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English Translation

Remark on the Integral Representation of Riemann Xi-Function

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The Riemann ξ-function, defined by the formula

$$\xi(\mathrm{i}z) = \frac{1}{2}\Big(z^2 - \frac{1}{4}\Big)\,\pi^{\frac{1}{2}z-\frac{1}{4}}\,\Gamma\Big(\frac{z}{2}+\frac{1}{4}\Big)\,\zeta\Big(\frac{1}{2}+z\Big), \tag{1}$$

had been presented by Riemann through an improper integral on trigonometric function, i.e.[1],

$$\xi(z) = 2\int\limits_0^\infty \Phi(u)\cos(zu)\mathrm{d}u \tag{2}$$

$$\Phi(u) = 2\pi e^{5u/2}\sum_{n=1}^{\infty}\Big(2\pi e^{2u}n^2 - 3\Big)n^2 e^{-n^2\pi e^{2u}}. \tag{3}$$

Apparently,

$$\Phi(u) \sim 4\pi^2 e^{9u/2-\pi e^{2u}}, \qquad \text{for } u \to +\infty. \tag{4}$$

Furthermore, $\Phi(u)$ is an even function (see Section 4 below). Consequently, there is

$$\Phi(u) \sim \Big(e^{9u/2} + e^{-9u/2}\Big)e^{-\pi(e^{2u}+e^{-2u})}, \qquad \text{for } u \to \pm\infty. \tag{5}$$

With respect to the Riemann hypothesis one can raise the following question[2]: When $\Phi(u)$ in (2) is replaced by the right-hand-side of (4), would the resulting function $\xi(z)$ have only real zeros?

[1] B. Riemann, Werke (1876), P.138.

[2] This question was mentioned occasionally by Prof. Landau in a conversation in 1913.

The answer is No (see Section 4 below): the resulting function has infinitely many imaginary zeros. However, if right-hand-side of (5) is used to substitute into (2), then a new function is created

$$\xi^*(z) = 8\pi^2 \int_0^\infty \Big(e^{9u/2} + e^{-9u/2}\Big) e^{-\pi(e^{2u}+e^{-2u})} \cos(zu) \mathrm{d}u, \tag{6}$$

which can be named as a *falsified* ξ*-function*, and in fact $\xi^*(z)$ has only real zeros. By the way, one can set up a completed angular domain form from the vertex O, which does not contain the real axis. When z is approaching ∞ in this domain, there is

$$\xi(z) \sim \xi^*(z).$$

If the number of zeros of $\xi(z)$ in the circle $|z| \leq r$ is denoted by $N(r)$, and the appropriate number of $\xi^*(z)$ is denoted by $N^*(r)$, there will be

$$N(r) \sim N^*(r),$$

and thus,

$$N(r) - N^*(r) = \mathcal{O}(\log r).$$

At follows, I will provide a proof to show that all zeros of $\xi^*(z)$ are real. The work will concentrate on another entire function, i.e.,

$$\mathcal{G}(z) = \mathcal{G}(z; a) = \int_{-\infty}^\infty e^{-a(e^u+e^{-u})+zu} \mathrm{d}u, \tag{7}$$

which has simpler expression and property. In the formula the parameter a will be set to a positive value. In fact, $\xi^*(z)$ can be expressed by $\mathcal{G}(z)$. From (6) and (7) one can derive easily

$$\xi^*(z) = 2\pi^2 \Big[\mathcal{G}\Big(\frac{\mathrm{i}z}{2} - \frac{9}{4}; \pi\Big) + \mathcal{G}\Big(\frac{\mathrm{i}z}{2} + \frac{9}{4}; \pi\Big)\Big]. \tag{8}$$

Thus, I will show that $\mathcal{G}(\mathrm{i}z)$ *has only real zeros*. Then, this properties will be equally applied to $\xi^*(z)$ as well.

