THE INTEGRAL OF THE RIEMANN ξ-FUNCTION

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ABSTRACT. This paper studies the integral of the Riemann ξ-function defined by $\xi^{(-1)}(s) = \int_{1/2}^{s} \xi(w)dw$. More generally, it studies a one-parameter family of functions given by Fourier integrals and satisfying a functional equation. Members of this family are shown to have only finitely many zeros on the critical line, with $\xi^{(-1)}(s)$ having exactly one zero on the critical line, at $s = \frac{1}{2}$. It is also shown there are zeros of $\xi^{(-1)}(s)$ that lie arbitrarily far away from the critical line. An analogue of the de-Bruijn-Newman constant is introduced for this family, and shown to be infinite.

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1. INTRODUCTION

The Riemann ξ-function is the entire function defined by the formula

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s). \tag{1.1}$$

The ξ-function satisfies the functional equation $\xi(s) = \xi(1-s)$, and its zeros are exactly the non-trivial zeros of the Riemann zeta function $\zeta(s)$, those that lie in the critical strip $0 < Re(s) < 1$. The rescaled function $\Xi(z) := \xi(\frac{1}{2} + iz) = \xi(\frac{1}{2} - iz)$, obtained using the variable change $s = \frac{1}{2} + iz$ which sends the critical line $Re(s) = \frac{1}{2}$ to the real z-axis, has the Fourier integral representation

$$\Xi(z) = 2\int_{0}^{\infty} \Phi(u)\cos zu\, du, \tag{1.2}$$

in which

$$\Phi(u) := \sum_{n=1}^{\infty}(4\pi^2 n^4 e^{\frac{9}{2}u} - 6\pi n^2 e^{\frac{5}{2}u})\ \exp(-\pi n^2 e^{2u}),\ 0 < u < \infty,$$

is a rapidly decreasing function. Here $\Xi(z)$ is the original function introduced by Riemann, see Edwards [18, p. 18].

We define the integral of the ξ-function to be

$$\xi^{(-1)}(s) := \int_{\frac{1}{2}}^{s} \xi(z)dz. \tag{1.3}$$

It satisfies the functional equation

$$\xi^{(-1)}(s) = -\xi^{(-1)}(1-s). \tag{1.4}$$

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The rescaled function[1] $\Xi^{(-1)}(z) := -i\,\xi^{(-1)}(\frac{1}{2}+iz)$ has the Fourier integral representation

$$\Xi^{(-1)}(z) = 2\int_0^\infty \Phi(u)(\frac{\sin zu}{u})\,du. \tag{1.5}$$

This paper studies the locations of zeros of this function, and of other entire functions related to $\xi^{(-1)}(s)$, defined below.

It is a pleasure to dedicate this paper to Akio Fujii, in view of his long-standing interest in location of the zeros of the Riemann zeta function (e.g. [20], [21], [22]).

1.1. **Background.** There have been many studies of properties of the Riemann ξ-function. This function motivated the study of functions in the Laguerre-Pólya class (see Pólya [37], Levin [30, Chap. 8]), to which the function $\Xi(z)$ would belong if the Riemann hypothesis were true. It motivated the study of properties of entire functions represented by Fourier integrals that are real and bounded on the real axis (see Pólya [34], [35], [36], Titchmarsh [42, Chap. X], Cardon [4]) and related Fourier transforms (Wintner [44, Theorems III, IV]). It led to the study of the effect of various operations on entire functions, including differential operators and convolution integral operators, preserving the property of having zeros on a line (e.g. Craven, Csordas and Smith [11], [12], Craven and Csordas [10], Cardon and Nielsen [6], Cardon and de Gaston [5]). Various necessary conditions for the Ξ-function to to have real zeros have been verified (Csordas, Norfolk and Varga [13], Csordas and Varga [16]).

In 1976 Newman [31] introduced a one-parameter family of Fourier cosine integrals, given for real λ by

$$\Xi_\lambda(z) := 2\int_0^\infty e^{\lambda u^2}\Phi(u)\cos zu\,du. \tag{1.6}$$

Here $\Xi_0(z) = \Xi(z)$, so this family of functions can be viewed as deformations of the Ξ-function. It follows from a 1950 result of de Bruijn [3, Theorem 13] that the entire function $\Xi_\lambda(z)$ has only real zeros for $\lambda \geq \frac{1}{8}$. Newman [31] proved that there exists a real number λ_0 such that[2] $\Xi_\lambda(t)$ has all real zeros for $\lambda \geq \lambda_0$, and has some nonreal zeros for each $\lambda < \lambda_0$. The Riemann hypothesis holds if and only if $\lambda_0 \leq 0$, and Newman conjectured that the converse inequality $\lambda_0 \geq 0$ holds. Newman [31, Remark 2] stated that his conjecture represents a quantitative version of the assertion that the Riemann hypothesis, if true, is just barely true. The rescaled value $\Lambda := 4\lambda_0$ was later named by Csordas, Norfolk and Varga [14] the *de Bruijn-Newman constant,* and they proved that $-50 \leq \Lambda$. Successive authors obtained better bounds obtaining by finding two zeros of the Riemann zeta function that were unusually close together. Successive improvements of examples on close zeta zeros led to the lower bound

$$-2.7 \times 10^{-9} < \Lambda.$$

obtained by Odlyzko [33]. Recently Ki, Kim and Lee [28, Theorem 1] established that $\Lambda < \frac{1}{2}$. The conjecture that $\Lambda = 0$ is now termed the de Bruijn-Newman conjecture. Odlyzko [33, Sect. 5] observed that the existence of very close spacings of zeta zeros, would imply the truth of the de Bruijn-Newman conjecture.

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[1]The factor of $-i$ is included here since $\frac{d}{ds} = -i\,\frac{d}{dz}$, to make $\Xi^{(-1)}(z)$ real-valued on the real axis.

[2]Our definition of $\Xi_\lambda(t)$ corresponds to the function $\Xi_b(t)$ with $b = -\lambda$ in Newman's paper.

In another direction, one may consider the effects of differentiation on the location and spacing of zeros of an entire function $F(z)$. In 1943 Pólya [39, p. 182] conjectured that an entire function $F(z)$ of order less than 2 that has only a finite number of zeros off the real axis, has the property that there exists a finite $m_0 \geq 0$ such that all successive derivatives $F^{(m)}(z)$ for $m \geq m_0$ have only real zeros. This was proved by Craven, Csordas and Smith [12] in 1987, with a new proof given by Ki and Kim [27] in 2000. Farmer and Rhoades [19] have shown (under certain hypotheses) that differentiation of an entire function with only real zeros will yield a function having real zeros whose zero distribution on the real line is "smoothed." Their results apply to the Riemann ξ-function, and imply that if the Riemann hypothesis holds, then the same will be true for all derivatives $\xi^{(m)}(s) = \frac{d^m}{ds^m}\xi(s)$, $m \geq 1$. Various general results are given in Cardon and de Gaston [5].

Passing to results on derivatives of the ξ -function, in 1983 Conrey [9] unconditionally showed that the m-th derivative $\xi^{(m)}(s)$ of the ξ-function necessarily has a positive fraction of its zeros falling on the critical line, and his lower bound for this fraction increases towards 1 as m increases. In 2006 Ki [26] proved a conjecture of Farmer and Rhoades, showing that there exist positive sequences A_m, C_m, with $C_m \to 0$ slowly with m, such that

$$\lim_{m\to\infty} A_m \, \Xi^{(2m)}(C_m z) = \cos z.$$

This result can be viewed as quantitative version of the assertion that for the ξ-function differentiation smooths out the spacings of the zeros, since $\cos z$ has perfectly spaced zeros. (See Coffey [8] for a related result.)

In 2009 Ki, Kim and Lee [28] combined differentiation with the de Bruijn-Newman constant. For each integer $m \geq 0$ they introduced the family of functions

$$\Xi_\lambda^{(m)}(z) := \frac{d^m}{dz^m}\Xi_\lambda(z),$$

depending on the real parameter λ. These are given by the Fourier integrals

$$\Xi_\lambda^{(m)}(z) = \begin{cases} \int_0^\infty e^{\lambda u^2} u^{2n}\, \Phi(u)((-1)^n \cos zu)\, du & \text{for } m = 2n,\ n \geq 0, \\ \int_0^\infty e^{\lambda u^2} u^{2n-1} \Phi(u)((-1)^n \sin zu)\, du & \text{for } m = 2n-1,\ n \geq 1. \end{cases} \tag{1.7}$$

To each of these families they associated a de Bruijn-Newman-like constant, first defining

$$\lambda_m := \inf\{\lambda : \Xi_\lambda^{(m)}(z) \text{ has all zeros real}\}, \tag{1.8}$$

and then setting $\Lambda^{(m)} := 4\lambda_m$. The case $\Lambda^{(0)} = \Lambda$ recovers the original de Bruijn-Newman constant. They proved that

$$\Lambda^{(0)} \geq \Lambda^{(1)} \geq \Lambda^{(2)} \geq \cdots,$$

and that

$$\lim_{m\to\infty} \Lambda^{(m)} \leq 0.$$

Finally we remark that $\xi(s)$ is an even function around the point $s = \frac{1}{2}$, having there a Taylor series expansion

$$\xi(s) = \sum_{j=0}^{\infty} \frac{c_{2j}}{(2j)!} (s - \frac{1}{2})^{2j},$$

with coefficients $c_{2j} = \xi^{(2j)}(\frac{1}{2})$ that are real and positive. The maximum modulus $M(r) := \max\{|\xi(\frac{1}{2} + iz)| : |z| = r\}$ is therefore attained for iz on the real axis. In 1945 Haviland [24] obtained an asymptotic expansion for $M(r)$ of the shape

$$M(r) \sim (\frac{1}{2}\pi)^{\frac{1}{4}} (2\pi e)^{-\frac{1}{2}r} r^{\frac{1}{2}r + \frac{7}{4}} \Big(\sum_{n=0}^{\infty} \frac{C_n}{r^n} \Big),$$

having $C_0 = 1$. From the integral (1.3) we deduce the Taylor expansion

$$\xi^{(-1)}(s) = \sum_{j=0}^{\infty} \frac{c_{2j}}{(2j+1)!} (s - \frac{1}{2})^{2j+1},$$

manifestly showing that $\xi^{(-1)}(s)$ is an odd function around $s = \frac{1}{2}$. Coffey ([7], [8]) found integral formulas for the coefficients c_{2j} and determined their asymptotics as $j \to \infty$.

1.2. **Present Work.** To add perspective to the results above, we study the effect of the inverse operation of integration applied to the Riemann ξ-function on the zeros of the resulting function. Since differentiation seems to smooth the distribution of zero spacings, we may anticipate that integration will "roughen" their distribution, and even force zeros off the critical line. Our object is to obtain quantitative information in this direction. We study several variants of the function $\xi^{(-1)}(s)$, including a family of functions defined in analogy with $\Xi_\lambda(z)$ above.

