta-function are derived.

1. Introduction

We begin this section by introducing the standard notion

$$(a;q)_0 := 1, \tag{1.1}$$

$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k) \quad \text{for } n = 1, 2, \ldots, \tag{1.2}$$

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n) \quad \text{for } |q| < 1. \tag{1.3}$$

E-mail addresses: jinheeyi@math.uiuc.edu, jinheeyi@pusan.ac.kr.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2003.12.009

Furthermore, for any integer n, we set

$$(a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \quad |q| < 1.$$

It is tacitly assumed in the sequel that $|q| < 1$ always. We now give Ramanujan's definition of his general theta-function. For $|ab| < 1$, define

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{1.4}$$

We introduce two theta-functions that play central roles. In [1, Entry 22, p. 36], they are defined by, for $q = e^{2\pi i z}$,

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty (-q^2;q^2)_\infty} =: \theta_3(0,2z) \tag{1.5}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty =: q^{-1/24}\eta(z), \tag{1.6}$$

where $\operatorname{Im} z > 0$, $\eta(z)$ denotes the Dedekind eta-function, and θ_3 is the classical theta-function [3, p. 36]. The Jacobi triple product identity [1, Entry 19, p. 35] is used to obtain the third equality of Eq. (1.6).

In this paper, we introduce two parameterizations $h_{k,n}$ and $h'_{k,n}$ of the theta-function φ for any positive real numbers n and k. We then show how they are connected with two parameters of the Dedekind eta-function [12] and the Ramanujan–Weber class invariants. Explicit formulas for determining values of the theta-function are derived, and several examples will be given and using them, we give some complete explicit results for the complete elliptic integral of the first kind and the Gaussian hypergeometric function. Also, several new modular equations for the theta-function are derived.

Ramanujan derived 23 beautiful theta-function identities [2, p. 192, pp. 204–237], which are certain types of modular equations. We found more than 70 of certain types of modular equations, and some of them are given in [10,11]. There are many definitions of a modular equation in the literature. We now give the definition of a modular equation that Ramanujan employed and the one that we shall use in the sequel. First, the complete elliptic integral of the first kind $K(k)$ is defined by

$$\begin{aligned} K(k) &= \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \frac{\pi}{2}\sum_{n=0}^{\infty} \frac{(1/2)_n^2}{(n!)^2} k^{2n} = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right) \\ &= \frac{\pi}{2}\varphi^2\left(e^{-\pi K'/K}\right), \end{aligned} \tag{1.7}$$

where ${}_2F_1$ denotes the ordinary or Gaussian hypergeometric function, $0 < k < 1$, $K' = K(k')$, where $k' := \sqrt{1-k^2}$, and

$$(\alpha)_n = \alpha(\alpha+1)\dots(\alpha+n-1)$$

for each nonnegative integer n. The number k is called the modulus of K, and k' is called the complementary modulus.

Definition 1.1. Let K, K', L, and L' denote complete elliptic integrals of the first kind associated with the moduli k, k', l, and l', respectively, where $0 < k, l < 1$. Suppose that

$$\frac{L'}{L} = n\frac{K'}{K} \tag{1.8}$$

holds for some positive integer n. A relation between k and l induced by (1.8) is called a modular equation of degree n.

If we set

$$q = \exp\left(-\pi\frac{K'}{K}\right) \quad \text{and} \quad q' = \exp\left(-\pi\frac{L'}{L}\right),$$

we see that (1.8) is equivalent to the relation $q^n = q'$. Thus, a modular equation can be viewed as an identity involving theta-functions at the arguments q and q^n. Let $K'/K = \sqrt{m}$ and $K = K[\sqrt{m}\,]$. Then once K is known for any given m it is comparatively simple to find K for $n\sqrt{m}$, where n is a positive integer, using the modular equation of degree n. Indeed $K[n\sqrt{m}\,] =$ algebraic number $\times$ $K[\sqrt{m}\,]$. Hence the value of k when m is square-free is of interest.

It is well known that for $m = 1, 3$, and 4, we may express K in terms of Γ-functions, and the results are given explicitly in [8]. The result for $m = 2$ is given explicitly in [6]. Selberg and Chowla [5] gave an explicit result for K in terms of Γ-functions when $h(-m) = 1$, where $h(-\Delta)$ is the number of classes of the binary quadratic form $ax^2 + bxy + cy^2 = (a, b, c)$ of negative fundamental discriminant

$$-\Delta = b^2 - 4ac$$

with a, b, c, x, y all integers. They then calculate K explicitly for $m = 7$. They further show how to evaluate K for when $h(-4m) = 2$ and thus find K for $m = 5$. Finally they prove that K is expressible in terms of Γ-functions for all rational m but do not give a complete explicit result. Zucker [13] gave the explicit results for $m = 1, 2, \dots, 13$, 15, and 16. From (1.7), if we know the value of $\varphi(e^{-\pi\sqrt{n}})$ for some number n, we can derive the explicit result of $K[n]$ and k will be evaluated in a simple fashion for certain n.

We introduce the gamma-function $\Gamma(x)$. For all complex numbers $x \neq 0, -1, -2, \dots,$ the gamma-function $\Gamma(x)$ is defined by

$$\Gamma(x) = \lim_{k\to\infty}\frac{k!k^{x-1}}{(x)_k}.$$

Next we introduce the Ramanujan–Weber class invariants. For $q = e^{-\pi\sqrt{n}}$, where n is a positive real number, define the two class invariants G_n and g_n by

$$G_n = 2^{-1/4}q^{-1/24}\chi(q) \quad \text{and} \quad g_n = 2^{-1/4}q^{-1/24}\chi(-q), \tag{1.9}$$

where

$$\chi(q) = (-q; q^2)_\infty. \tag{1.10}$$

The term "class invariant" is due to Weber. In the notation of Weber [9], $G_n := 2^{-1/4} f(\sqrt{-n})$ and $g_n := 2^{-1/4} f_1(\sqrt{-n})$. If $Q(\omega)$ is the algebraic number field generated by the complex quadratic integer ω, such that $\{1, \omega\}$ is a basis for the algebraic integral domain, which is also called the maximal order of $Q(\omega)$, then the absolute class field of $Q(\omega)$ is generated by the modular invariant $j(\omega)$, which Weber calls a class invariant. It is well known that g_n and G_n are algebraic and frequently units [3, p. 184], [4, p. 214]. Ramanujan calculated a total of 116 class invariants [3, pp. 189–204], and the table at the end of Weber's book [9, pp. 721–726] contains the values of 105 class invariants. Weber primarily was motivated to calculate class invariants so that he could construct Hilbert class fields. Ramanujan apparently had no knowledge of class field theory and independently calculated class invariants for different reasons. We now introduce the definitions of $r_{k,n}$ and $r'_{k,n}$ from [12]. For all positive real numbers n and k, define $r_{k,n}$ by

$$r_{k,n} := \frac{f(-q)}{k^{1/4} q^{(k-1)/24} f(-q^k)} = \frac{\eta(i\sqrt{n/k})}{k^{1/4}\eta(i\sqrt{nk})}, \tag{1.11}$$

where $q = e^{-2\pi\sqrt{n/k}}$, and define $r'_{k,n}$ by

$$r'_{k,n} := \frac{f(q)}{k^{1/4} q^{(k-1)/24} f(q^k)} = \frac{\eta((1 + i\sqrt{n/k})/2)}{k^{1/4}\eta((1 + i\sqrt{nk})/2)}, \tag{1.12}$$

where $q = e^{-\pi\sqrt{n/k}}$. We found more than 800 values of $r_{k,n}$ and $r'_{k,n}$ in [12] by using eta-function identities and the properties of the two parameterizations. Using them, we derived some new values for G_n and g_n in [12].

