Pure and Applied Mathematics

A Series of Monographs and Textbooks

Editors **Paul A. Smith and Samuel Eilenberg**

Columbia University, New York

RECENT TITLES

Ernst August Behrens. Ring Theory

Morris Newman. Integral Matrices

Glen E. Bredon. Introduction to Compact Transformation Groups

Werner Greub, Stephen Halperin, and Ray Vanstone. Connections, Curvature, and Cohomology: Volume I, De Rham Cohomology of Manifolds and Vector Bundles. Volume II, Lie Groups, Principal Bundles, and Characteristic Classes

Xia Dao-xing. Measure and Integration Theory of Infinite-Dimensional Spaces: Abstract Harmonic Analysis

Ronald G. Douglas. Banach Algebra Techniques in Operator Theory

Willard Miller, Jr. Symmetry Groups and Their Applications

Arthur A. Sagle and Ralph E. Walde. Introduction to Lie Groups and Lie Algebras

T. Benny Rushing. Topological Embeddings

James W. Vick. Homology Theory: An Introduction to Algebraic Topology

E. R. Kolchin. Differential Algebra and Algebraic Groups

Gerald J. Janusz. Algebraic Number Fields

A. S. B. Holland. Introduction to the Theory of Entire Functions

Wayne Roberts and Dale Varberg. Convex Functions

A. M. Ostrowski. Solution of Equations in Euclidean and Banach Spaces, Third Edition of Solution of Equations and Systems of Equations

Samuel Eilenberg. Automata, Languages, and Machines: Volume A

H. M. Edwards. Riemann's Zeta Function

In preparation

Wilhelm Magnus. Non-Euclidean Tesselations and Their Groups

J. Dieudonné. Treatise on Analysis: Volume IV

Morris Hirsch and Stephen Smale. Differential Equations, Dynamical Systems, and Linear Algebra

Riemann's Zeta Function

H. M. Edwards

New York University

New York, New York

ACADEMIC PRESS New York and London 1974

A Subsidiary of Harcourt Brace Jovanovich, Publishers

ACADEMIC PRESS, INC.
111 Fifth Avenue, New York, New York 10003

United Kingdom Edition published by
ACADEMIC PRESS, INC. (LONDON) LTD.
24/28 Oval Road, London NW1

LIBRARY OF CONGRESS CATALOG CARD NUMBER: 73-794

AMS(MO) 1970 Subject Classifications: 10-02, 10-03, 10H05, 10H15

PRINTED IN THE UNITED STATES OF AMERICA

Contents

Preface

My primary objective in this book is to make a point, not about analytic number theory, but about the way in which mathematics is and ought to be studied. Briefly put, I have tried to say to students of mathematics that they should *read the classics* and beware of secondary sources.

This is a point which Eric Temple Bell makes repeatedly in his biographies of great mathematicians in *Men of Mathematics*. In case after case, Bell points out that the men of whom he writes learned their mathematics not by studying in school or by reading textbooks, but by going straight to the sources and reading the best works of the masters who preceded them. It is a point which in most fields of scholarship at most times in history would have gone without saying.

No mathematical work is more clearly a classic than Riemann's memoir *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse,* published in 1859. Much of the work of many of the great mathematicians since Riemann—men like Hadamard, von Mangoldt, de la Vallée Poussin, Landau, Hardy, Littlewood, Siegel, Polya, Jensen, Lindelöf, Bohr, Selberg, Artin, Hecke, to name just a few of the most important—has stemmed directly from the ideas contained in this eight-page paper. According to legend, the person who acquired the copy of Riemann's collected works from the library of Adolph Hurwitz after Hurwitz's death found that the book would automatically fall open to the page on which the Riemann hypothesis was stated.

Yet it is safe to say both that the dictum "read the classics" is not much heard among contemporary mathematicians and that few students read *Ueber die Anzahl* . . . today. On the contrary, the mathematics of previous generations is generally considered to be unrigorous and naïve, stated in obscure terms which can be vastly simplified by modern terminology, and full of false starts and misstatements which a student would be best advised to avoid. Riemann in particular is avoided because of his reputation for lack of rigor (his "Dirichlet principle" is remembered more for the fact that Weierstrass pointed out that its proof was inadequate than it is for the fact that it was after all correct and that with it Riemann revolutionized the study of Abelian functions), because of his difficult style, and because of a general impression that the valuable parts of his work have all been gleaned and incorporated into subsequent more rigorous and more readable works.

These objections are all valid. When Riemann makes an assertion, it may be something which the reader can verify himself, it may be something which Riemann has proved or intends to prove, it may be something which was not proved rigorously until years later, it may be something which is still unproved, and, alas, it may be something which is not true unless the hypotheses are strengthened. This is especially distressing for a modern reader who is trained to digest each statement before going on to the next. Moreover, Riemann's style is extremely difficult. His tragically brief life was too occupied with mathematical creativity for him to devote himself to elegant exposition or to the polished presentation of all of his results. His writing is extremely condensed and *Ueber die anzahl . . .* in particular is simply a resumé of very extensive researches which he never found the time to expound upon at greater length; it is the only work he ever published on number theory, although Siegel found much valuable new material on number theory in Riemann's private papers. Finally, it is certainly true that most of Riemann's best ideas have been incorporated in later, more readable works.

Nonetheless, it is just as true that one should read the classics in this case as in any other. No secondary source can duplicate Riemann's insight. Riemann was so far ahead of his time that it was 30 years before anyone else began really to grasp his ideas—much less to have their own ideas of comparable value. In fact, Riemann was so far ahead of his time that the results which Siegel found in the private papers were a major contribution to the field when they were published in 1932, seventy years after Riemann discovered them. Any simplification, paraphrasing, or reworking of Riemann's ideas runs a grave risk of missing an important idea, of obscuring a point of view which was a source of Riemann's insight, or of introducing new technicalities or side issues which are not of real concern. There is no mathematician since Riemann whom I would trust to revise his work.

The perceptive reader will of course have noted the paradox here of a secondary source denouncing secondary sources. I might seem to be saying, "Do not read this book." But he will also have seen the answer to the paradox. What I am saying is: Read the classics, not just Riemann, but all the major contributions to analytic number theory that I discuss in this book. The purpose of a secondary source is to make the primary sources accessible to you. If you can read and understand the primary sources without reading this book, more power to you. If you read this book without reading the primary sources you are like a man who carries a sack lunch to a banquet.

Acknowledgments

It is a pleasure to acknowledge my indebtedness to many individuals and institutions that have aided me in the writing of this book. The two institutions which have supported me during this time are the Washington Square College of New York University and the School of General Studies of the Australian National University; I have been proud to be a part of these particular schools because of their affirmation of the importance of undergraduate teaching in an age when the pressures toward narrow over-specialization are so great.

I am grateful to many libraries for the richness of the resources which they make available to us all. Of the many I have used during the course of the preparation of this book, I think particularly of the following institutions which provided access to relatively rare documents: The New York Public Library, the Courant Institute of Mathematical Sciences, the University of Illinois, Sydney University, the Royal Society of Adelaide, the Linnean Society of New South Wales, the Public Library of New South Wales, and the Australian National University. I am especially grateful to the University Library in Göttingen for giving me access to Riemann's *Nachlass* and for permitting me to photocopy the portions of it relevant to this book.

Among the individuals I would like to thank for their comments on the manuscript are Gabriel Stolzenberg of Northeastern University, David Lubell of Adelphi, Bruce Chandler of New York, Ian Richards of Minnesota, Robert Spira of Michigan State, and Andrew Coppel of the Australian National University. J. Barkley Rosser and D. H. Lehmer were very helpful in providing information on their researches. Carl Ludwig Siegel was very hospitable and generous with his time during my brief visit to Göttingen. And, finally, I am deeply grateful to Wilhelm Magnus for his understanding of my objectives and for his encouragement, which sustained me through many long days when it seemed that the work would never be done.

Chapter 1

Riemann's Paper

1.1 THE HISTORICAL CONTEXT OF THE PAPER

This book is a study of Bernhard Riemann's epoch-making 8-page paper "On the Number of Primes Less Than a Given Magnitude,"† and of the subsequent developments in the theory which this paper inaugurated. This first chapter is an examination and an amplification of the paper itself, and the remaining 11 chapters are devoted to some of the work which has been done since 1859 on the questions which Riemann left unanswered.

The theory of which Riemann's paper is a part had its beginnings in Euler's theorem, proved in 1737, that the sum of the reciprocals of the prime numbers

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\cdots \tag{1}$$

is a divergent series. This theorem goes beyond Euclid's ancient theorem that there are infinitely many primes [E2] and shows that the primes are rather *dense* in the set of all integers—denser than the squares, for example, in that the sum of the reciprocals of the square numbers converges.

Euler in fact goes beyond the mere statement of the divergence of (1) to say that since $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ diverges like the logarithm and since the series (1) diverges like‡ the logarithm of $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\cdots$, the series (1)

†The German title is *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.* An English translation of the paper is given in the Appendix.

‡This is true by dint of the Euler product formula which gives $\sum(1/n)=\prod(1-p^{-1})^{-1}$ (see Section 1.2); hence $\log\sum(1/n)=-\sum\log(1-p^{-1})=\sum(p^{-1}+\frac{1}{2}p^{-2}+\frac{1}{3}p^{-3}+\cdots)=\sum(1/p)+$ convergent.

must diverge like the log of the log, which Euler writes [E4] as

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\cdots=\log(\log\infty). \tag{2}$$

It is not clear exactly what Euler understood this equation to mean—if indeed he understood it as anything other than a mnemonic—but an obvious interpretation of it would be

$$\sum_{p<x} 1/p \sim \log(\log x) \qquad (x \longrightarrow \infty), \tag{2'}$$

where the left side denotes the sum of $1/p$ over all primes p less than x and where the sign $\sim$ means that the relative error is arbitrarily small for x sufficiently large or, what is the same, that the ratio of the two sides approaches one as $x \longrightarrow \infty$. Now

$$\log(\log x)=\int_1^{\log x}\frac{du}{u}=\int_e^x\frac{dv}{v\log v}$$

so (2′) says that the integral of $1/v$ relative to the measure $dv/\log v$ diverges in the same way as the integral of $1/v$ relative to the point measure which assigns weight 1 to primes and weight 0 to all other points. In this sense (2′) can be regarded as saying that the density of primes is roughly $1/\log v$. However, there is no evidence that Euler thought about the density of primes, and his methods were not adequate to prove the formulation (2′) of his statement (2).

Gauss states† in a letter [G2] written in 1849 that he had observed as early as 1792 or 1793 that the density of prime numbers appears on the average to be $1/\log x$ and he says that each new tabulation of primes which was published in the ensuing years had tended to confirm his belief in the accuracy of this approximation. However, he does not mention Euler's formula (2) and he gives no analytical basis for the approximation, which he presents solely as an empirical observation. He gives, in particular, Table I.

TABLE I[a]

x	Count of primes $< x$	$\int \frac{dn}{\log n}$	Difference
500,000	41,556	41,606.4	50.4
1,000,000	78,501	78,627.5	126.5
1,500,000	114,112	114,263.1	151.1
2,000,000	148,883	149,054.8	171.8
2,500,000	183,016	183,245.0	229.0
3,000,000	216,745	216,970.6	225.6

[a]From Gauss [G2].

†For some corroboration of Gauss's claim see his collected works [G3].

Gauss does not say exactly what he means by the symbol $\int(dn/\log n)$, but the data given in Table II, taken from D.N. Lehmer [L9], would indicate that he means n to be a continuous variable integrated from 2 to x, that is, $\int_2^x (dt/\log t)$. Note that Lehmer's count† of primes, which can safely be assumed to be accurate, differs from Gauss's information and that the difference is in *favor* of Gauss's estimate for the larger values of x.

TABLE II[a]

x	Count of primes $< x$	$\int_2^x \frac{dt}{\log t}$	Difference
500,000	41,538	41,606	68
1,000,000	78,498	78,628	130
1,500,000	114,155	114,263	108
2,000,000	148,933	149,055	122
2,500,000	183,072	183,245	173
3,000,000	216,816	216,971	155

[a]Data from Lehmer [L9].

Around 1800 Legendre published in his *Theorie des Nombres* [L11] an empirical formula for the number of primes less than a given value which amounted more or less to the same statement, namely, that the density of primes is $1/\log x$. Although Legendre made some slight attempt to prove his formula, his argument amounts to nothing more than the statement that if the density of primes is assumed to have the form

$$1/(A_1x^{m_1}+A_2x^{m_2}+\cdots)$$

where $m_1 > m_2 > \ldots$, then m_1 cannot be positive [because then the sum (1) would converge]; hence m_1 must be "infinitely small" and the density must be of the form

$$1/(A\log x + B).$$

He then determines A and B empirically. Legendre's formula was well known in the mathematical world and was mentioned prominently by Abel [A2], Dirichlet [D3], and Chebyshev [C2] during the period 1800–1850.

The first significant results beyond Euler's were obtained by Chebyshev around 1850. Chebyshev proved that the relative error in the approximation

$$\pi(x) \sim \int_2^x \frac{dt}{\log t}, \tag{3}$$

†Lehmer insists on counting 1 as a prime. To conform to common usage his counts have therefore been reduced by one in Table II.

where $\pi(x)$ denotes the number of primes less than x, is less than 11% for all sufficiently large x; that is, he proved† that

$$(0.89) \int_2^x \frac{dt}{\log t} < \pi(x) < (1.11) \int_2^x \frac{dt}{\log t}$$

for all sufficiently large x. He proved, moreover, that no approximation of Legendre's form

$$\pi(x) \sim x/(A \log x + B)$$

can be better than the approximation (3) and that if the ratio of $\pi(x)$ to $\int_2^x (dt/\log t)$ approaches a limit as $x \longrightarrow \infty$, then this limit must be 1. It is clear that Chebyshev was attempting to prove that the relative error in the approximation (3) approaches zero as $x \longrightarrow \infty$, but it was not until almost 50 years later that this theorem, which is known as the "prime number theorem," was proved. Although Chebyshev's work was published in France well before Riemann's paper, Riemann does not refer to Chebyshev in his paper. He does refer to Dirichlet, however, and Dirichlet, who was acquainted with Chebyshev (see Chebyshev's report on his trip to Western Europe [C5, Vol. 5, p. 245 and pp. 254–255]) would probably have made Riemann aware of Chebyshev's work. Riemann's unpublished papers do contain several of Chebyshev's formulas, indicating that he had studied Chebyshev's work, and contain at least one direct reference to Chebyshev (see Fig. 1).

The real contribution of Riemann's 1859 paper lay not in its results but in its methods. The principal result was a formula‡ for $\pi(x)$ as the sum of an infinite series in which $\int_2^x (dt/\log t)$ is by far the largest term. However, Riemann's proof of this formula was inadequate; in particular, it is by no means clear from Riemann's arguments that the infinite series for $\pi(x)$ even *converges*, much less that its largest term $\int_2^x (dt/\log t)$ dominates it for large x. On the other hand, Riemann's methods, which included the study of the function $\zeta(s)$ as a function of a complex variable, the study of the complex zeros of $\zeta(s)$, Fourier inversion, Möbius inversion, and the representation of functions such as $\pi(x)$ by "explicit formulas" such as his infinite series, all have had important parts in the subsequent development of the theory.

For the first 30 years after Riemann's paper was published, there was

†Chebyshev did not state his result in this form. This form can be obtained from his estimate of the number of primes between l and L (see Chebyshev [C3, Section 6]) by fixing l, letting $L \longrightarrow \infty$, and using $\int_2^L (\log t)^{-1}\, dt \sim L/\log L$.