Based on the Fourier integral representation (1.5), we define an analogue for $m = -1$ of the one-parameter families of functions $\Xi_\lambda^{(m)}$ studied by Ki et al. [28], as follows. Given a real λ, set

$$\Xi_\lambda^{(-1)}(z) := 2 \int_0^\infty e^{\lambda u^2} \Phi(u) \left(\frac{\sin zu}{u} \right) du. \tag{1.9}$$

The functions $\Xi_\lambda^{(-1)}(z)$ are odd functions, are real on the real axis, and they satisfy $\frac{d}{dz}\Xi_\lambda^{(-1)}(z) = \Xi_\lambda(z)$. For this family we may define a de Bruijn-Newman constant for $m = -1$, by analogy with the definition above: we first set

$$\lambda_{-1} = \inf\{\lambda : \ \Xi_\lambda^{(-1)}(z) \text{ has all zeros real}\} \tag{1.10}$$

and then set $\Lambda^{(-1)} := 4\lambda_{-1}$. In the paper we will show that $\Lambda^{(-1)} = +\infty$. That is, we show that for each real λ the function $\Xi_\lambda^{(-1)}(z)$ has at least one non-real zero; in fact, it has infinitely many non-real zeros.

In another direction, concering the function $\xi^{(-1)}(s)$ defined by (1.3), we introduce a constant of integration $\alpha_0 \in \mathbb{C}$, and define

$$\xi^{(-1)}(s; \alpha_0) := \xi^{(-1)}(s) + \alpha_0 = \int_{\frac{1}{2}}^{s} \xi(w) dw + \alpha_0. \tag{1.11}$$

The functional equation for $\xi^{(-1)}(s)$ then yields

$$\xi^{(-1)}(s;\alpha_0) = -\xi^{(-1)}(1-s, -\alpha_0).$$

The problem of determining the zero set of $\xi^{(-1)}(s;-\alpha_0)$ with integration constant $-\alpha_0$ is the same as that of detemining the set of points where $\xi^{(-1)}(s) = \alpha_0$, which we call the α_0-value set of $\xi^{(-1)}$, and denote $V(\xi^{(-1)};\alpha_0)$. We obtain detailed information on the value sets, showing that for all but two values of α_0 only finitely many zeros are on the critical line, and for all values of α_0 there are zeros arbitrarily far off the critical line.

We give precise statements of results in Section 2; we then discuss consequences of these results. Sections 3 to 5 give proofs. In Section 3 we collect preliminary results needed for proofs of these results In Sections 4 and 5 we give proofs of the main theorems. In the final Section 6 we present numerical results on zeros of $\xi^{(-1)}(s)$ and related functions, and raise some open questions.

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2. Results

We study the set of real zeros of functions in the family $\Xi_\lambda^{(-1)}(z)$, and determine information on the zero sets of $\xi^{(-1)}(s) - \alpha_0$ for arbitrary values $\alpha_0 \in \mathbb{C}$. We obtain two main results.

2.1. **Behavior of $\Xi_\lambda^{(-1)}(z)$.**

The first result concerns the behavior of the function $\Xi_\lambda^{(-1)}(z)$ defined in (1.9) on the real axis.

Theorem 2.1. *For real λ, the functions $\Xi_\lambda^{(-1)}(z)$ have the following properties.*

(1) *For each real λ, one has*

$$\lim_{t\to\infty} \Xi_\lambda^{(-1)}(t) = A_0,$$

where A_0 is a nonzero constant independent of λ given by

$$A_0 := \pi\Phi(0) \approx 2.80668.$$

The value $A_0 = \frac{\pi}{2}\left(4\theta''(1) + 6\theta'(1)\right)$, taking $\theta(z) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 z}$.

(2) *For each real λ, the function $\Xi_\lambda^{(-1)}(t)$ has finitely many zeros on the real t-axis, and has infinitely many non-real zeros. The zeros on the real axis always include a zero at $t=0$, and for $\lambda \le 0$ this is the only real zero of $\Xi_\lambda^{(-1)}(t)$.*

Since $\Xi_\lambda^{(-1)}(z)$ is an odd function, we have

$$\lim_{t\to-\infty} \Xi_\lambda^{(-1)}(t) = -A_0.$$

This theorem shows that the "just barely true" heuristic for the Riemann hypothesis holds in a particularly strong fashion for the operation of integration. Namely, integration drives all but finitely many zeros off the real axis for every function in the family $\Xi_\lambda^{(-1)}(z)$.

The result (1) above is derived directly from the oscillatory integral representation for $\Xi_\lambda^{(-1)}(t)$; it corresponds to a Fourier sine integral (on a half line) against a function having a singularity at the endpoint $u = 0$. The proof of Theorem 2.1 obtains the bound, valid for $t \geq 3$,

$$\Xi_\lambda^{(-1)}(t) = A_0 + O\left(t^{-2/3}\right),$$

in which the implied constant in the error term depends on λ.

The result (2) above shows that the analogue of the de Bruijn-Newman constant for this integral of the ξ-function fails to exist; that is, it establishes

$$\Lambda^{(-1)} = +\infty.$$

In 1947 Wintner [45] proved that $\Xi_0^{(-1)}(t) > 0$ when $t > 0$; this fact together with the functional equation (1.4) implies that $\xi^{(-1)}(s)$ has no zeros on the critical line except for a zero at $s = \frac{1}{2}$. Wintner's approach extends to cover the case $\lambda \leq 0$, as stated in (2) above.

2.2. **Value Distribution of $\xi^{(-1)}(s)$.** The second result concerns the location of zeros of the function $\xi^{(-1)}(s)$ given in (1.3). More generally it studies the distribution of values $\xi^{(-1)}(s) = \alpha$ of this function. For an entire function $f(z)$, let $Z(f)$ denote the set of zeros of f, where we count zeros with multiplicity; thus $Z(f)$ is a multiset. We define the *value set* of f at value α by

$$V(f;\alpha) := Z(f - \alpha) = \{z \in \mathbb{C} :\ f(z) - \alpha = 0\}.$$

Theorem 2.1 gives two exceptional limiting values

$$\lim_{t\to\pm\infty} \xi^{(-1)}(\frac{1}{2} + it) = \pm iA_0.$$

We show that, aside from these two values $\alpha = \pm iA_0$, the locations of all values in $V(\xi^{(-1)};\alpha)$ are qualitatively similar in asymptotics relating the real part of zeros to their imaginary part. In what follows $4\pi e \approx 34.1588$ serves as as a useful cutoff value.

Theorem 2.2. *For the function $\xi^{(-1)}(s)$ and $\alpha_0 \in \mathbb{C}$, consider the set of points where $\xi^{(-1)}(s) = \alpha_0$. All such points $\rho = \sigma + it$ having $|t| \leq 4\pi e$ lie in a bounded region, which depends on α_0.*

(1) For each $\alpha_0 \neq \pm iA_0$, all members ρ having $|t| \geq 4\pi e$ satisfy

$$|\sigma| = \frac{\pi}{2}\frac{|t|}{\log|t|} + O\left(\frac{|t|}{(\log|t|)^2}\right), \tag{2.1}$$

in which the implied constant in the O-symbol depends on $|\alpha_0|$. For $\alpha_0 = iA_0$ (resp. $\alpha_0 = -iA_0$) this bound applies when $t \leq -4\pi e$ (resp. $t \geq 4\pi e$).

(2) For $\alpha_0 = iA_0$ (resp. $\alpha_0 = -iA_0$) the upper bound applies when $t \geq 4\pi e$ (resp. $t \leq -4\pi e$) :

$$|\sigma| \leq \frac{\pi}{2}\frac{|t|}{\log|t|} + O\left(\frac{|t|}{(\log|t|)^2}\right), \tag{2.2}$$

This result shows that the value distribution of $\xi^{(-1)}(s)$ is qualitatively the same for all values $\alpha_0 \neq \pm iA_0$, as well as for the values $\alpha_0 = iA_0$ in the lower half plane, and $\alpha_0 = -iA_0$ in the upper half-plane. The two remaining cases in (2) appear to have a different distribution; numerical evidence given in §6 supports this possibility. In the exceptional cases in (2) we suspect that the values remain closer to the critical line, at least to the extent that $|\sigma| = O(\frac{|t|}{(\log|t|)^2})$ might hold in these two cases.

Since the functions $\xi^{(-1)}(s) - \alpha_0$ have infinitely many zeros (because they are entire functions of order 1 of maximal type, cf. Lemma 3.2), we deduce that all of them have zeros arbitrarily far away from the critical line. (In the case $\alpha_0 = iA_0$, resp. $-iA_0$, one must additionally show that they have infinitely many zeros with negative real part, resp. positive real part.)

2.3. **Discussion.** First, Theorem 2.1 is obtained by viewing the functions $\Xi_\lambda^{(-1)}(t)$ as Fourier integrals of functions $\Xi^{(-1)}(z) = \frac{1}{i}\int_{-\infty}^{\infty} \Psi(u) e^{izu} du$, where the function $\Psi(u) = \frac{1}{u}\Phi(u)$ has a singularity at the point $u = 0$. Since $\Psi(u)$ is an odd function, this integral can be rewritten as an absolutely convergent integral $2\int_0^\infty \Phi(u)(\frac{\sin zu}{u})du$. The singularity at $u = 0$, which occurs since $\Phi(0) \neq 0$, results in a nonzero integral on the critical line, and this is the mechanism that forces zeros off the real axis. In this regard, one may consider more generally an oscillatory integral $F(z) = \int_0^\infty \Psi(u)\cos(zu)du$ in which $\Psi(u)$ is a smooth function with very rapid decay as $u \to \infty$, such that $F(z)$ is an entire function. Then the order k of the zero at $x = 0$ of a smooth function $\Psi(u)$ in an oscillatory integral places an absolute limit on the number of integrations $F^{(-j)}(z)$ of $F(z)$ that can be taken (with any choices of constant of integration) to have the property that $F^{(-j)}(t) \to 0$ holds as $t \to \pm\infty$ for each $1 \leq j \leq m$; it requires that $m \leq k$.

Second, one may ask how the zeros on the real axis of $\Xi_\lambda^{(-1)}(t)$ behave as $\lambda \to \infty$. Theorem 2.1 implies there are a finite number of zeros for each λ. It may be that this number increases as $\lambda \to \infty$, and that new zeros are created in pairs at the origin at certain values of λ as it increases, and afterwards have a regular behavior as a function of λ.

Third, one may ask whether the constant $A_0 = \frac{\pi}{2}\left(4\theta''(1) + 6\theta'(1)\right)$ appearing in Theorem 2.1 may possibly have an arithmetic interpretation. It is known that the value $\theta(1) = \frac{\pi^{1/4}}{\Gamma(\frac{3}{4})}$ has an arithmetic interpretation in the context of the Arakelov zeta function studied in Lagarias and Rains [29, Appendix].

