ELEMENTARY READING NOTES ON THE RIEMANN ξ-FUNCTION

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Abstract. The $\xi(s)$ integral function, an essential tool in the first proofs of the prime number theorem, is also the subject of the Riemann hypothesis and has been continuously studied since its discovery by Riemann in 1859. In this document, we collect and discuss specific results that have been published about it since the late 19th century, some of which have just now become readily available through the conversion to digital format of out-of-print books and rare journals in English, French, and German.

"Your manuscript is both good and original. But the part that is good is not original, and the part that is original is not good."

(attributed to Samuel Johnson, London, 1709–1784)

"This is an expository paper about the Hurwitz-Routh criterion on the Hurwitz (or stable) polynomials. It contains a detailed discussion of the works of Routh, Hurwitz, Schur, Pontryagin and others. There are occasionally obtained new proofs of classical results."

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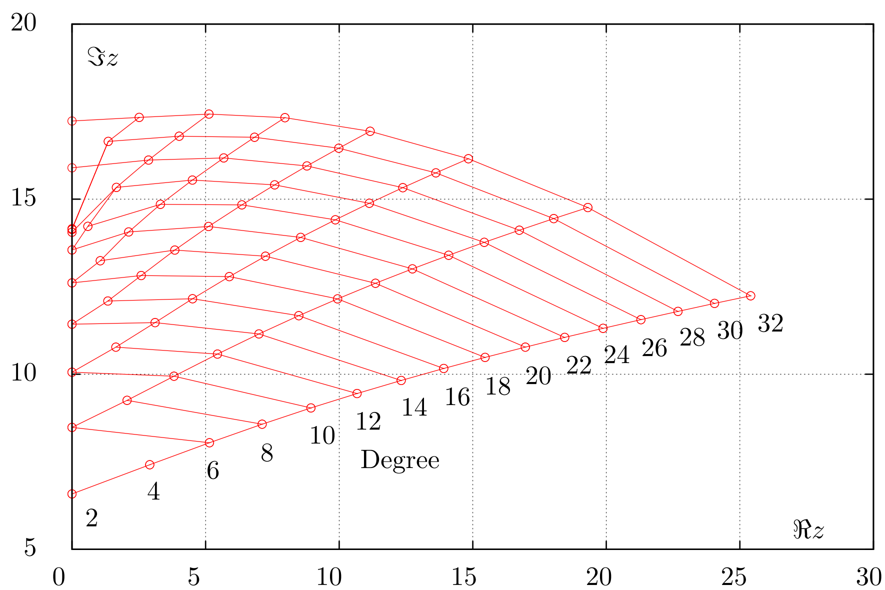
Zeros of Taylor polynomials of $\xi(1/2+z)$, $\Re z \geq 0$, $\Im z > 0$



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1. Definition and properties of the Riemann $\xi(s)$ function

In this section, we collect some important results directly related to the entire function $\xi(s)$ discovered by Riemann in his study of the distribution of prime numbers, including many formulas that have been used to define and represent it. We follow the chronological order of the publications.

1.1. **Original definitions.** In his 1859 memoir on the distribution of prime numbers [75], Riemann proves the functional equation of the complex analytic function satisfying

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad (\Re s > 1),$$

by using Cauchy's contour integration in the integral representation of the second Eulerian function

$$\Gamma(s) = \int_0^\infty e^{-t}\, t^{s-1}\, dt, \qquad (\Re s > 0), \tag{1.1.1}$$

leading to[1]

$$\Gamma(s)\,\zeta(s) = \int_0^\infty \frac{t^{s-1}}{e^t - 1}\, dt, \qquad (\Re s > 1),$$

a formula found by Plana [67] and by Abel ([1], [44, II, p. 222], [61, I, p. 22]) to represent with $s = 2n$ the Bernoulli numbers. Riemann then notes that this functional equation is equivalent to the invariance under the transformation $s \mapsto 1 - s$ of the "completed zeta function" [65, p. 422]

$$\xi_M(s) := \pi^{-s/2}\,\Gamma(s/2)\zeta(s), \qquad (s \neq 0, s \neq 1), \tag{1.1.2}$$

