ON THE RIEMANN ZETA FUNCTION

ROBERT SPIRA

In this paper we discuss a conjectured inequality which implies both the Riemann hypothesis and that the left half of the critical strip is free of zeros of $\zeta'(s)$. Moreover, it is proved that on the critical line $\zeta'(s)$ cannot vanish except possibly at the zeros of $\zeta(s)$.

Let $s = \sigma + it$, $d/ds = D$, $\partial/\partial\sigma = D_\sigma$. The possibility that $|\zeta(s)|$ increases as one moves to the left from the critical line was noticed by the author, and also independently (and earlier) by D. H. Lehmer. This tendency can be seen in the Jahnke and Emde figure, and also in other calculations as far as $t = 100$ (Spira [**1**]). The empirical fact that zeros of $\zeta'(s)$ lie to the right of $\sigma = \frac{1}{2}$ (Spira [**2**]), also dimly suggests such an increase, but the values of $\zeta(s)$ off the critical line previously calculated were below the first violations of Gram's Law. At such places, the first of which is near $t = 282$, one might expect various conjectures to break down, so it was decided to study more intensively the behaviour of $\zeta(s)$ near the Gram disturbances.

A series of computations was undertaken, first of $|\zeta(s)|$, and then of $D_\sigma|\zeta(s)|$ and $D_\sigma \log|\zeta(s)|$, using the formulas:

$$\begin{aligned} D_\sigma \log|\zeta(s)| &= \operatorname{Re} D \log \zeta(s) = \operatorname{Re}\left(\zeta'(s)/\zeta(s)\right) \\ &= (\operatorname{Re}\zeta \operatorname{Re}\zeta' + \operatorname{Im}\zeta \operatorname{Im}\zeta')/|\zeta(s)|^2 \\ &= \left(D_\sigma|\zeta(s)|\right)/|\zeta(s)|. \end{aligned}$$

With the aid of routines for ζ and ζ' (Spira [**3**]), the partial derivatives are easily computed. (The log on the left above is the real log function, and that on the right is a log defined for $\sigma > 1$, and extended with cut slits, around zeros of $\zeta(s)$, into the critical strip.)

These calculations revealed that both $D_\sigma|\zeta(s)|$ and $D_\sigma \log|\zeta(s)|$ were less than 0 for $0 < \sigma < \frac{1}{2}$. While there is little scientific pattern for $D_\sigma|\zeta(s)|$, the values for $D_\sigma \log|\zeta(s)|$ showed remarkable regularity. We write $D_\sigma \log|\zeta(s)|\big|_{\frac{1}{2}+it}$ as

$$D_\sigma \log|\zeta(\tfrac{1}{2}+it)|.$$

On the line $\sigma = \frac{1}{2}$, the value of $D_\sigma \log|\zeta(s)|$ was slightly greater than $-\frac{1}{2}\log|t/2\pi|$, and indeed the author was able to show that $|D_\sigma \log|\zeta(\frac{1}{2}+it)| + \frac{1}{2}\log|t/2\pi|| \leqq \cdot 27/t$, for $t \geqq 10$. Lowell Schoenfeld improved this result to a smaller constant times $1/t^2$, and also showed that for $|t| \geqq 4$, $D_\sigma \log|\zeta(\frac{1}{2}+it)| > -\frac{1}{2}\log|t/2\pi|$, bringing the theory into agreement with the observed phenomena. The significance of this last result is that, in a possible application to the Riemann hypothesis, we know that the $O(1/|t|)$ term cannot help us, no matter how large t is. Professor Schoenfeld's theorem is included here with his kind permission.

Received 16 May, 1966.