‡See Section 1.17. Note that $Li(x) = \int_2^x (dt/\log t) + \text{const.}$

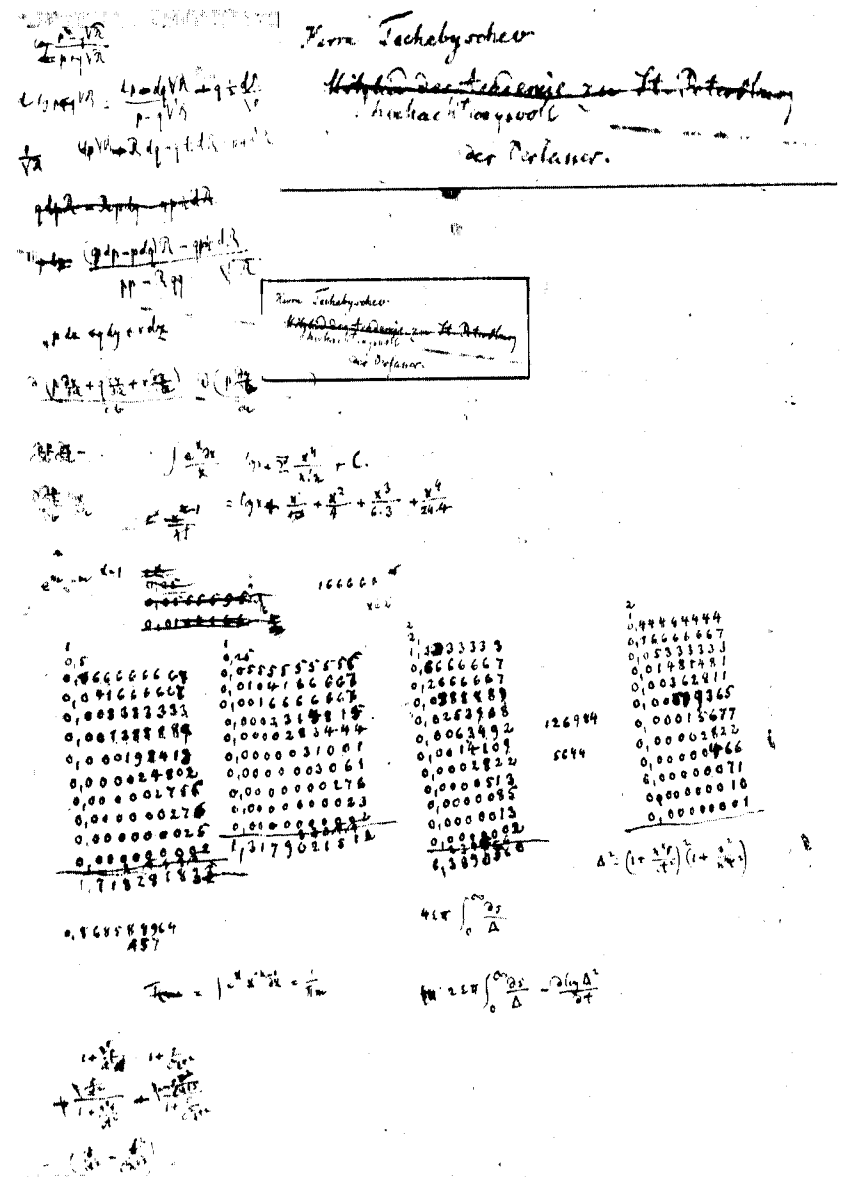

Fig. 1. A scrap sheet used to hold some other loose sheets in Riemann's papers. The note seems to prove that Riemann was aware of Chebyshev's work and intended to send him an offprint of his own paper. In all likelihood Riemann was practicing his penmanship in forming Roman, rather than German, letters to write a dedication to Chebyshev. (Reproduced with the permission of the Niedersächsische Staats- und Universitätsbibliothek, Handschriftenabteilung, Göttingen.)

virtually no progress† in the field. It was as if it took the mathematical world that much time to digest Riemann's ideas. Then, in a space of less than 10 years, Hadamard, von Mangoldt, and de la Vallée Poussin succeeded in proving both Riemann's main formula for $\pi(x)$ and the prime number theorem (3), as well as a number of other related theorems. In all these proofs Riemann's ideas were crucial. Since that time there has been no shortage of new problems and no shortage of progress in analytic number theory, and much of this progress has been inspired by Riemann's ideas.

Finally, no discussion of the historical context of Riemann's paper would be complete without a mention of the Riemann hypothesis. In the course of the paper, Riemann says that he considers it "very likely" that the complex zeros of $\zeta(s)$ all have real part equal to $\frac{1}{2}$, but that he has been unable to prove that this is true. This statement, that the zeros have real part $\frac{1}{2}$, is now known as the "Riemann hypothesis." The experience of Riemann's successors with the Riemann hypothesis has been the same as Riemann's—they also consider its truth "very likely" and they also have been unable to prove it. Hilbert included the problem of proving the Riemann hypothesis in his list [H9] of the most important unsolved problems which confronted mathematics in 1900, and the attempt to solve this problem has occupied the best efforts of many of the best mathematicians of the twentieth century. It is now unquestionably the most celebrated problem in mathematics and it continues to attract the attention of the best mathematicians, not only because it has gone unsolved for so long but also because it appears tantalizingly vulnerable and because its solution would probably bring to light new techniques of far-reaching importance.

1.2 THE EULER PRODUCT FORMULA

Riemann takes as his starting point the formula

(1) $$\sum_n \frac{1}{n^s} = \prod_p \frac{1}{\left(1 - \frac{1}{p^s}\right)}$$

of Euler. Here n ranges over all positive integers ($n = 1, 2, 3, \ldots$) and p ranges over all primes ($p = 2, 3, 5, 7, 11, \ldots$). This formula, which is now known as the "Euler product formula," results from expanding each of the

†A major exception to this statement was Mertens's theorem [M5] of 1874 stating that (2′) is true in the strong sense that the difference of the two sides approaches a limit as $x \longrightarrow \infty$, namely, Euler's constant plus $\sum_p [\log(1 - p^{-1}) + p^{-1}]$. Another perhaps more natural statement of Mertens's theorem is

$$\lim_{x\to\infty} \log x \prod_{p<x} (1 - p^{-1}) = e^{-\gamma},$$

where γ is Euler's constant. See, for example, Hardy and Wright [H7].

factors on the right

$$\frac{1}{\left(1 - \frac{1}{p^s}\right)} = 1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} + \frac{1}{(p^3)^s} + \cdots$$

and observing that their product is therefore a sum of terms of the form

$$\frac{1}{(p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r})^s},$$

where $p_1, \ldots, p_r$ are distinct primes and $n_1, n_2, \ldots, n_r$ are natural numbers, and then using the fundamental theorem of arithmetic (every integer can be written in essentially only one way as a product of primes) to conclude that this sum is simply $\sum(1/n^s)$. Euler used this formula principally as a formal identity and principally for integer values of s (see, for example, Euler [E5]).

Dirichlet also based his work† in this field on the Euler product formula. Since Dirichlet was one of Riemann's teachers and since Riemann refers to Dirichlet's work in the first paragraph of his paper, it seems certain that Riemann's use of the Euler product formula was influenced by Dirichlet. Dirichlet, unlike Euler, used the formula (1) with s as a real variable and, also unlike Euler, he proved‡ rigorously that (1) is true for all real $s > 1$.

Riemann, as one of the founders of the theory of functions of a complex variable, would naturally be expected to consider s as a *complex* variable. It is easy to show that both sides of the Euler product formula converge for complex s in the halfplane Re $s > 1$, but Riemann goes much further and shows that even though both sides of (1) diverge for other values of s, the function they define is meaningful for *all* values of s except for a pole at $s = 1$. This extension of the range of s requires a few facts about the factorial function which will be covered in the next section.

1.3 THE FACTORIAL FUNCTION

Euler extended the factorial function $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$ from the natural numbers n to all real numbers greater than -1 by observing that¶

(1) $$n! = \int_0^\infty e^{-x} x^n \, dx \qquad (n = 1, 2, 3, \ldots)$$

†Dirichlet's major contribution to the theory was his proof that if m is relatively prime to n, then the congruence $p \equiv m \pmod n$ has infinitely many prime solutions p. He was also interested in questions concerning the density of the distribution of primes, but he did not have significant success with these questions.

‡Dirichlet [D3]. Since the terms p^{-s} are all positive, there is nothing subtle or difficult about this proof—it is essentially a reordering of absolutely convergent series—but it has the important effect of transforming (1) from a formal identity true for various values of s to an analytical formula true for all real $s > 1$.

¶However Euler wrote the integral in terms of $y = e^{-x}$ as $n! = \int_0^1 (\log 1/y)^n \, dy$ (see Euler [E3]).

(integration by parts) and by observing that the integral on the right converges for noninteger values of n, provided only that $n > -1$. Gauss [G1] introduced the notation†

(2) $$\Pi(s) = \int_0^\infty e^{-x}x^s\,dx \qquad (s > -1)$$

for Euler's integral on the right side of (1). Thus $\Pi(s)$ is defined for all real numbers s greater than -1, in fact for all complex numbers s in the halfplane $\operatorname{Re} s > -1$, and $\Pi(s) = s!$ whenever s is a natural number. There is another representation of $\Pi(s)$ which was also known‡ to Euler, namely,

(3) $$\Pi(s) = \lim_{N\to\infty} \frac{1\cdot 2\cdots N}{(s+1)(s+2)\cdots(s+N)}(N+1)^s.$$

This formula is valid for all s for which (2) defines $\Pi(s)$, that is, for all s in the halfplane $\operatorname{Re} s > -1$. On the other hand, it is not difficult to show [use formula (4) below] that the limit (3) exists for *all* values of s, real or complex, provided only that the denominator is not zero, that is, provided only that s is not a negative integer. In short, formula (3) extends the definition of $\Pi(s)$ to all values of s other than $s = -1, -2, -3, \ldots$.

In addition to the fact that the two definitions (2) and (3) of $\Pi(s)$ coincide for real $s > -1$, the following facts will be used without proof:

(4) $$\Pi(s) = \prod_{n=1}^\infty \frac{n^{1-s}(n+1)^s}{s+n} = \prod_{n=1}^\infty \left(1+\frac{s}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^s,$$

(5) $$\Pi(s) = s\Pi(s-1),$$

(6) $$\frac{\pi s}{\Pi(s)\Pi(-s)} = \sin \pi s,$$

(7) $$\Pi(s) = 2^s\Pi\left(\frac{s}{2}\right)\Pi\left(\frac{s-1}{2}\right)\pi^{-1/2}.$$

For the proofs of these facts the reader is referred to any book which deals with factorial function or the "Γ-function," for example, Edwards [E1, pp. 421–425]. Identity (4) is a simple reformulation of formula (3). Using it one can prove that $\Pi(s)$ *is an analytic function of the complex variable s which has simple poles at $s = -1, -2, -3, \ldots$. It has no zeros.* Identity (5) is called the "functional equation of the factorial function"; together with $\Pi(0) = 1$ [from (4)] it gives $\Pi(n) = n!$ immediately. Identity (6) is essentially the product formula for the sine; when $s = \frac{1}{2}$ it combines with (5) to give the important value $\Pi(-\frac{1}{2}) = \pi^{1/2}$. Identity (7) is known as the *Legendre relation*. It is the case $n = 2$ of a more general identity

$$\frac{\Pi(s)}{n^s\Pi\left(\frac{s}{n}\right)\Pi\left(\frac{s-1}{n}\right)\cdots\Pi\left(\frac{s-n+1}{n}\right)} = \left[\frac{2\pi n}{(2\pi)^n}\right]^{1/2}$$

which will not be needed.

†Unfortunately, Legendre subsequently introduced the notation $\Gamma(s)$ for $\Pi(s-1)$. Legendre's reasons for considering $(n-1)!$ instead of $n!$ are obscure (perhaps he felt it was more natural to have the first pole occur at $s = 0$ rather than at $s = -1$) but, whatever the reason, this notation prevailed in France and, by the end of the nineteenth century, in the rest of the world as well. Gauss's original notation appears to me to be much more natural and Riemann's use of it gives me a welcome opportunity to reintroduce it.

‡See Euler [E3, E8].

1.4 THE FUNCTION $\zeta(s)$

It is interesting to note that Riemann does not speak of the "analytic continuation" of the function $\sum n^{-s}$ beyond the halfplane $\operatorname{Re} s > 1$, but speaks rather of finding a formula for it which "remains valid for all s." This indicates that he viewed the problem in terms more analogous to the extension of the factorial function by formula (3) of the preceding section than to a piece-by-piece extension of the function in the manner that analytic continuation is customarily taught today. The view of analytic continuation in terms of chains of disks and power series convergent in each disk descends from Weierstrass and is quite antithetical to Riemann's basic philosophy that analytic functions should be dealt with *globally*, not locally in terms of power series.

Riemann derives his formula for $\sum n^{-s}$ which "remains valid for all s" as follows. Substitution of nx for x in Euler's integral for $\Pi(s-1)$ gives

$$\int_0^\infty e^{-nx}x^{s-1}\,dx = \frac{\Pi(s-1)}{n^s}$$

($s > 0$, $n = 1, 2, 3, \ldots$). Riemann sums this over n and uses $\sum_{n=1}^\infty r^{-n} = (r-1)^{-1}$ to obtain†

(1) $$\int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx = \Pi(s-1)\sum_{n=1}^\infty \frac{1}{n^s}$$

($s > 1$). Convergence of the improper integral on the left and the validity of

†This formula, with $s = 2n$, occurs in a paper [A1] of Abel which was included in the 1839 edition of Abel's collected works. It seems very likely that Riemann would have been aware of this. A very similar formula

$$\int_0^\infty (e^x-1)^{-1}e^{-x}x^\rho\,dx = \Pi(\rho)\sum_{n=2}^\infty n^{-1-\rho}$$

is the point of departure of Chebyshev's 1848 paper [C2].

the interchange of summation and integration are not difficult to establish.

Next he considers the contour integral

$$\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x}.$$

The limits of integration are intended to indicate a path of integration which begins at $+\infty$, moves to the left down the positive real axis, circles the origin once in the positive (counterclockwise) direction, and returns up the positive real axis to $+\infty$. The definition of $(-x)^s$ is $(-x)^s = \exp[s \log(-x)]$, where the definition of $\log(-x)$ conforms to the usual definition of $\log z$ for z not on the negative real axis as the branch which is real for positive real z; thus $(-x)^s$ is not defined on the positive real axis and, strictly speaking, the path of integration must be taken to be slightly above the real axis as it descends from $+\infty$ to 0 and slightly below the real axis as it goes from 0 back to $+\infty$. When this integral is written in the form

$$\int_{+\infty}^{\delta} \frac{(-x)^s\, dx}{(e^x - 1)x} + \int_{|x|=\delta} \frac{(-x)^s\, dx}{(e^x - 1)x} + \int_{\delta}^{+\infty} \frac{(-x)^s\, dx}{(e^x - 1)x},$$

the middle term is $2\pi i$ times the average value of $(-x)^s(e^x-1)^{-1}$ on the circle $|x| = \delta$ [because on this circle $i\, d\theta = (dx/x)$]. Thus the middle term approaches zero as $\delta \to 0$ provided $s > 1$ [because $x(e^x - 1)^{-1}$ is nonsingular near $x = 0$]. The other two terms can then be combined to give

$$\begin{aligned}\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \cdot \frac{dx}{x} &= \lim_{\delta \to 0} \left\{ \int_{+\infty}^{\delta} \frac{\exp[s(\log x - i\pi)]\, dx}{(e^x - 1)x} \right. \\ &\qquad \left. + \int_{\delta}^{+\infty} \frac{\exp[s(\log x + i\pi)]\, dx}{(e^x - 1)x} \right\} \\ &= (e^{i\pi s} - e^{-i\pi s}) \int_0^{\infty} \frac{x^{s-1}\, dx}{e^x - 1}\end{aligned}$$

which combines with the previous formula (1) to give

$$\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \cdot \frac{dx}{x} = 2i \sin(\pi s) \Pi(s - 1) \sum_{n=1}^{\infty} \frac{1}{n^s}$$

or, finally, when both sides are multiplied by $\Pi(-s)s/2\pi is$ and identity (6) of the preceding section is used,

$$\frac{\Pi(-s)}{2\pi i} \int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \cdot \frac{dx}{x} = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{2}$$

In other words, if $\zeta(s)$ is defined by the formula†

$$\zeta(s) = \frac{\Pi(-s)}{2\pi i} \int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \cdot \frac{dx}{x}, \tag{3}$$

†This formula is misstated by the editors of Riemann's works in the notes; they put the factor π on the wrong side of their equation.

then, for real values of s greater than one, $\zeta(s)$ is equal to Dirichlet's function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{4}$$

However, formula (3) for $\zeta(s)$ "remains valid for all s." In fact, since the integral in (3) clearly converges for all values of s, real or complex (because e^x grows much faster than x^s as $x \to \infty$), and since the function it defines is complex analytic (because convergence is uniform on compact domains), the function $\zeta(s)$ of (3) is defined and analytic at all points with the possible exception of the points $s = 1, 2, 3, \ldots$, where $\Pi(-s)$ has poles. Now at $s = 2, 3, 4, \ldots$, formula (4) shows that $\zeta(s)$ has no pole [hence the integral in (3) must have a zero which cancels the pole of $\Pi(-s)$ at these points, a fact which also follows immediately from Cauchy's theorem], and at $s = 1$ formula (4) shows that $\lim \zeta(s) = \infty$ as $s \downarrow 1$, hence that $\zeta(s)$ has a simple [because the pole of $\Pi(-s)$ is simple] pole at $s = 1$. Thus *formula (3) defines a function $\zeta(s)$ which is analytic at all points of the complex s-plane except for a simple pole at $s = 1$.* This function coincides with $\sum n^{-s}$ for real values of $s > 1$ and in fact, by analytic continuation, throughout the halfplane $\operatorname{Re} s > 1$.