This property leads him to start instead from the integral for $\Gamma(s/2)$ and thus obtain integral representations of the *meromorphic* function

$$\xi_M(s) = \int_0^\infty \psi(x)\, x^{s/2-1}\, dx, \quad (\Re s > 1) \tag{1.1.3}$$

$$= \frac{1}{s(s-1)} + \int_1^\infty \psi(x)\, \left(x^{s/2} + x^{(1-s)/2}\right) \frac{dx}{x}, \quad (s \neq 0, s \neq 1). \tag{1.1.4}$$

Here,

$$\psi(x) := \sum_{n=1}^{\infty} e^{-n^2 \pi x}, \qquad (\Re x > 0), \tag{1.1.5}$$

can be defined via the third Jacobi theta function ([53], [86, ch. XXI]) by

$$\psi(x) = \frac{\theta_3(0, e^{-\pi x}) - 1}{2}, \tag{1.1.6}$$

$$\theta_3(z, q) := \sum_{n=-\infty}^{\infty} q^{n^2}\, \cos(2nz), \qquad (|q| < 1). \tag{1.1.7}$$

Riemann then integrates twice by parts, and he finally proves two integral representations of a new entire function

$$\xi(s) := \frac{s(s-1)}{2}\, \xi_M(s) = \pi^{-s/2}\,\Gamma(1 + \frac{s}{2})(s-1)\zeta(s) \tag{1.1.8}$$

$$= \frac{1}{2} - \frac{s(1-s)}{2} \int_1^\infty \psi(x)\, \left(x^{s/2} + x^{(1-s)/2}\right) \frac{dx}{x} \tag{1.1.9}$$

$$= 4 \int_1^\infty x^{3/4} \left[x^{3/2}\psi'(x)\right]' \frac{x^{(s-1/2)/2} + x^{-(s-1/2)/2}}{2} \frac{dx}{x}. \tag{1.1.10}$$

This provides him with a second proof of the functional equation of his complex $\zeta(s)$ function from the invariance under $s \mapsto 1 - s$ of this auxiliary entire function $\xi(s)$. In fact, Riemann makes the

[1] The inversion of summation and integration is justified since the left-hand side is finite when s is replaced by $\Re s > 1$ to give the absolute value of the integrand: "if S_x and T_y denote summations or integrations with respect to the variables indicated, then $S_x T_y f(x, y) = T_y S_x f(x, y)$, provided that one side is finite when f is replaced by $|f|$" [45, p. 32]. See [38], [3, p. 451, ex. 11, Riemann integration], [4, p. 278, ex. 4, Lebesgue integration].

invariance yet more explicit by defining an even function of t, for which he uses the symbol ξ while Landau [57, p. 288] uses Ξ, as we do here:

$$\Xi(t) := \xi(\frac{1}{2}+it) \tag{1.1.11}$$

$$= \frac{1}{2} - (t^2+\frac{1}{4}) \int_1^\infty x^{1/4}\, \psi(x)\, \cos(\frac{1}{2}t\log x))\, \frac{dx}{x}. \tag{1.1.12}$$

$$= 4\int_1^\infty x^{3/4}\, \Big[x^{3/2}\psi'(x)\Big]'\, \cos(\frac{1}{2}t\log x)\, \frac{dx}{x}. \tag{1.1.13}$$

The *Riemann Hypothesis* is the statement that the real and even entire function $\Xi(t)$ has only real zeros, which would correspond, after a rotation and a translation, to the zeros of $\xi(s)$ and of $\zeta(s)$ on the *critical line* $\Re s = 1/2$.