[J. LONDON MATH. SOC., 44 (1969), 325-328]

THEOREM. *If* $|t| \geqq 3.8$ *is such that* $\zeta(\tfrac{1}{2}+it) \neq 0$, *then*

$$\left|\operatorname{Re}\frac{\zeta'(\frac{1}{2}+it)}{\zeta(\frac{1}{2}+it)} + \tfrac{1}{2}\log\frac{|t|}{2\pi} - \frac{1}{48t^2}\right| < \frac{1}{32t^4}.$$

Proof. The familiar functional equation is $\zeta(1-s) = g(s)\zeta(s)$, where

$$g(s) = 2(2\pi)^{-s}\,\Gamma(s)\cos \pi s/2 \quad \big(= \chi(1-s) = 1/\chi(s)\big).$$

Hence, if $s_0 = \tfrac{1}{2}+it$ and $\zeta(s_0) \neq 0$, we get

$$\frac{g'(s_0)}{g(s_0)} = -\frac{\zeta'(1-s_0)}{\zeta(1-s_0)} - \frac{\zeta'(s_0)}{\zeta(s_0)} = -2\operatorname{Re}\frac{\zeta'(s_0)}{\zeta(s_0)},$$

inasmuch as ζ'/ζ takes conjugate values at conjugate points. So

$$-2\operatorname{Re}\frac{\zeta'(s_0)}{\zeta(s_0)} = -\log 2\pi + \operatorname{Re}\frac{\Gamma'(s_0)}{\Gamma(s_0)} - \frac{\pi}{2}\operatorname{Re}\tan\frac{\pi}{2}s_0.$$

On using the line preceding (3) in Dixon–Schoenfeld [4], we find

$$-2\operatorname{Re}\frac{\zeta'(s_0)}{\zeta(s_0)} = -\log 2\pi + \operatorname{Re}\left\{\log s_0 - \frac{1}{2s_0} - \frac{1}{12{s_0}^2} + 6J(s_0)\right\} - \frac{\pi}{e^{\pi t}+e^{-\pi t}},$$

where, on integrating by parts,

$$J(s) = \int_0^\infty \frac{P_3(x)}{(s+x)^4}\,dx = \left[\frac{Q(x)}{(s+x)^4}\right]_0^\infty + 4\int_0^\infty \frac{Q(x)}{(s+x)^5}\,dx;$$

here $P_3(x)$ and $Q(x)$ have period 1, and on $[0, 1]$ they are given by

$$P_3(x) = x(2x^2-3x+1)/12, \quad Q(x) = x^2(1-x)^2/24.$$

Hence $|Q(x)| \leqslant 1/384$ for all x, so that if $\sigma \geqslant 0$ we have

$$|J(s)| \leqq \frac{4}{384}\int_0^\infty \frac{dx}{\{(x+\sigma)^2+t^2\}^{5/2}} \leqq \frac{1}{96}\int_0^\infty \frac{dy}{(y^2+t^2)^{5/2}}$$

$$\leqq \frac{1}{96t^4}\int_0^{\pi/2} \frac{\sec^2\xi}{\sec^5\xi}\,d\xi = \frac{1}{144t^4},$$

where we have set $y = |t|\tan\xi$. Now

$$\left|\operatorname{Re} 6J(s_0) - \frac{\pi}{e^{\pi t}+e^{-\pi t}}\right| < 6\cdot\frac{1}{144t^4} + \frac{\pi}{e^{\pi|t|}},$$

so for some θ in $(-1, 1)$

$$2\operatorname{Re}\frac{\zeta'(s_0)}{\zeta(s_0)} = \log 2\pi - \tfrac{1}{2}\log(t^2+\tfrac{1}{4}) + \left[\frac{1}{4t^2+1} - \frac{4t^2-1}{3(4t^2+1)^2}\right] + \theta\left\{\frac{1}{24t^4} + \frac{\pi}{e^{\pi|t|}}\right\}.$$

Now, the quantity in the brackets above equals

$$1/(6t^2) - 1/\{6t^2(4t^2+1)^2\} = 1/(6t^2) - \xi_2/(96t^6)$$

for some ξ_2 in (0, 1); also, by the series representation of the logarithm, if $|t| \geqslant 1$,

$$\log(t^2+\tfrac{1}{4}) = \log(t^2)+\log\left(1+\frac{1}{4t^2}\right)$$

$$= \log(t^2)+\frac{1}{4t^2}-\xi_1\cdot\tfrac{1}{2}\left(\frac{1}{4t^2}\right)^2,$$

for some ξ_1 in (0, 1).