Section 1

The most important property of the entire function $\mathcal{G}(z)$ is that it satisfies a simple difference equation, i.e.,

$$z\mathcal{G}(z) = a\Big(\mathcal{G}(z+1) - \mathcal{G}(z-1)\Big), \tag{9}$$

which can be obtained easily by taking integration by parts to (7). It is noted that $\mathcal{G}(z)$ is an even function,

$$\mathcal{G}(-z) = \mathcal{G}(z). \tag{10}$$

Furthermore, when z is not an integer,

$$\begin{aligned} \mathcal{G}(z) &= a^{-z}\Gamma(z)\left(1 + \sum_{n=1}^{\infty} \frac{a^{2n}}{n!(1-z)(2-z)\cdots(n-z)}\right) \\ &+ a^{z}\Gamma(-z)\left(1 + \sum_{n=1}^{\infty} \frac{a^{2n}}{n!(1+z)(2+z)\cdots(n+z)}\right). \end{aligned} \tag{11}$$

Some work is needed to prove (11). Since

$$\Gamma(z) = \int_0^a e^{-v} v^{z-1} \mathrm{d}v + \int_a^\infty e^{-v} v^{z-1} \mathrm{d}v = P(z) + Q(z), \tag{12}$$

where,

$$P(z) = P(z;a) = \sum_{n=0}^{\infty} \frac{(-1)^n a^{z+n}}{n!(z+n)}, \tag{13}$$

$$Q(z) = Q(z;a) = a^z \int_0^\infty e^{-ae^u + uz} \mathrm{d}u. \tag{14}$$

Now,

$$\begin{aligned} \mathcal{G}(z) &= \int_0^\infty e^{-a(e^u + e^{-u})}\Big(e^{uz} + e^{-uz}\Big)\mathrm{d}u \\ &= \int_0^\infty \sum_{n=0}^{\infty} e^{-ae^u} \frac{(-a)^n}{n!}\Big(e^{(-n+z)u} + e^{(-n-z)u}\Big)\mathrm{d}u. \end{aligned}$$

According to (14), the above series and integral converge absolutely. Thus, the order of summation and integration can be exchanged, and it becomes

$$\mathcal{G}(z) = a^{-z}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}}{n!}Q(-n+z) + a^{z}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}}{n!}Q(-n-z). \tag{15}$$

Then again from an equality

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{k+l}\,a^{k+l}}{k!\,l!}\left(\frac{1}{-k+z+l}+\frac{1}{k-z-l}\right) \\ &= a^{-z}\sum_{k=0}^{\infty}\frac{(-1)^k\,a^{2k}}{k!}\sum_{l=0}^{\infty}\frac{(-1)^l\,a^{-k+z+l}}{l!\,(-k+z+l)} \\ &\quad + a^{z}\sum_{l=0}^{\infty}\frac{(-1)^l\,a^{2l}}{l!}\sum_{k=0}^{\infty}\frac{(-1)^k\,a^{-l-z+k}}{l!\,(-l-z+k)}, \end{aligned}$$

also from (13), the above equation becomes

$$0 = a^{-z}\sum_{k=0}^{\infty}\frac{(-1)^k\,a^{2k}}{k!}P(-k+z) + a^{z}\sum_{l=0}^{\infty}\frac{(-1)^l\,a^{2l}}{l!}P(-l-z). \tag{16}$$

According to (12), (15), (16), it follows

$$\mathcal{G}(z) = a^{-z}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}}{n!}\Gamma(-n+z) + a^{z}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}}{n!}\Gamma(-n-z),$$

which is equivalent to (11).