1.2. **Notations.** It is clear that $\zeta(s)$ is of primary importance for the prime number theorem, from the Euler relation that Riemann first points out in his memoir [75],

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}, \qquad (\Re s > 1).$$

Thus it has become customary to define $\Xi(t)$ in terms of $\xi(s)$, whose zeros are the complex zeros of $\zeta(s)$, following Landau [57, p. 288], Jensen [47], Hardy [39], Wilton [88], Titchmarsh [82, p. 3] and Ingham [45, p. 46]. Already in 1915, Ramanujan [74] writes "changing s to $(1+it)/2$ and writing as usual

$$\xi(\frac{1}{2}+i\frac{t}{2}) = \Xi(\frac{t}{2})."$$

Older publications by authors such as Scheibner [76], Genocchi [33], Hermite [42], Hadamard [37], and even by Pólya reporting on the work of Jensen [70], follow Riemann instead and study the function $\Xi(t)$, which they denote by $\xi(t)$. The Landau convention (1.1.11) will be followed throughout the present work.

We will also use the notation

$$\theta(x) = \theta_3(0, e^{-\pi x}) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi x}, \qquad (\Re x > 0) \tag{1.2.1}$$

for which the transformation equation for the Jacobi theta function θ_3 used by Riemann to get (1.1.4) reduces to a simple but important identity [73, p. 76]

$$\theta(x) = \frac{1}{\sqrt{x}}\,\theta(\frac{1}{x}), \qquad (\Re x > 0, \Re\sqrt{x} > 0). \tag{1.2.2}$$

1.3. **Genocchi 1860.** An Italian translation of Riemann's 1859 memoir appeared one year later in [33], correcting the misprint where $\Xi(0)$ should be replaced by $\xi(0) = 1/2$ and providing details for the integration by parts by using a theorem of Cauchy [16, p. 607–609] instead of the Jacobi θ_3 transformation formula: if $ab = \pi^2$, then

$$a^{1/4}(1+2e^{-4a}+2e^{-9a}+\ldots) = b^{1/4}(1+2e^{-4b}+2e^{-9b}+\ldots). \tag{1.3.1}$$

The substitutions $a = \pi x, b = \pi/x$ show the equivalence of (1.3.1) and (1.2.2). For the justification of the product representation of $\Xi(t)$, Genocchi [33, p. 55] refers to a book by Briot and Bouquet [15, p. 135–140] where a "very clever" method of Cauchy is used to justify the infinite product expansions of even functions; following this analysis based on the residue theorem, in order to justify the convergence of the product, it is sufficient to bound the logarithmic derivative of the function as for the Bessel functions in [85, p. 497]; Riemann could possibly have derived this from his study of the number of zeros via $\int d\log\Xi(t)$ [75, p. 139], but "he gives no hint whatsoever of the method he used to bound the integral" [30, p. 19].

1.4. **Scheibner 1860.** A longer article in German [76] also appeared one year after Riemann's memoir, where Scheibner uses the integration variable substitution $x \mapsto x^2$ to obtain somewhat simpler formulas which can be written as

$$\xi(s) = \frac{1}{2} - s(1-s)\int_1^\infty \psi(x^2)\,(x^s + x^{1-s})\,\frac{dx}{x}$$

and [76, eq. 33]

$$\xi(\frac{1}{2}+z) = \int_1^\infty \rho(x)\,\frac{x^z + x^{-z}}{2}\,\frac{dx}{\sqrt{x}},$$

where [76, eq. 32] (with ρ here replacing Ξ there)

$$\rho(x) = 2\frac{d}{dx}(x^2\psi'(x^2)) = 4\sum_{n=1}^\infty [2(n^2\pi x^2)^2 - 3n^2\pi x^2]\,\exp(-n^2\pi x^2).$$

For the transformation equation of θ_3, Scheibner refers to works of Cauchy [16, 17], Abel, Poisson, Jacobi, "u.s.w."; he proves the relation of Jacobi

$$x\sum_{n=-\infty}^\infty \exp(-\pi(nx+h)^2) \;=\; \sum_{n=-\infty}^\infty \exp(\frac{2n\pi ih}{x} - \frac{n^2\pi}{x^2}),$$

after deducing the Poisson summation formula with the help of the Dirichlet integral [28] – a method used by Schlömilch two years earlier for the Euler functional equation of $\beta(s) = 1 - 1/3^s + 1/5^s - 1/7^s + \cdots$ ([31, 77, 21, 78]) first proven by Malmstèn [62, eq. 51]. Scheibner adds to those of $\zeta(s)$ and $\beta(s)$ yet another functional equation for $L(s) = 1 - 1/5^s - 1/7^s + 1/11^s + 1/13^s \cdots$.