Thus

$$2\,\mathrm{Re}\,\frac{\zeta'(s_0)}{\zeta(s_0)} = \log 2\pi-\log|t|-\tfrac{1}{2}\left\{\frac{1}{4t^2}-\xi_1\cdot\tfrac{1}{2}\left(\frac{1}{4t^2}\right)^2\right\}+\frac{1}{6t^2}-\frac{\xi_2}{96t^6}+\theta\left\{\frac{1}{24t^4}+\frac{\pi}{e^{\pi|t|}}\right\}$$

$$= -\log\frac{|t|}{2\pi}+\frac{1}{24t^2}+\left\{\frac{\xi_1}{64t^4}-\frac{\xi_2}{96t^6}\right\}+\theta\left\{\frac{1}{24t^4}+\frac{\pi}{e^{\pi|t|}}\right\}.$$

Hence, if $|t| \geqslant 3.8$,

$$\left|\mathrm{Re}\,\frac{\zeta'(s_0)}{\zeta(s_0)}+\tfrac{1}{2}\log\frac{|t|}{2\pi}-\frac{1}{48t^2}\right| \leqslant \frac{1}{128t^4}+\frac{1}{48t^4}+\frac{\pi}{2e^{\pi|t|}}$$

$$\leqslant \frac{3}{384t^4}+\frac{8}{384t^4}+\frac{1}{384t^4}=\frac{1}{32t^4}.$$

The estimate for $\pi/(2e^{\pi|t|})$ is easily shown by taking logarithms.

COROLLARY 1. *If* $|t| \geqslant 4$ *is such that* $\zeta(\tfrac{1}{2}+it) \neq 0$, *then*

$$\mathrm{Re}\,\frac{\zeta'(\tfrac{1}{2}+it)}{\zeta(\tfrac{1}{2}+it)} > -\tfrac{1}{2}\log\frac{|t|}{2\pi}+\frac{1}{53t^2}.$$

Proof. For $|t| \geqslant 4$ we have that

$$\frac{1}{48t^2}-\frac{1}{32t^4}=\frac{1}{48t^2}\left(1-\frac{3}{2t^2}\right) \geqslant \frac{1}{48t^2}\left(1-\frac{3}{32}\right) > \frac{1}{53t^2}.$$

COROLLARY 2. *If* $|t| \geqslant 4$ *and* $\zeta(\tfrac{1}{2}+it) \neq 0$, *then*

$$\left|\mathrm{Re}\,\frac{\zeta'(\tfrac{1}{2}+it)}{\zeta(\tfrac{1}{2}+it)}+\tfrac{1}{2}\log\frac{|t|}{2\pi}\right| < \frac{1}{43t^2}.$$

Proof. For $|t| \geqslant 4$, $\dfrac{1}{48t^2}+\dfrac{1}{32t^4}=\dfrac{1}{48t^2}\left(1+\dfrac{3}{2t^2}\right) \leqslant \dfrac{35}{48}\cdot\dfrac{1}{32t^2} \leqslant \dfrac{1}{43t^2}.$

COROLLARY 3. *If* $\zeta(\tfrac{1}{2}+it) \neq 0$, *then* $\zeta'(\tfrac{1}{2}+it) \neq 0$.

Proof. If $\zeta(\tfrac{1}{2}+it) \neq 0$ for some t such that $|t| \geqslant 6{\cdot}3$, then Corollary 2 gives

$$0 = \mathrm{Re}\,\frac{\zeta'(\tfrac{1}{2}+it)}{\zeta(\tfrac{1}{2}+it)} < \frac{1}{43t^2}-\tfrac{1}{2}\log\frac{|t|}{2\pi} < {\cdot}001-\tfrac{1}{2}\log\frac{6{\cdot}3}{2\pi} < 0,$$

which is a contradiction. If $|t| < 6{\cdot}3$, we have previously shown that $\zeta'(s) \neq 0$ by calculation (Spira [**2**]).