The function $\zeta(s)$ is known as the Riemann zeta function.

1.5 VALUES OF $\zeta(s)$

The function $x(e^x - 1)^{-1}$ is analytic near $x = 0$; therefore it can be expanded as a power series

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!} \tag{1}$$

valid near zero [in fact valid in the disk $|x| < 2\pi$ which extends to the nearest singularities $x = \pm 2\pi i$ of $x(e^x - 1)^{-1}$]. The coefficients B_n of this expansion are by definition the *Bernoulli numbers;* the first few are easily determined to be

$$\begin{aligned} B_0 &= 1, & B_1 &= -\tfrac{1}{2}, \\ B_2 &= \tfrac{1}{6}, & B_3 &= 0, \\ B_4 &= -\tfrac{1}{30}, & B_5 &= 0, \\ B_6 &= \tfrac{1}{42}, & B_7 &= 0, \\ B_8 &= -\tfrac{1}{30}, & B_9 &= 0. \end{aligned}$$

The odd Bernoulli numbers B_{2n+1} are all zero† after the first, and the even Bernoulli numbers B_{2n} can be determined successively, but there is no simple

†This can be proved directly by noting that $(-t)(e^{-t} - 1)^{-1} + (-t/2) = (-te^t + t - t)(1 - e^t)^{-1} - (t/2) = t(e^t - 1)^{-1} + (t/2)$, that is, $t(e^t - 1)^{-1} + (t/2)$ is an even function. For alternative proofs see the note of Section 1.6 and formula (10) of Section 6.2.

computational formula for them. (See Euler [E6] for a list of the values of $(-1)^{n-1}B_{2n}$ up to B_{30}.)

When $s=-n$ ($n=0,1,2,\ldots$), this expansion (1) can be used in the defining equation of $\zeta(s)$ to obtain

$$\begin{aligned}\zeta(-n) &= \frac{\Pi(n)}{2\pi i}\int_{+\infty}^{\infty}\frac{(-x)^{-n}}{e^x-1}\cdot\frac{dx}{x}\\ &= \frac{\Pi(n)}{2\pi i}\int_{|x|=\delta}\left(\sum_m\frac{B_m x^m}{m!}\right)\frac{(-x)^{-n}}{x}\cdot\frac{dx}{x}\\ &= \sum_m \Pi(n)\frac{B_m}{m!}(-1)^n\cdot\frac{1}{2\pi}\int_0^{2\pi}x^{m-n-1}\,d\theta\\ &= n!\frac{B_{n+1}}{(n+1)!}(-1)^n=(-1)^n\frac{B_{n+1}}{n+1}.\end{aligned}$$

Riemann does not give this formula for $\zeta(-n)$, but he does state the particular consequence $\zeta(-2)=\zeta(-4)=\zeta(-6)=\cdots=0$. He was surely aware, however, not only of the values†

$$\zeta(0)=-1/2,\qquad \zeta(-1)=-1/12,\qquad \zeta(-3)=1/120,$$

etc., which it implies, but also of the values

$$\zeta(2)=\pi^2/6,\ \zeta(4)=\pi^4/90,\ldots,$$

and, in general,

$$\zeta(2n)=\frac{(2\pi)^{2n}(-1)^{n+1}B_{2n}}{2\cdot(2n)!} \tag{2}$$

which had been found by Euler [E6]. There is no easy way to deduce this famous formula of Euler's from Riemann's integral formula for $\zeta(s)$ [(3) of Section 1.4] and it may well have been this problem of deriving (2) anew which led Riemann to the discovery‡ of the functional equation of the zeta function which is the subject of the next section.

1.6 FIRST PROOF OF THE FUNCTIONAL EQUATION

For negative real values of s, Riemann evaluates the integral

$$\zeta(s)=\frac{\Pi(-s)}{2\pi i}\int_{+\infty}^{+\infty}\frac{(-x)^s}{e^x-1}\cdot\frac{dx}{x} \tag{1}$$

†The editors of Riemann's collected works give the erroneous value $\zeta(0)=\frac{1}{2}$.

‡Actually the functional equation occurs in Euler's works [E7] in a slightly different form, and it is entirely possible that Riemann found it there. (See also Hardy [H5, pp. 23–26].) In any case, Euler had nothing but an empirical (!) proof of the functional equation and Riemann, in a reversal of his usual role, gave the first rigorous proof of a statement which had been made, but not adequately proved, by someone else.

as follows. Let D denote the domain in the s-plane which consists of all points other than those which lie within ϵ of the positive real axis or within ϵ of one of the singularities $x=\pm 2\pi in$ of the integrand of (1). Let ∂D be the boundary of D oriented in the usual way. Then, ignoring for the moment the fact that D is not compact, Cauchy's theorem gives

$$\frac{\Pi(-s)}{2\pi i}\int_{\partial D}\frac{(-x)^s}{e^x-1}\cdot\frac{dx}{x}=0. \tag{2}$$

Now one component of this integral is the integral (1) with the orientation reversed, whereas the others are integrals over the circles $|x\pm 2\pi in|=\epsilon$ oriented clockwise. Thus when the circles are oriented in the usual counterclockwise sense, (2) becomes

$$-\zeta(s)-\sum\frac{\Pi(-s)}{2\pi i}\int_{|x\pm 2\pi in|=\epsilon}\frac{(-x)^s}{e^x-1}\cdot\frac{dx}{x}=0. \tag{3}$$

The integrals over the circles can be evaluated by setting $x=2\pi in+y$ for $|y|=\epsilon$ to find

$$\begin{aligned}&\frac{\Pi(-s)}{2\pi i}\int_{|y|=\epsilon}\frac{(-2\pi in-y)^s}{e^{2\pi in+y}-1}\frac{dy}{2\pi in+y}\\ &\quad=-\frac{\Pi(-s)}{2\pi i}\int_{|y|=\epsilon}(-2\pi in-y)^{s-1}\cdot\frac{y}{e^y-1}\cdot\frac{dy}{y}\\ &\quad=-\Pi(-s)(-2\pi in)^{s-1}\end{aligned}$$

by the Cauchy integral formula. Summing over all integers n other than $n=0$ and using (3) then gives

$$\begin{aligned}\zeta(s)&=\sum_{n=1}^{\infty}\Pi(-s)[(-2\pi in)^{s-1}+(2\pi in)^{s-1}]\\ &=\Pi(-s)(2\pi)^{s-1}[i^{s-1}+(-i)^{s-1}]\sum_{n=1}^{\infty}n^{s-1}.\end{aligned}$$

Finally, using the simplification

$$\begin{aligned}i^{s-1}+(-i)^{s-1}&=\frac{1}{i}[e^{s\log i}-e^{s\log(-i)}]\\ &=\frac{1}{i}[e^{s\pi i/2}-e^{-s\pi i/2}]=2\sin\frac{s\pi}{2},\end{aligned}$$

one obtains the desired formula

$$\zeta(s)=\Pi(-s)(2\pi)^{s-1}\,2\sin(s\pi/2)\zeta(1-s). \tag{4}$$

This relationship between $\zeta(s)$ and $\zeta(1-s)$ is known as the *functional equation of the zeta function.*

In order to prove rigorously that (4) holds for $s<0$, it suffices to modify the above argument by letting D_n be the intersection of D with the disk $|s|\le(2n+1)\pi$ and letting $n\to\infty$; then the integral (2) splits into two parts, one being an integral over the circle $|s| = (2n+1)\pi$ with the points within ϵ of the positive real axis deleted, and the other being an integral whose limit as $n \to \infty$ is the left side of (3). The first of these two parts approaches zero because the length of the path of integration is less than $2\pi(2n+1)\pi$, because the factor $(e^x - 1)^{-1}$ is bounded on the circle $|s| = (2n+1)\pi$, and because the modulus of $(-x)^s/x$ on this circle is $|x|^{s-1} \le [(2n+1)\pi]^{-\delta-1}$ for $s \le -\delta < 0$. Thus the second part, which by Cauchy's theorem is the negative of the first part, also approaches zero, which implies (3) and hence (4).

This completes the proof of the functional equation (4) in the case $s < 0$. However, both sides of (4) are analytic functions of s, so this suffices to prove (4) for all values of s [except for $s = 0, 1, 2, \ldots$, where† one or more of the terms of (4) have poles].

For $s = 1 - 2n$ the functional equation plus the identity

$$\zeta(-(2n-1)) = (-1)^{2n-1}\frac{B_{2n}}{2n}$$

of the previous section gives

$$(-1)^{2n-1}\frac{B_{2n}}{2n} = \Pi(2n-1)(2\pi)^{-2n}2(-1)^n\zeta(2n)$$

and hence Euler's famous formula for $\zeta(2n)$ [(2) of Section 1.5].

Riemann uses two of the basic identities of the factorial function [(6) and (7) of Section 1.3] to rewrite the functional equation (4) in the form

$$\zeta(s) = \pi^{-1/2}2^{-s}\Pi\left(-\frac{s}{2}\right)\Pi\left(-\frac{s+1}{2}\right)2^s\pi^{s-1}\frac{\pi s/2}{\Pi\left(\frac{s}{2}\right)\Pi\left(-\frac{s}{2}\right)}\zeta(1-s)$$

and hence in the form

$$(5) \qquad \Pi\left(\frac{s}{2} - 1\right)\pi^{-s/2}\zeta(s) = \Pi\left(\frac{1-s}{2} - 1\right)\pi^{-(1-s)/2}\zeta(1-s).$$

In words, then, *the function on the left side of (5) is unchanged by the substitution $s = 1 - s$.*

Riemann appears to consider this symmetrical statement (5) as the natural statement of the functional equation, because he gives‡ an alternative proof

†When $s = 2n+1$, the fact that $\zeta(s)$ has no pole at $2n+1$ implies, since Π has a pole at $-2n-1$ and $\sin(s\pi/2)$ has no zero at $2n+1$, that $\zeta(-2n) = 0$ and hence, by the formula for $\zeta(-2n)$ of the preceding section, that the odd Bernoulli numbers $B_3, B_5, B_7, \ldots$ are all zero.

‡Since the second proof renders the first proof wholly unnecessary, one may ask why Riemann included the first proof at all. Perhaps the first proof shows the argument by which he originally discovered the functional equation or perhaps it exhibits some properties which were important in his understanding of ζ.

which exhibits this symmetry in a more satisfactory way. This second proof is given in the next section.

1.7 SECOND PROOF OF THE FUNCTIONAL EQUATION

Riemann first observes that the change of variable $x = n^2\pi x$ in Euler's integral for $\Pi(s/2 - 1)$ gives

$$\frac{1}{n^s}\pi^{-s/2}\Pi\left(\frac{s}{2} - 1\right) = \int_0^\infty e^{-n^2\pi x}x^{s/2}\frac{dx}{x} \qquad (\operatorname{Re} s > 1).$$

Thus summation over n gives

$$(1) \qquad \Pi\left(\frac{s}{2} - 1\right)\pi^{-s/2}\zeta(s) = \int_0^\infty \psi(x)x^{s/2}\cdot\frac{dx}{x} \qquad (\operatorname{Re} s > 1),$$

where† $\psi(x) = \sum_{n=1}^\infty \exp(-n^2\pi x)$. The symmetrical form of the functional equation is the statement that the function (1) is unchanged by the substitution $s = 1 - s$. To prove directly that the integral on the right side of (1) is unchanged by this substitution Riemann uses the *functional equation of the theta function* in a form taken from Jacobi,‡ namely, in the form

$$(2) \qquad \frac{1 + 2\psi(x)}{1 + 2\psi(1/x)} = \frac{1}{\sqrt{x}}.$$

[Since $\psi(x)$ approaches zero very rapidly as $x \to \infty$, this shows in particular that $\psi(x)$ is like $\frac{1}{2}(x^{-1/2} - 1)$ for x near zero and hence that the integral on the right side of (1) is convergent for $s > 1$. Once this has been established, the validity of (1) for $s > 1$ can be proved by an elementary argument using absolute convergence to justify the interchange of summation and integration.] Using (2), Riemann reformulates the integral on the right side of (1) as

$$\begin{aligned}
\int_0^\infty \psi(x)x^{s/2}\frac{dx}{x} &= \int_1^\infty \psi(x)x^{s/2}\cdot\frac{dx}{x} - \int_\infty^1 \psi\left(\frac{1}{x}\right)x^{-s/2}\cdot\frac{dx}{x} \\
&= \int_1^\infty \psi(x)x^{s/2}\cdot\frac{dx}{x} + \int_1^\infty \left[x^{1/2}\psi(x) + \frac{x^{1/2}}{2} - \frac{1}{2}\right]x^{-s/2}\frac{dx}{x} \\
&= \int_1^\infty \psi(x)[x^{s/2} + x^{(1-s)/2}]\frac{dx}{x} \\
&\quad + \frac{1}{2}\int_1^\infty [x^{-(s-1)/2} - x^{-s/2}]\frac{dx}{x}.
\end{aligned}$$

†This function $\psi(x)$ has nothing whatsoever to do with the function $\psi(x)$ which appears in Chapter 3.

‡Riemann refers to Section 65 of Jacobi's treatise "Fundmenta Nova Theoriae Functionum Ellipticarum." Although the needed formula is not given explicitly there, Jacobi in another place [J1] shows how the needed formula follows from formula (6) of Section 65. Jacobi attributes the formula to Poisson. For a proof of the formula see Section 10.4.

Now $\int_1^\infty x^{-a}(dx/x) = 1/a$ for $a > 0$ so the second integral is

$$\frac{1}{2}\left[\frac{1}{(s-1)/2} - \frac{1}{s/2}\right] = \frac{1}{s(s-1)}$$

for $s > 1$. Thus for $s > 1$ the formula

(3) $$\Pi\left(\frac{s}{2}-1\right)\pi^{-s/2}\zeta(s) = \int_1^\infty \psi(x)[x^{s/2} + x^{(1-s)/2}]\frac{dx}{x} - \frac{1}{s(1-s)}$$

holds. But, because $\psi(x)$ decreases more rapidly than any power of x as $x \to \infty$, the integral in this formula converges for all† s. Since both sides are analytic, the same equation holds for all s. Because the right side is obviously unchanged by the substitution $s = 1 - s$, this proves the functional equation of the zeta function.

1.8 THE FUNCTION $\xi(s)$

The function $\Pi((s/2)-1)\pi^{-s/2}\zeta(s)$, which occurs in the symmetrical form of the functional equation, has poles at $s = 0$ and $s = 1$. [This follows immediately from (3) of the preceding section.] Riemann multiplies it by $s(s-1)/2$ and defines‡

(1) $$\xi(s) = \Pi(s/2)(s-1)\pi^{-s/2}\zeta(s).$$

Then $\xi(s)$ is an entire function—that is, an analytic function of s which is defined for all values of s—and the functional equation of the zeta function is equivalent to $\xi(s) = \xi(1-s)$.