I would mention also, but not give a proof, the following two expressions:

$$\begin{aligned} \mathcal{G}(z) &= \frac{1}{2\pi\mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty}\Gamma\Big(s-\frac{z}{2}\Big)\Gamma\Big(s+\frac{z}{2}\Big)a^{-2s}\mathrm{d}s, \\ \mathcal{G}(z) &= \frac{\pi}{\sin(\pi z)}e^{\mathrm{i}\pi z/2}J_{-z}(\mathrm{i}2a) - \frac{\pi}{\sin(\pi z)}e^{-\mathrm{i}\pi z/2}J_{z}(\mathrm{i}2a). \end{aligned}$$

where, in the first equation, $2a > |\Re z|$; and in the second equation, both two terms in the right-hand-side are the solutions of the same difference equation (9), and hence so does $\mathcal{G}(z)$.

Section 2

The asymptotic expression and the estimation of $\mathcal{G}(z)$ will be obtained by a further work on (11). The z-plane must be divided into several domains for study separately. Let $z =$

$x + \mathrm{i}y = re^{\mathrm{i}\varphi}$, where x, y, r, φ are real numbers and $r > 0$; and $y > 0$ is corresponding to $0 < \varphi < \pi$; and $x - \mathrm{i}y = \bar{z}$.

Claim I. For $|y| \geq 1$, $a \leq A$, there is

$$\mathcal{G}(z) = a^{-z}\Gamma(z)\Big(1 + \frac{\chi(z)}{z}\Big) + a^{z}\Gamma(-z)\Big(1 - \frac{\chi(-z)}{z}\Big), \tag{17}$$

where, the function $\chi(z) = \chi(z; a)$ is bounded by A.

Proof: Compared to (11), there is

$$\chi(z) = \frac{za^2}{1-z}\Big(1 + \sum_{n=2}^{\infty} \frac{a^{2n-2}}{n!\,(2-z)(3-z)\cdots(n-z)}\Big),$$

from which the Claim holds and will not be shown further.

Claim II. Let ε and A be positive numbers such that

$$\frac{a^z\mathcal{G}(z)}{\Gamma(z)} - 1 = \psi(z) = \psi(z; a). \tag{18}$$

Then, in the half space $x \geq \varepsilon$, there is

$$\lim_{|z|\to\infty} \psi(z) = 0, \tag{19}$$

and the limit is uniform for z (in the domain $x \geq \varepsilon$) and a if $a \leq A$.

Proof: According to the Stirling's formula, there exist a constant C such that for $y \geq 1$, $x \geq \varepsilon$, $r \geq ae$,

$$\Big|a^{2z}\frac{\Gamma(-z)}{\Gamma(z)}\Big| \leq a^{2x}Cr^{-2x}e^{-(\pi-2\varphi)y+2x} \leq C\Big(\frac{ae}{r}\Big)^{2\varepsilon}. \tag{20}$$

In one part of the half-space $x \geq \varepsilon$ in which $y \geq 1$, the Claim II comes directly from (17), (18), and (20). For another part of the half-space where $y \leq -1$, the Claim II also becomes valid through the symmetry. Then, there is a remaining half stripe

$$x \geq \varepsilon, \qquad -1 \leq y \leq 1.$$

At the boundary to the already proven part of Claim II, the function (18) approaches to 0 when $z \to \infty$. Since the function (18) is entire and of finite order, according to a well-known general theorem[3], it must uniformly converge to zero when $z \to \infty$ in the half-strip. This

[3]See about G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis* (Berlin 1925), Bd. 1, Aufgaben III 333, III 339.

completes the proof to Claim II. Of course, instead of using the general theorem, one also can directly use (11) to estimate the function (18), by forcing z to approach ∞ from a curve in the half-strip, e.g., a curve that pass a point $x = k + \frac{1}{2}$, where $k = 0, 1, 2, 3, \cdots$. In the case $k = 0$, we will do this estimation work for another purpose, see following (22).

Claim III. When $a \leq \frac{1}{4}$, there is no zeros of $\mathcal{G}(z, a)$ outside the strip $-\frac{1}{2} < x < \frac{1}{2}$.