In Section 2, we introduce two parameters $h_{k,n}$ and $h'_{k,n}$ involving theta-functions $\varphi(q)$ for any positive real numbers k and n. We establish some properties of these parameters.

In Section 3, we establish relations among $h_{k,n}$, $h'_{k,n}$, $r_{k,n}$, $r'_{k,n}$ and the class invariants.

In Section 4, we give some new theta-function identities. In addition, we establish some formulas for $h_{k,n}$ and $h'_{k,n}$ by using those theta-function identities. Particular values of theta-functions are determined.

In Section 5, we offer explicit formulas for $\varphi(-e^{-n\pi})$ and $\varphi(e^{-n\pi})$ for any positive real numbers n. We give some examples.

2. Two parameterizations $h_{k,n}$ and $h'_{k,n}$ for the theta-function φ and some properties of them

In this section, we introduce two useful parameterizations $h_{k,n}$ and $h'_{k,n}$ of theta-function for all positive real numbers k and n. We then establish some of their properties.

Definition 2.1. For any positive real numbers k and n, define $h_{k,n}$ by

$$h_{k,n} = \frac{\varphi(e^{-\pi\sqrt{n/k}})}{k^{1/4}\varphi(e^{-\pi\sqrt{nk}})} = \frac{\theta_3(0, i\sqrt{n/k})}{k^{1/4}\theta_3(0, i\sqrt{nk})} \tag{2.1}$$

and define $h'_{k,n}$ by

$$h'_{k,n}=\frac{\varphi(-e^{-2\pi\sqrt{n/k}})}{k^{1/4}\varphi(-e^{-2\pi\sqrt{nk}})}=\frac{\theta_3(0,1+2i\sqrt{n/k})}{k^{1/4}\theta_3(0,1+2i\sqrt{nk})}. \tag{2.2}$$

For the proofs of the following theorems, we recall the transformation formula for φ [1, Entry 27(i), p. 43]. If $a,b>0$ with $ab=\pi$, then

$$\sqrt{a}\varphi\bigl(e^{-a^2}\bigr)=\sqrt{b}\varphi\bigl(e^{-b^2}\bigr). \tag{2.3}$$

In particular, if $a^2=\pi/\sqrt{n}$, then, from (1.1),

$$\varphi(e^{-\pi/\sqrt{n}})=n^{1/4}\varphi(e^{-\pi\sqrt{n}}). \tag{2.4}$$

Theorem 2.2. *For all positive real numbers k and n,*

(i) $h_{k,1}=1$,

(ii) $h_{k,1/n}=h^{-1}_{k,n}$,

(iii) $h_{k,n}=h_{n,k}$.

Proof. By using the definition of $h_{k,n}$ and (2.4), we find that

$$h_{k,1}=\frac{\varphi(e^{-\pi/\sqrt{k}})}{k^{1/4}\varphi(e^{-\pi\sqrt{k}})}=1.$$

By using the definition of $h_{k,n}$ and replacing n in (2.4) by k/n and nk, respectively, we find that

$$h_{k,n}h_{k,1/n}=\frac{\varphi(e^{-\pi\sqrt{n/k}})}{k^{1/4}\varphi(e^{-\pi\sqrt{nk}})}\frac{\varphi(e^{-\pi/\sqrt{nk}})}{k^{1/4}\varphi(e^{-\pi\sqrt{k/n}})}=1.$$

Using again the definition of $h_{k,n}$ and (2.4) with n replaced by k/n, we find that

$$\frac{h_{k,n}}{h_{n,k}}=\frac{\varphi(e^{-\pi\sqrt{n/k}})}{k^{1/4}\varphi(e^{-\pi\sqrt{nk}})}\frac{n^{1/4}\varphi(e^{-\pi\sqrt{nk}})}{\varphi(e^{-\pi\sqrt{k/n}})}=1.$$

So we complete the proof. □

Remark 2.3. By using the definitions of $\varphi(q)$ and $h_{k,n}$ we notice that $h_{k,n}$ decreases as n increases when $k>1$. From (i) of Theorem 2.2, $h_{k,n}<1$ for all $n>1$ if $k>1$.

Lemma 2.4. *For all positive real numbers k, m, and n,*

$$h_{k,n/m}=h_{mk,n}h^{-1}_{nk,m}.$$

Proof. By using the definition of $h_{k,n}$, we find that

$$h_{mk,n}h^{-1}_{nk,m}=\frac{\varphi(e^{-\pi\sqrt{n/mk}})}{(mk)^{1/4}\varphi(e^{-\pi\sqrt{nmk}})}\frac{(nk)^{1/4}\varphi(e^{-\pi\sqrt{nmk}})}{\varphi(e^{-\pi\sqrt{m/nk}})}=h_{m/n,1/k}.$$

Using (ii) and (iii) of Theorem 2.2, we complete the proof. □

Theorem 2.5. *For all positive real numbers a, b, c, and d,*

$$h_{a/b,c/d} = \frac{h_{ad,bc}}{h_{ac,bd}}.$$

Proof. By using Theorem 2.2(iii) in Lemma 2.4, we deduce that, for all positive real numbers a, b, and n,

$$h_{a/b,n} = h_{a,bn} h_{b,an}^{-1}. \tag{2.5}$$

By using (2.5), Lemma 2.4, and Theorem 2.2(iii), we find that

$$h_{a/b,c/d} = h_{a,bc/d} h_{b,ac/d}^{-1} = \left(h_{ad,bc} h_{abc,d}^{-1}\right)\left(h_{bd,ac} h_{abc,d}^{-1}\right)^{-1} = \frac{h_{ad,bc}}{h_{ac,bd}}.$$

So we complete the proof. □

Corollary 2.6. *For all positive real numbers k and n,*

$$h_{k^2,n} = h_{k,nk} h_{k,n/k}.$$

Proof. Letting $a = k$, $b = 1/k$, $c = n$, and $d = 1$ in Theorem 2.5, we deduce that

$$h_{k^2,n} = \frac{h_{k,n/k}}{h_{kn,1/k}}.$$

By using (ii) and (iii) of Theorem 2.2, we complete the proof. □

Corollary 2.7. *For all positive real numbers a and b,*

(i) $h_{a/b,a/b} = h_{b,b} h_{a,a/b^2}$,

(ii) $h_{a,a} h_{a,b^2/a} = h_{b,b} h_{b,a^2/b}$,

(iii) $h_{a,a} h_{b,a^2 b} = h_{b,b} h_{a,ab^2}$.