Fourth, in connection with the smoothing property on the zeros distribution of taking derivatives of $\xi(s)$, one may inquire concerning the level spacing distribution of $\xi^{(n)}(s)$ for $n \geq 1$. The GUE conjecture asserts that the level spacing distributions of k consecutive normalized zeros of $\xi(s)$ has a limiting distribution specified by the GUE distribution. It seems plausible to expect that $\xi^{(n)}(s)$ will have its own level spacing distribution $\mathrm{GUE}^{(n)}$ which will differ from that of the GUE. It would be interesting to make a prediction for $\mathrm{GUE}^{(n)}$, and if possible, to find a random matrix model for it.

Fifth, Theorem 2.2 gives a functional bound relating the horizontal and vertical coordinates of individual values. It remains to determine the asymptotics of the vertical distribution of the zeros of $\xi^{(-1)}(s)$, or more generally of any fixed value set $\xi^{(-1)}(s) = \alpha_0$. We might expect these values to obey approximately the same asymptotics as that of the Riemann ξ-function, which is ([17, Chap. 15])

$$N(T) = \frac{1}{\pi} T \log T - \frac{1}{2\pi} \log(\frac{1}{2\pi e}) T + O(\log T),$$

as least in the main term $\frac{T}{\pi} \log T$ in the asymptotics. For $\xi^{(-1)}(s)$ we note the weak result $N(T) = O(T \log T)$ follows from a Jensen's formula estimate from growth of its maximum modulus.

3. Basic Observations

The ξ-function can be expressed in terms of a Mellin transform of derivatives of the Jacobi theta function $\vartheta_3(0, q) = \sum_{n \in \mathbb{Z}} q^{n^2}$. We make the variable change $q = e^{-\pi z}$, and on the half-plane $Re(z) > 0$ define the *theta function*

$$\theta(z) := \sum_{n=-\infty}^{\infty} e^{-\pi n^2 z}$$

We write $z = x + iy$, and mainly consider $\theta(x)$ restricted to the real axis. For the derivatives of θ, we have the formulas

$$\theta'(x) = \sum_{n=-\infty}^{\infty} -\pi n^2 e^{-\pi n^2 x} \qquad \theta''(x) = \sum_{n=-\infty}^{\infty} \pi^2 n^4 e^{-\pi n^2 x}.$$

The function $\Phi(u)$ given in Sect. 1 is expressible in terms of derviatives of the theta function as given in (3.3) below.

Lemma 3.1. *The function $\xi(s)$ is given by the Fourier cosine transform*

$$\Xi(z) = \xi(\frac{1}{2} + iz) = 2 \int_0^{\infty} \Phi(u) \cos zu \, du \tag{3.1}$$

in which

$$\Phi(u) = \sum_{n=1}^{\infty} \left(4\pi^2 n^4 e^{\frac{9}{2}u} - 6\pi n^2 e^{\frac{5}{2}u} \right) \exp\left(-\pi n^2 e^{2u} \right). \tag{3.2}$$

The function $\Phi(u)$ has the following properties.

(1) $\Phi(u) = \frac{1}{2} \left(\frac{d^2}{du^2} - \frac{1}{4} \right) \left(e^{\frac{1}{2}u} \theta(e^{2u}) \right)$.
(2) $\Phi(u)$ *is an even function:* $\Phi(u) = \Phi(-u)$.
(3) $\Phi(u)$ *decays extremely rapidly on the real axis as* $u \to \pm\infty$, *with*

$$\Phi(u) \ll \exp\left(-e^{|u|} \right) \text{ as } u \to \pm\infty.$$

(4) $\Phi(u)$ *analytically continues to the strip* $|Im(u)| < \frac{\pi}{4}$. *For integer* $m \geq 0$, *it satisfies, allowing only real* t *in the limit,*

$$\lim_{t \to \pi/4} \Phi^{(m)}(it) = 0.$$

(5) $\Phi(u)$ *is a strictly decreasing function on* $[0, \infty)$.

Proof. We start from Riemann's formula ([41])

$$\xi(\frac{1}{2}+it) = 4\int_1^\infty \frac{d[x^{3/2}(\psi'(x)]}{dx} x^{-\frac{1}{4}} \cos(\frac{t}{2}\log x)dx,$$

in which $\psi(x) = \frac{1}{2}(\theta(x)-1)$, so that $\psi'(x) = \frac{1}{2}\theta'(x)$, and make the variable change $x = e^{2u}$ to obtain (3.1), cf. Edwards [18, Sec. 1.8], or Titchmarsh [42, Sec. 10.1]. The expansion of the theta function yields

$$\Phi(u) = [3x^{\frac{5}{4}}\theta'(x) + 2x^{\frac{9}{4}}\theta''(x)]\,|_{x=e^{2u}}\ . \tag{3.3}$$

We now consider properties of $\Phi(u)$.

(1) This formula was noted by Pólya [34] in 1926. It can be directly verified by comparison of the right side with (3.3).

(2) The functional equation $\theta(x) = \sqrt{\frac{1}{x}}\theta(\frac{1}{x})$ yields $e^{\frac{u}{2}}\theta(e^{2u}) = e^{-\frac{u}{2}}\theta(e^{-2u})$. Substituting this in (1) yields $\Phi(u) = \Phi(-u)$.

(3) For $u \to +\infty$ this follows by inspection of (3.2). For $u \to -\infty$ it follows using (2).

(4) The theta function $\theta(x)$ defines an analytic function on the right half plane $\mathrm{Re}(x) > 0$. Under the change of variable $x = e^{2u}$, this region corresponds to the strip $|\mathrm{Im}(u)| < \frac{\pi}{4}$. The limiting values were noted by Pólya [34]. (Fact (4) is not used in this paper.)

(5) The decreasing property of $\Phi(u)$ was proved in 1935 by Wintner [43]. Note that $\Phi(0) = \sum_{n=1}^{\infty}(4\pi^2n^4 - 6\pi n^2)e^{-\pi n^2} \approx 0.89339$. □

We next give basic properties of the family of functions $\Xi_\lambda^{(-1)}(z)$.

Lemma 3.2. *For real λ the functions $\Xi_\lambda^{(-1)}(z)$ have the following properties.*

(1) *Each function $\Xi_\lambda^{(-1)}(z)$ is an entire function of z of order 1 and maximal type.*
(2) *Each function is real on the real axis and is an odd function, i.e.*

$$\Xi_\lambda^{(-1)}(z) = -\,\Xi_\lambda^{(-1)}(-z).$$

Thus it is pure imaginary on the imaginary axis $z = it$.
(3) *One has $\frac{d}{dz}\Xi_\lambda^{(-1)}(z) = \Xi_\lambda(z)$. Thus, for $\lambda = 0$,*

$$\Xi_0^{(-1)}(z) = -i\,\xi^{(-1)}(\frac{1}{2}+iz) = -i\int_{1/2}^{1/2+iz}\xi(w)dw.$$

Proof. (1) The integral representation $\Xi_\lambda^{(-1)}(z) = 2\int_0^\infty e^{\lambda u^2}\Phi(u)\left(\frac{\sin zu}{u}\right)du$ shows that it is an entire function of z. The results of Pólya [36, pp. 9-10] imply it is of order 1. Viewing the representation as a Fourier integral, the Paley-Wiener theorem shows it cannot have growth of order 1 and finite type, whence it has maximal type.

(2) The integral representation shows that $\Xi_\lambda^{(-1)}(z)$ is real on the real axis, and it is clearly an odd function of z. Since $\sin(izu) = -i\sinh(zu)$ we conclude this function is pure imaginary on the imaginary axis.

(3) The rapid decay of $\Phi(u)$ as $|u| \to \infty$ permits differentiation under the integral sign. Now the identity $\Xi_0(z) = \xi(\frac{1}{2}+iz)$ and the fact that $\Xi_\lambda^{(-1)}(0) = 0$ by (2) yield the last equation. (Note that $ds = i\,dz$.) □

Lemma 3.2(2) above implies that the zeros of $\Xi_\lambda^{(-1)}(z)$ necessarily have a four-fold symmetry about the real and imaginary axes: If ρ is a zero of any $\Xi_\lambda^{(-1)}(z)$, then so are $\bar{\rho}$, $-\rho$ and $-\bar{\rho}$. Similarly $\xi_\lambda^{(-1)}(s) = i\Xi_\lambda^{(-1)}(i(\frac{1}{2}-s))$ necessarily has zeros obeying the same four-fold symmetry as those of the ξ-function: If ρ is a zero of any $\xi_\lambda^{(-1)}(s)$, then so are $\bar{\rho}$, $1-\rho$ and $1-\bar{\rho}$.

To prove Theorem 2.2 we will use estimates on the size of $\xi(s)$, derived using the factorization (1.1) for $\xi(s)$. To state these it is convenient to introduce the function $F(\sigma, t)$ of two real variables defined for $\sigma \ge 0$, $t > 0$ by

$$F(\sigma,t) := \sqrt{\pi}(2\pi e)^{-\frac{\sigma}{2}}\,(\sigma^2+t^2)^{\frac{\sigma+3}{4}}\exp\left(-\frac{t}{2}\arctan(\frac{t}{\sigma})\right). \tag{3.4}$$

We have the following estimates.

Lemma 3.3. *There are positive constants C_1, C_2, C_3 with the following properties.*

(1) For $\frac{1}{2} \le Re(s) \le 2$, the function $\xi(s)$ satisfies

$$|\xi(s)| \le C_1 e^{-\frac{\pi}{4}|t|}(|t|+1)^{5/2}. \tag{3.5}$$

(2) For $s = \sigma + it$ with $\sigma \ge 2$ and all real t, there holds

$$F(\sigma,t)\left(1 - C_2(\frac{1}{|\sigma+it|}+2^{-\sigma})\right) \le |\xi(s)| \le F(\sigma,t)\left(1 + C_3(\frac{1}{|\sigma+it|}+2^{-\sigma})\right), \tag{3.6}$$

with $F(\sigma,t)$ given by (3.4).