Riemann next derives the following representation of $\xi(s)$. Equation (3) of the preceding section gives

$$\begin{aligned}\xi(s) &= \frac{1}{2} - \frac{s(1-s)}{2}\int_1^\infty \psi(s)(x^{s/2} + x^{(1-s)/2})\frac{dx}{x} \\ &= \frac{1}{2} - \frac{s(1-s)}{2}\int_1^\infty \frac{d}{dx}\left\{\psi(x)\left[\frac{x^{s/2}}{s/2} + \frac{x^{(1-s)/2}}{(1-s)/2}\right]\right\} dx \\ &\quad + \frac{s(1-s)}{2}\int_1^\infty \psi'(x)\left[\frac{x^{s/2}}{s/2} + \frac{x^{(1-s)/2}}{(1-s)/2}\right] dx\end{aligned}$$

†Note that this gives, therefore, another formula for $\zeta(s)$ which is "valid for all s" other than $s = 0, 1$; that is, it gives an alternative proof of the fact that $\zeta(s)$ can be analytically continued.

‡Actually Riemann uses the letter ξ to denote the function which it is now customary to denote by Ξ, namely, the function $\Xi(t) = \xi(\frac{1}{2} + it)$, where ξ is defined as above. I follow Landau, and almost all subsequent writers, in rejecting Riemann's change of variable $s = \frac{1}{2} + it$ in formula (1) as being confusing. In fact, there is reason to believe that Riemann himself was confused by it [see remarks concerning $\xi(0)$ in Section 1.16].

$$\begin{aligned}&= \frac{1}{2} + \frac{s(1-s)}{2}\psi(1)\left[\frac{2}{s} + \frac{2}{1-s}\right] \\ &\quad + \int_1^\infty \psi'(x)[(1-s)x^{s/2} + sx^{(1-s)/2}]\,dx \\ &= \frac{1}{2} + \psi(1) + \int_1^\infty x^{3/2}\psi'(x)[(1-s)x^{[(s-1)/2]-1} + sx^{-(s/2)-1}]\,dx \\ &= \frac{1}{2} + \psi(1) + \int_1^\infty \frac{d}{dx}[x^{3/2}\psi'(x)(-2x^{(s-1)/2} - 2x^{-s/2})]\,dx \\ &\quad - \int_1^\infty \frac{d}{dx}[x^{3/2}\psi'(x)][-2x^{(s-1)/2} - 2x^{-s/2}]\,dx \\ &= \frac{1}{2} + \psi(1) - \psi'(1)[-2-2] \\ &\quad + \int_1^\infty \frac{d}{dx}[x^{3/2}\psi'(x)](2x^{(s-1)/2} + 2x^{-s/2})\,dx.\end{aligned}$$

Now differentiation of

$$2\psi(x) + 1 = x^{-1/2}[2\psi(1/x) + 1]$$

easily gives

$$\tfrac{1}{2} + \psi(1) + 4\psi'(1) = 0,$$

and using this puts the formula in the final form

(2) $$\xi(s) = 4\int_1^\infty \frac{d[x^{3/2}\psi'(x)]}{dx} x^{-1/4}\cosh\left[\frac{1}{2}\left(s-\frac{1}{2}\right)\log x\right]dx$$

or, as Riemann writes it,

$$\xi\left(\frac{1}{2}+it\right) = 4\int_1^\infty \frac{d[x^{3/2}\psi'(x)]}{dx} x^{-1/4}\cos\left(\frac{t}{2}\log x\right)dx.$$

If $\cosh[\frac{1}{2}(s-\frac{1}{2})\log x]$ is expanded in the usual power series $\cosh y = \frac{1}{2}(e^y + e^{-y}) = \sum y^{2n}/(2n)!$, formula (2) shows that

(3) $$\xi(s) = \sum_{n=0}^\infty a_{2n}(s-\tfrac{1}{2})^{2n}$$

where

$$a_{2n} = 4\int_1^\infty \frac{d[x^{3/2}\psi'(x)]}{dx} x^{-1/4}\frac{(\frac{1}{2}\log x)^{2n}}{(2n)!}\,dx.$$

Riemann states that this series representation of $\xi(s)$ as an even function of $s - \frac{1}{2}$ "converges very rapidly," but he gives no explicit estimates and he does not say what role this series plays in the assertions which he makes next.

The two paragraphs which follow the formula (2) for $\xi(s)$ are the most difficult portion of Riemann's paper. Their goal is essentially to prove that

$\xi(s)$ can be expanded as an infinite product

$$\xi(s) = \xi(0) \prod_\rho \left(1 - \frac{s}{\rho}\right), \tag{4}$$

where ρ ranges† over the roots of the equation $\xi(\rho) = 0$. Now any *polynomial* $p(s)$ can be expanded as a finite product $p(s) = p(0) \prod_\rho [1 - (s/\rho)]$, where ρ ranges over the roots of the equation $p(\rho) = 0$ [except that the product formula for $p(s)$ is slightly different if $p(0) = 0$]; hence the product formula (4) states that $\xi(s)$ is *like a polynomial of infinite degree*. (Similarly, Euler thought of $\sin x$ as a "polynomial of infinite degree" when he conjectured, and finally proved, the formula $\sin \pi x = \pi x \prod_{n=1}^\infty [1 - (x/n)^2]$.) On the other hand, the statement that the series (3) converges "very rapidly" is also a statement that $\xi(s)$ is like a polynomial of infinite degree—a finite number of terms gives a very good approximation in any finite part of the plane. Thus there is some relationship between the series (3) and the product formula (4)—in fact it is *precisely* the rapid decrease of the coefficients a_n which Hadamard (in 1893) proved was necessary and sufficient for the validity of the product formula—but the steps of the argument by which Riemann went from the one to the other are obscure, to say the very least.

The next section contains a discussion of the distribution of the roots ρ of $\xi(\rho) = 0$, and the following section returns to the discussion of the product formula for $\xi(s)$.

1.9 THE ROOTS ρ OF ξ

In order to prove the convergence of the product $\xi(s) = \xi(0) \prod_\rho [1 - (s/\rho)]$, Riemann needed, of course, to investigate the distribution of the roots ρ of $\xi(\rho) = 0$. He begins by observing that the Euler product formula

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \qquad (\operatorname{Re} s > 1)$$

shows immediately that $\zeta(s)$ has no zeros in the halfplane $\operatorname{Re} s > 1$ (because a convergent infinite product can be zero only if one of its factors is zero). Since $\xi(s) = \Pi(s/2)(s-1)\pi^{-s/2}\zeta(s)$ and since the factors other than $\zeta(s)$ have only the simple zero at $s = 1$, it follows that none of the roots ρ of $\xi(\rho) = 0$ lie in the halfplane $\operatorname{Re} s > 1$. Since $1 - \rho$ is a root if and only if ρ is, this implies that none of the roots lie in the halfplane $\operatorname{Re} s < 0$ either, and hence that *all the roots ρ of $\xi(\rho) = 0$ lie in the strip* $0 \le \operatorname{Re} \rho \le 1$.

He then goes on to say that the number of roots ρ whose imaginary parts

†Here, and in the many formulas in the remainder of the book which involve sums or products over the roots ρ, it is understood that multiple roots—if there are any—are to be counted with multiplicities.

lie between 0 and T is approximately

$$\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} \tag{1}$$

and that the relative† error in this approximation is of the order of magnitude $1/T$. His "proof" of this is simply to say that the number of roots in this region is equal to the integral of $\xi'(s)\,ds/2\pi i \xi(s)$ around the boundary of the rectangle $\{0 \le \operatorname{Re} s \le 1, 0 \le \operatorname{Im} s \le T\}$ and that this integral is equal to (1) with a relative error T^{-1}. Unfortunately he gives no hint whatsoever of the method he used to estimate the integral. He himself was a master at evaluating and estimating definite integrals (see, for example, Section 1.14 or 7.4) and it is quite possible that he assumed that his readers would be able to carry out their own estimation of this integral, but if so he was wrong; it was not until 1905 that von Mangoldt succeeded in proving that Riemann's estimate was correct (see Section 6.7).

Riemann's next statement is even more baffling. He states that the number of roots *on the line* $\operatorname{Re} s = \frac{1}{2}$ is also "about" (1). He does not make precise the sense in which this approximation is true, but it is generally assumed that he meant that the relative error in the approximation of the number of zeros of $\xi(\frac{1}{2} + it)$ for $0 \le t \le T$ by (1) approaches zero as $T \to \infty$. He gives no indication of a proof at all, and no one since Riemann has been able to prove (or disprove) this statement. It was proved in 1914 that $\xi(\frac{1}{2} + it)$ has infinitely many real roots (Hardy [H3]), in 1921 that the number of real roots between 0 and T is at least KT for some positive constant K and all sufficiently large T (Hardy and Littlewood [H6]), in 1942 that this number is in fact at least $KT \log T$ for some positive K and all large T (Selberg, [S1]), and in 1914 that the number of complex roots t of $\xi(\frac{1}{2} + it) = 0$ in the range $\{0 \le \operatorname{Re} t \le T, -\epsilon \le \operatorname{Im} t \le \epsilon\}$ is equal, for any $\epsilon > 0$, to (1) with a relative error which approaches zero as $T \to \infty$ (Bohr and Landau, [B8]). However, these partial results are still far from Riemann's statement. We can only guess what lay behind this statement (see Siegel [S4 p. 67], Titchmarsh [T8, pp. 213–214], or Section 7.8 of this book), but we do know that it led Riemann to conjecture an even stronger statement, namely, that *all* the roots lie on $\operatorname{Re} s = \frac{1}{2}$.

This is of course the famous "Riemann hypothesis." He says he considers it "very likely" that the roots all do lie on $\operatorname{Re} s = \frac{1}{2}$, but says that he was not able to prove it (which would seem to imply, incidentally, that he did feel he had rigorous proofs of the preceding two statements). Since it is not necessary for his main goal, which is the proof of his formula for the number of primes less than a given magnitude, he simply leaves the matter there—where it has remained ever since—and goes on to the product formula for $\xi(s)$.

†Titchmarsh, in an unfortunate lapse which he did not catch in the 21 years between the publication of his two books on the zeta function, failed to realize that Riemann meant the *relative* error and believed that Riemann had made a mistake at this point. See Titchmarsh [T8, p. 213].

1.10 THE PRODUCT REPRESENTATION OF $\xi(s)$

A recurrent theme in Riemann's work is the *global characterization of analytic functions by their singularities.*† Since the function $\log \xi(s)$ has logarithmic singularities at the roots ρ of $\xi(s)$ and no other singularities, it has the same singularities as the formal sum

$$(1) \qquad \sum_\rho \log\left(1 - \frac{s}{\rho}\right).$$

Thus if this sum converges and if the function it defines is in some sense as well behaved near ∞ as $\log \xi(s)$ is, then it should follow that the sum (1) differs from $\log \xi(s)$ by at most an additive constant; setting $s = 0$ gives the value $\log \xi(0)$ for this constant, and hence exponentiation gives

$$(2) \qquad \xi(s) = \xi(0) \prod_\rho \left(1 - \frac{s}{\rho}\right)$$

as desired. This is essentially the proof of the product formula (2) which Riemann sketches.

There are two problems associated with the sum (1). The first is the determination of the imaginary parts of the logarithms it contains. Riemann passes over this point without comment and, indeed, it is not a very serious problem. For any fixed s the ambiguity in the imaginary part of $\log[1 - (s/\rho)]$ disappears for large ρ; hence the sum (1) is defined except for a (finite) multiple of $2\pi i$ which drops out when one exponentiates (2). Furthermore, one can ignore the imaginary parts altogether; the real parts of the terms of (1) are unambiguously defined and their sum is a harmonic function which differs from Re $\log \xi(s)$ by a harmonic function without singularities, and if this difference function can be shown to be constant, it will follow that its harmonic conjugate is constant also.

The second problem associated with the sum (1) is its convergence. It is in fact a conditionally convergent sum, and the *order* of the series must be specified in order for the sum to be well determined. Roughly speaking the natural order for the terms would be the order of increasing $|\rho|$, or perhaps of increasing $|\rho - \frac{1}{2}|$, but specifically it suffices merely to stipulate that each

†See, for example, the Inauguraldissertation, especially article 20 (*Werke*, pp. 37–39) or part 3 of the introduction to the article "Theorie der Abel'schen Functionen," which is entitled "Determination of a function of a complex variable by boundary values and singularities [R1]." See also Riemann's introduction to Paper XI of the collected works, where he writes "... our method, which is based on the determination of functions by means of their singularities (*Unstetigkeiten und Unendlichwerden*) ... [R1]." Finally, see Ahlfors [A3], the section at the end entitled "Riemann's point of view."

term be paired with its "twin" $\rho \leftrightarrow 1 - \rho$,

$$(3) \qquad \sum_{\operatorname{Im}\rho>0} \left[\log\left(1 - \frac{s}{\rho}\right) + \log\left(1 - \frac{s}{1-\rho}\right)\right],$$

because this sum converges absolutely. The proof of the absolute convergence of (3) is roughly as follows.

To prove the absolute convergence of

$$\sum_{\operatorname{Im}\rho>0} \log\left[\left(1 - \frac{s}{\rho}\right)\left(1 - \frac{s}{1-\rho}\right)\right] = \sum_{\operatorname{Im}\rho>0} \log\left[1 - \frac{s(1-s)}{\rho(1-\rho)}\right],$$

it suffices to prove the absolute convergence of

$$(4) \qquad \sum_{\operatorname{Im}\rho>0} \frac{1}{\rho(1-\rho)}.$$

(In other words, to prove the absolute convergence of a product $\prod(1 + a_i)$, it suffices to prove the absolute convergence of the sum $\sum a_i$.) But the estimate of the distribution of the roots ρ given in the preceding section indicates that their density is roughly

$$d\left(\frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi}\right) = \frac{1}{2\pi}\log\frac{T}{2\pi}\,dT.$$

Hence

$$\sum_{\operatorname{Im}\rho>0} \frac{1}{\rho(1-\rho)} \sim \int^\infty \frac{1}{T^2}\frac{1}{2\pi}\log\frac{T}{2\pi}\,dT < \infty,$$

or, in short, the terms are like T^{-2} and their density is like $\log T$ so their sum converges. As will be seen in Chapter 2, the only serious difficulty in making this into a rigorous proof of the absolute convergence of (3) is the proof that the vertical density of the roots ρ is in some sense a constant times $\log(T/2\pi)$. Riemann merely states this fact without proof.

Riemann then goes on to say that the function defined by (3) grows only as fast as $s \log s$ for large s; hence, because it differs from $\log \xi(s)$ by an even function of $s - \frac{1}{2}$ [and because $\log \xi(s)$ also grows like $s \log s$ for large s], this difference must be constant because it can contain no terms in $(s - \frac{1}{2})^2$, $(s - \frac{1}{2})^4, \ldots$. It will be shown in Chapter 2 that the steps in this argument can all be filled in more or less as Riemann indicates, but it must be admitted that Riemann's sketch is so abbreviated as to make it virtually useless in constructing a proof of (2).

The first proof of the product representation (2) of $\xi(s)$ was published by Hadamard [H1] in 1893.

1.11 THE CONNECTION BETWEEN $\zeta(s)$ AND PRIMES

The essence of the relationship between $\zeta(s)$ and prime numbers is the Euler product formula

(1) $$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}} \qquad (\operatorname{Re} s > 1)$$

in which the product on the right is over all prime numbers p. Taking the log of both sides and using the series $\log(1 - x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots$ puts this in the form

$$\log \zeta(s) = \sum_p \left[\sum_n (1/n) p^{-ns} \right] \qquad (\operatorname{Re} s > 1).$$

Since the double series on the right is absolutely convergent for $\operatorname{Re} s > 1$, the order of summation is unimportant and the sum can be written simply

(2) $$\log \zeta(s) = \sum_p \sum_n (1/n) p^{-ns} \qquad (\operatorname{Re} s > 1).$$

It will be convenient in what follows to write this sum as a Stieltjes integral

(3) $$\log \zeta(s) = \int_0^\infty x^{-s}\, dJ(x) \qquad (\operatorname{Re} s > 1)$$

where $J(x)$ is† the function which begins at 0 for $x = 0$ and increases by a jump of 1 at primes p, by a jump of $\frac{1}{2}$ at prime squares p^2, by a jump of $\frac{1}{3}$ at prime cubes, etc. As is usual in the theory of Stieltjes integrals, the value of $J(x)$ at each jump is defined to be halfway between its new value and its old value. Thus $J(x)$ is zero for $0 \le x < 2$, is $\frac{1}{2}$ for $x = 2$, is 1 for $2 < x < 3$, is $1\frac{1}{2}$ for $x = 3$, is 2 for $3 < x < 4$, is $2\frac{1}{4}$ for $x = 4$, is $2\frac{1}{2}$ for $4 < x < 5$, is 3 for $x = 5$, is $3\frac{1}{2}$ for $5 < x < 7$, etc. A formula for $J(x)$ is

$$J(x) = \frac{1}{2}\left[\sum_{p^n < x} \frac{1}{n} + \sum_{p^n \le x} \frac{1}{n} \right].$$

Riemann did not, of course, have the vocabulary of Stieltjes integration available to him, and he stated (3) in the slightly different form

(4) $$\log \zeta(s) = s \int_0^\infty J(x) x^{-s-1}\, dx \qquad (\operatorname{Re} s > 1)$$

which can be obtained from (3) by integration by parts. [As $x \downarrow 0$, clearly $x^{-s} J(x) = 0$ because $J(x) \equiv 0$ for $x < 2$. On the other hand, $J(x) < x$ for all x, so $x^{-s} J(x) \to 0$ as $x \to \infty$ for $\operatorname{Re} s > 1$.] The integral in (4) can be considered to be an ordinary Riemann integral and the formula itself can be derived without using Stieltjes integration by setting

$$p^{-ns} = s \int_{p^n}^\infty x^{-s-1}\, dx \qquad (\operatorname{Re} s > 1)$$

in (2), which is Riemann's derivation of (4).

