A NOTE ON THE RIEMANN ξ-FUNCTION

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It has been noticed by Jensen, Hurwitz, and others that the Riemann integral representation of the ξ-function may be written in the form

$$\xi(\tfrac{1}{2}+\tfrac{1}{2}it)=\int_{-\infty}^{+\infty}\Phi(x)\cos xt\,dx,$$

where

$$\Phi(x)=4\sum_{n=1}^{\infty}(2n^4\pi^2e^{9x}-3n^2\pi e^{5x})\exp(-n^2\pi e^{4x})$$

and $\Phi(x)=\Phi(-x)$. Hence $\Phi'(0)=0$ and $\Phi(x)$ is everywhere *positive*†. The object of the present note is to point out that $\Phi'(x)<0$ for every $x>0$. Hence $\Phi(x)$ is a *monotonic* function of $|x|$. Thus

$$(-x)^m\Phi^{(m)}(x)\geqslant 0$$

for every x if $m=0$ and $m=1$ but not if m is arbitrary. In fact, it is easy to see that the symmetrically monotonic curve $y=\Phi(x)$ has, like the Gauss curve, points x such that $\Phi''(x)=0$ and $\Phi'''(x)\neq 0$; this implies that $\Phi''(x)$ takes both positive and negative values for positive values of x‡.

Differentiation of $\Phi(x)$ gives

$$\Phi'(x)=4\pi e^{5x}\sum_{n=1}^{\infty}f_n(x)\,n^2\exp(-n^2\pi e^{4x}),$$

where
$$f_n(x)=-15+30\pi n^2e^{4x}-8\pi^2n^4e^{8x}.$$

The quadratic polynomial $f_n(x)$ in e^{4x} is negative for sufficiently large values of x (>0) and vanishes at

$$x=\tfrac{1}{4}\log\frac{15\pm\sqrt{(105)}}{8\pi n^2}.$$

Since this x is positive only when $\pm$ is $+$ and $n=1$, every $f_n(x)$ is less than 0 if $x>x_0$, where x_0 denotes the small positive number

$$x_0=\tfrac{1}{4}\log\frac{15+\sqrt{(105)}}{8\pi}=0\cdot001\ldots.$$

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* Received 2 October, 1934; read 15 November, 1934.

† *Cf.* E. C. Titchmarsh, "The zeta-function of Riemann", *Cambridge Tracts*, No. 26 (1930), 43–45.

‡ Compare, for example, G. Pólya, *Danske Videns. Selskab. Math.-fys. Meddelelser*, 7 (1927), 14.

Hence $\Phi'(x_0) < 0$ and every term of the series $\Phi'(x)$ is negative if $x > x_0$. Thus it is sufficient to prove that $\Phi'(x) < 0$ holds also when $0 < x < x_0$, *although the highest term of the series $\Phi'(x)$ is then positive.*

Differentiation of the series $\Phi'(x)$ gives

$$\{e^{-5x}\Phi'(x)\}' = 16\pi^2 e^{4x} \sum_{n=1}^{\infty} g_n(x)\, n^4 \exp(-n^2 \pi e^{4x}),$$

where $$g_n(x) = 45 - 46\pi n^2 e^{4x} + 8\pi^2 n^4 e^{8x}.$$

Since $23 < 8\pi n^2$, the derivative

$$g_n'(x) = (-23 + 8\pi n^2 e^{4x})\, 8\pi n^2 e^{4x}$$

cannot vanish for positive values of x, so that every $g_n(x)$ is monotonic for $x > 0$. Also, $g_n(x)$ is positive for large positive values of x and vanishes at

$$x = \tfrac{1}{4} \log \frac{23 \pm 13}{8\pi n^2}.$$

This x is positive only when $\pm$ is $+$ and $n = 1$; the resulting number $\frac{1}{4}\log\{9/(2\pi)\}$ is greater than $x_0 = 0{\cdot}001 \ldots$. It follows that the increasing function $g_n(x)$ is, at every point of the interval $0 < x < x_0$, positive or negative according as $n > 1$ or $n = 1$. Hence, putting

$$A = g_1(x_0)\exp(-\pi e^{4x_0}), \quad B = 16 g_2(x_0)\, e^{-4\pi}, \quad C = \sum_{n=3}^{\infty} g_n(x_0)\, n^4 e^{-n^2\pi},$$

we have $$\{e^{-5x}\Phi'(x)\}' < 16\pi^2 e^{4x}(A + B + C)$$

when $0 < x < x_0$. Substituting $x_0 = 0{\cdot}001 \ldots$ in A and B, we find that $A + B < -\frac{1}{3}$, and that $C < 10^{-6}$ since

$$C < \sum_{n=3}^{\infty} (45n^4 + 8\pi^2 n^8 e^{8x_0})\, e^{-n^2\pi} < 100 \sum_{n=3}^{\infty} n^8 e^{-n^2\pi},$$

so that $A + B + C < 0$. Thus, for $0 < x < x_0$, we have

$$\{e^{-5x}\Phi'(x)\}' < 0,$$

which implies that

$$\Phi'(x) < e^{5x}\Phi'(0) = 0.$$

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