(3) For each $\delta > 0$, there is a positive constant $C = C(\delta)$ such that for all $\sigma_0 > C$ and all real t, the function $\xi(\sigma+it)$ satisfies

$$\int_{\sigma_0}^{\sigma_0+2} |\frac{d\arg\xi(\sigma+it)}{d\sigma}|d\sigma \le \frac{1+\pi}{2} + \delta.$$

Proof. (1) For $\frac{1}{2} \le Re(s) \le 2$, and $|t| \ge 2$, we have the estimates

$$|\frac{1}{2}s(s-1)| = O(|t|^2),$$

$$|\pi^{-s/2}| = O(1),$$

and

$$|\Gamma(s/2)| = O(e^{-\frac{\pi|t|}{4}}).$$

The convexity bound $|\zeta(s)| \le C(\epsilon)|t|^{\frac{1}{2}-\frac{1}{2}\sigma+\epsilon}$ ([42, Chap. V]), valid uniformly for $|t| \ge 2$, yields for all $\sigma \ge \frac{1}{2}$ and $|t| \ge 2$,

$$|\zeta(s)| \le C|t|^{\frac{1}{2}}.$$

Combining all these estimates, we easily obtain, for $\frac{1}{2} \le \sigma \le 2$ and all real t,

$$|\xi(s)| = O(e^{-\pi t/4}(|t|+1)^{\frac{5}{2}}).$$

(2) By definition, $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$. Suppose $\sigma \ge 2$ and t is arbitrary. On this domain

$$|s(s-1)| = |\sigma+it|^2(1+O(\frac{1}{|\sigma+it|})).$$

and

$$|\zeta(s)| = 1 + O(|2^{-s}|) = 1 + O(2^{-\sigma})$$

and $|\pi^{-\frac{s}{2}}| = e^{-\frac{\sigma}{2}\log\pi}$. Now Stirling's formula gives, for $Re(s) \geq \frac{1}{2}$,

$$\begin{aligned}
|\Gamma(\frac{s}{2})| &= \exp\left(Re\left(\frac{s}{2}\log\frac{s}{2} - \frac{s}{2} + \frac{1}{2}\log(\frac{4\pi}{s}) + O(\frac{1}{s})\right)\right) \\
&= \exp\left(\frac{\sigma}{2}\log|\frac{s}{2}| - \frac{t}{2}\arctan(\frac{t}{\sigma}) - \frac{\sigma}{2} + \frac{1}{2}\log 4\pi - \frac{1}{2}\log|s| + O(\frac{1}{|s|})\right)
\end{aligned}$$

Combining all of the above estimates, we obtain

$$\begin{aligned}
|\xi(s)| &= |\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)| \\
&= \frac{1}{2}|\sigma+it|^2\left(1 + O(\frac{1}{|\sigma+it|})\right)\left(1 + O(2^{-\sigma})\right)e^{-\frac{\sigma}{2}\log\pi} \cdot \\
&\quad \cdot \exp\left(\frac{\sigma}{2}\log|\frac{\sigma+it}{2}| - \frac{t}{2}\arctan(\frac{t}{\sigma}) - \frac{\sigma}{2} + \frac{1}{2}\log 4\pi - \frac{1}{2}\log|s| + O(\frac{1}{|s|})\right) \\
&= \sqrt{\pi}|\sigma+it|^{\frac{3}{2}}e^{-\frac{\sigma}{2}\log(2\pi e)}\exp\left(\frac{\sigma}{4}\log(|\sigma+it|^2) - \frac{t}{2}\arctan(\frac{t}{\sigma})\right) \cdot \\
&\quad \cdot \left(1 + O(\frac{1}{|\sigma+it|} + \frac{1}{2^\sigma})\right) \\
&= F(\sigma,t)\left(1 + O(\frac{1}{|\sigma+it|} + \frac{1}{2^\sigma})\right).
\end{aligned}$$

(3) We derive estimates related to $\arg(\xi(s))$, taken to be 0 on the real axis for $\sigma > 1$. Then, for $\sigma > 1$,

$$\begin{aligned}
\arg\xi(s) &= \arg(s) + \arg(s-1) + \arg(\pi^{-s/2}) + \arg(\zeta(s)) + \arg(\Gamma(s/2)) \\
&= \arctan(\frac{t}{\sigma}) + \arctan(\frac{t}{\sigma-1}) - \frac{t\log\pi}{2} + \arg(\zeta(s)) + \arg(\Gamma(s/2)).
\end{aligned}$$

First, note that

$$\frac{d\arg\zeta(\sigma+it)}{d\sigma} = \frac{d\,\mathrm{Im}(\log\zeta(\sigma+it))}{d\sigma} = \mathrm{Im}\left(\frac{\zeta'}{\zeta}(\sigma+it)\right),$$

and since we have uniformly in t that

$$\lim_{\sigma\to\infty}\zeta(\sigma+it) = 1, \qquad \lim_{\sigma\to\infty}\zeta'(\sigma+it) = 0,$$

we can choose C sufficiently large so that $\sigma_0 > C$ implies that $|\frac{d\arg\zeta(\sigma+it)}{d\sigma}| < \delta/4$. If we also choose C large enough so that $|\arctan(\frac{t}{\sigma_0+2}) - \arctan(\frac{t}{\sigma_0-1})| < \delta/4$, then

$$\begin{aligned}\int_{\sigma_0}^{\sigma_0+2} \left|\frac{d\arg\xi(\sigma+it)}{d\sigma}\right| d\sigma &\leq 2\left|\arctan\left(\frac{t}{\sigma_0+2}\right) - \arctan\left(\frac{t}{\sigma_0-1}\right)\right| \\ &\quad + \int_{\sigma_0}^{\sigma_0+2} \left|\frac{d\arg\zeta(\sigma+it)}{d\sigma}\right| d\sigma + \int_{\sigma_0}^{\sigma_0+2} \left|\frac{d\arg\Gamma(\frac{\sigma+it}{2})}{d\sigma}\right| d\sigma \\ &\leq \delta + \int_{\sigma_0}^{\sigma_0+2} \left|\frac{d\arg\Gamma(\frac{\sigma+it}{2})}{d\sigma}\right| d\sigma.\end{aligned}$$

By Stirling's formula, we have

$$\arg(\Gamma(\frac{s}{2})) = \frac{t}{2}\log\left|\frac{\sigma+it}{2}\right| + \frac{\sigma}{2}\arctan(\frac{t}{\sigma}) - \frac{t}{2} - \frac{1}{2}\arctan(t/\sigma) + O(\frac{1}{|\sigma+it|}).$$

Therefore, for $\sigma > 1$,

$$\begin{aligned}\frac{d\arg\Gamma(\frac{\sigma+it}{2})}{d\sigma} &= \frac{d}{d\sigma}\left(\frac{t}{4}\log(\frac{\sigma^2+t^2}{4}) + \frac{\sigma-1}{2}\arctan(\frac{t}{\sigma}) + O(\frac{1}{|\sigma+it|})\right) \\ &= \frac{t}{4}\left(\frac{2\sigma}{\sigma^2+t^2}\right) + \frac{1}{2}\arctan(\frac{t}{\sigma}) + O(\frac{1}{|\sigma+it|}). \qquad (3.7)\end{aligned}$$

As $\sigma^2 + t^2 \geq 2\sigma t$, we can bound the first term of (3.7) by $1/4$, and the second term by $\pi/4$, giving

$$\left|\frac{d\arg\Gamma(\frac{\sigma+it}{2})}{d\sigma}\right| \leq \frac{1+\pi}{4} + O\left(\frac{1}{|\sigma+it|}\right).$$

Thus, for any $\delta > 0$, if $C = C(\delta)$ is chosen large enough, then for all $\sigma_0 > C$,

$$\int_{\sigma_0}^{\sigma_0+2} \left|\frac{d\arg\xi(\sigma+it)}{d\sigma}\right| d\sigma \leq \frac{1+\pi}{2} + \delta.$$

□

In a region where $\sigma/t \to 0$, the parameter range relevant to this paper, the first term on the right in (3.7) goes to zero, and the upper bound in Lemma 3.3 (3) above can be further improved to $\frac{\pi}{2} + \delta$. This latter bound cannot be improved, since when $s = \sigma + it$ has σ much smaller than t the argument must necessarily change by nearly $\pi/2$; this variation comes from the change in argument of the factor $\Gamma(\frac{s}{2})$ by $\arg(\frac{s_0}{2})$ between $s = s_0$ and $s_0 + 2$.

For later use we collect some properties of the function $F(\sigma, t)$ defined in (3.4) above.

Lemma 3.4. *On the region $\sigma \geq 0$ the function $F(\sigma, t)$ has the following properties.*
(1) For fixed $t \geq 2\pi e$, the function $F(\sigma, t)$ is a strictly increasing function of σ.
(2) For fixed $t \geq 4\pi e$, and any positive σ_1,

$$\int_0^{\sigma_1} F(\sigma, t)\, d\sigma \leq 4F(\sigma_1, t). \qquad (3.8)$$

Proof. (1) Rewrite

$$F(\sigma,t) = \sqrt{\pi}(\sigma^2+t^2)^{\frac{3}{4}} \exp\left(\frac{\sigma}{4}\left(\log(\sigma^2+t^2) - \log(4\pi^2e^2)\right)\right) \exp\left(-\frac{\pi t}{4} + \frac{t}{2}\arctan(\frac{\sigma}{t})\right). \qquad (3.9)$$

For fixed $t \geq 2\pi e$ all terms separately in this product are constant or increasing functions of σ.

(2) Since $t \geq 2\pi e$, by (1) the integrand on the left side of (3.8) is increasing. We obtain for $t \geq 4\pi e$,

$$\begin{aligned}\int_0^{\sigma_1} F(\sigma,t)d\sigma &\leq \sqrt{\pi}(\sigma_1^2+t^2)^{\frac{3}{4}}e^{-\frac{\pi t}{4}+\frac{t}{2}\arctan(\frac{\sigma_1}{t})}\int_0^{\sigma_1}\exp\left(\frac{\sigma}{4}\log(\frac{\sigma_1^2+t^2}{4\pi^2e^2})\right)d\sigma \\ &\leq \frac{4}{\log(\frac{\sigma_1^2+t^2}{4\pi^2e^2})}F(\sigma_1,t) \leq 4\,F(\sigma_1,t).\end{aligned}$$

□

4. Integrals of the ξ-Function: Proof of Theorem 2.1

We consider the family of functions $\Xi_\lambda^{(-1)}(z)$ given by (1.9), which has $\Xi_0^{(-1)}(z) = -i\xi^{(-1)}(\frac{1}{2}+iz)$.

(1) To show $\lim_{t\to\infty}\Xi_\lambda^{(-1)}(t) = \pi\Phi(0)$ we will establish the stronger result that for $t \geq 3$ one has

$$\Xi_\lambda^{(-1)}(t) = \pi\Phi(0) + O\left(\frac{1}{t^{2/3}}\right), \tag{4.1}$$

where the implied constant in the O-symbol depends on λ. We start from

$$\begin{aligned}\Xi_\lambda^{(-1)}(t) &= 2\int_0^\infty e^{\lambda u^2}\Phi(u)\frac{\sin tu}{u}du \\ &= 2\int_0^\infty e^{\lambda(\frac{v}{t})^2}\Phi(\frac{v}{t})\,\frac{\sin v}{v}dv\end{aligned}$$

We estimate the latter integral by splitting the integration region into three pieces, the first integrating over the interval $[0, 2\pi\lfloor t^{2/3}\rfloor]$, the second integrating over the interval $[2\pi\lfloor t^{2/3}\rfloor, 2\pi\lfloor t^{4/3}\rfloor]$, and the third integrating over $[2\pi\lfloor t^{4/3}\rfloor, +\infty)$. The first integral will give the main contribution $\frac{\pi}{2}\Phi(0)+O(\frac{1}{t^{2/3}})$, the second will be bounded by $O(\frac{1}{t^{2/3}})$, and the third will be shown negligibly small, of size $O(e^{-2t})$.