Formulas (2)–(4) should all be thought of as minor variations of the Euler product formula (1) which is the basic idea connecting $\zeta(s)$ and primes.

†Riemann denotes this function $f(x)$, and most other writers denote it $\Pi(x)$. Since $f(x)$ now is commonly used to denote a generic function and since $\Pi(x)$ in this book denotes the factorial function, I have taken the liberty of introducing a new notation $J(x)$ for this function.

1.12 FOURIER INVERSION

Riemann was a master of Fourier analysis and his work in developing this theory must certainly be counted among his greatest contributions to mathematics. It is not surprising, therefore, that he immediately applies Fourier inversion to the formula

(1) $$\frac{\log \zeta(s)}{s} = \int_0^\infty J(x) x^{-s-1}\, dx \qquad (\operatorname{Re} s > 1)$$

to conclude

(2) $$J(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log \zeta(s) x^s \frac{ds}{s} \qquad (a > 1).$$

Then using an alternative formula for $\log \zeta(s)$, he obtains an alternative formula for $J(x)$ which is the main result of the paper.

[The improper integral in (2) is only conditionally convergent and an "order of summation" must be specified. Here it is understood that the integral in (2) means the limit as $T \to \infty$ of the integral over the vertical line segment from $a - iT$ to $a + iT$. More generally, conditionally convergent integrals and series are very common in Fourier analysis, and it is always understood that such integrals and series are summed in their "natural order"; for example,

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{means} \quad \lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{inx},$$

$$\int_{-\infty}^{\infty} f(y) e^{iyx}\, dy \quad \text{means} \quad \lim_{T\to\infty} \int_{-T}^{T} f(y) e^{iyx}\, dy,$$

etc. This is analogous to the convention that discontinuous functions such as $J(x)$ assume the middle value $J(x) = \frac{1}{2}[J(x - \epsilon) + J(x + \epsilon)]$ at any jump x, that divergent integrals such as $\operatorname{Li}(x)$ (see Section 1.14 below) are taken to mean the Cauchy principal value, and that the product $\prod [1 - (s/\rho)]$ is ordered in such a way as to pair ρ with $1 - \rho$, or, later on, ordered by $|\operatorname{Im} \rho|$.]

In deriving (2) from (1) Riemann makes use of "Fourier's theorem," by which he means† the Fourier inversion formula

$$\phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\phi(\lambda)e^{i(x-\lambda)\mu}\,d\lambda\right]d\mu. \tag{3}$$

Otherwise stated, "Fourier's theorem" is the statement that in order to write a given function $\phi(x)$ as a superposition of exponentials

$$\phi(x) = \int_{-\infty}^{\infty}\Phi(\mu)e^{i\mu x}\,d\mu,$$

it is necessary and sufficient (under suitable conditions) that the "coefficients" $\Phi(\mu)$ of the expansion be defined by

$$\Phi(\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(\lambda)e^{-i\lambda\mu}\,d\lambda.$$

This statement of Fourier's theorem brings out the analogy with Fourier series

$$f(x) = \sum_{-\infty}^{\infty}a_n e^{inx} \Longleftrightarrow a_n = \frac{1}{2\pi}\int_0^{2\pi}f(\lambda)e^{-in\lambda}\,d\lambda,$$

and in fact theorem (3) for Fourier integrals follows formally from a passage to the limit in the theorem for Fourier series.

To derive (2) from (1), let $s = a + i\mu$, where a is a constant $a > 1$ and μ is a real variable, let $\lambda = \log x$, and let $\phi(x) = 2\pi J(e^x)e^{-ax}$. Then (1) becomes

$$\begin{aligned}\frac{\log\zeta(a+i\mu)}{a+i\mu} &= \int_{-\infty}^{\infty}J(e^\lambda)e^{-(a+i\mu)\lambda}\,d\lambda\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(\lambda)e^{-i\mu\lambda}\,d\lambda \qquad (a>1),\end{aligned}$$

and when this function is taken to be $\Phi(\mu)$, Fourier's theorem gives

$$\begin{aligned}2\pi J(e^x)e^{-ax} &= \int_{-\infty}^{\infty}\frac{\log\zeta(a+i\mu)}{a+i\mu}e^{i\mu x}\,d\mu,\\ J(y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\log\zeta(a+i\mu)}{a+i\mu}y^{a+i\mu}\,d\mu,\end{aligned}$$

from which (2) follows immediately.

Riemann completely ignores the question of the applicability of Fourier's theorem to the function $J(e^x)e^{-ax}$ and states simply that (2) holds in "complete generality." However, $J(e^x)e^{-ax}$ is a very well-behaved function—it has simple well-behaved jumps, it is identically zero for $x < 0$, and it goes to zero

†See Riemann [R2, p. 86].

faster than $e^{-(a-1)x}$ as $x \to \infty$—and the very simplest theorems† on Fourier integrals suffice to prove rigorously Riemann's statement that (2) holds in complete generality.

1.13 METHOD FOR DERIVING THE FORMULA FOR $J(x)$

The two formulas for $\xi(s)$, namely,

$$\xi(s) = \Pi\left(\frac{s}{2}\right)\pi^{-s/2}(s-1)\zeta(s) \qquad \text{and} \qquad \xi(s) = \xi(0)\prod_\rho\left(1-\frac{s}{\rho}\right),$$

combine to give

$$\begin{aligned}\log\zeta(s) &= \log\xi(s) - \log\Pi\left(\frac{s}{2}\right) + \frac{s}{2}\log\pi - \log(s-1)\\ &= \log\xi(0) + \sum_\rho\log\left(1-\frac{s}{\rho}\right) - \log\Pi\left(\frac{s}{2}\right)\\ &\quad + \frac{s}{2}\log\pi - \log(s-1).\end{aligned}$$

Riemann's formula for $J(x)$, which is the main result of his paper, is obtained essentially by substituting this formula for $\log\zeta(s)$ in the formula

$$J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\log\zeta(s)x^s\frac{ds}{s} \qquad (a>1)$$

of the preceding section and integrating termwise. However, because a direct substitution leads to divergent integrals [the term $(s/2)\log\pi$, for example, leads to an integral which is a constant times $(i)^{-1}\int x^s\,ds = e^a\int e^{iu\log x}\,du$ which oscillates and does not converge even conditionally], Riemann first integrates by parts to obtain

$$J(x) = -\frac{1}{2\pi i}\cdot\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\log\zeta(s)}{s}\right]x^s\,ds \qquad (a>1) \tag{1}$$

before substituting the above expression for $\log\zeta(s)$. The validity of the integration by parts by which (1) is obtained depends merely on showing that

$$\lim_{T\to\infty}\frac{\log\zeta(a\pm iT)}{a\pm iT}x^{a\pm iT} = 0, \tag{2}$$

†See, for example, Taylor [T2]. The particular form of Fourier inversion that Riemann uses here—which is essentially Fourier analysis on the multiplicative group of positive reals rather than on the additive group of all real numbers—is often called Mellin inversion. Riemann's work precedes that of Mellin by 40 years.

which follows easily from the inequality

$$|\log \zeta(a \pm iT)| = \left|\sum_n \sum_p (1/n)p^{-n(a\pm iT)}\right|$$

(3) $$\leq \sum_n \sum_p (1/n)p^{-na} = \log \zeta(a) = \text{const}$$

because this shows that the numerator in (2) is bounded while the denominator goes to infinity.

The substitution of

$$\log \zeta(s) = \log \xi(0) + \sum_\rho \log\left(1 - \frac{s}{\rho}\right) - \log \Pi\left(\frac{s}{2}\right) + \frac{s}{2}\log \pi - \log(s-1)$$

into (1) expresses $J(x)$ as a sum of five terms (the integral of a finite sum is always the sum of the integrals provided the latter converge) and the derivation of Riemann's formula for $J(x)$ depends now on the evaluation of these five definite integrals.

It should be noted that for any fixed s there is some ambiguity in the definition of $\log[1 - (s/\rho)]$ for those roots ρ which are not large relative to s. In order to remove this ambiguity in $\operatorname{Re} s > 1$ let $\log[1 - (s/\rho)]$ be defined to be $\log(s - \rho) - \log(-\rho)$; this is meaningful because none† of the ρ's are real and greater than or equal to 0. In this way $\log[1 - (s/\rho)]$ is unambiguously defined throughout $\operatorname{Re} s > 1$ and, in particular, on the path of integration $\operatorname{Re} s = a > 1$.

1.14 THE PRINCIPAL TERM OF $J(x)$

It will be seen below that the principal term in the formula for $J(x)$ is the term corresponding to the term $-\log(s-1)$ of the expansion of $\log \zeta(s)$. This term is

$$\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty} \frac{d}{ds}\left[\frac{\log(s-1)}{s}\right]x^s\, ds \qquad (a > 1).$$

Riemann shows that for $x > 1$ the value of this definite integral is the logarithmic integral

$$\operatorname{Li}(x) = \lim_{\epsilon \downarrow 0}\left[\int_0^{1-\epsilon}\frac{dt}{\log t} + \int_{1+\epsilon}^x \frac{dt}{\log t}\right];$$

†See Section 2.3, or observe that the series $1 - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - \cdots$ converges to a positive number for $s > 0$ and that this number is

$$\zeta(s) - 2\cdot 2^{-s}(1 + 2^{-s} + 3^{-s} + \cdots) = (1 - 2^{1-s})\zeta(s).$$

that is, it is the Cauchy principal value of the divergent integral $\int_0^x (dt/\log t)$. His argument is as follows:

Fix $x > 1$ and consider the function of β defined by

$$F(\beta) = \frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left\{\frac{\log[(s/\beta) - 1]}{s}\right\}x^s\, ds$$

so that the desired number is $F(1)$. The definition of $F(\beta)$ can be extended to all real or complex numbers β other than real numbers $\beta \leq 0$ by taking $a > \operatorname{Re}\beta$ and defining $\log[(s/\beta) - 1]$ to be $\log(s - \beta) - \log \beta$, where, as usual, $\log z$ is defined for all z other than real $z \leq 0$ by the condition that it be real for real $z > 0$. The integral $F(\beta)$ converges absolutely because

$$\left|\frac{d}{ds}\left\{\frac{\log[(s/\beta) - 1]}{s}\right\}\right| \leq \frac{|\log[(s/\beta) - 1]|}{|s|^2} + \frac{1}{|s(s-\beta)|}$$

is integrable while x^s oscillates on the line of integration. Because

$$\frac{d}{d\beta}\left\{\frac{\log[(s/\beta) - 1]}{s}\right\} = \frac{1}{(\beta - s)\beta},$$

differentiation under the integral sign and integration by parts give

$$F'(\beta) = \frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{1}{(\beta - s)\beta}\right]x^s\, ds = -\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{x^s}{(\beta - s)\beta}\, ds.$$

This last integral can be evaluated by applying Fourier inversion to the formula

$$\frac{1}{s-\beta} = \int_1^\infty x^{-s}x^{\beta-1}\, dx \qquad [\operatorname{Re}(s-\beta) > 0],$$

$$\frac{1}{a + i\mu - \beta} = \int_0^\infty e^{-i\lambda\mu}e^{\lambda(\beta-a)}\, d\lambda \qquad [a > \operatorname{Re}\beta],$$

to obtain

(1) $$\int_{-\infty}^\infty \frac{1}{a + i\mu - \beta}e^{i\mu x}\, d\mu = \begin{cases} 2\pi e^{x(\beta-a)} & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$$

from which it follows that

(2) $$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{1}{s-\beta}y^s\, ds = \begin{cases} y^\beta & \text{if } y > 1, \\ 0 & \text{if } y < 1, \end{cases}$$

provided $a > \operatorname{Re}\beta$. Since $x > 1$ by assumption, this gives $F'(\beta) = x^\beta/\beta$.

Now let C^+ denote the contour in the complex t-plane which consists of the line segment from 0 to $1 - \epsilon$ (where ϵ is a small positive number), followed by the semicircle in the upper halfplane $\operatorname{Im} t \geq 0$ from $1 - \epsilon$ to $1 + \epsilon$, followed by the line segment from $1+\epsilon$ to x, and let

$$G(\beta)=\int_{C+}\frac{t^{\beta-1}}{\log t}\,dt.$$

Then

$$G'(\beta)=\int_{C+}t^{\beta-1}\,dt=\frac{t^\beta}{\beta}\Big|_0^x=F'(\beta).$$

Now $G(\beta)$ is defined and analytic for $\operatorname{Re}\beta>0$ (if $\operatorname{Re}\beta<0$, then the integral which defines G diverges at $t=0$) as is $F(\beta)$; hence they differ by a constant (which might depend on x) throughout $\operatorname{Re}\beta>0$. Riemann states that this constant can be evaluated by holding $\operatorname{Re}\beta$ fixed and letting $\operatorname{Im}\beta\to+\infty$ in both $F(\beta)$ and $G(\beta)$, but he does not carry out this evaluation.

To evaluate the limit of $G(\beta)$, set $\beta=\sigma+i\tau$, where σ is fixed and $\tau\to\infty$. The change of variable $t=e^u$, $u=\log t$ puts $G(\beta)$ in the form

$$\int_{i\delta-\infty}^{i\delta+\log x}\frac{e^{\beta u}}{u}\,du+\int_{i\delta+\log x}^{\log x}\frac{e^{\beta u}}{u}\,du,$$

where the path of integration has been altered slightly using Cauchy's theorem. The changes of variable $u=i\delta+v$ in the first integral and $u=\log x+iw$ in the second put this in the form

$$G(\beta)=e^{i\delta\sigma}e^{-\delta\tau}\int_{-\infty}^{\log x}\frac{e^{\sigma v}}{i\delta+v}e^{i\tau v}\,dv-ix^\beta\int_0^\delta\frac{e^{-\tau w}e^{\sigma i w}}{\log x+iw}\,dw.$$

Both integrals in this expression approach zero as $\tau\to\infty$, the first because $e^{-\delta\tau}\to 0$ and the second because $e^{-\tau w}\to 0$ except at $w=0$. Thus *the limit of $G(\beta)$ as $\tau\to\infty$ is zero.* (Note, however, that this argument would not be valid if C^+ were changed to follow the lower semicircle because then $e^{-\delta\tau}$ would be replaced by $e^{\delta\tau}$ and $e^{-\tau w}$ would be replaced by $e^{\tau w}$.)