1.5. **Hadamard 1893.** From the estimates in §**??** of the growth of the Taylor coefficients obtained by expanding the cosine in (1.1.11),

$$\Xi(t) = \frac{1}{2} - (t^2 + \frac{1}{4})\sum_{n=0}^\infty (-1)^n c_{2n}\frac{t^{2n}}{(2n)!}, \tag{1.5.1}$$

$$c_{2n} = \frac{1}{2^{2n}}\int_1^\infty x^{-\frac{3}{4}}\psi(x)(\log x)^{2n}\,dx, \tag{1.5.2}$$

Hadamard proves in [37, p. 210] that $\Xi(t)$, considered as a function of t^2, is of genus zero as defined by Laguerre [56], and can be represented by "the product of primary factors and a simple constant, without any exponential factor":

$$\Xi(t) = \Xi(0)\prod_{k=1}^\infty \left(1 - \frac{t^2}{\rho_k^2}\right). \tag{1.5.3}$$

Hadamard notes that this result had been stated by Riemann in his 1859 memoir, "but without sufficient proof". One modern proof [30, p. 20, §1.10] uses the Jensen's formula in [46] where Jensen stated that he was preparing a proof of the "Riemann Hypothesis" showing that $\Xi(t)$ does not have zeros inside $|t - ri| = r$, a project disputed later by Pólya [70, p. 9–10].

1.6. **Jensen 1911.** In his 1911 Copenhagen conference [47], Jensen defines Ξ, which he denotes by ξ, and represents it as a Fourier cosine transform :

$$\Xi(t) := \pi^{-s/2}\,\Gamma(1+s/2)(s-1)\zeta(s) = \int_0^\infty \phi_J(x)\,\cos xt\,dx, \quad (s = \frac{1}{2} + it),$$

where what we will label the "Jensen function ϕ_J" is given by

$$\begin{aligned}\phi_J(x) &= 2e^{5x/2}\left[2e^{2x}\theta''(e^{2x}) + 3\theta'(e^{2x})\right] \\ &= 4\sum_{n=1}^\infty \left(2n^4\pi^2 e^{9x/2} - 3n^2\pi e^{5x/2}\right)\exp(-n^2\pi e^{2x}).\end{aligned} \tag{1.6.1}$$

These elegant formulas can be considered as classical, and they are used by subsequent authors such as Titchmarsh [82], Wintner [89], Haviland [40], Spira [80], Matiyasevich [63], and many others. Removing the “inconsequential” [23] fractions of two and the factor of four as in [24, 25, 84],

$$\frac{1}{8}\,\Xi(\frac{t}{2}) = \int_0^\infty \Phi(x)\,\cos xt\,dx, \tag{1.6.2}$$

where what we will call the “Jensen function Φ” is given by [82, p. 45]

$$\Phi(x) = \frac{\phi_J(2x)}{4} = \sum_{n=1}^\infty \left(2n^4\pi^2 e^{9x} - 3n^2\pi e^{5x}\right)\,\exp(-n^2\pi e^{4x}). \tag{1.6.3}$$

The following equivalent representations, which come from Riemann’s double integration by parts, are “not hard (although somewhat tedious)” [80, p. 498] to verify directly by differentiation :

$$16\,\Phi(x) = 2(D^2-1)\left\{e^x\psi(e^{4x})\right\} = (D^2-1)\left\{e^x\theta(e^{4x})\right\}, \qquad (D = \frac{d}{dx}).$$

The second relation and (1.2.2) show that $\Phi(x)$ is even [69, p. 12]. Moreover

$$\Xi(\frac{t}{2}) = \int_0^\infty (D^2-1)\left\{e^x\psi(e^{4x})\right\}\,\cos xt\,dx. \tag{1.6.4}$$