Proof: Since $\mathcal{G}(z)$ is even function, it is suffices to consider it in the half plane $x \geq \frac{1}{2}$. On the straight line $x = \frac{1}{2}$,

$$z + \bar{z} = 1, \qquad \bar{z} = 1 - z,$$

and because x-axis is a symmetric line,

$$|\Gamma(1 - z)| = |\Gamma(z)|,$$

from which one gets

$$\left|\frac{\Gamma(-z)}{\Gamma(z)}\right| = \left|\frac{\Gamma(1 - z)}{-z\Gamma(z)}\right| = \frac{1}{|z|} \leq 2, \qquad \left(x = \frac{1}{2}\right). \tag{21}$$

Using (21) and (11), on the straight line $x = \frac{1}{2}$,

$$\begin{aligned}
|\psi(z)| &= \left|\frac{a^z\mathcal{G}(z)}{\Gamma(z)} - 1\right| \\
&= \left|\sum_{n=1}^{\infty} \frac{a^{2n}}{n!(1 - z)\cdots(n - z)} + \frac{a^{2z}\Gamma(-z)}{\Gamma(z)}\left(1 + \sum_{n=1}^{\infty} \frac{a^{2n}}{n!(1 + z)\cdots(n + z)}\right)\right| \\
&\leq \sum_{n=1}^{\infty} \frac{a^{2n}}{n!\,\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2}} + 2a\left(1 + \sum_{n=1}^{\infty} \frac{a^{2n}}{n!\,\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{2n+1}{2}}\right) \\
&= \sum_{n=1}^{\infty} \frac{(2a)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(2a)^{2n+1}}{(2n+1)!} \;=\; e^{2a} - 1.
\end{aligned} \tag{22}$$

It follows from (22) that

$$|\psi(z)| = |\psi(z; a)| < 1, \qquad \text{for } x = \frac{1}{2},\ a \leq \frac{1}{4}. \tag{23}$$

From Claim II, there exists a number R such that

$$|\psi(z)| = |\psi(z; a)| < 1, \qquad \text{for } x \geq \frac{1}{2},\ r \geq R,\ a \leq \frac{1}{4}. \tag{24}$$

Furthermore, the formula $|\psi(z)| < 1$ holds in a circle part region $x \geq \frac{1}{2}$, $r \leq R$, in which there is still a question on boundary of the circle part region when the inequality signs are applied to (23) and (24). Also, $|\psi(z)| < 1$ holds in the whole half-plane $x \geq \frac{1}{2}$. Thus, by (18), $\mathcal{G}(z)$ cannot be zero in the half-plane.

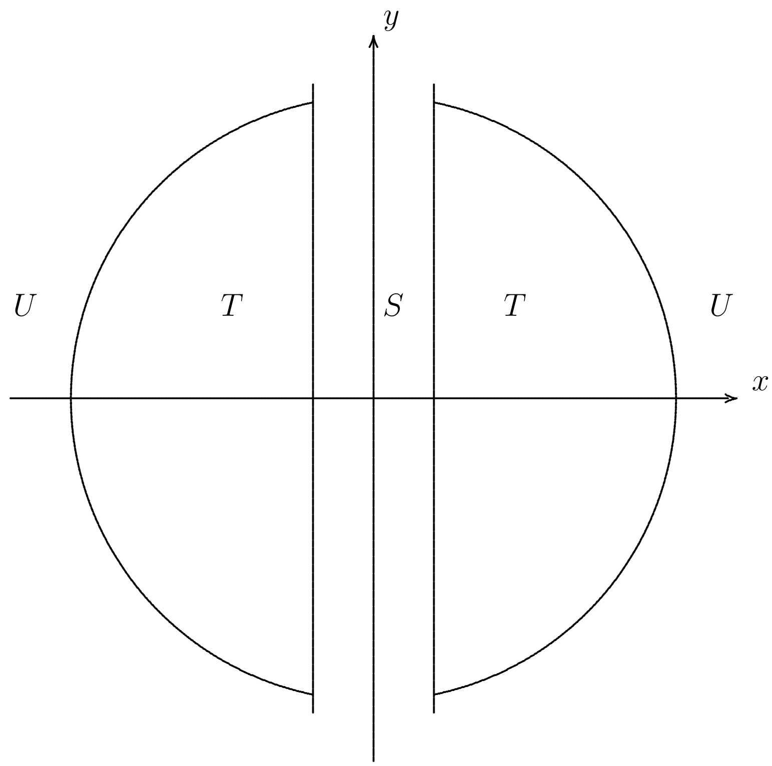

Figure 1:

