Académie Serbe des Sciences et des Arts, Bulletin CXVIII Classe des Sciences Mathématiques et naturelles, Sciences Mathématiques No. 24, Belgrade 1999, 119–132

## ON THE MULTIPLICITY OF ZEROS OF THE ZETA-FUNCTION

# A. IVIĆ

(Presented at the 1st Meeting, held on February 26, 1999)

A b s t r a c t. Several results are obtained concerning multiplicities of zeros of the Riemann zeta-function  $\zeta(s)$ . They include upper bounds for multiplicities, showing that zeros with large multiplicities have to lie to the left of the line  $\sigma=1$ . A zero-density counting function involving multiplicities is also discussed.

AMS Subject Classification (1991): 11M06

Key Words: Riemann zeta-function, bounds for multiplicities of zeros, Lindelöf Hypothesis, Riemann Hypothesis, isolation hypothesis

### 1. Introduction

Let  $r=m(\rho)$  denote the multiplicity of the complex zero  $\rho=\beta+i\gamma$  of the Riemann zeta-function  $\zeta(s)$ . This means that  $\zeta(\rho)=\zeta'(\rho)=\ldots=\zeta^{(r-1)}(\rho)=0$ , but  $\zeta^{(r)}(\rho)\neq 0$ . All known zeros  $\rho$  are simple (i.e.  $m(\rho)=1$ ), and it may well be that they are all simple, although the proof of this is certainly beyond reach at present. In estimating  $m(\rho)$  one may suppose that  $\beta\geq \frac{1}{2}$  and that  $\gamma>0$ , since  $1-\rho$  and  $\overline{\rho}$  are zeros of  $\zeta(s)$  if  $\rho$  is a zero.

It seems that there exist no good upper bounds in the literature for  $m(\rho)$ . All that appears to be known unconditionally is

$$m(\beta + i\gamma) \ll \log \gamma.$$
 (1.1)

On the Lindelöf Hypothesis (LH) that  $\zeta(\frac{1}{2}+it)\ll |t|^{\varepsilon}$  this bound can be improved to

$$m(\beta + i\gamma) = o(\log \gamma) \qquad (\gamma \to \infty),$$
 (1.2)

and on the Riemann Hypothesis (RH) that  $\rho = \frac{1}{2} + i\gamma$  to

$$m(\beta + i\gamma) \ll \frac{\log \gamma}{\log \log \gamma}.$$
 (1.3)

Furthermore, on the RH H.L. Montgomery [4] proved that at least 2/3 of the zeros are simple. It transpires that the estimation of  $m(\beta + i\gamma)$  is a very difficult problem, and one

2 Aleksandar Ivić

which is not satisfactorily solved even under the assumption of the LH or the RH. To see how one obtains (1.1)–(1.3) recall that for N(T), the number of zeros  $\beta + i\gamma$  for which  $0 < \gamma \le T$ , one has the classical Riemann-von Mangoldt formula (see [1] and [6])

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right), \quad S(T) = \frac{1}{\pi} \arg\zeta(\frac{1}{2} + iT), \quad (1.4)$$

where  $\arg \zeta(\frac{1}{2}+iT)$  is obtained by continuous variation along the straight lines joining the points  $2, 2+iT, \frac{1}{2}+iT$ , starting with the value 0. If T is the ordinate of a zero lying on the critical line, then S(T) = S(T+0). One has (see [6]) the bounds

$$S(T) \ll \log T$$
,  $S(T) = o(\log T)$  (LH),  $S(T) \ll \frac{\log T}{\log \log T}$  (RH),

and these bounds combined with the trivial inequality

$$m(\beta + i\gamma) \le N(\gamma + H) - N(\gamma - H) \qquad (0 < H \le 1) \tag{1.5}$$

easily yield (1.1)–(1.3), respectively. It seems, however, that these estimates are much too large, and that perhaps one even has

$$m(\beta + i\gamma) \ll_{\varepsilon} (\log \log \gamma)^{1+\varepsilon},$$
 (1.6)

which is of course still much weaker than the conjecture that all zeros are simple. The use of pointwise estimates for S(T) certainly cannot give anything close to (1.6), since one has