Proof. Let a and b be any positive real numbers. By using Theorems 2.5 and 2.2(ii), we find that

$$h_{a/b,a/b} = h_{a/b,1/(b/a)} = h_{b,b} h_{a,b^2/a}^{-1} = h_{b,b} h_{a,a/b^2}. \tag{2.6}$$

Thus we prove (i). Similarly, we find that

$$h_{b/a,b/a} = h_{a,a} h_{b,a^2/b}^{-1}. \tag{2.7}$$

From (2.6), (2.7), and Theorem 2.2(ii), we derive (ii). By using (i) and Lemma 2.4, we find that

$$h_{a/b,a/b} = h_{b,b} h_{a,a/b^2} = h_{b,b} h_{ab^2,a} h_{a^2,b^2}^{-1}. \tag{2.8}$$

Similarly, we find that

$$h_{b/a,b/a} = h_{a,a} h_{b,b/a^2} = h_{a,a} h_{ba^2,b} h_{b^2,a^2}^{-1}. \tag{2.9}$$

From (2.8), (2.9), and Theorems 2.2(ii) and (iii), we complete the proof of (iii). □

Theorem 2.8. *For all positive real numbers k, a, b, c, and d, with $ab = cd$,*

$$h_{a,b}h_{kc,kd} = h_{ka,kb}h_{c,d}.$$

Proof. From the definition of $h_{k,n}$, we find that, for all positive real numbers k, a, b, c, and d,

$$h_{ka,kb}h_{a,b}^{-1} = \frac{\varphi(e^{-\pi\sqrt{ab}})}{k^{1/4}\varphi(e^{-\pi k\sqrt{ab}})} \tag{2.10}$$

and

$$h_{kc,kd}h_{c,d}^{-1} = \frac{\varphi(e^{-\pi\sqrt{cd}})}{k^{1/4}\varphi(e^{-\pi k\sqrt{cd}})}. \tag{2.11}$$

The result follows from (2.10), (2.11), and the hypothesis $ab = cd$. □

Corollary 2.9. *For all positive real numbers n and p, we have*

$$h_{np,np} = h_{n,np^2}h_{p,p}.$$

Proof. The result follows immediately from Theorems 2.2(i), (iii), and 2.8 with $a = p^2$, $b = 1$, $c = d = p$, and $k = n$. □

Theorem 2.10. *For any positive real number n, we have*

$$\frac{\varphi(e^{-\pi})}{\varphi(e^{-2^n\pi})} = 2^{n/4}\prod_{i=1}^{n} h_{2,2^{2i-1}}.$$

Proof. Using the definition of $h_{k,n}$ with $n = 1$, we find that the result is true for $n = 1$. Suppose that it is true for $n = k$; then we have

$$\frac{\varphi(e^{-\pi})}{\varphi(e^{-2^k\pi})} = 2^{k/4}\prod_{i=1}^{k} h_{2,2^{2i-1}}. \tag{2.12}$$

Using the definition of $h_{k,n}$ and (2.12), we deduce that

$$\begin{aligned}\frac{\varphi(e^{-\pi})}{\varphi(e^{-2^{k+1}\pi})} &= \left(2^{k/4}\prod_{i=1}^{k} h_{2,2^{2i-1}}\right)\frac{\varphi(e^{-2^k\pi})}{\varphi(e^{-2^{k+1}\pi})} = \left(2^{k/4}\prod_{i=1}^{k} h_{2,2^{2i-1}}\right)2^{1/4}h_{2,2^{2k+1}} \\ &= 2^{(k+1)/4}\prod_{i=1}^{k+1} h_{2,2^{2i-1}}.\end{aligned}$$

So we complete the proof. □

3. Relations among $h_{k,n}$, h'_{k_1,n_1}, $r_{k,n}$ and $r'_{k,n}$

In this section, we state relations among $h_{k,n}$, h'_{k_1,n_1}, $r_{k,n}$, and $r'_{k,n}$. We then can determine the values of $h_{k,n}$ and $h'_{k,n}$ by using known values of $r_{k,n}$ and $r'_{k,n}$ in [12].

Theorem 3.1. *Let k and n be any positive real numbers. Then*

$$\text{(i)} \quad h_{k,n} = \frac{(r'_{k,n})^2}{r_{k,n}},$$

$$\text{(ii)} \quad h'_{k,n} = \frac{r_{2,2n/k}}{r_{2,2nk}} r_{k,n}.$$

Proof. (i) Let $q = e^{-\pi\sqrt{n/k}}$. From [1, Entry 24(iii), p. 39], we have

$$\varphi(q) = \chi(q) f(q). \tag{3.1}$$

Using (3.1) and the definitions of $h_{k,n}$ and $r'_{k,n}$, we find that

$$h_{k,n} = \frac{\varphi(q)}{k^{1/4}\varphi(q^k)} = \frac{\chi(q) f(q)}{k^{1/4}\chi(q^k) f(q^k)} = \frac{\chi(q)}{\chi(q^k)} q^{(k-1)/24} r'_{k,n}.$$

Using the definition of G_n in (1.9) and Theorem 3.1 in [12], we deduce that

$$h_{k,n} = \frac{G_{n/k}}{G_{nk}} r'_{k,n} = \frac{(r'_{k,n})^2}{r_{k,n}}. \tag{3.2}$$

So we complete the proof of (i).

(ii) Similarly, let $q = e^{-2\pi\sqrt{n/k}}$. Using (3.1) and the definitions of $h'_{k,n}$, $r_{k,n}$, and g_n, we find that

$$\begin{aligned} h'_{k,n} &= \frac{\varphi(-q)}{k^{1/4}\varphi(-q^k)} = \frac{\chi(-q) f(-q)}{k^{1/4}\chi(-q^k) f(-q^k)} = \frac{\chi(-q)}{\chi(-q^k)} q^{(k-1)/24} r_{k,n} \\ &= \frac{g_{4n/k}}{g_{4nk}} r_{k,n}, \end{aligned} \tag{3.3}$$

which, upon using Theorem 3.3(i) in [12], proves (ii). □

Corollary 3.2. *For every positive real number k, we have*

$$h'_{k,1} = \frac{r_{2,2/k}}{r_{2,2k}}.$$

Proof. The result follows directly from Theorem 3.1(ii) with $n = 1$ and the value $r_{k,1} = 1$, for any positive real number k. □