It is good to divide the z-plane into several domains, as shown in Figure 1. The strip $-1 \leq x \leq 1$ is denoted by S. The circle parts where $r \leq R$ and $|x| > 1$ are denoted by T. The remaining outside domains linked to infinity are denoted by U. We choose the number $R = R(A)$ according to a certain positive number A such that in U and on its boundary the inequality $|\psi(z)| < 1$ holds for $a \leq A$. This is possible according to Claim II. Thus, it follows from (18) that $G(z)$ will neither vanish in U nor on the boundary of U.

Section 3

Now we will show the results from the difference equation (9), which can be connected to the those which we just obtained from the expression (11).

Claim IV. There is no zeros of $\mathcal{G}(z)$ on the straight lines $x = 1$ and $z = -1$, i.e., the boundary of S.

Proof: We take two steps to consider the difference equation: the first step is from the symmetric property of $\mathcal{G}(z)$, the second step is from the asymptotic property of $\mathcal{G}(z)$.

First: $\mathcal{G}(z)$ is an even function and take real value for real z. Thus, when y is real, so is the $\mathcal{G}(\mathrm{i}y)$, and the both $\mathcal{G}(1 + \mathrm{i}y)$ and $\mathcal{G}(-1 + \mathrm{i}y)$ are conjugates complex. When $z = \mathrm{i}y$, from (9),

$$y\mathcal{G}(\mathrm{i}y) = 2a\,\Im\left(\mathcal{G}(1 + \mathrm{i}y)\right). \tag{25}$$

Second: It is not possible that $\mathcal{G}(z)$ take zeros simultaneously in the two points which are connected by a line parallel to the real axis with length of 1. If it is, i.e., $\mathcal{G}(c) = 0$ and also $\mathcal{G}(c-1) = 0$, from (9) there is $\mathcal{G}(c+1) = 0$, the and also $\mathcal{G}(c+2) = 0$, $\mathcal{G}(c+3) = 0$, $\cdots$ and so on, Thus, $\mathcal{G}(z)$ will have zero in the domain U (see Figure 1). This is a contradiction.

From (7), $\mathcal{G}(z)$ is positive for real z. If $\mathcal{G}(1 + \mathrm{i}y) = 0$ but $y \neq 0$, from (25), one has $\Im\left(\mathcal{G}(1 + \mathrm{i}y)\right) = 0$ and $\mathcal{G}(\mathrm{i}y) = 0$. This means that $\mathcal{G}(1 + \mathrm{i}y)$ and $\mathcal{G}(\mathrm{i}y)$ take zeros simultaneously, which is a contradiction. Thus, $\mathcal{G}(1 + \mathrm{i}y) \neq 0$.

Claim V. There is no zeros of $\mathcal{G}(z)$ outside the strip S.

Proof: This has been shown for $a \le \frac{1}{4}$ in Claim III. It is only needed to consider

$$\frac{1}{4} \le a \le A. \tag{26}$$

As we already know that there is no zero in the domain U and its boundary (see Figure 1). It is known from Claim IV that there is also no zero of $\mathcal{G}(z)$ on the boundary of T. When z varies on the boundary of T and a varies on the closed interval (26), $\mathcal{G}(z, a)$ is an entire function of z and a, and has no zeros. Therefore the counter integral

$$\frac{1}{2\pi\mathrm{i}} \int \frac{\mathcal{G}'(z; a)\,\mathrm{d}z}{\mathcal{G}(z; a)} \tag{27}$$

where, the counter is in the positive direction of the boundary of T, is an entire function of a. The value of counter integral (27) is an integer, i.e., the the number of zeros of $\mathcal{G}(z; a)$

inside the domain T. An entire function which takes only an integral value is a constant. Therefore, from Claim III, set $a = \frac{1}{4}$ in (27) we get this constant as 0.