$$S(T) = \Omega_{\pm} \left( \left( \frac{\log T}{\log \log T} \right)^{1/3} \right), \quad S(T) = \Omega_{\pm} \left( \left( \frac{\log T}{\log \log T} \right)^{1/2} \right)$$
 (RH),

proved by K.-M. Tsang [7] and H.L. Montgomery [5], respectively. (As usual,  $f = \Omega_{\pm}(g)$  means that  $\limsup_{x\to\infty} f(x)/g(x) = +\infty$  and  $\liminf_{x\to\infty} f(x)/g(x) = -\infty$  both hold). One could use (1.5) with H = o(1) ( $\gamma \to \infty$ ) to try to improve (1.1)–(1.3). In view of (1.4) this is equivalent to obtaining bounds for  $S(\gamma + H) - S(\gamma - H)$ , but no satisfactory results seem to be known for this problem.

In this note we shall seek other approaches to the estimation of  $m(\beta + i\gamma)$ , and several bounds will be proved in the sequel. We shall also discuss a zero-density counting function involving multiplicities. In what follows C will denote positive, absolute constants, not necessarily the same ones at each occurrence.

#### 2. Bounds for multiplicities

In this section we shall formulate and prove the results involving upper bounds for  $m(\beta + i\gamma)$ . We start with

THEOREM 1. If  $\zeta(\beta + i\gamma) = 0, \frac{1}{2} < \beta < 1 \text{ and } \gamma \ge \gamma_0 > 0, \text{ then}$ 

$$m(\beta + i\gamma) \le \frac{1}{\log \frac{1}{2 - 2\beta}} \left( \max_{\sigma \ge \frac{1}{2}, |t| \le \frac{1}{2}} \log |\zeta(\sigma + i\gamma + it)| + O(\log \log \gamma) \right). \tag{2.1}$$

**Proof of Theorem 1.** We shall use Jensen's classical formula (see e.g. [2], pp 257-258). Namely if f(z) is regular in  $|z| \le R$  and  $f(0) \ne 0$ , then

$$\sum_{f(\rho)=0, |\rho| \le R} \log \frac{R}{|\rho|} + \log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| \, \mathrm{d}\theta, \tag{2.2}$$

where the zeros  $\rho$  are counted according to their multiplicities. In (2.2) we take  $f(z) = \zeta(1+i\gamma+z), R = \frac{1}{2}, \gamma \geq \gamma_0, \frac{1}{2} < \beta < 1$ . Then  $f(0) \neq 0$ , in fact (see Lemma 12.3 of [1])

$$\zeta(\sigma + it) \gg \log^{-2/3} t (\log \log t)^{-1/3} \quad (\sigma \ge 1 - C(\log t)^{-2/3} (\log \log t)^{-1/3}, t \ge t_0 > 0). (2.3)$$

If  $r = m(\beta + i\gamma)$ , then (2.2) gives

$$r\log\frac{\frac{1}{2}}{1-\beta} + \log|\zeta(1+i\gamma)| \le \frac{1}{2\pi} \int_0^{2\pi} \log|\zeta(1+i\gamma+\frac{1}{2}e^{i\theta})| d\theta$$
$$\le \max_{\sigma \ge \frac{1}{2}, |t| \le \frac{1}{2}} \log|\zeta(\sigma+i\gamma+it)|,$$

and using (2.3) we immediately obtain (2.1). A slight improvement is possible with the choice  $f(z) = \zeta(1 - \delta(\gamma) + z), R = \frac{1}{2} - \delta(\gamma), \delta(\gamma) = C(\log \gamma)^{-2/3} (\log \log \gamma)^{-1/3}$ .

THEOREM 2. If  $\zeta(\beta + i\gamma) = 0, \frac{1}{2} < \beta < 1, \gamma \ge \gamma_0 > 0$  and c is a constant satisfying  $c > 1 - \beta$ , then

$$m(\beta + i\gamma) \leq \frac{\frac{c+\beta-1}{c+\beta-\frac{1}{2}} \max_{|t| \leq \log^2 \gamma} \log |\zeta(\frac{1}{2} + i\gamma + it)| + O(\log \log \gamma)}{\log \left\{ \frac{c}{1-\beta} \left( \frac{\beta-\frac{1}{2}}{c} \right)^{\frac{c+\beta-1}{c+\beta-\frac{1}{2}}} \right\}}.$$
 (2.4)