Similarly, since $\phi_J(x)$ is also even (1899 lectures of Hurwitz [69, p. 11], [68, p. 99], [69, p. 12, p. 15], [82, p. 45]), de Bruijn writes in [26, p. 220]

$$\Xi(2t) = \frac{1}{4}\int_{-\infty}^\infty \phi_J(\frac{x}{2})\,e^{ixt}\,dx.$$

1.7. **Hardy 1914.** In the important note [39], Hardy uses the $\xi(s)$ function to show that $\zeta(s)$ has an infinite number of zeros on the critical line $\Re s = 1/2$. Starting from the inverse Mellin transform

$$e^{-n^2\pi x} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s)\,\left(n^2\pi x\right)^{-s}\,ds, \quad (\Re x > 0, \sigma > 0),$$

and then moving the integration path left from $\sigma > 1/2$ to $\sigma = 1/4$, he obtains with Cauchy’s integral theorem

$$\theta(x) = 1 + \frac{1}{\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \xi_M(2s)x^{-s}\,ds, \quad (\Re x > 0, \sigma > \frac{1}{2}),$$

$$x\theta(x^4) = x + \frac{1}{x} - \frac{2}{\pi}\int_{-\infty}^\infty \frac{\Xi(t/2)}{1+t^2}x^{-it}\,dt, \quad (\Re(x^4) > 0).$$

Setting $x = e^{i\alpha}$ and differentiating $2p$ times with respect to the real variable $\alpha \in (-\pi/8, \pi/8)$ yields

$$\frac{2}{\pi}\int_0^\infty \frac{\Xi(\frac{t}{2})}{1+t^2}\,t^{2p}\,\cosh\alpha t\,dt = (-1)^p\cos\alpha - D^{2p}\left[\frac{1}{2}e^{i\alpha}\theta(e^{4i\alpha})\right], \qquad (p = 0, 1.2, \ldots). \tag{1.7.1}$$

With the help of a series of advanced theorems from real analysis, Hardy finally obtains a contradiction when $\alpha \to \pi/8$ if it is assumed that $\Xi(t)$ has a constant sign over an infinite interval. Landau published one year later an article on Hardy’s theorem, including elementary proofs of all the needed identities and convergence results for improper integrals [58]. In particular, *if* $\Xi(t)$ had a constant sign on an infinite interval, *then* for any fixed positive integer p the integral $\int_0^\infty \Xi(t/2)t^{2p}\cosh(\alpha t)\,dt$ would be uniformly convergent for $0 \le \alpha \le \pi/8$, so that in the limit [86, p. 280]

$$\frac{2}{\pi}\int_0^\infty \frac{\Xi(\frac{t}{2})}{1+t^2}\,t^{2p}\,\cosh(\frac{\pi t}{8})\,dt = (-1)^p\cos\frac{\pi}{8}, \quad (p = 0, 1.2, \ldots).$$

In fact, by taking linear combinations as in [13], we can also deduce the absurd conclusion that for any polynomial P, including the supposedly finite product formed by the real zeros in (1.5.3),

$$\int_0^\infty \Xi(t)\,P(t^2)\,\cosh(\frac{\pi t}{4})\,dt = 0.$$

Landau points out in a footnote that Hardy had not proven the unconditional convergence to zero of this last improper integral, as stated incorrectly in [13].

1.8. **Ramanujan 1915.** Some cosine transform formulas involving the $\Xi(t)$ and $\xi(s)$ functions were published by Ramanujan [74] such as, for $|\Im(z)| < \frac{3}{8}\pi$,

$$\frac{1}{4\pi\sqrt{\pi}}\int_0^\infty \left|\Gamma(\frac{-1+it}{4})\right|^2 \Xi(\frac{t}{2})\cos(zt)\,dt = e^{-z} - 4\pi e^{-3z}\int_0^\infty \frac{x\exp(-\pi x^2 e^{-4z})}{e^{2\pi x}-1}\,dx.$$

Hardy added to [74] a short comment with "another formula not unlike Mr Ramanujan's formula",

$$8\int_0^\infty \frac{\Xi(\frac{t}{2})}{1+t^2}\frac{\cos zt}{\cosh\frac{1}{2}\pi t}\,dt = (4z+\gamma+\log 4\pi)e^{-z} + 4e^z\int_0^\infty \Psi(1+x)\exp(-\pi x^2 e^{4z})\,dx,$$

where $\Psi(s)$ is the logarithmic derivative of $\Gamma(s)$ and γ is Euler's constant (with correct sign [29]).