To obtain the estimates, we view λ as fixed and let $F(u) = e^{\lambda u^2}\Phi(u)$. In the following estimates, all O-symbols will depend on λ unless otherwise noted. Now $|F(u)|$, $|F^{'}(u)|$, $|F^{''}(u)|$ are all absolutely bounded on $[0,\infty)$, using the very rapid decrease of $\Phi(u)$ and its first two derivatives; this follows from results in Lemma 3.1. Next, Lemma 3.1 (2) shows $F(u)$ is an even function, whence its power series expansion at $u = 0$ gives, for $0 \leq u \leq 2\pi$,

$$F(u) = \Phi(0) + O\left(\Phi^{''}(0)u^2\right). \tag{4.2}$$

For any $u_0 \geq 0$ and $0 \leq x \leq 2\pi$ we have

$$F(u_0+x) = F(u_0) + F'(u_0)x + O\left(|F''(u_0)|x^2\right), \tag{4.3}$$

with the O-constant depending on λ but not on u_0.

For the first integral, on the range $v \in [0, 2\pi\lfloor t^{2/3}\rfloor]$, (4.2) gives

$$F\left(\frac{v}{t}\right) = \Phi(0) + O\left((\frac{v}{t})^2\right).$$

We obtain

$$\begin{aligned}\int_0^{2\pi\lfloor t^{2/3}\rfloor} F\left(\frac{v}{t}\right)\frac{\sin v}{v}dv &= \Phi(0)\int_0^{2\pi\lfloor t^{2/3}\rfloor}\frac{\sin v}{v}dv + O\left(\int_0^{2\pi\lfloor t^{2/3}\rfloor}\frac{v}{t^2}|\sin v|dv\right)\\ &= \Phi(0)\int_0^{2\pi\lfloor t^{2/3}\rfloor}\frac{\sin v}{v}dv + O\left(\frac{1}{t^{2/3}}\right).\end{aligned}$$

Next we use the evaluation of the improper integral

$$\int_0^\infty \frac{\sin u}{u}du := \lim_{T\to\infty}\int_0^T \frac{\sin u}{u}du = \frac{\pi}{2}.$$

We use the quantitative estimate that for real $T \geq 1$

$$\int_0^{2\pi T}\frac{\sin u}{u}du = \frac{\pi}{2} + O\left(\frac{1}{T}\right),$$

which can be proved by integration by parts. Substituting this in the last equation, taking $T = 2\pi\lfloor t^{2/3}\rfloor$, we obtain

$$\int_0^{2\pi\lfloor t^{2/3}\rfloor} F\left(\frac{v}{t}\right)\frac{\sin v}{v}dv = \frac{\pi}{2}\Phi(0) + O\left(\frac{1}{t^{2/3}}\right).$$

For the second integral, we have

$$\begin{aligned}\int_{2\pi\lfloor t^{2/3}\rfloor}^{2\pi\lfloor t^{4/3}\rfloor} F\left(\frac{v}{t}\right)\frac{\sin v}{v}dv &= \sum_{n=\lfloor t^{2/3}\rfloor}^{\lfloor t^{4/3}\rfloor-1}\int_{2\pi n}^{2\pi(n+1)} F\left(\frac{v}{t}\right)\frac{\sin v}{v}dv\\ &= \sum_{n=\lfloor t^{2/3}\rfloor}^{\lfloor t^{4/3}\rfloor-1} 2\pi\int_0^1 F\left(\frac{2\pi(n+x)}{t}\right)\frac{\sin 2\pi x}{2\pi(n+x)}dx.\end{aligned}$$

For $n \geq 2$ and $0 \leq x \leq 1$ we have

$$\frac{1}{2\pi(n+x)} = \frac{1}{2\pi n}\left(1 + O\left(\frac{x}{n}\right)\right),$$

where the O-constant is absolute. This yields

$$\int_0^1 F\left(\frac{2\pi(n+x)}{t}\right)\frac{\sin 2\pi x}{2\pi(n+x)}dx = \frac{1}{2\pi n}\int_0^1 F\left(\frac{2\pi(n+x)}{t}\right)\sin 2\pi x\, dx + O\left(\frac{1}{n^2}\right),$$

where the O-constant depends on λ but not on $n \geq 2$. We now put in the integral on the right the bound, obtained from (4.3), that for $0 \leq x \leq 1$,

$$F\left(\frac{2\pi(n+x)}{t}\right) = F\left(\frac{2\pi n}{t}\right) + O\left(|\frac{x}{t}| + |\frac{x}{t}|^2\right).$$

Substituting this in the integral, the constant term $F(\frac{2\pi n}{t})$ integrates to 0, and we obtain

$$\frac{1}{2\pi n}\int_0^1 F\left(\frac{2\pi(n+x)}{t}\right)\sin 2\pi x dx = O\left(\frac{1}{nt}\right).$$

We conclude that

$$\int_{2\pi\lfloor t^{2/3}\rfloor}^{2\pi\lfloor t^{4/3}\rfloor} F\left(\frac{v}{t}\right)\frac{\sin v}{v}dv = O\left(\sum_{n=\lfloor t^{2/3}\rfloor}^{\lfloor t^{4/3}\rfloor}(\frac{1}{nt}+\frac{1}{n^2})\right) = O\left(\frac{1}{t^{2/3}}\right).$$

For the third integral, we use the rapid decrease of $\Phi(u) = O(e^{-e^u})$ to conclude, with much to spare, that

$$\left|\int_{\lfloor t^{4/3}\rfloor} F\left(\frac{v}{t}\right)\frac{\sin v}{v}dv\right| \leq \int_{t^{1/3}}^{\infty} F(u)du \leq e^{-2t}.$$

Combining these three integral estimates, we obtain the desired bound (4.1).

(2) First, the fact that $\lim_{t\to\infty}\Xi_\lambda^{(-1)}(t) = A_0 \neq 0$ implies that the function $\Xi_\lambda^{(-1)}$ has at most finitely many zeros on the positive t axis. The functional equation $\Xi_\lambda^{(-1)}(-t) = -\Xi_\lambda^{(-1)}(t)$ gives the result on the negative t axis as well, and shows $\Xi_\lambda^{(-1)}(0) = 0$. Since these functions are entire of order 1 and maximal type by Lemma 3.2(1), they necessarily have infinitely many zeros, whence all but finitely many are complex zeros.

Secondly, we recall that in 1947 Wintner [45] proved directly that

$$\Xi^{(-1)}(t) := \Xi_0^{(-1)}(t) > 0 \text{ when } t > 0; \tag{4.4}$$

this fact implies that $\Xi^{(-1)}(z) = \Xi_0^{(-1)}(z)$ has no zeros on the positive real axis, and the functional equation gives the same on the negative real axis. Here we note in passing that $\Xi^{(-1)}(z)$ has a simple zero at $z = 0$, since $\Xi(0) = \xi(\frac{1}{2}) \approx 0.49712 \neq 0$. Wintner's proof is based on the following assertion.

Claim. *If a function $\Psi(u)$ is positive and decreasing on the positive real axis, then for each positive t,*

$$\int_0^\infty \Psi(u)\left(\frac{\sin tu}{u}\right)du = \lim_{X\to\infty}\int_0^X \Psi(u)\left(\frac{\sin tu}{u}\right)du > 0.$$

To prove the claim, the existence of the limit is seen by writing $\psi(u) = \frac{1}{u}\Psi(u)$ and noting it is positive and decreases to 0 at ∞. On choosing values $X = X_n = \frac{n\pi}{t}$ one has

$$\int_0^{X_n}\Psi(u)\frac{\sin tu}{u}du = \frac{1}{t}\sum_{k=1}^{n}\int_{(k-1)\pi}^{k\pi}\psi(\frac{v}{t})\sin v\,dv.$$

Observing that the terms of the series have alternating signs, are decreasing and go to zero, one can let $n \to \infty$, get a convergent series, which has a positive limit since its first term is positive. The limit exists over all X since the variation between $X_n \leq X \leq X_{n+1}$ goes to zero as well. This proves the claim.

We now choose $\Psi(u) = \Phi(u)$ in the claim, noting that Lemma 3.1(5) asserts the positive decreasing hypothesis holds, and the claim gives Wintner's result (4.4). The functional equation in Lemma 3.1(2) then gives $\Xi_0^{(-1)}(t) < 0$ when $t < 0$, which proves assertion (2) in the case $\lambda = 0$. It is immediate that $e^{\lambda u^2}\Phi(u)$ is also positive and decreasing for $\lambda \leq 0$, whence the claim applies similarly to establish (2). $\square$

5. Value Sets of $\xi^{(-1)}(s)$: Proof of Theorem 2.2

The basic idea behind the lower bound in this theorem is given by the following two facts.

(1) As $t \to \infty$ the function $\xi^{(-1)}(\sigma + it)$ approaches iA_0 uniformly on any vertical strip $\sigma_1 \le \sigma \le \sigma_2$, where $-\infty < \sigma_1 < \sigma_2 < \infty$. Thus for any value $\alpha_0 \neq iA_0$ all the solutions to $\xi^{(-1)}(s) = \alpha_0$ in the strip must lie below some finite bound $t \le C$, where C depends on σ_1, σ_2.

(2) As $t \to -\infty$ the function $\xi^{(-1)}(\sigma + it)$ approaches $-iA_0$ uniformly on any vertical strip $\sigma_1 \le \sigma \le \sigma_2$. Thus for any value $\alpha_0 \neq -iA_0$ all the solutions to $\xi^{(-1)}(s) = \alpha_0$ in the strip must lie above some finite bound $t \ge -C$, where $-C$ depends on the values σ_1, σ_2.

These two facts follow directly from Theorem 2.1, using the well known fact that $|\xi(s)| \to 0$ as $|t| \to \infty$, uniformly on any vertical strip. (This may be proved following Lemma 3.3(1).) We fix a vertical strip, which without loss of generality includes the line $Re(s) = \frac{1}{2}$ in its interior. Then for $\sigma_0 \in [\sigma_1, \sigma_2]$ we have

$$\xi^{(-1)}(\sigma_0 + it) = \xi^{(-1)}(\frac{1}{2} + it) + \int_{1/2}^{\sigma_0} \xi(\sigma + it) d\sigma.$$

Theorem 2.1 now gives $\xi^{(-1)}(\frac{1}{2} + it) \to iA_0$, as $t \to \infty$, and $\xi^{(-1)}(\frac{1}{2} + it) \to -iA_0$ as $t \to -\infty$. The uniform bound on $|\xi(s)| \to 0$ as $|t| \to \infty$ in the strip then shows that the integral on the right can be bounded by ϵ in absolute value for large enough $|t|$ (depending on σ_1, σ_2), and the facts follow.