Claim VI. All zeros of $\mathcal{G}(z)$ inside the strip S are simple and pure imaginary.

Proof: According to (17) and the Stirling's formula, in the strip $-1 \le x \le 1$, for $y \to +\infty$ and uniformly in x,

$$\mathcal{G}(x+\mathrm{i}y) = \frac{1}{\sqrt{2\pi y}} e^{-\frac{\pi}{2}y+\mathrm{i}\frac{\pi}{2}} \left[\left(\frac{y}{a}\right)^{x} e^{\mathrm{i}\Phi} + \left(\frac{y}{a}\right)^{-x} e^{-\mathrm{i}\Phi} \right] + \mathcal{O}\left(e^{-\frac{\pi}{2}y} y^{|x|-\frac{3}{2}}\right), \tag{28}$$

where,

$$\Phi = y \log \frac{y}{a} - y - \frac{\pi}{4}. \tag{29}$$

One can get a weakened form of Claim VI from (28), (29), in which "all zeros" is replaced by "all zeros, if not a finite number". The proof to complete results needs a little more work.

Let us consider a branch of $\log \mathcal{G}(z)$ which take real value at $z = 1$. According to Claim IV, the blanch function is infinitely extendable along the straight line $x = 1$. Let y_n, $(n = 0, 1, 2, 3, \cdots)$ be those minimum y where the function $\Im \log \mathcal{G}(1 + \mathrm{i}y)$ takes the positive values

$$\Im \log G(1 + \mathrm{i}y_n) = (2n+1)\frac{\pi}{2}, \qquad \text{for } y \to \infty.$$

Since the big brace in the right-hand-side of (28) is denominated by the term $a^{-x}y^{x}e^{\mathrm{i}\Phi}$ when $x = 1$, and from (29) one knows

$$\Im \log \mathcal{G}(1 + \mathrm{i}y) \to +\infty, \qquad \text{for } y \to \infty,$$

this confirms the existence of y_n, $n = 0, 1, 2, 3, \cdots$.

Let R_n be a rectangle, whose four corners are denoted by

$$1 + \mathrm{i}y_n, \quad -1 + \mathrm{i}y_n, \quad -1 - \mathrm{i}y_n, \quad 1 - \mathrm{i}y_n$$

According to Claim IV, there is no zeros of $\mathcal{G}(z)$ on the vertical boundaries of R_n. It will be shown soon that there is no zeros of $\mathcal{G}(z)$ on the horizontal boundaries when n is sufficient large. Let us call

H_n – the change amount of $\Im \log \mathcal{G}(z)$ when z varies from $1 + \mathrm{i}y_n$ to $-1 + \mathrm{i}y_n$.

N_n – the number of pure imaginary zeros of $\mathcal{G}(z)$ inside R_n, without counting the multiplicity.

$N_n + N_n^*$ – the number of all zeros of $\mathcal{G}(z)$ inside R_n, counting on the multiplicity.

Where, N_n^* has included the number of any non-pure imaginary zeros of $\mathcal{G}(z)$ inside R_n. In addition, each pure imaginary zero of multiple order m in R_n is counted by $m-1$ into N_n^*. N_n and N_n^* cannot decrease when n increases.