Two more integrals involving $\Xi(t)$ are evaluated directly via complex integration and transform inversion in [83, §2.16] and can be expressed as:

$$\frac{4}{\pi}\int_0^\infty \frac{\Xi(\frac{t}{2})}{t^2+1}\cos xt\,dt = 2\cosh x - e^x\theta(e^{4x});$$
$$\frac{4}{\pi}\int_0^\infty \Xi(\frac{t}{2})\cos xt\,dt = (D^2-1)\left\{e^x\theta(e^{4x})\right\}.$$

Obviously, the variable x can be replaced by $-x$ throughout from (1.2.2); the second formula can be derived from the first since $(D^2-1)\cosh x = 0$, and is simply the inverse Fourier cosine transform of (1.6.2); the first formula follows from Riemann's integral representation of $\xi_M(s)$ (1.1.3) as shown by Wilton (see §1.9).

1.9. **Wilton 1915.** In the short note [88], with the inverse transform method used by Hardy a year earlier [39], Wilton shows that both the real and the imaginary parts of $\xi(s)$ have an infinity of zeros on any line $\Re s = \sigma$, for $0 < \sigma < 1$; the case $\sigma = 1/2$ yields Hardy's result since $\xi(s)$ is real on the critcal line. Wilton gives the Fourier cosine transform formula (1.6.4)

$$\Xi(\frac{t}{2}) = \int_0^\infty (D^2-1)\left\{e^x\psi(e^{4x})\right\}\cos tx\,dx, \qquad (\psi(x) = \sum_{n=1}^\infty e^{-n^2\pi x}),$$

but he starts from Riemann's integral (1.1.4) for $\xi_M(s)$, substitutes $x \mapsto e^{4x}$, sets $s = 1/2 + it/2$, and uses the integration formula

$$\int_0^\infty e^{-x}\cos(xt)\,dx = \frac{1}{1+t^2}, \quad (|\Im t| < 1),$$

to obtain, for $|\Im t| < 1$ or $0 < \Re s < 1$,

$$\frac{\Xi(t/2)}{1+t^2} = \frac{1}{1+t^2} - 2\int_0^\infty e^x\,\psi(e^{4x})\cos(xt)\,dx = \int_0^\infty \phi_W(x)\cos(xt)\,dx \tag{1.9.1}$$

where the Wilton function

$$\phi_W(x) := e^{-x} - 2e^x\,\psi(e^{4x}) = 2\cosh(x) - e^x\,\theta(e^{4x})$$

is even from (1.2.2). Using properties of the Fourier cosine transform [17], contour integration and finally Fourier's integral theorem, Wilton proves his result by following Hardy's argument in [39] with the parametric function

$$\psi_\sigma(x) := \cosh(2\sigma-1)x\,\phi_W(x), \qquad (0 < \sigma < 1).$$

Avoiding the pitfall in [13] pointed out by Landau [58, p. 221], Hardy had proceeded by contradiction, showing that *if* $\Xi(t)$ had a finite number of real zeros, *then* a crucial improper integral was convergent, *and* this convergence was incompatible with a known singularity of $\theta(x)$ [70, p. 13], [30, p. 226–229]. This method works as well here, but Wilton also tried to get the convergence of his corresponding parametric improper integral directly from the properties of his function ψ_σ. However, he fails to take into account the possibility of more than one root of the derivative between two consecutive roots of a real function in Rolle's theorem: while he states that "$\psi_\sigma^{(2n)}(x)$ vanishes for n values of x between 0 and ∞", a simple graph shows that this is false for $\sigma = 1/2$ since $\phi_W^{(12)}(x)$ has 14 real roots.

1.10. **Titchmarsh 1930.** The following properties are proven in the first pages (3–5) of the first edition of the book by Titchmarsh on the zeta function of Riemann.