To evaluate the limit of $F(\beta)$ let

$$H(\beta)=\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left\{\frac{\log[1-(s/\beta)]}{s}\right\}x^s\,ds$$

where $a>\operatorname{Re}\beta$ and where $\log[1-(s/\beta)]$ is defined for all complex numbers β other than real numbers $\beta\geq 0$ to be $\log(s-\beta)-\log(-\beta)$. The difference $H(\beta)-F(\beta)$ is defined for all complex numbers β other than the real axis, and in the upper halfplane $\operatorname{Im}\beta>0$ it is equal to

$$\begin{aligned}H(\beta)-F(\beta)&=\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\log\beta-\log(-\beta)}{s}\right]x^s\,ds,\\&=\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{i\pi}{s}\right]x^s\,ds\\&=-\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{i\pi}{s}x^s\,ds=-i\pi\end{aligned}$$

by the case $\beta=0$ of (2). Thus $F(\beta)=H(\beta)+i\pi$ throughout the upper halfplane, and it will suffice to evaluate the limit of $H(\beta)$ as $\tau\to\infty$ $(\beta=\sigma+i\tau)$. Now $1-(s/\beta)\to 1$; hence its log goes to zero and it appears plausible therefore that $H(\beta)$ also goes to zero. This can be proved by carrying out the differentiation

$$\begin{aligned}\frac{d}{ds}\left\{\frac{\log[1-(s/\beta)]}{s}\right\}&=-\frac{\log[1-(s/\beta)]}{s^2}+\frac{1}{s(s-\beta)}\\&=-\frac{\log[1-(s/\beta)]}{s^2}+\frac{1}{\beta(s-\beta)}-\frac{1}{\beta s},\end{aligned}$$

multiplying by $x^s\,ds/2\pi i$, and integrating from $a-i\infty$ to $a+i\infty$ (in the usual sense, namely, the limit as $T\to\infty$ of the integral from $a-iT$ to $a+iT$). Because of the s^2 in the denominator of the first integral, it is not difficult to show, using the Lebesgue bounded convergence theorem (see Edwards [E1]), that the limit of this integral as $\tau\to\infty$ is the integral of the limit, namely, zero. The remaining two integrals can be evaluated using (2) to find they are $x^\beta/\beta-x^0/\beta=(x^\beta-1)/\beta$. Since the numerator is bounded and $|\beta|\to\infty$, this approaches zero; hence $H(\beta)$ approaches zero and $F(\beta)$ therefore approaches $i\pi$. Hence $F(\beta)=G(\beta)+i\pi$ in the halfplane $\operatorname{Re}\beta>0$. Thus the desired number $F(1)$ is

$$F(1)=\int_0^{1-\epsilon}\frac{dt}{\log t}+\int_{1-\epsilon}^{1+\epsilon}\frac{(t-1)}{\log t}\cdot\frac{dt}{t-1}+\int_{1+\epsilon}^{x}\frac{dt}{\log t}+i\pi,$$

where the second integral is over the semicircle in the upper halfplane; as $\epsilon\downarrow 0$, the quotient $(t-1)/\log t$ approaches 1 along this semicircle, and hence the integral approaches $\int_{1-\epsilon}^{1+\epsilon}dt/(t-1)=-i\pi$. Thus the limit as $\epsilon\downarrow 0$ of the above formula is

$$F(1)=\operatorname{Li}(x)$$

as was to be shown.

1.15 THE TERM INVOLVING THE ROOTS ρ

Consider next the term in the formula for $J(x)$ arising from the term $\sum\log[1-(s/\rho)]$ in the formula for $\log\zeta(s)$, namely,

$$-\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left\{\frac{\sum\log[1-(s/\rho)]}{s}\right\}x^s\,ds. \tag{1}$$

If the operation of summation over ρ can be interchanged with the differentiation and the integration, *then* this is equal to $-\Sigma H(\rho)$, where $H(\rho)$ is defined as in the preceding section. Now it was shown that $H(\rho)\equiv G(\rho)$ for ρ in the first quadrant ($\operatorname{Re}\rho>0$, $\operatorname{Im}\rho>0$) and in exactly the same way it can be shown that for ρ in the fourth quadrant ($\operatorname{Re}\rho > 0$, $\operatorname{Im}\rho < 0$) the value of $H(\rho)$ is equal to the integral $G(\rho)$ except that the integral must be over the contour C^- which goes over the lower semicircle from $1-\epsilon$ to $1+\epsilon$ rather than over the upper semicircle as C^+ did. Thus, pairing terms of the sum over ρ in the usual way, the integral (1) would be

$$(2) \qquad -\sum_{\operatorname{Im}\rho>0}\left(\int_{C^+}\frac{t^{\rho-1}}{\log t}\,dt + \int_{C^-}\frac{t^{-\rho}}{\log t}\,dt\right)$$

if it could be evaluated termwise. Now if β is real and positive, then the change of variable $u = t^\beta$, which implies $\log t = \log u/\beta$, $dt/t = du/u\beta$, gives

$$\int_{C^+}\frac{t^{\beta-1}}{\log t}\,dt = \int_0^{x^\beta}\frac{du}{\log u} = \operatorname{Li}(x^\beta) - i\pi,$$

where the second integral is over a path which passes above the singularity at $u = 1$. Since the integral on the left converges throughout the halfplane $\operatorname{Re}\beta > 0$, this formula gives the analytic continuation of $\operatorname{Li}(x^\beta)$ to this halfplane (when x is, as always, a fixed number $x > 1$). In the same way

$$\int_{C^-}\frac{t^{\beta-1}}{\log t}\,dt = \operatorname{Li}(x^\beta) + i\pi,$$

and (2) becomes

$$(3) \qquad -\sum_{\operatorname{Im}\rho>0}[\operatorname{Li}(x^\rho) + \operatorname{Li}(x^{1-\rho})].$$

Thus, if termwise evaluation is valid, the desired integral (1) is equal to (3).

Riemann states that termwise evaluation is valid and that (3) is indeed the desired value (1) but that the series (3) is only conditionally convergent—even though the terms ρ, $1-\rho$ are paired—and that it must be summed in the order of increasing† $\operatorname{Im}\rho$. He concedes that the validity of this termwise evaluation of (1) requires "a more exact discussion of the function ξ," but says that this is "easy" and passes on to the next point.

One other small remark about the sum (3) is necessary. The computations above assume $\operatorname{Re}\rho > 0$, but it has not been shown that this is true for all roots ρ. Although Hadamard later proved that there are no roots ρ on the line $\operatorname{Re}\rho = 0$ (see Section 4.2), Riemann has not excluded this possibility and he is therefore not justified in ignoring the point as he does.

†It is interesting to note that Riemann writes $\rho = \frac{1}{2} + i\alpha$ and says first that the sum (3) is over all *positive* values of α in order of size before then adding parenthetically that it is over all α's with $\operatorname{Re}(\alpha) > 0$ in order of size. Thus he admits, albeit parenthetically, the possibility that the Riemann hypothesis is false.

1.16 THE REMAINING TERMS

One of the three remaining terms in the formula for $J(x)$, namely, the term arising from $(s/2)\log\pi$, drops out when it is divided by s and differentiated with respect to s. The term arising from the constant $\log\xi(0)$ is

$$-\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left(\frac{\log\xi(0)}{s}\right)x^s\,ds$$
$$= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\log\xi(0)}{s}x^s\,ds = \log\xi(0)$$

using (2) of Section 1.14 in the case $\beta = 0$. Now $\xi(0) = \Pi(0)\pi^{-0}(0-1)\zeta(0) = -\zeta(0) = \frac{1}{2}$ so $\log\xi(0) = -\log 2$ is the numerical value of this term.

Riemann writes $\log\xi(0)$ instead of $-\log 2$, but since he uses ξ to denote a different function—namely, the function $\xi(\frac{1}{2}+it)$ of t—his $\xi(0)$ denotes $\xi(\frac{1}{2}) \neq \frac{1}{2}$ and thus his formula is in error. It is hard to guess what the source of this trivial error might be, other than to say that it arises from some confusion between the product formula

$$\xi(s) = \xi(0)\prod_\rho[1-(s/\rho)]$$

in the form it is given above and the product formula

$$\begin{aligned}\xi\left(\frac{1}{2}+it\right) &= \xi(0)\prod\left(1-\frac{\frac{1}{2}+it}{\frac{1}{2}+i\alpha}\right)\\ &= \xi(0)\prod\left(\frac{i\alpha-it}{\frac{1}{2}+i\alpha}\right)\\ &= \xi(0)\prod\left(\frac{i\alpha}{\frac{1}{2}+i\alpha}\right)\prod\left(1-\frac{it}{i\alpha}\right)\\ &= \xi(0)\prod\left(1-\frac{\frac{1}{2}}{\frac{1}{2}+i\alpha}\right)\prod\left(1-\frac{t}{\alpha}\right)\\ &= \xi\left(\frac{1}{2}\right)\prod_{\operatorname{Re}\alpha>0}\left(1-\frac{t^2}{\alpha^2}\right)\end{aligned}$$

in the form given by Riemann, and a concomitant confusion of the integral

$$\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left\{\frac{\log[1-(s/\rho)]}{s}\right\}x^s\,ds$$

which he evaluates, with the integral

$$\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left\{\frac{\log[1-(s-\frac{1}{2})/i\alpha]}{s}\right\}x^s\,ds$$

which differs from it by a constant. Whatever the source of the error, Riemann makes the same error in the letter quoted by the editors in the notes which follow the paper in the collected works, and his unpublished papers [R1a] include a computation of $\log\xi(\frac{1}{2})$ to several decimal places, so it was definitely not a typographical error as the editors of the collected works suppose. The error was noticed by Genocchi [G4] during Riemann's lifetime.

This leaves only one term

$$(1) \qquad \frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\log \Pi(s/2)}{s}\right]x^s\,ds$$

to be evaluated. Now by formula (4) of Section 1.3

$$\log \Pi\left(\frac{s}{2}\right) = \sum_{n=1}^{\infty}\left[-\log\left(1+\frac{s}{2n}\right)+\frac{s}{2}\log\left(1+\frac{1}{n}\right)\right].$$

Using this formula in (1) and assuming that termwise integration is valid puts (1) in the form

$$-\sum_{n=1}^{\infty}\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left\{\frac{\log[1+(s/2n)]}{s}\right\}x^s\,ds = -\sum_{n=1}^{\infty}H(-2n),$$

where H is as in Section 1.14. The previous formulas for $H(\beta)$ apply only in the halfplane $\operatorname{Re}\beta > 0$. To obtain a formula for H in $\operatorname{Re}\beta < 0$ set

$$E(\beta) = -\int_x^{\infty}\frac{t^{\beta-1}}{\log t}\,dt.$$

Then $E(\beta)$ converges for $\operatorname{Re}\beta < 0$ and satisfies

$$E'(\beta) = -\int_x^{\infty}t^{\beta-1}\,dt = \frac{x^\beta}{\beta} = F'(\beta) = H'(\beta)$$

so $E(\beta)$ differs from $H(\beta)$ by a constant throughout $\operatorname{Re}\beta < 0$. Since both E and H approach zero as $\beta \longrightarrow -\infty$, the constant is zero and $E \equiv H$. Thus (1) becomes

$$\sum_{n=1}^{\infty}\int_x^{\infty}\frac{t^{-2n-1}}{\log t}\,dt = \int_x^{\infty}\frac{1}{t\log t}(\textstyle\sum t^{-2n})\,dt = \displaystyle\int_x^{\infty}\frac{dt}{t(t^2-1)\log t}\cdot$$

provided termwise integration is valid. The proof that termwise integration is valid, which Riemann (tacitly) leaves to the reader, can be given as follows.

Note first that the series

$$\frac{d}{ds}\left[\frac{\log \Pi(s/2)}{s}\right] = -\sum_{n=1}^{\infty}\frac{d}{ds}\left\{\frac{\log[1+(s/2n)]}{s}\right\}$$

converges uniformly in any disk $|s| \le K$. [For large n the series expansion $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$ can be used, and the summand on the right contains only terms in $n^{-2}, n^{-3}, \ldots$.] This justifies the termwise differentiation and also justifies termwise integration over any finite interval

$$(2) \qquad \frac{1}{2\pi i}\frac{1}{\log x}\int_{a-iT}^{a+iT}\frac{d}{ds}\left[\frac{\log \Pi(s/2)}{s}\right]x^s\,ds$$
$$= -\sum_{n=1}^{\infty}\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-iT}^{a+iT}\frac{d}{ds}\left\{\frac{\log[1+(s/2n)]}{s}\right\}x^s\,ds.$$

To estimate the nth term of the sum on the right set $v = (s-a)/2n$, $b = a/2n$,

$c = T/2n$, so $s = 2n(v+b)$ and the nth term is minus

$$\frac{1}{2\pi i}\frac{1}{\log x}\int_{-ic}^{ic}\frac{d}{2n\,dv}\left[\frac{\log(1+v+b)}{2n(v+b)}\right]x^{2nv+a}2n\,dv$$
$$= \frac{1}{2\pi i}\frac{x^a}{2n\log x}\int_{-ic}^{ic}\frac{d}{dv}\left[\frac{\log(1+v+b)}{v+b}\right]x^{2nv}\,dv.$$

Integration by parts puts this in the form

$$= \frac{1}{2\pi i}\frac{x^a}{2n\log x}\cdot\frac{1}{2n\log x}\left(\left\{\frac{d}{dv}\left[\frac{\log(1+v+b)}{v+b}\right]x^{2nv}\right|_{v=-ic}^{v=ic}\right.$$
$$\left.-\int_{-ic}^{ic}\frac{d^2}{dv^2}\left[\frac{\log(1+v+b)}{v+b}\right]x^{2nv}\,dv\right)\cdot$$

Now b is a real number $0 \le b \le a$, the function

$$\frac{d}{dv}\left[\frac{\log(1+v+b)}{v+b}\right] = \frac{1}{(v+b)(v+b+1)} - \frac{\log(1+v+b)}{(v+b)^2}$$

is bounded on the imaginary axis, and its derivative is absolutely integrable over $(-i\infty, i\infty)$, from which it follows that the modulus of the nth term of the series on the right side of (2) is at most a constant times n^{-2} *for all* T. Thus the series converges uniformly in T and one can pass to the limit $T \longrightarrow \infty$ termwise, as was to be shown.

This completes the evaluation of the terms in the formula for $J(x)$. Combining them gives the final result

$$(3) \qquad J(x) = \operatorname{Li}(x) - \sum_{\operatorname{Im}\rho>0}[\operatorname{Li}(x^\rho) + \operatorname{Li}(x^{1-\rho})]$$
$$+ \int_x^{\infty}\frac{dt}{t(t^2-1)\log t} + \log\xi(0) \qquad (x > 1)$$

which is Riemann's formula [except that, as noted above, $\log\xi(0)$ equals $\log(\frac{1}{2})$ and not $\log\xi(\frac{1}{2})$ as in Riemann's notation it should]. This analytic formula for $J(x)$ is the principal result of the paper.

1.17 THE FORMULA FOR $\pi(x)$

Of course Riemann's goal was to obtain a formula not for $J(x)$ but for the function $\pi(x)$, that is, for the number of primes less than any given magnitude x. Since the number of prime squares less than x is obviously equal to the number of primes less than $x^{1/2}$, that is, equal to $\pi(x^{1/2})$, and since in the same way the number of prime nth powers p^n less than x is $\pi(x^{1/n})$, it follows that J and π are related by the formula

$$(1) \qquad J(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots + \frac{1}{n}\pi(x^{1/n}) + \cdots.$$

The series in this formula is finite for any given x because $x^{1/n} < 2$ for n sufficiently large, which implies $\pi(x^{1/n}) = 0$. Riemann inverts this relationship by means of the Möbius inversion formula† (see Section 10.9) to obtain

$$\pi(x) = J(x) - \frac{1}{2}J(x^{1/2}) - \frac{1}{3}J(x^{1/3}) - \frac{1}{5}J(x^{1/5}) + \frac{1}{6}J(x^{1/6}) + \cdots + \frac{\mu(n)}{n}J(x^{1/n}) + \cdots, \tag{2}$$

where $\mu(n)$ is 0 if n is divisible by a prime square, 1 if n is a product of an even number of distinct primes, and -1 if n is a product of an odd number of distinct primes. The series (2) is a finite series for any fixed x and when combined with the analytical formula for $J(x)$

$$J(x) = \mathrm{Li}(x) - \sum_{\rho} \mathrm{Li}(x^{\rho}) - \log 2 + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t} \qquad (x > 1), \tag{3}$$

it gives an analytical formula for $\pi(x)$ as desired.