**Proof of Theorem 2.** Let  $r = m(\beta + i\gamma)$  and  $\mathcal{D}$  be the rectangle with vertices  $\frac{1}{2} - \beta \pm i \log^2 \gamma$ ,  $c \pm i \log^2 \gamma$ . If  $X(0 < X \ll \gamma^C)$  is a parameter which will be suitably chosen, then by the residue theorem we obtain

$$\frac{\zeta(1-\beta+\rho)}{(1-\beta)^r} = \frac{1}{2\pi i} \int_{\mathcal{D}} X^{s-1+\beta} \Gamma(s-1+\beta) \frac{\zeta(s+\rho)}{s^r} \, \mathrm{d}s \quad (\rho = \beta + i\gamma). \tag{2.5}$$

By using (2.3) and Stirling's formula for the gamma-function it follows from (2.5) that

$$\frac{1}{(1-\beta)^r \log \gamma} \ll e^{-\log^2 \gamma} + X^{-\frac{1}{2}} \max_{|t| \le \log^2 \gamma} \frac{|\zeta(\frac{1}{2} + i\gamma + it)|}{(\beta - \frac{1}{2})^r} 
+ X^{c-1+\beta} \int_{-\log^2 \gamma}^{\log^2 \gamma} |\Gamma(c-1+\beta+it)| \frac{|\zeta(c+\beta+i\gamma+it)|}{|c+it|^r} dt 
\ll (\beta - \frac{1}{2})^{-r} X^{-\frac{1}{2}} M + X^{c-1+\beta} c^{-r} + e^{-\log^2 \gamma},$$
(2.6)

where

$$M := \max_{|t| < \log^2 \gamma} |\zeta(\frac{1}{2} + i\gamma + it)|.$$

Aleksandar Ivić

We choose

$$X = M^{\frac{1}{c+\beta-\frac{1}{2}}} \left(\frac{c}{\beta-\frac{1}{2}}\right)^{\frac{r}{c+\beta-\frac{1}{2}}}$$

to equalize the terms containing X in (2.6), noting that in view of (1.1)  $X \ll \gamma^C$  will hold. Then after taking logarithms we obtain (2.4) from (2.6).

To assess the strength of (2.4) and compare it with (2.1) of Theorem 1, take  $c = \frac{3}{2} - \beta$ . Then (2.4) gives

$$m(\beta + i\gamma) \leq \frac{\log M}{\log\left\{\frac{\left(\frac{3}{2} - \beta\right)(\beta - \frac{1}{2}\right)}{(1 - \beta)^2}\right\}} + O_{\beta}(\log\log\gamma). \tag{2.7}$$

But we have

$$\frac{1}{\log\left\{\frac{\left(\frac{3}{2}-\beta\right)(\beta-\frac{1}{2})}{(1-\beta)^2}\right\}} \le \frac{1}{\log\frac{1}{2-2\beta}}$$

for  $\beta \ge \frac{5-\sqrt{5}}{4} = 0.69058...$ , hence in this range essentially (2.7) improves (2.1), and the range can be further increased by an appropriate choice of the constant c.

Both Theorem 1 and Theorem 2 could be modified to cover the case  $\beta = \frac{1}{2}$ , i.e. the zeros on the critical line. In that case the maxima in (2.1) and (2.4) would have to include zeta-values on the lines left of  $\sigma = \frac{1}{2}$ , which would not yield good results in view of the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s), \qquad \chi(s) \approx t^{\frac{1}{2}-\sigma} \quad (s = \sigma + it, \ t > t_0 > 0).$$

The main merit of Theorem 1 and Theorem 2 is that they show that  $m(\beta + i\gamma)$  is small if  $\beta$  is close to the line  $\sigma = 1$ . Namely, if  $\beta \ge 1 - \log^{-\delta} \gamma$  ( $0 < \delta \le \frac{2}{3}$ ;  $\delta > \frac{2}{3}$  is not possible in view of (2.3)), we obtain either from Theorem 1 or Theorem 2 that

$$m(\beta + i\gamma) \ll_{\delta} \frac{\log \gamma}{\log \log \gamma},$$

which is the same as (1.3). Therefore zeros close to the line  $\sigma = 1$ , if they exist, must have small multiplicities. On the other hand, if there exist zeros with large multiplicities ("large" in the sense that  $m(\beta + i\gamma)/\log\log\gamma \to \infty$ ), then they must be far away from the line  $\sigma = 1$ . This is precisely given by