The following corollaries show the relations among the parameters and the class invariants G_n and g_n. Ramanujan recorded several simple formulas relating these invariants, and we state below, one of them that is needed in our proofs. For $n > 0$, [3, Entry 2.1, p. 187]

$$g_{4n} = 2^{1/4} g_n G_n. \tag{3.4}$$

Corollary 3.3. *For all positive real numbers k and n, we have*

$$\text{(i)}\quad h_{k,n} = \frac{G_{n/k}}{G_{nk}} r'_{k,n},$$

$$\text{(ii)}\quad h'_{k,n} = \frac{g_{n/k}}{g_{nk}} r'_{k,n}.$$

Proof. Part (i) follows directly from (3.2). Using (3.4) and Theorem 3.1 of [12] in (3.3), we find that

$$h'_{k,n} = \frac{g_{n/k} G_{n/k}}{g_{nk} G_{nk}} r_{k,n} = \frac{g_{n/k}}{g_{nk}} r'_{k,n}.$$

So we complete the proof (ii). □

Corollary 3.4. *For every positive real number n, we have*

$$\text{(i)}\quad h_{n,n} = \frac{r_{n,n}}{G^2_{n^2}},$$

$$\text{(ii)}\quad h'_{n,n} = \frac{2^{1/8} r_{n,n}}{r_{2,2n^2}}.$$

Proof. (i) With $k = n$ in Theorem 3.1(i), we find that

$$h_{n,n} = \frac{(r'_{n,n})^2}{r_{n,n}}.$$

By using Corollary 3.2 in [12], we complete the proof.

(ii) Letting $k = n$ in Theorem 3.1(ii) and using the value $r_{2,2} = 2^{1/8}$ from Theorem 5.2(i) in [12], we complete the proof of (ii). □

4. Theta-function identities and values of $h_{k,n}$ and $h'_{k,n}$

This section is devoted to stating and proving certain modular equations, P–Q modular equations, which we will use in the sequel. Ramanujan discovered 23 P–Q modular equations [2, p. 192, pp. 204–237]. We have found about 70 new P–Q modular equations by using Garvan's Maple q-series package. We state some of them below, providing proofs for the new ones.

See Definition 1.1 in Section 1 for the definition of modular equation.

Definition 4.1. The multiplier m for a modular equation of degree n is defined by

$$m := \frac{K}{L} = \frac{{}_2F_1(1/2, 1/2; 1; \alpha)}{{}_2F_1(1/2, 1/2; 1; \beta)} = \frac{z_1}{z_n},$$

where

$$z_r = \varphi^2(q^r).$$

We prove some new identities for the theta-function φ. We then use them for evaluating some values of $h_{k,n}$ and $h'_{k,n}$.

Theorem 4.2. *Let*

$$P=\frac{\varphi(q)}{\varphi(q^2)} \quad \textit{and} \quad Q=\frac{\varphi(q^2)}{\varphi(q^4)}.$$

Then

$$PQ+\frac{2}{PQ}=\frac{Q}{P}+2.$$

Proof. From Entries 10(i), (iv), and (v) in [1, p. 122], we deduce that

$$\begin{aligned}\varphi^2(q^4)&=\frac{1}{4}z_1\bigl(1+(1-\alpha)^{1/4}\bigr)^2\\&=\frac{1}{4}z_1(1+\sqrt{1-\alpha})+\frac{1}{2}z_1(1-\alpha)^{1/4}\\&=\frac{1}{4}z_1(1+\sqrt{1-\alpha})+\frac{1}{2}z_1\bigl(1+(1-\alpha)^{1/4}\bigr)-\frac{1}{2}z_1\\&=\frac{1}{2}\varphi^2(q^2)+\varphi(q)\varphi(q^4)-\frac{1}{2}\varphi^2(q).\end{aligned} \tag{4.1}$$

Dividing (4.1) by $\varphi(q)\varphi(q^4)$ and multiplying by 2, we complete the proof. □

Theorem 4.3. *Let*

$$P=\frac{\varphi(q)}{\varphi(q^3)} \quad \textit{and} \quad Q=\frac{\varphi(q^3)}{\varphi(q^9)}.$$

Then

$$PQ+\frac{3}{PQ}=\left(\frac{Q}{P}\right)^2+3.$$

Proof. From Entry 1(iii) in Chapter 20 of [1, p. 345], we deduce that

$$\frac{\varphi^4(q^3)}{\varphi^4(q^9)}=\frac{\varphi^3(q)}{\varphi^3(q^9)}-3\frac{\varphi^2(q)}{\varphi^2(q^9)}+3\frac{\varphi(q)}{\varphi(q^9)}. \tag{4.2}$$

Multiplying both sides of (4.2) by $\varphi^2(q^9)/\varphi^2(q)$, we complete the proof. □

Theorem 4.4. *Let*

$$P=\frac{\varphi(q)}{\varphi(q^3)} \quad \textit{and} \quad Q=\frac{\varphi(q^5)}{\varphi(q^{15})}.$$

Then

$$(PQ)^2+\left(\frac{3}{PQ}\right)^2=\left(\frac{Q}{P}\right)^3+5\left(\frac{Q}{P}\right)^2+5\left(\frac{Q}{P}\right)-5\left(\frac{P}{Q}\right)+5\left(\frac{P}{Q}\right)^2-\left(\frac{P}{Q}\right)^3.$$

Proof. We set

$$t=\left(\frac{z_3z_5}{z_1z_{15}}\right)^{1/2} \quad \text{or} \quad m'=mt^2,$$

where $m=z_1/z_3$ and $m'=z_5/z_{15}$. From [1, Eq. (11.17), p. 389],

$$m^2+\frac{9}{m^2t^4}=\frac{t^6+5t^5+5t^4-5t^2+5t-1}{t^5}.$$

Multiplying both sides by t^2, we find that

$$m^2t^2+\frac{9}{m^2t^2}=t^3+5t^2+5t-\frac{5}{t}+\frac{5}{t^2}-\frac{1}{t^3}.$$

Using Entry 10(i) of Chapter 17 in [1, p. 122], we complete the proof. □

Theorem 4.5 [2, Entry 67, p. 235]. *Let*

$$P=\frac{\varphi(q)}{\varphi(q^5)} \quad \textit{and} \quad Q=\frac{\varphi(q^3)}{\varphi(q^{15})}.$$

Then

$$PQ+\frac{5}{PQ}=\left(\frac{Q}{P}\right)^2+3\left(\frac{Q}{P}\right)+3\left(\frac{P}{Q}\right)-\left(\frac{P}{Q}\right)^2.$$

Theorem 4.6. *We have*

$$\sqrt{2}\left(h_{2,n}h_{2,4n}+\frac{1}{h_{2,n}h_{2,4n}}\right)=\frac{h_{2,4n}}{h_{2,n}}+2$$

for any positive real number n.