The formula for $\pi(x)$ which results from substituting (3) in the (finite) series (2) consists of three kinds of terms, namely, those which do not grow as x grows [arising from the last two terms of (3)], those which grow as x grows but which oscillate in sign [the terms arising from $\mathrm{Li}(x^{\rho})$ which Riemann calls "periodic"], and those which grow steadily as x grows [the terms arising from $\mathrm{Li}(x)$]. If all but the last type are ignored, the terms in the formula for $\pi(x)$ are just

$$\mathrm{Li}(x) - \tfrac{1}{2}\mathrm{Li}(x^{1/2}) - \tfrac{1}{3}\mathrm{Li}(x^{1/3}) - \tfrac{1}{5}\mathrm{Li}(x^{1/5}) + \tfrac{1}{6}\mathrm{Li}(x^{1/6}) - \tfrac{1}{7}\mathrm{Li}(x^{1/7}) + \cdots.$$

Now *empirically* this is found to be a good approximation to $\pi(x)$. In fact, the first term alone is essentially Gauss's approximation

$$\pi(x) \sim \int_2^x \frac{dt}{\log t} = \mathrm{Li}(x) - \mathrm{Li}(2)$$

[$\mathrm{Li}(2) = 1.04\ldots$] and the first two terms indicate that

$$\pi(x) \sim \mathrm{Li}(x) - \tfrac{1}{2}\mathrm{Li}(x^{1/2})$$

†Very simply this inversion is effected by performing successively for each prime $p = 2, 3, 5, 7, 11, \ldots$ the operation of replacing the functions $f(x)$ on each side of the equation with the functions $f(x) - (1/p)f(x^{1/p})$. This gives successively

$$J(x) - \tfrac{1}{2}J(x^{1/2}) = \pi(x) + \tfrac{1}{3}\pi(x^{1/3}) + \tfrac{1}{5}\pi(x^{1/5}) + \cdots,$$

$$J(x) - \tfrac{1}{2}J(x^{1/2}) - \tfrac{1}{3}J(x^{1/3}) + \tfrac{1}{6}J(x^{1/6}) = \pi(x) + \tfrac{1}{5}\pi(x^{1/5}) + \tfrac{1}{7}\pi(x^{1/7}) + \cdots,$$

etc., where at each step the sum on the left consists of those terms of the right side of (2) for which the factors of n contain *only* the primes already covered and the sum on the right consists of those terms of the right side of (1) for which the factors of n contain *none* of the primes already covered. Once p is sufficiently large, the latter are all zero except for $\pi(x)$.

which gives, for example,

$$\pi(10^6) \sim 78{,}628 - \tfrac{1}{2}\cdot 178 = 78{,}539$$

which is better than Gauss's approximation and which becomes still better if the third term is used. The extent to which Riemann's suggested approximation

$$\pi(x) \sim \mathrm{Li}(x) + \sum_{n=2}^{\infty} \frac{\mu(n)}{n} \mathrm{Li}(x^{1/n}) \tag{4}$$

is better than $\pi(x) \sim \mathrm{Li}(x)$ is stunningly illustrated by one of Lehmer's tables [L9], an extract of which is given in Table III.

TABLE III[a]

x	Riemann's error	Gauss's error
1,000,000	30	130
2,000,000	−9	122
3,000,000	0	155
4,000,000	33	206
5,000,000	−64	125
6,000,000	24	228
7,000,000	−38	179
8,000,000	−6	223
9,000,000	−53	187
10,000,000	88	339

[a]From Lehmer [L9].

Of course Riemann did not have such extensive empirical data at his disposal, but he seems well aware of the fact that (4) is a better approximation, as well as a more natural approximation, to $\pi(x)$.

Riemann was also well aware, however, of the defects of the approximation (4) and of his analysis of it. Although he has succeeded in giving an exact analytical formula for the error

$$\pi(x) - \sum_{n=1}^{N} \frac{\mu(n)}{n} \mathrm{Li}(x^{1/n}) = \sum_{n=1}^{N} \sum_{\rho} \mathrm{Li}(x^{\rho/n}) + \text{lesser terms}$$

(where N is large enough that $x^{1/(N+1)} < 2$) he has no estimate at all of the size of these "periodic" terms $\sum\sum \mathrm{Li}(x^{\rho/n})$. Actually, the empirical fact that they are as small as Lehmer found them to be is somewhat surprising in view of the fact that the series $\sum [\mathrm{Li}(x^{\rho}) + \mathrm{Li}(x^{1-\rho})]$ is only conditionally convergent—hence the smallness of its sum for any x depends on wholesale cancellation of signs among the terms—and in view of the fact that the individual terms $\mathrm{Li}(x^\rho)$ grow in magnitude like $|x^\rho/\log x^\rho| = x^{\mathrm{Re}\,\rho}/|\rho| \log x$ (see Section 5.5) so that many of them grow at least as fast as $x^{1/2}/\log x \sim 2\,\mathrm{Li}(x^{1/2}) > \mathrm{Li}(x^{1/3})$ and would therefore be expected to be as significant for large x as the term $-\frac{1}{2}\mathrm{Li}(x^{1/2})$ and more significant than any of the following terms of (4). On these subjects Riemann restricts himself to the statement that it would be interesting in later counts of primes to study the effect of the particular "periodic" terms on their distribution.

In short, although formulas (2) and (3) combine to give an analytical formula for $\pi(x)$, the validity of the new approximation (4) to $\pi(x)$ to which it leads is based, like that of the old approximation $\pi(x) \sim \mathrm{Li}(x)$, solely on empirical evidence.

1.18 THE DENSITY *dJ*

A simple formulation of the main result

$$J(x) = \mathrm{Li}(x) - \sum_\rho \mathrm{Li}(x^\rho) - \log 2 + \int_x^\infty \frac{dt}{t(t^2-1)\log t} \tag{1}$$

can be obtained by differentiating to find

(2)

$$dJ = \left[\frac{1}{\log x} - \sum_{\mathrm{Re}\,\alpha>0} \frac{2\cos(\alpha\log x)}{x^{1/2}\log x} - \frac{1}{x(x^2-1)\log x}\right]dx \qquad (x>1),$$

where α ranges over all values such that $\rho = \frac{1}{2} + i\alpha$—in other words $\alpha = -i(\rho - \frac{1}{2})$, where ρ ranges over the roots—so that

$$x^{\rho-1} + x^{-\rho} = x^{-1/2}[x^{i\alpha} + x^{-i\alpha}] = 2x^{-1/2}\cos(\alpha\log x).$$

[The Riemann hypothesis is that the α's are all real. In writing formula (2) in this form Riemann is clearly thinking of the α's as being real since otherwise the natural form would be $x^{\rho-1} + x^{\bar\rho-1} = 2x^{\beta-1}\cos(\gamma\log x)$, where $\rho = \beta + i\gamma$.]

By the definition of J, the measure dJ is dx times the density of primes plus $\frac{1}{2}$ the density of prime squares, plus $\frac{1}{3}$ the density of prime cubes plus, etc. Thus $1/\log x$ should be considered to be an approximation not to the density of primes as Gauss suggested but rather to dJ, that is, to the density of primes *plus* $\frac{1}{2}$ the density of prime squares, plus, etc.

Given two large numbers $a < b$ the approximation obtained by taking a finite number of the α's

$$J(b) - J(a) \sim \int_a^b \frac{dt}{\log t} - 2\sum \int_a^b \frac{\cos(\alpha\log t)\,dt}{t^{1/2}\log t} \tag{3}$$

should be a fairly good approximation because the omitted term $\int dx/x(x^2-1)\log x$ is entirely negligible and because the integrals involving the large α's oscillate very rapidly for large x and therefore should make very small contributions. In fact, the basic formula (1) implies immediately that the error in (3) approaches the negligible omitted term as more and more of the α's are included in the sum.

It is in the sense of investigating the number of α's which are significant in (3) that Riemann meant to investigate empirically the influence of the "periodic" terms on the distribution of primes. So far as I know, no such investigation has ever been carried out.

1.19 QUESTIONS UNRESOLVED BY RIEMANN

Riemann himself, in a letter quoted in the notes which follow this paper in his collected works, singles out two statements of the paper as not having been fully proved as yet, namely, the statement that the equation $\xi(\frac{1}{2} + i\alpha) = 0$ has approximately $(T/2\pi)\log(T/2\pi)$ *real* roots α in the range $0 < \alpha < T$ and the statement that the integral of Section 1.15 can be evaluated termwise. He expresses no doubt about the truth of these statements, however, and says that they follow from a new representation of the function ξ which he has not yet simplified sufficiently to publish. Nonetheless, as was stated in Section 1.9, the first of these two statements—at least if it is understood to mean that the relative error in the approximation approaches zero as $T \to \infty$—has never been proved. The second was proved by von Mangoldt in 1895, but by a method completely different from that suggested by Riemann, namely by proving first that Riemann's formula for $J(x)$ is valid and by concluding from this that the termwise value of the integral in Section 1.15 must be correct.

Riemann evidently believed that he had given a proof of the product formula for $\xi(s)$, but, at least from the reading of the paper given above, one cannot consider his proof to be complete, and, in particular, one must question Riemann's estimate of the number of roots ρ in the range $\{0 \le \mathrm{Im}\,\rho \le T\}$ on which this proof is based. It was not until 1893 that Hadamard proved the product formula, and not until 1905 that von Mangoldt proved the estimate of the number of roots in $\{0 \le \mathrm{Im}\,\rho \le T\}$.

Next, the original question of the validity of the approximation $\pi(x) \sim \int_2^x (dt/\log t)$ remained entirely unresolved by Riemann's paper. It can be shown that the relative error of this approximation approaches zero as $x \to \infty$ if and only if the same is true of the relative error in Riemann's approximation $J(x) \sim \mathrm{Li}(x)$, so the original question is equivalent to the question of whether $\sum \mathrm{Li}(x^\rho)/\mathrm{Li}(x) \to 0$, but this unfortunately does not bring the problem any nearer to a solution. It was not until 1896 that Hadamard and, independently, de la Vallée Poussin proved the prime number theorem to the effect that the relative error in $\pi(x) \sim \int_2^x (dt/\log t)$ does approach zero as $x \to \infty$.

Finally, the paper raised a question much greater than any question it answered, the question of the truth or falsity of the Riemann hypothesis.

The remainder of this book is devoted to the subsequent history of these six questions. In summary, they are as follows:

(a) Is Riemann's estimate of the number of roots ρ on the line segment from $\frac{1}{2}$ to $\frac{1}{2} + iT$ correct as $T \to \infty$? (Unknown.)

(b) Is termwise evaluation of the integral of Section 1.15 valid? (Yes, von Mangoldt, 1895.)

(c) Is the product formula for $\xi(s)$ valid? (Yes, Hadamard, 1893.)

(d) Is Riemann's estimate of the number of roots ρ in the strip $\{0 \le \operatorname{Im} \rho \le T\}$ correct? (Yes, von Mangoldt, 1905.)

(e) Is the prime number theorem true? [Yes, Hadamard and de la Vallée Poussin (independently), 1896.]

(f) Is the Riemann hypothesis true? (Unknown.)

Chapter 2

The Product Formula for ξ

2.1 INTRODUCTION

In 1893 Hadamard published a paper [H1] in which he studied entire functions (functions of a complex variable which are defined and analytic at all points of the complex plane) and their representations as infinite products. One consequence of the general theory which he developed in this paper is the fact that the product formula

$$\xi(s) = \xi(0) \prod_\rho \left(1 - \frac{s}{\rho}\right) \tag{1}$$

is valid; here ξ is the entire function defined in Section 1.8, ρ ranges over all roots ρ of $\xi(\rho) = 0$, and the infinite product is understood to be taken in an order which pairs each root ρ with the corresponding root $1 - \rho$. Hadamard's proof of the product formula for ξ was called by von Mangoldt [M1] "the first real progress in the field in 34 years," that is, the first since Riemann's paper.

This chapter is devoted to the proof of the product formula (1). Since only the specific function ξ is of interest here, Hadamard's methods for general entire functions can, of course, be considerably specialized and simplified† for this case, and in the end the proof which results is closer to the one outlined by Riemann than to Hadamard's proof. The first step of the proof is to make an estimate of the distribution of the roots ρ. This estimate, which is that the number of roots ρ in the disk $|\rho - \frac{1}{2}| < R$ is less than a constant times $R \log R$ as $R \to \infty$, is based on Jensen's theorem and is much less exact than Riemann's estimate that the number of roots in the strip $\{0 < \operatorname{Im}$

†A major simplification is the use of Jensen's theorem, which was not known at the time Hadamard was writing.

$\rho < T\}$ is $(T/2\pi)\log(T/2\pi) - (T/2\pi)$ with a relative error which is of the order of magnitude of T^{-1}. It is exact enough, however, to prove the convergence of the product (1). Once it has been shown that this product converges, the rest of the proof can be carried out more or less as Riemann suggests.

2.2 JENSEN'S THEOREM

Theorem Let $f(z)$ be a function which is defined and analytic throughout a disk $\{|z| \le R\}$. Suppose that $f(z)$ has no zeros on the bounding circle $|z| = R$ and that inside the disk it has the zeros $z_1, z_2, \ldots, z_n$ (where a zero of order k is included k times in the list). Suppose, finally, that $f(0) \ne 0$. Then

$$\log\left| f(0)\cdot\frac{R}{z_1}\cdot\frac{R}{z_2}\cdot\cdots\cdot\frac{R}{z_n}\right| = \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|\,d\theta. \tag{1}$$

***Proof*†** If $f(z)$ has no zeros inside the disk, then the equation is merely

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|\,d\theta; \tag{2}$$

that is, the equation is the statement that the value of $\log|f(z)|$ at the center of the disk is equal to its average value on the bounding circle. This can be proved either by observing that $\log|f(z)|$ is the real part of the analytic function $\log f(z)$ and is therefore a harmonic function, or by taking the real part of the Cauchy integral formula

$$\log f(0) = \frac{1}{2\pi i}\int_{|z|=R}\frac{\log f(z)}{z}\,dz = \frac{1}{2\pi}\int_0^{2\pi}\log f(Re^{i\theta})\,d\theta,$$

where $\log f(z)$ is defined in the disk to be

$$\log|f(0)| + \int_0^z \frac{f'(t)}{f(t)}\,dt.$$

Applying this formula (2) to the function

$$F(z) = f(z)\frac{R^2 - \bar z_1 z}{R(z - z_1)}\cdot\frac{R^2 - \bar z_2 z}{R(z - z_2)}\cdots\frac{R^2 - \bar z_n z}{R(z - z_n)}$$

gives

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log|F(Re^{i\theta})|\,d\theta$$

because $F(z)$ is analytic and has no zeros in the disk. But this is the formula

†See Ahlfors [A3]. This method of proof of Jensen's threorem is to be found in Backlund's 1918 paper on the Lindelöf hypothesis [B3] (see Section 9.4).

of Jensen's theorem (1) because

$$\left|\frac{R^2 - \bar z_j\cdot 0}{R(0 - z_j)}\right| = \left|\frac{R}{z_j}\right|$$

and because by a basic formula in the theory of conformal mapping

$$\left|\frac{R^2 - \bar z_j z}{R(z - z_j)}\right| = 1 \qquad \text{when} \quad |z| = R.$$

(To prove this formula multiply the numerator by $\bar z/R$. This does not change the modulus if $|z| = R$ and it makes the numerator into the complex conjugate of the denominator.) This completes the proof of Jensen's theorem (1).

2.3 A SIMPLE ESTIMATE OF $|\xi(s)|$

Theorem For all sufficiently large values of R the estimate $|\xi(s)| \le R^R$ holds throughout the disk $|s - \frac12| \le R$.