According to the definition of y_n, H_n and from the symmetry property, when z has run one round over the boundary of the rectangle R_n along the positive direction, the overall changes of $\Im \log \mathcal{G}(z)$ is

$$2\pi(N_n + N_n^*) = 2H_n + (2n+1)2\pi. \tag{30}$$

According to the definition of $\Im \log \mathcal{G}(z)$, it takes the value $(\nu - \frac{1}{2})\pi$ for $y = y_{\nu-1}$, and the value $(\nu + \frac{1}{2})\pi$ for $y = y_{\nu+1}$. Therefore, there is at least one point η, $y_{\nu-1} < \eta < y_\nu$, such that

$$\Im \log \mathcal{G}(1 + \mathrm{i}\eta) = \nu\pi.$$

It follows

$$\Im \mathcal{G}(1 + \mathrm{i}\eta) = 0,$$

and from (25),

$$\mathcal{G}(\mathrm{i}\eta) = 0.$$

Since $\mathrm{i}\eta$ and $-\mathrm{i}\eta$ are zeros of $\mathcal{G}(z)$, if setting $\nu = 1, 2, 3, \cdots, n$, we have at least a minimum of $2n$ different imaginary zeros in the rectangle R_n. This means,

$$N_n \geq 2n. \tag{31}$$

From (30)and (31), one gets

$$2\pi N_n^* \leq 2H_n + 2\pi. \tag{32}$$

Let Φ_n be the value from (29) when $y = y_n$. Then, as we already seen, the big brace from (28) is dominated by the first term when $x = 1$, there is

$$\Im \log \mathcal{G}(1 + \mathrm{i}\eta) = (2n+1)\frac{\pi}{2} \equiv \Phi_n + \frac{\pi}{2} + \varepsilon_n \qquad (\mathrm{mod.}\ 2\pi)$$

where

$$\lim_{n \to \infty} \varepsilon_n = 0.$$

There, $\Phi_n \equiv n\pi - \varepsilon_n$, for $y = y_n$ with a big n, the value of $e^{\mathrm{i}\Phi n}$ approaches $(-1)^n$, and the big brace in (28) approaches to a real value. Then, there are two cases to occur. 1) The the high-order term $\mathcal{O}(\cdot)$ in right-hand-side of (28) for $y = y_n$ overweighs in the whole line $-1 \leq x \leq 1$, such that $\mathcal{G}(z)$ will not become zero on the horizontal sides of the rectangle R_n, as the cases we already discussed. 2) For the change of $\mathcal{G}(z)$ along these sides will add factor $e^{\mathrm{i}\pi x/2}$ in the formula (28). Or, in particular, for any positive ε, when n is sufficient large,

$$H_n < -\pi + \varepsilon, \tag{33}$$

Then, (32)and (33) give

$$2\pi N_n^* < 2\varepsilon.$$

Since N_n^* cannot be a negative integer, it is 0 for sufficient large n. Also N_n^* will not decrease when n increases, therefore, there always is $N_n^* = 0$. Then, Claim VI is proved.

Claim IV, V, VI have shown $\mathcal{G}(z)$ has only simple imaginary zeros in a complete classification of conditions. Incidentally, this study result in that the number of zeros between 0 and y is

$$\frac{y}{\pi}\log\frac{y}{a} - \frac{y}{\pi} + \mathcal{O}(1).$$

Section 4

I need to give two simple supplementary lemmas to support the previous assertion in the introduction.

Lemma I. Let $F(u)$ be an analytic function which takes only real values for non-negative real u. Furthermore, the limits

$$\lim_{u\to\infty} u^2 F^{(n)}(u) = 0 \tag{34}$$

hold for $n = 0, 1, 2, \cdots$ ($F^{(0)}(u) = f(u)$). If $F(u)$ is not an even function, then the function

$$G(x) = \int_0^\infty F(u)\cos(xu)\mathrm{d}u \tag{35}$$

will not get zeros when the real x is sufficient large.