Proof. The theorem follows directly from Theorem 4.2 and the definition of $h_{k,n}$. □

Theorem 4.7. *We have*

(i) $$h_{2,2}=\frac{\varphi(e^{-\pi})}{2^{1/4}\varphi(e^{-2\pi})}=\sqrt{2\sqrt{2}-2},$$

(ii) $$h_{2,1/2}=\frac{\varphi(e^{-\pi/2})}{2^{1/4}\varphi(e^{-\pi})}=\sqrt{\frac{\sqrt{2}+1}{2}},$$

(iii) $$h_{2,4}=\frac{\varphi(e^{-\sqrt{2}\pi})}{2^{1/4}\varphi(e^{-2\sqrt{2}\pi})}=\sqrt{2}+1-\sqrt{\sqrt{2}+1},$$

(iv) $$h_{2,1/4}=\frac{\varphi(e^{-\pi/2\sqrt{2}})}{2^{1/4}\varphi(e^{-\pi/\sqrt{2}})}=\frac{1+\sqrt{\sqrt{2}-1}}{\sqrt{2}},$$

(v) $$h_{2,8}=\frac{\varphi(e^{-2\pi})}{2^{1/4}\varphi(e^{-4\pi})}=\frac{\sqrt{2+\sqrt{2}}}{\sqrt[4]{2}+1},$$

(vi) $$h_{2,1/8}=\frac{\varphi(e^{-\pi/4})}{2^{1/4}\varphi(e^{-\pi/2})}=\frac{2^{-1/4}+1}{\sqrt{1+\sqrt{2}}}.$$

Proof. For (i) and (ii), setting $n = 1/2$ in Theorem 4.6 and recalling that $h_{k,1/n} = 1/h_{k,n}$ from Theorem 2.2(ii), we find that

$$2\sqrt{2} = h_{2,2}^2 + 2,$$

which (i) follows immediately by using the fact that $h_{2,2} > 0$. Using the identity $h_{2,1/2} = h_{2,2}^{-1}$, we complete the proof of (ii).

For (iii) and (iv), setting $n = 1$ in Theorem 4.6 and recalling the value $h_{k,1} = 1$ from Theorem 2.2(i), we find that

$$\sqrt{2}\left(h_{2,4} + h_{2,4}^{-1}\right) = h_{2,4} + 2. \tag{4.3}$$

Multiplying both sides of (4.3) by $h_{2,4}$, we deduce that

$$(\sqrt{2} - 1)h_{2,4}^2 - 2h_{2,4} + \sqrt{2} = 0.$$

Solving for $h_{2,4}$ and using the fact from Remark 2.3 that $h_{2,4} < 1$, we prove (iii). Using the identity $h_{2,1/4} = h_{2,4}^{-1}$, we complete the proof of (iv).

For (v) and (vi), letting $n = 2$ in Theorem 4.6, we find that

$$\sqrt{2}\left(h_{2,2}h_{2,8} + \frac{1}{h_{2,2}h_{2,8}}\right) = \frac{h_{2,8}}{h_{2,2}} + 2. \tag{4.4}$$

Multiplying both sides of (4.4) by $h_{2,2}h_{2,8}$, rearranging terms, using (i), and multiplying both sides by $\sqrt{2} + 1$, we deduce that

$$(\sqrt{2} - 1)h_{2,8}^2 - 2\sqrt{2 + 2\sqrt{2}}\,h_{2,8} + 2 + \sqrt{2} = 0.$$

Solving for $h_{2,8}$ and using the fact from Remark 2.3, $h_{2,8} < 1$, we deduce that

$$h_{2,8} = \frac{\sqrt{2 + 2\sqrt{2}} - \sqrt{2 + \sqrt{2}}}{\sqrt{2} - 1} = \frac{\sqrt{2 + \sqrt{2}}(2^{1/4} - 1)}{\sqrt{2} - 1} = \frac{\sqrt{2 + \sqrt{2}}}{2^{1/4} + 1}.$$

Thus we have proved (v). Using the identity $h_{2,1/8} = h_{2,8}^{-1}$, we complete the proof of (vi). □

By using above results and properties $h_{k,n}$ in Section 2, we obtain the following result.

Corollary 4.8. *We have*

$$h_{4,4} = \frac{\varphi(e^{-\pi})}{\sqrt{2}\varphi(e^{-4\pi})} = \frac{2^{3/4}}{\sqrt[4]{2} + 1}.$$

Proof. From Corollary 2.6, we find that

$$h_{4,4} = h_{2,8}h_{2,2}. \tag{4.5}$$

Using Theorems 4.7(i) and (v), we complete the proof. □

Theorem 4.9. *For any positive real number n, we have*

$$\text{(i)}\quad \sqrt{3}\left(h_{3,n}h_{3,9n}+\frac{1}{h_{3,n}h_{3,9n}}\right)=\left(\frac{h_{3,9n}}{h_{3,n}}\right)^2+3,$$

$$\text{(ii)}\quad \sqrt{3}\left(h'_{3,n}h'_{3,9n}+\frac{1}{h'_{3,n}h'_{3,9n}}\right)=\left(\frac{h'_{3,9n}}{h'_{3,n}}\right)^2+3.$$

Proof. Part (i) follows directly from Theorem 4.3 and the definition of $h_{k,n}$. Replacing q by $-q$ in Theorem 4.3 and using the definition of $h'_{k,n}$, we complete the proof of (ii). □

Theorem 4.10. *We have*

$$\text{(i)}\quad h_{3,3}=\frac{\varphi(e^{-\pi})}{3^{1/4}\varphi(e^{-3\pi})}=(2\sqrt{3}-3)^{1/4}=\frac{3^{1/8}\sqrt{\sqrt{3}-1}}{2^{1/4}},$$

$$\text{(ii)}\quad h_{3,1/3}=\frac{\varphi(e^{-\pi/3})}{3^{1/4}\varphi(e^{-\pi})}=\left(\frac{2\sqrt{3}+3}{3}\right)^{1/4}=\frac{\sqrt{\sqrt{3}+1}}{2^{1/4}3^{1/8}},$$

$$\text{(iii)}\quad h_{3,9}=\frac{\varphi(e^{-\sqrt{3}\pi})}{3^{1/4}\varphi(e^{-3\sqrt{3}\pi})}=\frac{1}{\sqrt{3}}(1-\sqrt[3]{2}+\sqrt[3]{4}),$$

$$\text{(iv)}\quad h_{3,1/9}=\frac{\varphi(e^{-\pi/3\sqrt{3}})}{3^{1/4}\varphi(e^{-\pi/\sqrt{3}})}=\frac{1+\sqrt[3]{2}}{\sqrt{3}}.$$

Proof. For (i) and (ii), letting $n=1/3$ in Theorem 4.9(i) and using the identity $h_{3,1/3}=h_{3,3}^{-1}$ from Theorem 2.2(ii) with $k=n=3$, we find that

$$2\sqrt{3}=h_{3,3}^4+3.$$

Since $h_{3,3}$ has a positive real value,

$$h_{3,3}=(2\sqrt{3}-3)^{1/4}=3^{1/8}(2-\sqrt{3})^{1/4}=\frac{3^{1/8}(\sqrt{3}-1)^{1/2}}{2^{1/4}}.$$

So we complete the proof of (i). Using (i) and the identity $h_{3,1/3}=h_{3,3}^{-1}$ again, we prove (ii).