Proof It was shown in Section 1.8 that $\xi(s)$ can be expanded as a power series in $(s - \frac12)$:

$$\xi(s) = a_0 + a_2(s - \tfrac12)^2 + \cdots + a_{2n}(s - \tfrac12)^{2n} + \cdots,$$

where

$$a_{2n} = 4\int_1^\infty \frac{d}{dx}[x^{3/2}\psi'(x)]\,x^{-1/4}\frac{(\frac12\log x)^{2n}}{(2n)!}\,dx.$$

The fact that the coefficients a_n are *positive* follows immediately from

$$\begin{aligned}\frac{d}{dx}[x^{3/2}\psi'(x)] &= \frac{d}{dx}\left(-\sum_{n=1}^\infty x^{3/2}n^2\pi e^{-n^2\pi x}\right)\\ &= \sum_{n=1}^\infty \left(n^4\pi^2 x - \frac32 n^2\pi\right)x^{1/2}e^{-n^2\pi x}\end{aligned}$$

because this shows that the integrand in the integral for a_{2n} is positive for $x \ge 1$. Thus the largest value of $\xi(s)$ on the disk $|s - \frac12| \le R$ occurs at the point $s = \frac12 + R$, and to prove the theorem it suffices to show that $\xi(\frac12 + R) \le R^R$ for all sufficiently large R. Now

$$\xi(s) = \Pi(s/2)\pi^{-s/2}(s-1)\zeta(s)$$

and $\zeta(s)$ decreases to 1 as $s \to +\infty$, so if R is given and if N is chosen so that $\frac12 + R \le 2N < \frac12 + R + 2$, it follows that

$$\begin{aligned}\xi(\tfrac12 + R) \le \xi(2N) &= (N!)\pi^{-N}(2N-1)\zeta(2N)\\ &\le N^N\pi^{-0}(2N)\zeta(2)\\ &= \text{const}\ N^{N+1}\\ &\le \text{const}\ (\tfrac12 R + 2)^{(R/2)+3} < R^R\end{aligned}$$

for all sufficiently large R, which completes the proof of the theorem.

2.4 THE RESULTING ESTIMATE OF THE ROOTS ρ

Theorem Let $n(R)$ denote the number of roots ρ of $\xi(\rho) = 0$ which lie inside or on the circle $|s - \frac{1}{2}| = R$ (counted with multiplicities). Then $n(R) \leq 2R \log R$ for all sufficiently large R.

Proof Jensen's theorem applied to $\xi(s)$ on the disk $|s - \frac{1}{2}| \leq 2R$ gives

$$\log \xi\left(\frac{1}{2}\right) + \sum_{|\rho - 1/2| < 2R} \log \frac{2R}{|\rho - \frac{1}{2}|} \leq \log[(2R)^{2R}].$$

The terms of the sum over ρ are all positive and the terms corresponding to roots ρ inside the circle $|\rho - \frac{1}{2}| \leq R$ are all at least log 2; hence,

$$n(R) \log 2 \leq 2R \log 2R - \log \xi(\tfrac{1}{2})$$
$$n(R) \leq \frac{2}{\log 2} R \log R + 2R - \frac{\log \xi(\frac{1}{2})}{\log 2}$$
$$\leq 2R \log R$$

for all sufficiently large R, as was to be shown. If there are roots ρ on the circle $|s - \frac{1}{2}| = 2R$, so that Jensen's theorem is not applicable, one can apply the above to the circle with radius $R + \epsilon$ and let $\epsilon \to 0$.

2.5 CONVERGENCE OF THE PRODUCT

As was noted in Section 1.10, in order to prove the convergence of the product

$$\prod \left(1 - \frac{s}{\rho}\right) = \prod_{\operatorname{Im} \rho > 0} \left[1 - \frac{s(1-s)}{\rho(1-\rho)}\right] \tag{1}$$

for all s, it suffices to prove the convergence of the sum $\sum |\rho(1 - \rho)|^{-1}$. Since all but a finite number of roots ρ satisfy the inequality

$$\frac{1}{|\rho(1-\rho)|} = \frac{1}{|(\rho - \frac{1}{2})^2 - \frac{1}{4}|} < \frac{1}{|\rho - \frac{1}{2}|^2},$$

it suffices therefore to prove the convergence of the sum $\sum |\rho - \frac{1}{2}|^{-2}$; here the sum can be considered either as a sum over roots ρ in the upper halfplane $\operatorname{Im} \rho > 0$ or as a sum over all roots since the first of these is merely twice the second. The convergence of the product (1) is therefore a consequence of the case $\epsilon = 1$ of the following theorem.

Theorem For any given $\epsilon > 0$ the series

$$\sum \frac{1}{|\rho - \frac{1}{2}|^{1+\epsilon}}$$

converges, where ρ ranges over all roots ρ of $\xi(\rho) = 0$.

[Note that this theorem would follow immediately from Riemann's observation that the vertical density of the roots ρ is a constant times $(\log T)\, dT$ and from the fact that $\int^\infty T^{-1-\epsilon}(\log T)\, dT$ converges. This is Riemann's first step in his "proof" of the product formula for ξ.]

Proof Let the roots ρ be numbered $\rho_1, \rho_2, \rho_3, \ldots$ in order of increasing $|\rho - \frac{1}{2}|$. Furthermore, let $R_1, R_2, R_3, \ldots$ be the sequence of positive real numbers defined implicitly by the equation $3R_n \log R_n = n$. Then by the theorem of the preceding section there are at most $2n/3$ roots ρ inside the circle $|s - \frac{1}{2}| = R_n$; hence the nth root is not in this circle, that is, $|\rho_n - \frac{1}{2}| > R_n$. Thus

$$\sum \frac{1}{|\rho_n - \frac{1}{2}|^{1+\epsilon}} \leq \sum \frac{1}{R_n^{1+\epsilon}} = \sum \frac{(3 \log R_n)^{1+\epsilon}}{n^{1+\epsilon}}$$
$$= \sum \frac{1}{n^{1+(\epsilon/2)}} \cdot \frac{(3 \log R_n)^{1+\epsilon}}{n^{\epsilon/2}}.$$

Now $\log n = \log R_n + \log 3 + \log \log R_n > 3 \log R_n$ for large n; hence $(3 \log R_n)^{1+\epsilon} < (\log n)^2$ which is much less than $(n^{\epsilon/4})^2$ for large n so

$$\sum \frac{1}{|\rho - \frac{1}{2}|^{1+\epsilon}} < \text{const} + \sum \frac{1}{n^{1+(\epsilon/2)}} < \infty$$

as was to be shown.

2.6 RATE OF GROWTH OF THE QUOTIENT

Riemann states that $\log \xi(s) - \sum \log [1 - (s/\rho)]$ grows no faster than $s \log s$, from which he concludes, since it is an even function, that it must be a constant. In this section the weaker result that the growth of its real part is no faster than $|s|^{1+\epsilon}$ will be proved. This still permits one to conclude, as will be shown in the next section, that it is constant.

Theorem Let $\epsilon > 0$ be given. Then

$$\operatorname{Re} \log \frac{\xi(s)}{\prod_\rho \left(1 - \frac{s - \frac{1}{2}}{\rho - \frac{1}{2}}\right)} \leq \left|s - \frac{1}{2}\right|^{1+\epsilon}$$

for all sufficiently large $|s - \frac{1}{2}|$.

Proof Let R be given and let the function being estimated be written as a sum of two functions

$$\operatorname{Re} \log \frac{\xi(s)}{\prod \left(1 - \frac{s - \frac{1}{2}}{\rho - \frac{1}{2}}\right)} = u_R(s) + v_R(s),$$

where

$$u_R(s) = \operatorname{Re}\log \frac{\xi(s)}{\prod_{|\rho-1/2|\le 2R}\left(1-\frac{s-\frac12}{\rho-\frac12}\right)}$$

$$v_R(s) = \operatorname{Re}\log \frac{1}{\prod_{|\rho-1/2|>2R}\left(1-\frac{s-\frac12}{\rho-\frac12}\right)}.$$

These logarithms are defined only up to multiples of $2\pi i$, but their real parts are well defined except at the points $s=\rho$ for $|\rho-\frac12|>2R$ (at which points u_R is $-\infty$ and v_R is $+\infty$). It will suffice to show that for large R both $u_R(s)$ and $v_R(s)$ are at most $R^{1+\epsilon}$ on $|s-\frac12|=R$ since then, when ϵ is decreased slightly to ϵ', it follows that $u_R(s)+v_R(s)\le 2R^{1+\epsilon'}\le|s-\frac12|^{1+\epsilon}$ on $|s-\frac12|=R$ for all R large enough that $u_R\le R^{1+\epsilon'}$, $v_R\le R^{1+\epsilon'}$, and $2\le R^{\epsilon-\epsilon'}$.

First consider $u_R(s)$. On the circle $|s-\frac12|=4R$ the factors in the denominator are all at least 1; therefore

$$u_R(s)\le \operatorname{Re}\log\xi(s)=\log|\xi(s)|$$
$$\le \log[(4R)^{4R}]=4R\log 4R\le R^{1+\epsilon}$$

on the circle $|s-\frac12|=4R$, for large R (large enough that $4\log 4R<R^\epsilon$). Now u_R is a harmonic function on the disk $|s-\frac12|\le 4R$ except at the points $s=\rho$ in the range $2R<|s-\frac12|\le 4R$. But near these singular points $s=\rho$ the value of u_R is near $-\infty$, so the maximum value of the harmonic function u_R on the disk $|s-\frac12|\le 4R$ must occur on the outer boundary $|s-\frac12|=4R$. Thus the maximum of u_R on the disk, and in particular on the circle $|s-\frac12|=R$, is at most $R^{1+\epsilon}$ as was to be shown.

Now consider $v_R(s)$. For complex x in the disk $|x|\le\frac12$ the inequality

$$\operatorname{Re}\log\frac{1}{1-x}=-\operatorname{Re}\log(1-x)=\operatorname{Re}\int_0^x\frac{dt}{1-t}$$
$$\le\left|\int_0^x\frac{dt}{1-t}\right|\le|x|\max\frac{1}{|1-t|}=2|x|$$

holds. Thus for $|s-\frac12|=R$ the inequality

$$v_R(s)=\operatorname{Re}\log\frac{1}{\prod_{|\rho-1/2|>2R}\left(1-\frac{(s-\frac12)^2}{(\rho-\frac12)^2}\right)}$$
$$\le 2\sum_{|\rho-1/2|>2R}\frac{R^2}{|\rho-\frac12|^2}$$
$$=2\sum\left(\frac{R}{|\rho-\frac12|}\right)^{1-\epsilon}\left(\frac{R}{|\rho-\frac12|}\right)^{1+\epsilon}$$
$$\le 2\sum\left(\frac12\right)^{1-\epsilon}\frac{R^{1+\epsilon}}{|\rho-\frac12|^{1+\epsilon}}$$
$$=2^\epsilon R^{1+\epsilon}\sum_{|\rho-1/2|>2R}\frac{1}{|\rho-\frac12|^{1+\epsilon}}$$

holds. Now the sum in this expression converges by the theorem of Section 2.5, and it decreases to zero as R increases. Thus $v_R(s)\le R^{1+\epsilon}$ on $|s-\frac12|=R$ for all sufficiently large R as was to be shown. This completes the proof.

2.7 RATE OF GROWTH OF EVEN ENTIRE FUNCTIONS

Theorem Let $f(s)$ be an analytic function, defined in the entire s-plane, which is even in the sense that $f(-s)\equiv f(s)$ and which grows more slowly than $|s|^2$ in the sense that for every $\epsilon>0$ there is an R such that $\operatorname{Re}f(s)<\epsilon|s|^2$ at all points s satisfying $|s|\ge R$. Then f must be constant.

Proof The subtle point of the theorem is that only the *upward* growth of the *real part* of f is limited. The main step in the proof is the following lemma, which shows that this implies that the growth of the *modulus* of f is also limited.

Lemma Let $f(s)$ be an analytic function on the disk $\{|s|\le r\}$, let $f(0)=0$, and let M be the maximum value of $\operatorname{Re}f(s)$ on the bounding circle $|s|=r$ (and hence on the entire disk). Then for $r_1<r$ the modulus of f on the smaller disk $\{|s|\le r_1\}$ is bounded by

$$|f(s)|\le 2r_1M/(r-r_1)\qquad(|s|\le r_1).$$

Proof of the Lemma Consider the function

$$\phi(s)=f(s)/s[2M-f(s)].$$

If $u(s)$ and $v(s)$ denote the real and imaginary parts, respectively, of f, then $|2M-u(s)|\ge M\ge u(s)$ on the circle $|s|=r$; so the modulus of ϕ on this circle is at most

$$|\phi(s)|=\frac{(u^2+v^2)^{1/2}}{r[(2M-u)^2+v^2]^{1/2}}\le\frac{(u^2+v^2)^{1/2}}{r(u^2+v^2)^{1/2}}=\frac1r$$

which implies that $|\phi(s)|\le r^{-1}$ throughout the disk $\{|s|\le r\}$. But $f(s)$ can be expressed in terms of $\phi(s)$ as

$$\phi(s)s[2M-f(s)]=f(s),\qquad f(s)=\frac{2Ms\phi(s)}{1+s\phi(s)}$$

which shows that for $|s|=r_1$ the modulus of $f(s)$ is at most

$$|f(s)|\le 2Mr_1r^{-1}/(1-r_1r^{-1})=2Mr_1/(r-r_1).$$

Hence the same inequality holds throughout the disk $\{|s|\le r_1\}$ as was to be shown.

Now to prove the theorem let $f(s)=\sum_{n=0}^\infty a_ns^n$ be the power series expansion of a function $f(s)$ satisfying the conditions of the theorem. Note first that it can be assumed without loss of generality that $a_0 = 0$ because $f(s)$ satisfies the growth condition of the theorem if and only if $f(s) - f(0)$ does and because $f(s)$ is constant if and only if $f(s) - f(0)$ is. Now Cauchy's integral formula for the coefficients is

$$a_n = \frac{1}{2\pi i}\int_{\partial D} \frac{f(s)\,ds}{s^{n+1}},$$

where D is any domain containing the origin. Let ϵ, R be as in the statement of the theorem and let D be the disk $\{|s| \leq \frac{1}{2}R\}$. Then the above formula gives

$$\begin{aligned} |a_n| &= \left|\frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\frac{1}{2}Re^{i\theta})}{(\frac{1}{2}Re^{i\theta})^n}\, i\,d\theta\right| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \frac{2^n\,|f(\frac{1}{2}Re^{i\theta})|}{R^n}\,d\theta. \end{aligned}$$

The right side is the average value of a function whose value is by the lemma at most

$$\frac{2^n}{R^n}\frac{2(\epsilon R^2)(\frac{1}{2}R)}{R - (\frac{1}{2}R)} = \frac{2^{n+1}\epsilon}{R^{n-2}}.$$

If $n \geq 2$, this is at most $2^{n+1}\epsilon$, and since ϵ is arbitrary, a_n must be zero for $n \geq 2$. Thus $f(s) = a_1 s$. However a_1 must be zero by the evenness condition $f(s) \equiv f(-s)$. Therefore $f(s) \equiv 0$ which is constant, as was to be shown.

2.8 THE PRODUCT FORMULA FOR ξ

The function $F(s) = \xi(s)/\prod_\rho [1 - (s - \frac{1}{2})/(\rho - \frac{1}{2})]$ is analytic in the entire s-plane and is an even function of $s - \frac{1}{2}$. Moreover, it has no zeros, so its logarithm is well defined up to an additive constant $2\pi n i$ (n an integer) by the formula $\log F(s) = \int_0^s F'(z)\,dz/F(z) + \log F(0)$, where $\log F(0)$ is determined to within an additive constant $2\pi n i$. The results of the preceding two sections then combine to give $\log F(s) =$ const, and therefore upon exponentiation

$$\xi(s) = c\prod\left(1 - \frac{s - \frac{1}{2}}{\rho - \frac{1}{2}}\right),$$

where c is a constant. Dividing this by the particular value

$$\xi(0) = c\prod\left(1 - \frac{-\frac{1}{2}}{\rho - \frac{1}{2}}\right)$$

gives

$$\frac{\xi(s)}{\xi(0)} = \prod\left(1 - \frac{s - \frac{1}{2}}{\rho - \frac{1}{2}}\right)\left(1 - \frac{-\frac{1}{2}}{\rho - \frac{1}{2}}\right)^{-1}.$$

The factors on the right are linear functions of s which are 0 when $s = \rho$ and 1 when $s = 0$; hence they are $1 - (s/\rho)$ and the formula is the desired formula

$$\xi(s) = \xi(0)\prod_\rho\left(1 - \frac{s}{\rho}\right),$$

where, as always, it is understood that the factors ρ and $1 - \rho$ are paired.†

†The same argument proves the validity of the product formula for the sine

$$\sin \pi s = \pi s \prod_{n=1}^{\infty}\left(1 - \frac{s^2}{n^2}\right)$$

mentioned in Section 1.3. The only other unproved statement in Section 1.3 which is not elementary is the equivalence of the two definitions (2) and (3) of $\Pi(s)$.