Proof: Since $F(u)$ is not an even function, there exists an integer q such that

$$F'(0) = F'''(0) = \cdots = F^{(2q-3)}(0) = 0, \qquad F^{(2q-1)}(0) \neq 0.$$

Taking integration by parts to (35) repeatedly and using (34),

$$G(x) = (-1)^q \frac{F^{(2q-1)}(0)}{x^{2q}} + \frac{(-1)^{q+1}}{x^{2q+1}} \int_0^\infty F^{2q+1}(u) \sin(xu) \mathrm{d}u.$$

Again from (34), it follows

$$\lim_{x \to \infty} x^{2q} G(x) = (-1)^q F^{(2q-1)}(0) \neq 0.$$

If the function $F(u)$ in (35) is taken as the right-hand-side of (4) or a finite term of (3), it is easy to prove from Hadamard's Theorem that the obtained entire function $G(x)$ has infinite number of zeros. But according to Lemma I, only finite number of those zeros are real, the selected function $F(u)$ is apparently not even. This explains why the function $\Phi(u)$ appeared in the integral representation (2) must be an even function. This of course can be confirmed directly.

Lemma II. Let a be a positive constant, and $G(z)$ an entire function which takes only real values between 0 and 1 for z being real. Then, the function

$$G(z - \mathrm{i}a) + G(z + \mathrm{i}a) \tag{36}$$

has only real zeros[4].

Proof: The condition for $G(z)$ means

$$G(z) = c z^q e^{\alpha z} \prod \Big(1 - \frac{z}{\alpha_n}\Big) e^{z/\alpha_n},$$

where, c, α, $\alpha_1, \alpha_2, \cdots$ are real constants, $\alpha_n \neq 0$ for $n = 1, 2, \cdots$, $\alpha_1^{-2} + \alpha_2^{-2} + \cdots$ converges, and q is a non-negative integer. (When $G(z)$ has no zeros, i.e., (36) will degenerate into $ce^{\alpha z}$.) If $z = x + \mathrm{i}y$ is a zero of function (36), then

$$|G(z - \mathrm{i}a)| = |G(z + \mathrm{i}a)|.$$

[4]The equivalent polynomial is a special case of CH. Biehler's Theorem. (See, e.g., G.Pólya and G. Szegö a.a.O., Assignment III 25.)

Therefore,

$$1 = \left|\frac{G(z-\mathrm{i}a)}{G(z+\mathrm{i}a)}\right|^2 = \left(\frac{x^2+(y-a)^2}{x^2+(y+a)^2}\right)^q \prod \left[\frac{(x-\alpha_n)^2+(y-a)^2}{(x-\alpha_n)^2+(y+a)^2}\right].$$

If $y > 0$, then all factors in the right-hand-side < 1; if $y < 0$, then all factors in the right-hand-side > 1. Therefore, there must be $y = 0$.

It is know from the previous proof that $\mathcal{G}(\frac{1}{2}\mathrm{i}z)$ has only real zeros. The Hadamard's Theorem can be used to deduce that $\mathcal{G}(\frac{1}{2}\mathrm{i}z)$ also satisfies the prerequisites of Lemma II. Then, applying the Lemma II to $G(z) = \mathcal{G}(\frac{1}{2}\mathrm{i}z;\pi)$ and setting $a = \frac{9}{2}$, from (8), it shows all zeros of $\xi^*(z)$ are real.

By the way, from (8), (17), (19), one finds that in the halfspace $y \geq \varepsilon$, (fixed $\varepsilon > 0$) for $z \to \infty$,

$$\xi^*(\mathrm{i}z) \sim \frac{1}{2}z^2\pi^{-\frac{z}{2}-\frac{1}{4}}\Gamma\left(\frac{z}{2}+\frac{1}{4}\right).$$

This formula can be compared to (1).

A remark by the correction (5.2.26): One extension to Lemma II can be shown that the function

$$\xi^{**}(z) = 4\pi\int_0^\infty \left[2\pi\left(e^{9u/2}+e^{-9u/2}\right) - 3\left(e^{5u/2}-e^{-5u/2}\right)\right]e^{-\pi(e^{2u}+e^{-2u})}\cos(zu)\mathrm{d}u,$$

is better closed to the ξ-function, has also only real zeros.