1.10.1. *Definition.* $\xi(s) := 1/2s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$, $s \in \mathbb{C}$

1.10.2. *Functional equation.* $\xi(s) = \xi(1-s)$.

1.10.3. *Order.* $\xi(s)$ is an integral function of growth order 1.

1.10.4. *Symmetry.* $\Xi(t) := \xi(\frac{1}{2} + it)$ is a real and even function of t.

1.10.5. *Zeros1.* $\Xi(\sqrt{z})$ is an integral function of order $\frac{1}{2}$ with an infinity of zeros.

1.10.6. *Zeros2.* $\xi(s)$ has an infinity of complex zeros ρ inside the strip $0 < \Re(s) < 1$.

1.10.7. *Weierstrass product.* $\xi(s) = \frac{1}{2}e^{bs}\prod_\rho(1-\frac{s}{\rho})e^{\frac{s}{\rho}}$, where ρ runs through the complex zeros of $\zeta(s)$, and

$$b = \frac{\xi'(0)}{\xi(0)} = -\sum_\rho \frac{1}{\rho} = -\frac{1}{2}\sum_\rho \left(\frac{1}{\rho} + \frac{1}{\bar\rho}\right) = \log\sqrt{4\pi} - 1 - \frac{\gamma}{2} \approx -0.023095.$$

1.10.8. *Zeros3.* The number of zeros of $\Xi(t)$ between $t = 0$ and $t = T$ is approximately

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + \frac{7}{8} + O(\log T).$$

The third chapter “The distribution of the zeros” presents the second proof of the functional equation of $\zeta(s)$ by Riemann, including the integral representations of $\Xi(t)$ by Riemann (1.1.11) and by Jensen (1.6.2, 1.6.3). The results of Hardy and of Hardy & Littlewood on the number of zeros on the critical line are also proven.

1.10.9. *Zeros4.* $\Xi(t)$ has asymptotically at least AT real zeros on $[0, T]$, where $A > 0$.

1.11. **Wintner 1935.** The two-page note [89] contains an elementary proof that the Jensen function $\Phi(t)$ is strictly decreasing on $(0, \infty)$. Crediting “Jensen, Hurwitz and others”, Wintner first obtains from (1.6.3) what can be expressed as

$$-\frac{\Phi'(t)}{ue^t} = \sum_{n=1}^{\infty} e^{-n^2u} n^2 p_2(n^2u), \qquad (u = \pi e^{4t}),$$

where the quadratic polynomial $p_2(u) = 8u^2 - 30u + 15$ is positive for $u > u_0 := (15 + \sqrt{105})/8 \approx 3.156$. All the terms of the series are thus positive for $t > 0$, except the first one which changes sign from negative to positive at $u = u_0$ or $t = t_0 := \log(u_0/\pi)/4 \approx 0.0011$, and the sum of the series is indeed positive on $[t_0, \infty)$. For the remaining finite interval $(0, t_0)$, a clever method involves differentiation :

$$\frac{1}{4u}\left(\frac{\Phi'(t)}{ue^t}\right)' = \sum_{n=1}^{\infty} e^{-n^2u} n^4(8n^4u^2 - 46n^2u + 45), \qquad (u = \pi e^{4t}).$$

The quadratic polynomial $8u^2 - 46u + 45$ being positive for $u > 9/2$, all the terms of this series are positive for $t > 0$, except the first one which is negative for $5/4 < u < 9/2$, and thus for $0 < t < t_0$. Wintner then shows that the negative first term dominates the sum of the others on $(0, t_0)$, using the numerical bound $\sum_{n\geq 3} e^{-n^2\pi} n^8 < 10^{-4}$ without explanation. Finally, he concludes that

$$\text{“for } 0 < t < t_0, \text{ we have} \quad \left(e^{-5t}\Phi'(t)\right)' < 0,$$
$$\text{which implies} \quad \Phi'(t) < e^{5t}\Phi'(0) = 0.\text{”}$$

This result was also proven independently by Spira in 1971, using $\Phi''(t)$ instead of $\Phi''(t) - 5\Phi'(t)$ for the finite interval step which then involves a cubic polynomial (see §**??**).

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