Titchmarsh [**5**; p. 221] showed a weaker form of Corollary 2, with a term of $O(1/t)$ rather than $O(1/t^2)$. His theorem is also stated a little differently, but can be transformed to the present form using the Cauchy–Riemann equations.

The empirical behaviour of $D_\sigma \log|\zeta(s)|$ in the left half of the critical strip is as follows: On lines parallel to the real axis, there may be an increase or decrease as σ moves between $\frac{1}{2}$ and 0. On such a line, slightly above or slightly below a t-ordinate of a zero of $\zeta(s)$ on the critical line, there is a deep minimum near $\sigma = \frac{1}{2}$, and then, usually, a slow increase towards $\sigma = 0$, though the function can also peak and start to decrease before coming to $\sigma = 0$. At t-ordinates centrally between zeros, there appears to be a slight decrease as σ moves toward 0.

On lines parallel and to the left of $\sigma = \frac{1}{2}$, $D_\sigma \log|\zeta(s)|$ takes dips at t-ordinates of zeros and rises to maxima between these t-ordinates. The maxima appear to be always less than $-\frac{1}{2}\log|t/2\pi| + O(1/t)$, but the maxima neither steadily increase nor steadily decrease. Related maxima (similar t's), appear to decrease as σ moves left. These empirical facts are based on a calculation made for $\sigma = 0(\cdot 1)\cdot 4$ of the nearest maxima to $t = 100(100)1000$. With this evidence in mind, we arrive at the following:

Conjecture. For $0 \leqq \sigma < \frac{1}{2}$,

$$D_\sigma \log|\zeta(s)| \leqslant -\log\sqrt{\left(\frac{t}{2\pi}\right)} + O\left(\frac{1}{t}\right).$$

This conjecture implies both the Riemann hypothesis and that $\zeta'(s) \neq 0$ for $0 \leqslant \sigma < \frac{1}{2}$ and t sufficiently large. If we replace $0 \leqslant \sigma < \frac{1}{2}$ by $-\varepsilon \leqslant \sigma < \frac{1}{2}$ $(\varepsilon > 0)$, then, in conjunction with Spira [2], one can show (using the method of proof of Corollary 3) that for t sufficiently large $\zeta'(s) \neq 0$ for $\sigma < \frac{1}{2}$. This is only slightly weaker than a previous conjecture that $\zeta'(s) \neq 0$ for $\sigma \leqslant \frac{1}{2}$, $t > 0$.

Moreover, these things can also be shown from the weaker condition $D_\sigma \log|\zeta(s)| < 0$, so that, if true, the conjecture provides us with the needed leeway in proving inequalities using asymptotic formulas.

The calculations were done on an IBM 7040 at the University of Tennessee Computing Center, which was aided by grants NSF-G 13581 and NSF-GP 4046.

References

1. R. Spira, " Table of the Riemann zeta function ", *Math. Comp.*, 18 (1964), 519–521.
2. ———, " Zero-free regions of $\zeta^{(k)}(s)$ ", *J. London Math. Soc.*, 40 (1965), 677–682.
3. ———, " Check values, zeros and FORTRAN programs for the Riemann zeta function and its first three derivatives ", Technical Report No. 1, University Computing Center, University of Tennessee.
4. R. D. Dixon and L. Schoenfeld, " On the size of the Riemann zeta-function at places symmetric with respect to the point $\frac{1}{2}$ ", *Duke Math. J.*, 33 (1966), 291–292.
5. E. C. Titchmarsh, *Theory of the Riemann zeta function* (Oxford, 1951).

Michigan State University,
East Lansing,
Michigan 48823,
U.S.A.