For proofs of (iii) and (iv), letting $n=1$ in Theorem 4.9(i) and using the value $h_{3,1}=1$ from Theorem 2.2(i), we find that

$$\sqrt{3}\big(h_{3,9}+h_{3,9}^{-1}\big)=h_{3,9}^2+3. \tag{4.6}$$

Multiplying both sides of (4.6) by $h_{3,9}$ and rearranging terms, we deduce that

$$h_{3,9}^3-\sqrt{3}h_{3,9}^2+3h_{3,9}-\sqrt{3}=0.$$

Since $h_{3,9}$ has a real value, $h_{3,9}=(1/\sqrt{3})(1-\sqrt[3]{2}+\sqrt[3]{4})$. So we complete the proof of (iii). Using (iii) and the fact $h_{3,1/9}=h_{3,9}^{-1}$, we prove (iv). □

Theorem 4.11. *We have*

$$h'_{3,1}=\frac{\varphi(-e^{-2\pi/\sqrt{3}})}{3^{1/4}\varphi(-e^{-2\sqrt{3}\pi})}=2^{-1/4}\sqrt{\sqrt{3}-1}.$$

Proof. Using Corollary 3.2 with $k=3$, we find that

$$h'_{3,1}=\frac{r_{2,2/3}}{r_{2,6}}. \tag{4.7}$$

Using the fact $r_{2,2/3}=r^{-1}_{2,3/2}$, $r_{2,3/2}=g_3=(1+\sqrt{3})^{1/4}/2^{7/24}$, and $r_{2,6}=g_{12}=2^{1/24}\times(\sqrt{3}+1)^{1/4}$ from [12], we complete the proof. □

Theorem 4.12. *For any positive real number n, we have*

$$\begin{aligned}
\text{(i)}\quad & 3\{(h_{3,n}h_{3,25n})^2+(h_{3,n}h_{3,25n})^{-2}\}\\
&=\left\{\left(\frac{h_{3,25n}}{h_{3,n}}\right)^3-\left(\frac{h_{3,25n}}{h_{3,n}}\right)^{-3}\right\}+5\left\{\left(\frac{h_{3,25n}}{h_{3,n}}\right)^2+\left(\frac{h_{3,25n}}{h_{3,n}}\right)^{-2}\right\}\\
&\quad+5\left\{\left(\frac{h_{3,25n}}{h_{3,n}}\right)-\left(\frac{h_{3,25n}}{h_{3,n}}\right)^{-1}\right\},\\
\text{(ii)}\quad & 3\{(h'_{3,n}h'_{3,25n})^2+(h'_{3,n}h'_{3,25n})^{-2}\}\\
&=\left\{\left(\frac{h'_{3,25n}}{h'_{3,n}}\right)^3-\left(\frac{h'_{3,25n}}{h'_{3,n}}\right)^{-3}\right\}+5\left\{\left(\frac{h'_{3,25n}}{h'_{3,n}}\right)^2+\left(\frac{h'_{3,25n}}{h'_{3,n}}\right)^{-2}\right\}\\
&\quad+5\left\{\left(\frac{h'_{3,25n}}{h'_{3,n}}\right)-\left(\frac{h'_{3,25n}}{h'_{3,n}}\right)^{-1}\right\}.
\end{aligned}$$

Proof. Part (i) follows directly from Theorem 4.4 and the definition of $h_{k,n}$. Replacing q by $-q$ in Theorem 4.4 and using the definition of $h'_{k,n}$, we complete the proof of (ii). □

Theorem 4.13. *We have*

$$\begin{aligned}
\text{(i)}\quad & h_{3,5}=\frac{\varphi(e^{-\sqrt{5}\pi/\sqrt{3}})}{3^{1/4}\varphi(e^{-\sqrt{15}\pi})}=\frac{\sqrt{\sqrt{5}-1}}{\sqrt{2}},\\
\text{(ii)}\quad & h_{3,1/5}=\frac{\varphi(e^{-\pi/\sqrt{15}})}{3^{1/4}\varphi(e^{-\sqrt{3}\pi/\sqrt{15}})}=\frac{\sqrt{\sqrt{5}+1}}{\sqrt{2}},\\
\text{(iii)}\quad & h_{3,25}=\frac{\varphi(e^{-5\pi/\sqrt{3}})}{3^{1/4}\varphi(e^{-5\sqrt{3}\pi})}=\frac{5^{1/6}-\sqrt{5^{1/3}-2^{2/3}}}{2^{1/3}},\\
\text{(iv)}\quad & h_{3,1/25}=\frac{\varphi(e^{-\pi/5\sqrt{3}})}{3^{1/4}\varphi(e^{-\sqrt{3}\pi/5})}=\frac{5^{1/6}+\sqrt{5^{1/3}-2^{2/3}}}{2^{1/3}}.
\end{aligned}$$

Proof. From Theorem 3.1(i), we find that

$$h_{3,5} = \frac{(r'_{3,5})^2}{r_{3,5}}.$$

Using

$$r_{3,5} = \left(\frac{1+\sqrt{5}}{2}\right)^{5/6} \quad \text{and} \quad r'_{3,5} = \left(\frac{1+\sqrt{5}}{2}\right)^{1/6}$$

from [12], we deduce that

$$h_{3,5} = \frac{\sqrt{2}}{\sqrt{\sqrt{5}+1}} = \frac{\sqrt{\sqrt{5}-1}}{\sqrt{2}}.$$

Thus we complete the proof of (i). To prove (ii), use the identity $h_{3,1/5} = h_{3,5}^{-1}$ in Theorem 2.2(ii).

For the proof of (iii), letting $n = 1$ in Theorem 4.12(i), using the value $h_{3,1} = 1$, and setting $A = h_{3,25}^2 + h_{3,25}^{-2}$, we find that

$$3A = (A+1)\sqrt{A-2} + 5A + 5\sqrt{A-2}. \tag{4.8}$$

Rearranging (4.8), we have

$$(A+6)\sqrt{A-2} = -2A. \tag{4.9}$$

Squaring both sides of (4.9) and rearranging terms, we deduce that

$$A^3 + 6A^2 + 12A - 72 = 0.$$

Since A has a real value, $h_{3,25}^2 + h_{3,25}^{-2} = A = 2(10^{1/3} - 1)$. So $h_{3,25} + h_{3,25}^{-1} = \sqrt{A+2} = \sqrt{2} \cdot 10^{1/6} = 2^{2/3}5^{1/6}$. Solving the equation

$$h_{3,25}^2 - 2^{2/3}5^{1/6}h_{3,25} + 1 = 0$$

for $h_{3,25}$ and using the fact from Remark 2.3 that $h_{3,25} < 1$, we complete the proof of (iii). To prove (iv), use the identity $h_{3,1/25} = h_{3,25}^{-1}$ in Theorem 2.2(ii). □