The proof of Theorem 2.2 obtains a lower bound using a quantitative version of the two facts above, determining the dependence of the constants C above on the width of the strip, chosen to have $Re(s) = \frac{1}{2}$ as its central line. The upper bound is obtained by analyzing the rapid growth of $\xi(s)$ on horizontal lines of constant t, which comes from the gamma factor in $\xi(s)$.

We commence the proof. Using the symmetries of the function $\xi^{(-1)}(s)$, it suffices to prove the results (1) and (2) for a zero $\xi^{(-1)}(\rho) = \alpha_0$ with $\rho = \sigma + it$ in the first quadrant region $\sigma \ge \frac{1}{2}$ and $t \ge 0$. In this proof we treat α_0 as fixed, and all constants C_j given in the proof will depend on α_0. We divide the first quadrant region into three subregions which we treat separately.

The first case considers the subregion $\frac{1}{2} \le \sigma \le 2$, and $t \ge 0$. We assert that $|\xi^{(-1)}(s) - iA_0| \to 0$ as $t \to \infty$ uniformly in this range of σ. From the assertion we may conclude that for any $\alpha_0 \neq iA_0$ the solutions to $\xi^{(-1)} = \alpha_0$ are confined to a compact region, which proves the theorem in this case. The assertion immediately follows from the result of Theorem 2.1(1) (for $\lambda = 0$) that gives $\xi^{(-1)}(\frac{1}{2} + it) \to iA_0$ as $t \to \infty$, combined with the bound

$$|\xi^{(-1)}(\sigma + it) - \xi^{(-1)}(\frac{1}{2} + it)| \le C_4 e^{-\frac{\pi}{4}|t|}(|t| + 1)^{\frac{5}{2}},$$

which follows from Lemma 3.3(1) by integration on a horizontal line.

The second case is the subregion $\sigma \ge 2$ and $t \ge 4\pi e$, which is the main case. To prove the bounds (1) and (2) for this case, we will obtain lower and upper bounds of the required form on σ as a function of t. The lower bound (for $\alpha_0 \neq iA_0$) asserts there is a constant C_5 (depending on α_0) such that $|\xi^{(-1)}(s_0) - \alpha_0| \neq 0$ for $s_0 = \sigma_0 + it$, whenever

$$\frac{1}{2} \leq \sigma_0 \leq \frac{\pi}{2}\left(\frac{t}{\log t}\right) - C_5 \frac{t}{(\log t)^2}. \tag{5.1}$$

We begin with

$$\xi^{(-1)}(s_0) = \xi^{(-1)}(\frac{1}{2} + it) + \int_{\frac{1}{2}}^{\sigma_0} \xi(\sigma + it) d\sigma.$$

We assume $\alpha_0 \neq iA_0$, and set $\delta = |\alpha_0 - iA_0| > 0$. We have $\xi^{(-1)}(\frac{1}{2} + iT) \to iA_0$ as $T \to \infty$, and thus one has $|\xi^{(-1)}(\frac{1}{2} + iT) - \alpha_0| \geq \frac{1}{2}\delta$ for all T larger than some constant T_0. Note, however, that we can assume that $|\xi^{(-1)}(\frac{1}{2} + it) - \alpha_0| \geq \frac{1}{2}\delta$ holds for all pairs $\sigma + it$ satisfying (5.1) with $t \geq 4\pi e$ since we can increase the size of C_5 such that (5.1) will have no solutions for $t < T_0$.

We next show one can pick C_5 large enough that when σ_0 satisfies (5.1) we have the estimate, valid for $t \geq 4\pi e$,

$$\int_{\frac{1}{2}}^{\sigma_0} |\xi(\sigma + it)|\, d\sigma \leq \frac{1}{4}\delta. \tag{5.2}$$

We use Lemma 3.3 (1) to bound the integral from $\sigma = \frac{1}{2}$ to $\sigma = 2$ by $\frac{1}{8}\delta$. For the remaining integral with σ satisfying (5.1) we use the upper bound in Lemma 3.3(2), noting that in this range

$$\arctan\left(\frac{t}{\sigma}\right) = \frac{\pi}{2} - \frac{\sigma}{t} + O\left(\frac{\sigma^2}{t^2}\right),$$

to obtain

$$\begin{aligned}
\int_{2}^{\sigma_0} |\xi(\sigma + it)| d\sigma &\leq C_6 |t|^{3/2} \int_{2}^{\sigma_0} \exp\left(\frac{\sigma}{2}\log|\frac{\sigma + it}{2\pi e}| - \frac{\pi}{4}t + O(t/\log t)\right) d\sigma \\
&\leq C_7 |t|^{3/2} e^{-\frac{\pi}{4}t} \int_{2}^{\sigma_0} \exp\left(\frac{\pi}{4}(\frac{t}{\log t})\log t - C_5 \frac{t}{\log t} + O(\frac{t}{\log t})\right) d\sigma \\
&\leq C_8 |t|^{5/2} \exp\left(-C_5 \frac{t}{\log t} + O(\frac{t}{\log t})\right).
\end{aligned}$$

Keeping in mind that we are free to choose C_5 as large as necessary, we note that it is possible to choose C_5 large enough, depending on α_0, to make the exponential term above smaller than $\exp(-C_9 \frac{t}{\log t})$, where C_9 is large enough that $\exp(-C_9 \frac{t}{\log t}) \leq \frac{1}{8}\delta$ for all $t \geq 4\pi e$. Thus we establish (5.2).

Now the triangle inequality gives

$$\begin{aligned}
|\xi^{(-1)}(s_0) - \alpha_0| &\geq |\alpha_0 - iA_0| - |iA_0 - \xi^{(-1)}(\frac{1}{2} + it)| - |\xi^{(-1)}(\frac{1}{2} + it) - \xi^{(-1)}(s_0)| \\
&\geq \delta - \frac{\delta}{2} - \frac{\delta}{4} > 0,
\end{aligned} \tag{5.3}$$

as asserted. This bound applies to all $\alpha_0 \neq iA_0$, for $t \geq 4\pi e$ and it similarly applies for $\alpha_0 \neq -iA_0$ in the lower half-plane region $t \leq -4\pi e$. Thus it gives the lower bound asserted in (1), for $\alpha_0 \neq \pm iA_0$. The upper bounds in (1) and (2) assert that there is a constant C_{10} (depending on α_0) such that for any fixed α_0 (including $\alpha_0 = \pm iA_0$) one has $|\xi^{(-1)}(s_0) - \alpha_0| \neq 0$ whenever

$$\sigma_0 \geq \frac{\pi}{2}\left(\frac{t}{\log t}\right) + C_{10}\frac{t}{(\log t)^2}. \tag{5.4}$$

It suffices to prove that $|\xi^{(-1)}(s_0)| > |\alpha_0|$ holds when (5.4) holds. To show this upper bound, we will use the following analytic lemma.

Lemma 5.1. *Suppose that* $f : [0,\infty) \to \mathbb{C} \setminus \{0\}$ *is continuous, and that the total variation of* $\arg(f)$ *on the interval* $[a,b]$ *is at most* $\theta < \pi$. *Then*

$$\left|\int_a^b f(x)dx\right| \geq \cos\left(\frac{\theta}{2}\right)\int_a^b |f(y)|\mathrm{d}y \tag{5.5}$$

Proof. Note that since the total variation of $\arg(f)$ on the interval $[a,b]$ is equal to $\theta < \pi$, there exists $\beta \in \mathbb{R}$ such that $\arg(f(x)) \in [\beta - \frac{\theta}{2}, \beta + \frac{\theta}{2}]$ for $x \in [a,b]$. Then for $v_1 = e^{i\beta}$ and $v_2 = e^{i(\beta+\pi/2)}$, there exist real valued functions u and w such that $f(x) = u(x)v_1 + w(x)v_2$. Since v_1 and v_2 are orthogonal,

$$\left|\int_a^b f(y)dy\right| = \left|\left(\int_a^b u(x)dx\right)v_1 + \left(\int_a^b w(y)dy\right)v_2\right| \geq \left|\int_a^b u(x)dx\right|.$$

Finally, since $\arg(f) \in [\beta - \frac{\theta}{2}, \beta + \frac{\theta}{2}]$, we have that $u(x) \geq \cos(\frac{\theta}{2})|f(x)|$, so

$$\left|\int_a^b f(y)dy\right| \geq \left|\int_a^b u(x)dx\right| \geq \cos\left(\frac{\theta}{2}\right)\int_a^b |f(x)|dx.$$

□

We write $\xi^{(-1)}(\sigma_0 + it) = \xi^{(-1)}(\frac{1}{2} + it) + \int_{1/2}^{\sigma_0} \xi(\sigma + it)d\sigma$, and will use the fact that the main contribution to the size of $\xi^{(-1)}(\sigma + it)$ will come from the integral over a small interval near its right endpoint, and the function will be very large when (5.4) holds. Thus we start from the inequality

$$|\xi^{(-1)}(\sigma_0 + it)| \geq \left|\int_{\sigma_0 - 2}^{\sigma_0} \xi(\sigma + it)\mathrm{d}\sigma\right| - \int_{1/2}^{\sigma_0 - 2} |\xi(\sigma + it)|\mathrm{d}\sigma - |\xi^{(-1)}(\frac{1}{2} + it)|,$$

and will show that the right side is positive when (5.4) holds. A total variation bound on $\arg(\xi(\sigma + it))$ is obtained via Lemma 3.3(3), taking $\delta = \frac{2\pi}{3} - (\frac{\pi+1}{2}) \approx 0.0236$, yielding

$$\int_{\sigma_0 - 2}^{\sigma_0} \left|\frac{d \arg \xi(\sigma + it)}{d\sigma}\right| \mathrm{d}\sigma \leq \frac{2\pi}{3}.$$

which is valid provided $\sigma_0 \geq C_{11}$ for a suitable constant C_{11}. Now Lemma 5.1 applies to give

$$\left|\int_{\sigma_0 - 2}^{\sigma_0} \xi(\sigma + it)\mathrm{d}\sigma\right| \geq \frac{1}{2}\int_{\sigma_0 - 2}^{\sigma_0} |\xi(\sigma + it)|\,\mathrm{d}\sigma.$$

We conclude that

$$|\xi^{(-1)}(\sigma_0 + it)| \geq \frac{1}{2}\int_{\sigma_0 - 2}^{\sigma_0} |\xi(\sigma + it)|\ \mathrm{d}\sigma - \int_{1/2}^{\sigma_0 - 2} |\xi(\sigma + it)|\mathrm{d}\sigma - |\xi^{(-1)}(\frac{1}{2} + it)|. \tag{5.6}$$

The last term on the right has $|\xi^{(-1)}(\frac{1}{2}+it)| = O(1)$, since $\xi^{(-1)}(s)$ is bounded on the critical line.