Theorem 4.14. *For any positive real number n, we have*

$$\begin{aligned}
\text{(i)}\quad & \sqrt{5}\left(h_{5,n}h_{5,9n} + \frac{1}{h_{5,n}h_{5,9n}}\right) \\
&= \left(\frac{h_{5,9n}}{h_{5,n}}\right)^2 - \left(\frac{h_{5,9n}}{h_{5,n}}\right)^{-2} + 3\left\{\left(\frac{h_{5,9n}}{h_{5,n}}\right) + \left(\frac{h_{5,9n}}{h_{5,n}}\right)^{-1}\right\}, \\
\text{(ii)}\quad & \sqrt{5}\left(h'_{5,n}h'_{5,9n} + \frac{1}{h'_{5,n}h'_{5,9n}}\right) \\
&= \left(\frac{h'_{5,9n}}{h'_{5,n}}\right)^2 - \left(\frac{h'_{5,9n}}{h'_{5,n}}\right)^{-2} + 3\left\{\left(\frac{h'_{5,9n}}{h'_{5,n}}\right) + \left(\frac{h'_{5,9n}}{h'_{5,n}}\right)^{-1}\right\}.
\end{aligned}$$

Proof. Part (i) follows directly from Theorem 4.5 and the definition of $h_{k,n}$. Replacing q by $-q$ in Theorem 4.5 and using the definition of $h'_{k,n}$, we complete the proof of (ii). $\square$

Theorem 4.15. *We have*

$$\text{(i)}\quad h_{5,9}=\frac{\varphi(e^{-3\pi/\sqrt{5}})}{5^{1/4}\varphi(e^{-3\sqrt{5}\pi})}=\frac{\sqrt{3}+1}{\sqrt{3}+\sqrt{5}},$$

$$\text{(ii)}\quad h_{5,1/9}=\frac{\varphi(e^{-\pi/3\sqrt{5}})}{5^{1/4}\varphi(e^{-\sqrt{5}\pi/3})}=\frac{\sqrt{3}+\sqrt{5}}{\sqrt{3}+1}.$$

Proof. To prove (i) and (ii), letting $n=1$ in Theorem 4.14(i), using the value $h_{5,1}=1$, and setting $A=h_{5,9}+h_{5,9}^{-1}$, we find that

$$\sqrt{5}A=A\sqrt{A^2-4}+3A.$$

Rearranging terms, squaring both sides, and rearranging terms again, we deduce that

$$A^2\bigl(A^2-(18-6\sqrt{5})\bigr)=0.$$

Since A has a positive real value,

$$A=\sqrt{18-6\sqrt{5}}=\sqrt{3}\sqrt{6-2\sqrt{5}}=\sqrt{3}(\sqrt{5}-1).$$

So we find that $h_{5,9}^2-(\sqrt{15}-\sqrt{3})h_{5,9}+1=0$. Solving for $h_{5,9}$ and using the fact from Remark 2.3, $h_{5,9}<1$, we complete the proof of (i). Using the identity $h_{5,1/9}=h_{5,9}$ in Theorem 2.2(ii), we prove (ii). $\square$

Theorem 4.16. *We have*

$$h'_{5,1}=\frac{\varphi(-e^{-2\pi/\sqrt{5}})}{5^{1/4}\varphi(-e^{-2\sqrt{5}\pi})}=\frac{(\sqrt{\sqrt{5}+1}-\sqrt{2})^{1/2}}{(\sqrt{5}-1)^{1/4}}.$$

Proof. Using Corollary 3.2 with $k=5$, we find that

$$h'_{5,1}=\frac{r_{2,2/5}}{r_{2,10}}. \tag{4.10}$$

Using the values

$$r_{2,5/2}=\frac{(\sqrt{\sqrt{5}+1}+\sqrt{2})^{1/4}}{2^{1/4}}\quad\text{and}\quad r_{2,10}=\left(\frac{1}{2}(1+\sqrt{5})\bigl(\sqrt{\sqrt{5}+1}+\sqrt{2}\bigr)\right)^{1/4}$$

and the fact that $r_{2,2/5}=r_{2,5/2}^{-1}$ from [12], we complete the proof. $\square$

Using known values of $r_{k,n}$ and $r'_{k,n}$ and their relations with $h_{k,n}$ and $h'_{k,n}$ in Section 3, we can find further values of $h_{k,n}$ and $h'_{k,n}$.

5. Explicit formulas for $\varphi(e^{-n\pi})$ and $\varphi(-e^{-n\pi})$

We prove explicit formulas for the theta-functions $\varphi(e^{-n\pi})$ and $\varphi(-e^{-n\pi})$ for any positive real number n and then give some examples. From each given value of $\varphi(e^{-\pi\sqrt{n}})$, one can derive the explicit result of $K[n]$ and k will be evaluated in a simple fashion for certain n.

Lemma 5.1. *Let*

$$a = \frac{\pi^{1/4}}{\Gamma(3/4)}.$$

Then

$$\text{(i)}\quad \varphi(e^{-\pi}) = a,$$
$$\text{(ii)}\quad \varphi(-e^{-2\pi}) = a2^{-1/8}.$$

For a proof of Lemma 5.1, see Entries 1(i) and (iv) in Chapter 35 of [3, p. 325].

Next is the main theorem of this section.

Theorem 5.2. *For every positive real number n,*

$$\text{(i)}\quad \varphi(e^{-n\pi}) = \frac{a}{n^{1/4}h_{n,n}} = \frac{aG_{n^2}^2}{n^{1/4}r_{n,n}},$$
$$\text{(ii)}\quad \varphi(-e^{-2n\pi}) = \frac{a}{2^{1/8}n^{1/4}h'_{n,n}} = \frac{ar_{2,2n^2}}{n^{1/4}2^{1/4}r_{n,n}}.$$

Proof. Using the definitions of $h_{n,n}$ and $h'_{n,n}$, Lemma 5.1, and Corollary 3.4, we complete the proofs of (i) and (ii). □

Corollary 5.3. *For every positive real number n,*

$$\text{(i)}\quad \varphi(e^{-\pi/n}) = \frac{an^{1/4}}{h_{n,n}} = \frac{an^{1/4}G_{n^2}^2}{r_{n,n}},$$
$$\text{(ii)}\quad \varphi(-e^{-2\pi/n}) = \frac{an^{1/4}r_{2,2/n^2}}{2^{1/4}r_{n,n}}.$$

Proof. From Corollary 2.7(i) and Theorem 2.2(i), (iii), we have

$$h_{1/n,1/n} = h_{n,n}h_{1,1/n^2} = h_{n,n}h_{1/n^2,1} = h_{n,n}.$$

Using the relations $h_{n,n} = h_{1/n,1/n}$, $r_{n,n} = r_{1/n,1/n}$ in [12], and $G_{1/n} = G_n$ [7, p. 23] in Theorem 5.2, we immediately obtain the results. □

First, we give some values of $h_{n,n}$ and $h'_{n,n}$, and then we use these values to determine some values of theta-functions.