We now obtain from Lemma 3.3 (2) and Lemma 3.4 (1) a lower bound for the first integral on the right hand side of (5.6). Namely, for all sufficiently large σ_0 and $t \geq 4\pi e$ there holds

$$\begin{aligned}\int_{\sigma_0-2}^{\sigma_0} |\xi(\sigma+it)| \, \mathrm{d}\sigma &\geq \frac{1}{2}\int_{\sigma_0-1}^{\sigma_0} F(\sigma,t)\, \mathrm{d}\sigma \\ &\geq \frac{1}{2}F(\sigma_0-1,t),\end{aligned}$$

where $F(\sigma,t)$ is given by (3.4). On the other hand, using Lemma 3.3(2) and Lemma 3.4(2), we can pick a constant C_{12} large enough that for all $t \geq 4\pi e$ and sufficiently large σ_0 (depending on C_{12}),

$$\begin{aligned}\int_{1/2}^{\sigma_0-2} |\xi(\sigma+it)| \, \mathrm{d}\sigma &\leq C_{12}\int_{1/2}^{\sigma_0-2} F(\sigma,t)\, \mathrm{d}\sigma \\ &\leq C_{13}F(\sigma_0-2,t).\end{aligned}$$

The function $F(\sigma,t)$ is rapidly increasing in σ. For $t \geq 4\pi e$ and $\sigma_0 > 2$ comparison of the terms in (3.9) yields

$$\frac{F(\sigma_0-1,t)}{F(\sigma_0-2,t)} \geq \frac{1}{\sqrt{2\pi e}}((\sigma_0-2)^2+t^2)^{\frac{1}{4}}. \tag{5.7}$$

This fact implies for all sufficiently large σ_0, $\frac{1}{4}F(\sigma_0-1,t) \geq 4C_{13}F(\sigma_0-2,t)$, whence half of the first term on the right in (5.6) already dominates the second integral. It remains to choose σ_0 large enough as a growing function of t that the remaining half of the absolute value of the first term also dominates $|\xi^{(-1)}(\frac{1}{2}+it)| + |\alpha_0| = O(1)$ on the right side of (5.6). We show the lower bound in (5.4) achieves this, taking $\sigma_0 \geq \frac{\pi}{2}(\frac{t}{\log t}) + C_5\frac{t}{(\log t)^2}$ with sufficiently large C_5. By the monotonicity property in Lemma 3.4(1) it suffices to consider $\sigma_0 = \frac{\pi}{2}(\frac{t}{\log t}) + C_5\frac{t}{(\log t)^2}$, for which we obtain

$$\begin{aligned}F(\sigma_0-1,t) &\geq \exp\left(-\frac{\pi\log(2\pi e)}{4}(\frac{t}{\log t}) + O(\frac{t}{(\log t)^2})\right)\cdot\exp\left(\frac{\pi t}{4} + C_5\frac{t}{(\log t)}\right)\cdot \\ &\quad \cdot\exp\left(-\frac{\pi t}{4} + \frac{\pi}{4}(\frac{t}{\log t}) + O(\frac{t}{(\log t)^2})\right) \\ &\geq \exp\left(C_{10}\frac{t}{\log t} + O(\frac{t}{(\log t)^2})\right).\end{aligned}$$

Here $C_{10} = C_5 - \frac{\pi}{4}\log(2\pi)$, so by choosing C_5 sufficiently large, we can overcome the O-constant in the last term and force

$$\exp\left(C_{10}\frac{t}{\log t} + O(\frac{t}{(\log t)^2})\right) > |\xi^{(-1)}(\frac{1}{2}+it)| + |\alpha_0|$$

to hold for all $t \geq 4\pi e$. We conclude that for proper choices of C_{12}, C_5 the right hand side of (5.6) is larger than$|\alpha_0|$ for $t \geq 4\pi e$ and σ satisfying (5.4). This gives, for any fixed $\alpha_0 \in \mathbb{C}$, an estimate establishing the upper bound case of both (1) and (2) in the range $t \geq 4\pi e$.

The third case is the subregion $0 \le t \le 4\pi e$ and $\sigma \ge 2$. We assert that $|\xi^{(-1)}(s)|$ becomes very large as σ increases, which for any constant $\alpha_0 \in \mathbb{C}$ will confine solutions to $\xi^{(-1)}(s) = \alpha_0$ with $0 \le t \le 4\pi e$ to a compact region $0 \le \sigma \le C_{14}$, and so complete the proof.

We proceed to estimate the size of $\xi^{(-1)}(s_0)$, with $s_0 = \sigma_0 + it$, using

$$\xi^{(-1)}(s_0) = \xi^{(-1)}(\sigma_0 - 2) + \int_0^t \xi(\sigma_0 - 2 + iy)dy + \int_{\sigma_0-2}^{\sigma_0} \xi(\sigma + it)d\sigma.$$

We obtain

$$\begin{aligned} |\xi^{(-1)}(s_0)| &\ge |\int_{\sigma_0-2}^{\sigma_0} \xi(\sigma+it)d\sigma| - |\xi^{(-1)}(\sigma_0-2)| - \int_0^t |\xi(\sigma_0 - 2 + iy)|dy \\ &\ge |\int_{\sigma_0-2}^{\sigma_0} \xi(\sigma+it)d\sigma| - \Big(|\xi^{(-1)}(\sigma_0-2)| + 4\pi e|\xi(\sigma_0-2)|\Big), \end{aligned} \tag{5.8}$$

with the last inequality based on the fact that for fixed σ, the function $|\xi(\sigma + it)|$ is maximized on the real axis. The argument now proceeds similarly to the case $t \ge 4\pi e$. Namely, the first integral can be bounded below by the use of Lemma 5.1. Together with Lemma 3.3(2) we obtain

$$\begin{aligned} |\int_{\sigma_0-2}^{\sigma_0} \xi(\sigma+it)d\sigma| &\ge \frac{1}{2}\int_{\sigma_0-2}^{\sigma_0} |\xi(\sigma+it)|d\sigma \\ &\ge \frac{1}{2}\int_{\sigma_0-1}^{\sigma_0} F(\sigma,t)\,\mathrm{d}\sigma \\ &\ge \frac{1}{2}F(\sigma_0-1,t), \end{aligned}$$

where $F(\sigma, t)$ is given by (3.4). Since $t \le 4\pi e$ and $\sigma_0 \ge 2$ we have

$$F(\sigma_0 - 1, t) \ge e^{-\pi^2 e} F(\sigma_0 - 1, 0)$$

directly from the definition (3.4). Additionally, Lemma 3.3(2) guarantees the existence of a constant C_{15} such that for all $\sigma_0 \ge C_{15}$,

$$4\pi e|\xi(\sigma_0 - 2)| \le (4\pi e + 1)F(\sigma_0 - 2, 0).$$

We also obtain

$$|\xi^{(-1)}(\sigma_0 - 2)| \le \int_{1/2}^{\sigma_0-2} |\xi(\sigma)|d\sigma \le 4F(\sigma_0 - 2, 0) + \int_{1/2}^{4\pi e} |\xi(\sigma)|d\sigma,$$

by appealing to the estimate

$$\int_{4\pi e}^{\sigma_1} F(\sigma, t)d\sigma \le 4F(\sigma_1, t)$$

which is seen to be valid for any $t \ge 0$, following the proof of Lemma 3.4(2).

Combining all of the above estimates with (5.8), we have

$$|\xi^{(-1)}(s_0)| \ge \frac{e^{-\pi^2 e}}{2} F(\sigma_0 - 1, 0) - \left(\int_{1/2}^{4\pi e} |\xi(\sigma)|d\sigma + (4\pi e + 5)F(\sigma_0 - 2, 0)\right).$$

Because the integral in the right hand side above is a constant and the function $F(\sigma, 0)$ is increasing without bound, we can choose constants $C_{16}, C_{17} > 0$ such that for all $\sigma_0 > C_{16}$, we have

$$|\xi^{(-1)}(s_0)| \geq \frac{e^{-\pi^2 e}}{2} F(\sigma_0 - 1, 0) - C_{17} F(\sigma_0 - 2, 0).$$

Finally, we may apply the bound (5.7) to conclude that for all $\sigma_0 \geq C_{18}$ and $0 \leq t \leq 4\pi e$ the first term on the right hand side above dominates the second enough to give

$$|\xi^{(-1)}(s_0)| \geq F(\sigma_0 - 2, 0).$$

Since the function on the right is unbounded as σ_0 increases, by choosing C_{14} sufficiently large we can guarantee that $|\xi^{(-1)}(s_0)| > |\alpha_0|$ holds on the region $\sigma_0 \geq C_{14}$, $0 \leq t \leq 4\pi e$. This completes the proof of Theorem 2.2.

Remark. In the exceptional case (2), for $\alpha_0 = iA_0$ and $t > 0$, it seems possible that a stronger upper bound than (2.2) may be valid. We cannot even rule out the possibility that a $|\sigma| \leq O(1)$ upper bound might be valid; see the numerical data in §5, plotted in Figure 3. To improve the upper bound significantly, one would like an improved error term in (4.1) that decreases exponentially in t.

6. Numerical Results

We report on numerical results on the zeros of $\xi^{(-1)}(s) - c$, kindly supplied to us by Henri Cohen. These results were computed using PARI.

Table 1 below gives values of the first few zeros of the function $\xi^{(-1)}(s)$. They are distributed in a very regular way, consistent with Theorem 2.2. For comparison purposes we include data on averaged position of pairs of consecutive zeros of $\xi(s)$ (i.e. zeta zeros in the critical strip). Each zero ρ of $\xi^{(-1)}(s)$ (with $Im(\rho) > 0$) off the critical line has a companion zero $1 - \bar{\rho}$ and we expect these to correspond to a pair of notrivial zeta zeros.

Figure 1 pictures a plot the first 100 zeros of $\xi^{(-1)}(s)$ in each quadrant; note the fourfold symmetry, and the fact that the zeros appear to fall on a smooth curve. (A suitable smooth curve that interpolates the points is given by a certain level set of the function $F(s) = \xi^{(-1)}(s) - iA_0$.)

Figure 2 plots, on a smaller scale, the first 500 zeros in the first quadrant. There is general agreement with the asymptotics of Theorem 2.2.