Theorem 5.4. *We have*

(i) $h_{1,1} = 1$,

(ii) $h_{2,2} = \sqrt{2\sqrt{2} - 2}$,

(iii) $h_{3,3} = (2\sqrt{3} - 3)^{1/4} = \dfrac{3^{1/8}\sqrt{\sqrt{3} - 1}}{2^{1/4}}$,

(iv) $h_{4,4} = \dfrac{2^{3/4}}{\sqrt[4]{2} + 1}$,

(v) $h_{5,5} = \sqrt{5 - 2\sqrt{5}}$,

(vi) $h_{6,6} = \dfrac{2^{3/4}3^{1/8}((\sqrt{2} - 1)(\sqrt{3} - 1))^{1/6}}{(-4 + 3\sqrt{2} + 3^{5/4} + 2\sqrt{3} - 3^{3/4} + 2\sqrt{2} \cdot 3^{3/4})^{1/3}}$.

Proof. Part (i) follows directly from Theorem 2.2(i). For part (ii), use Theorem 4.7(i). For parts (iii) and (iv), use Theorems 4.10(i) and 4.8, respectively.

To prove (v), using

$$r_{5,5} = \sqrt{\frac{5 + \sqrt{5}}{2}} \quad \text{and} \quad r'_{5,5} = \sqrt{\frac{5 - \sqrt{5}}{2}}$$

from Theorems 6.4(ii) and 6.7(iv) of [12] in Theorem 3.1, we find that

$$h_{5,5} = \frac{5 - \sqrt{5}}{\sqrt{2}\sqrt{5 + \sqrt{5}}} = \frac{\sqrt{5}(\sqrt{5} - 1)}{\sqrt{2} \cdot 5^{1/4}\sqrt{\sqrt{5} + 1}} = \frac{5^{1/4}(\sqrt{5} - 1)^{3/2}}{2^{3/2}}.$$

Since $((\sqrt{5} - 1)/2)^3 = \sqrt{5} - 2$, we complete the proof of (v).

To prove (vi), using

$$r_{6,6} = \frac{3^{1/8}\sqrt{1 + \sqrt{3}}(1 + \sqrt{3} + \sqrt{2} \cdot 3^{3/4})^{1/3}}{2^{13/24}}$$

and

$$r'_{6,6} = \frac{2^{11/16}3^{1/8}(\sqrt{2} - 1)^{1/12}(\sqrt{3} + 1)^{1/6}}{(2 - 3\sqrt{2} + 3 \cdot 3^{1/4} + 3^{3/4})^{1/3}}$$

from Theorems 6.4(iii) and 6.7(v) of [12] in Theorem 3.1, we prove (vi). □

Using the values of Theorem 5.4 in Theorem 5.2(i), we find the following results.

Theorem 5.5. *Let* $a = \pi^{1/4}/\Gamma(3/4)$. *Then we have*

(i) $\varphi(e^{-\pi}) = a$,

(ii) $\varphi(e^{-2\pi}) = a2^{-1}\sqrt{\sqrt{2} + 2}$,

(iii) $\varphi(e^{-3\pi}) = a2^{-1/4}3^{-3/8}\sqrt{\sqrt{3} + 1}$,

(iv) $\varphi(e^{-4\pi}) = a2^{-1}(2^{-1/4}+1)$,

(v) $\varphi(e^{-5\pi}) = a5^{-3/4}\sqrt{5+2\sqrt{5}}$,

(vi) $\varphi(e^{-6\pi}) = \dfrac{a(-4+3\sqrt{2}+3^{5/4}+2\sqrt{3}-3^{3/4}+2\sqrt{2}\cdot 3^{3/4})^{1/3}}{2\cdot 3^{3/8}((\sqrt{2}-1)(\sqrt{3}-1))^{1/6}}$.

See [3, p. 325] for another proof of (i), (ii), and (iv).

Theorem 5.6. *We have*

(i) $h'_{1,1} = 1$,

(ii) $h'_{2,2} = 2^{1/16}(\sqrt{2}-1)^{1/4}$,

(iii) $h'_{3,3} = \dfrac{2^{1/3}3^{1/8}(\sqrt{3}-1)^{1/6}}{(1+\sqrt{3}+\sqrt{2}\sqrt[4]{3^3})^{1/3}}$,

(iv) $h'_{4,4} = \dfrac{2^{1/4}}{(16+15\sqrt[4]{2}+12\sqrt{2}+9\sqrt[4]{2^3})^{1/8}}$,

(v) $h'_{5,5} = \dfrac{1}{2}(\sqrt[4]{5}-1)\sqrt{5+\sqrt{5}}$,

(vi) $h'_{6,6} = \dfrac{2^{1/48}3^{1/8}(\sqrt{2}-1)^{1/12}(\sqrt{3}+1)^{1/6}(-1-\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{1/3}}{(2-3\sqrt{2}+3^{5/4}+3^{3/4})^{1/3}}$.

Proof. We give proofs of (i) and (ii) only, because the other values follow by the same argument as in the proof of Theorem 5.4, by using values in Section 5, Theorems 5.15(i) and 6.4 in [12].

Part (i) follows directly by using the values $r_{1,1} = 1$ and $r_{2,2} = 2^{1/8}$ from [12] in Corollary 3.4(ii).

For the proof of (ii), using $r_{2,2} = 2^{1/8}$ and $r_{2,8} = 2^{3/16}(1+\sqrt{2})^{1/4}$ from [12] in Corollary 3.4(ii), we find that

$$h'_{2,2} = \frac{2^{1/4}}{2^{3/16}(\sqrt{2}+1)^{1/4}} = 2^{1/6}(\sqrt{2}-1)^{1/4}.$$

Thus we complete the proof of (ii). □

Using values of Theorem 5.6 in Theorem 5.2(ii), we find the following results.

Theorem 5.7. *We have*

(i) $\varphi(-e^{-2\pi}) = a2^{-1/8}$,

(ii) $\varphi(-e^{-4\pi}) = a2^{-7/16}(\sqrt{2}+1)^{1/4}$,

(iii) $\varphi(-e^{-6\pi}) = \dfrac{a(1+\sqrt{3}+\sqrt{2}\sqrt[4]{3^3})^{1/3}}{2^{11/24}3^{3/8}(\sqrt{3}-1)^{1/6}}$,

(iv) $\varphi(-e^{-8\pi}) = a2^{-7/8}(16+15\sqrt[4]{2}+12\sqrt{2}+9\sqrt[4]{2^3})^{1/8}$,

$$\text{(v)} \quad \varphi(-e^{-10\pi}) = \frac{a2^{7/8}}{(\sqrt[4]{5}-1)\sqrt{5\sqrt{5}+5}},$$

$$\text{(vi)} \quad \varphi(-e^{-12\pi}) = \frac{a2^{-19/48}3^{-3/8}(2-3\sqrt{2}+3^{5/4}+3^{3/4})^{1/3}}{(\sqrt{2}-1)^{1/12}(\sqrt{3}+1)^{1/6}(-1-\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{1/3}}.$$

See [3, p. 325] for another proof of (i) and (ii).

Acknowledgments

The author appreciates helpful advice and guidance from Professor Bruce C. Berndt. Also I thank Professor Frank G. Garvan of the University of Florida for sending me his Maple q-series package. In addition, the referee has kindly informed the author of some misprints in this paper.

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