Next we consider the distribution of zeros of $\xi^{(-1)}(s; -iA_0) := \xi^{(-1)}(s) - iA_0$. This function has

$$\lim_{t \to \infty} \xi^{(-1)}(\frac{1}{2} + it; -iA_0) = 0.$$

These are plotted in Figure 3 to height 180. This data hints that infinitely many zeros lie on the critical line. Perhaps this will be a positive proportion of all zeros. However, as the height increases more zeros seem to go off the line and up to height 500 only about $1/3$ of the zeros are on the critical line. (Note that Theorem 2.2 shows that $\xi^{(-1)}(s; -iA_0)$ has only finitely many zeros on the critical line in the lower half plane.)

These computational results suggest a number of further questions.

k	$\mathrm{Re}(\rho_k)$	$\mathrm{Im}(\rho_k)$	$\lvert\rho_k\rvert$	$\tilde{\gamma}_k$	γ_{2k-1}	γ_{2k}
0	0.50000	0.00000 i	0.00000			
1	12.26164	10.74143 i	16.30111	17.57838 i	14.13472 i	21.02203 i
2	16.59401	18.18824 i	24.62059	27.71787 i	25.01085 i	30.42487 i
3	19.91864	24.52433 i	31.59501	35.26062 i	32.93506 i	37.58617 i
4	22.76123	30.28316 i	37.88330	42.12290 i	40.91871 i	43.32707 i
5	25.30557	35.66576 i	43.73121	48.88949 i	48.00515 i	49.77383 i
6	27.64154	40.77783 i	49.26344	54.70828 i	52.97032 i	56.44624 i
7	29.82109	45.68184 i	54.55391	60.08941 i	59.34704 i	60.83177 i
8	31.87747	50.41877 i	59.65087	66.09617 i	65.11254 i	67.07981 i
9	33.83352	55.01727 i	64.58799	70.80678 i	69.54640 i	72.06715 i
10	35.70571	59.49838 i	69.38988	76.42477 i	75.70469 i	77.14484 i
11	37.50640	63.87809 i	74.07524	81.12388 i	79.33737 i	82.91038 i
12	39.24515	68.16894 i	78.65868	86.08038 i	84.73549 i	87.42527 i
13	40.92954	72.38096 i	83.15186	90.65050 i	88.80911 i	92.49189 i
14	42.56569	76.52235 i	87.56431	95.26098 i	94.65134 i	95.87063 i
15	44.15865	80.59992 i	91.90394	100.07452 i	98.83119 i	101.31785 i
16	45.71262	84.61941 i	96.17738	104.58608 i	103.72553 i	105.44662 i
17	47.23115	88.58569 i	100.39027	109.09907 i	107.16861 i	111.02953 i
18	48.71728	92.50297 i	104.54746	113.09744 i	111.87465 i	114.32022 i
19	50.17363	96.37488 i	108.65317	117.50873 i	116.22668 i	118.79078 i
20	51.60248	100.20464 i	112.71107	122.15847 i	121.37012 i	122.94682 i

TABLE 1. Initial zeros ρ_k of $\xi^{(-1)}(s)$ in the first quadrant. Averaged pairs of zeta zero ordinates $\tilde{\gamma}_k := \frac{1}{2}(\gamma_{2k-1} + \gamma_{2k})$ are included for comparison.

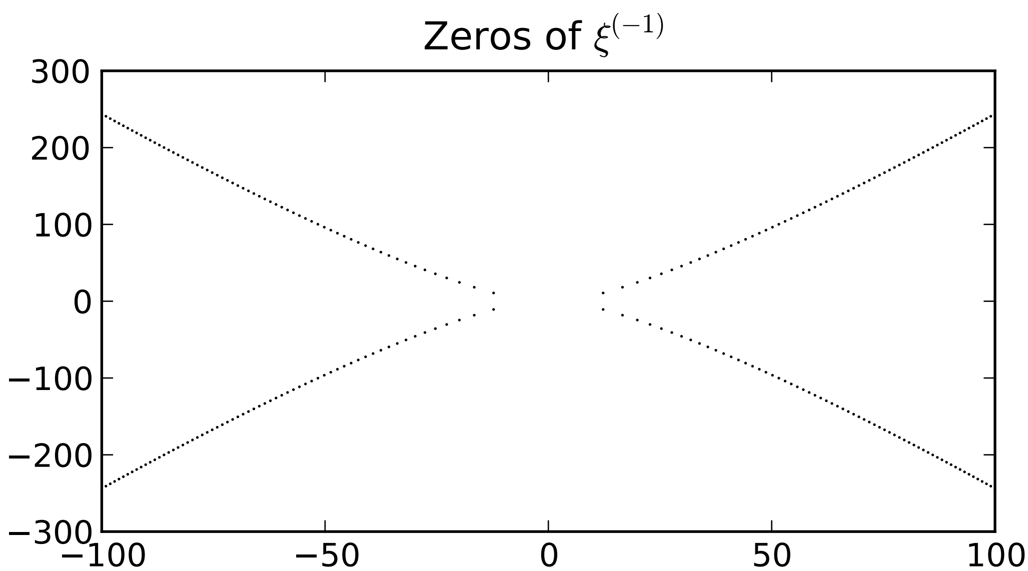

FIGURE 1. Plot of initial zeros of $\xi^{(-1)}(s)$ in all four quadrants.

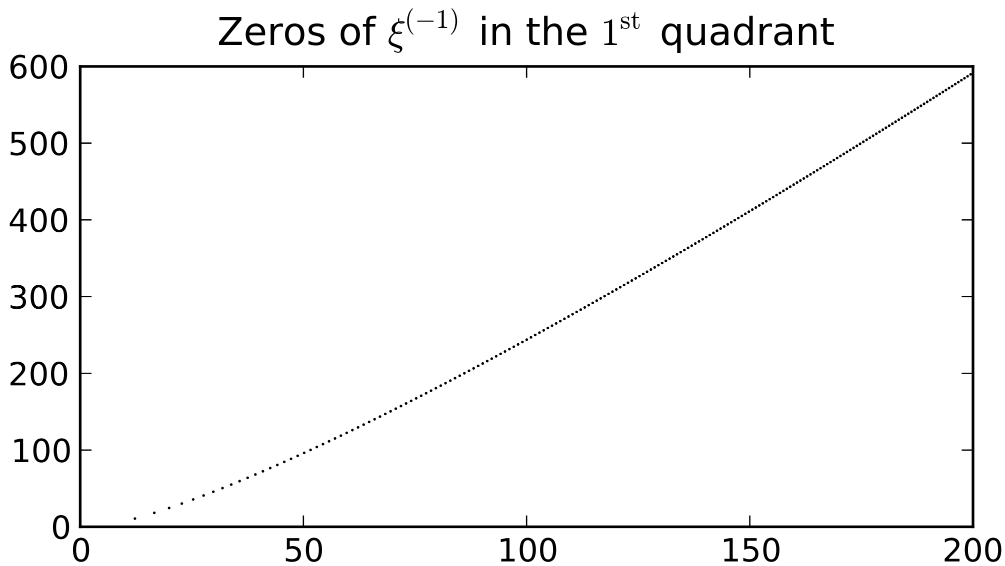

FIGURE 2. Plot of first 500 zeros of $\xi^{(-1)}(s)$ in the first quadrant.

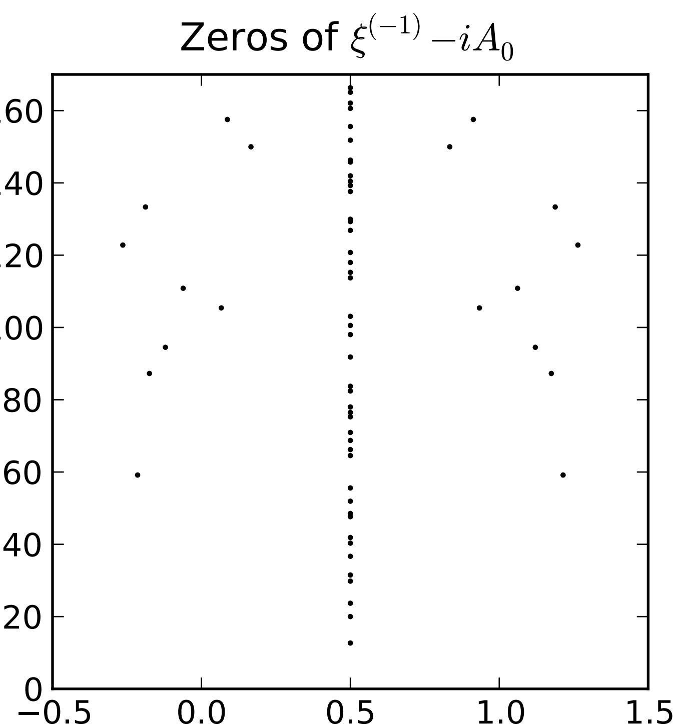

FIGURE 3. Plot of zeros of $\xi^{(-1)}(s) - A_0 i$, where $A_0 \approx 2.80668$, to height $T = 160$

Question 1. Let the zeros of $\xi^{(-1)}(s)$ have its zeros $\rho = \sigma + it$ in the quadrant $\sigma \geq \frac{1}{2}, t \geq 0$ arranged $\rho_n = \sigma_n + it_n$ arranged in order of increasing $|\rho_n|$ have the property that $\sigma_n > \sigma_{n-1}$ and $t_n > t_{n-1}$?

The numerical evidence supports a positive answer to this question. Furthermore, numerical differencing of the abscissas and ordinates of the first 500 zeros uncovers regular trends. One may expect that there is an asymptotic expansion in functions of n for the spacings.

Question 2. What properties of the zeros of a function like $\xi^{(-1)}(s)$, which do not lie on the critical line, would be sufficient to imply that the zeros of its derivative $\xi(s)$ would all lie on the critical line?

Would monotone increase of the imaginary parts of the zeros in the first quadrant as the real part increases, as in Question 1, be a sufficient condition? For an possibly related situation involving the ξ-function, where such a monotonicity implies the RH, see Haglund [23].

Question 3. How would the GUE spacing distribution of zeros of $\xi(s)$ manifest iteself in terms of the distribution of the zeros of $\xi^{(-1)}(s)$?

Recall that the GUE hypothesis (see Odlyzko [32], Berry and Keating [2], Katz and Sarnak [25]) asserts that when zeros are ordered by increasing ordinates, and zero spacings at height T are rescaled by a factor $\frac{1}{2\pi} \log T$ to have expected spacing 1, then the distribution of spacings from height $0 \leq T \leq X$ should as $X \to \infty$ approach a nontrivial continuous limiting distribution, called the GUE distribution; this distribution arises as an eigenvalue spacing distribution in random matrix theory for the Gaussian Unitary Ensemble.

The data above, in Table 1 and Figure 2, while rather limited, seems to suggest that the zero spacings of $\xi^{(-1)}(s)$ are extremely regular. No fluctuations in spacings analogous to GUE seem visible in this data. In contrast, fluctuations in zeta zero spacings are already evident by $T = 100$.

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