THEOREM 3. If  $\zeta(\beta + i\gamma) = 0$ , then for  $\gamma \ge \gamma_0 > 0$ 

$$m(\beta + i\gamma) \ll (1 - \beta)^{3/2} \log \gamma + \log \log \gamma,$$
 (2.8)

and if  $\lim_{\gamma \to \infty} \frac{m(\beta + i\gamma)}{\log \log \gamma} = +\infty$ , then there is a constant C > 0 such that for  $\gamma \ge \gamma_0 > 0$ 

$$\beta \le 1 - C \left( \frac{m(\beta + i\gamma)}{\log \gamma} \right)^{2/3}. \tag{2.9}$$

**Proof of Theorem 3.** Let  $\mathcal{D}$  be the rectangle with vertices  $-2(1-\beta) \pm i \log^2 \gamma$ ,  $1 \pm i \log^2 \gamma$ . We can suppose that  $\beta \geq \frac{3}{4}$ , for otherwise (2.8) is trivial in view of (1.1), and so is (2.9) if C is sufficiently small. For  $\beta \geq \frac{3}{4}$  formula (2.5) (with the above  $\mathcal{D}$ ) is valid, and to estimate the integrand on the line  $\sigma = -2(1-\beta)$  we shall use the bound

$$\Gamma(w) \ll \frac{e^{-|\operatorname{Im} w|}}{|w|}.$$

Since  $(1 - \beta)^{-1} \ll \log \gamma$  it follows from (2.5) that

$$\frac{(1-\beta)^{-r}}{\log \gamma} \ll e^{-\log^2 \gamma} + 2^{-r} (1-\beta)^{-r} X^{-3(1-\beta)} \log \gamma \max_{|\gamma| \le \log^2 \gamma} |\zeta(3\beta - 2 + i\gamma + it)| + X^{\beta}.$$
(2.10)

To bound the zeta-factor in (2.10) we use the estimate (see Ch. 6 of [1])

$$\zeta(\sigma + it) \ll t^{C(1-\sigma)^{3/2}} \log^{2/3} t \qquad (t \ge t_0, \frac{1}{2} \le \sigma \le 1, C \le 122),$$

which is the strongest known one when  $\sigma$  is sufficiently close to  $\sigma = 1$ . It follows that

$$\frac{1}{(1-\beta)^r} \ll \left(2^{-r}(1-\beta)^{-r}X^{-3(1-\beta)}\gamma^{C_1(1-\beta)^{3/2}} + X^{\beta}\right)\log^{C_2}\gamma \qquad (C_1, C_2 > 0).$$

To equalize the terms containing X we choose

$$X = 2^{-\frac{r}{3-2\beta}} (1-\beta)^{-\frac{r}{3-2\beta}} \gamma^{\frac{C_1(1-\beta)^{3/2}}{3-2\beta}}.$$

The condition  $X \ll \gamma^C$  will hold in view of (2.1), and we obtain

$$\left\{2^{\frac{\beta}{3-2\beta}}(1-\beta)^{\frac{3\beta-3}{3-2\beta}}\right\}^r \ll \gamma^{C(1-\beta)^{3/2}} \log^{C_2} \gamma.$$

Taking logarithms we have

$$r(\beta \log 2 + (3\beta - 3)\log(1 - \beta)) \le C(1 - \beta)^{3/2}\log \gamma + O(\log \log \gamma).$$
 (2.11)

Now since  $\beta \geq \frac{3}{4}$  and

$$(3\beta - 3)\log(1 - \beta) \ge 0$$
  $(0 < \beta < 1),$ 

6 Aleksandar Ivić

we obtain (2.8) from (2.11), and if additionally

$$\lim_{\gamma \to \infty} \frac{r}{\log \log \gamma} = \lim_{\gamma \to \infty} \frac{m(\beta + i\gamma)}{\log \log \gamma} = +\infty, \tag{2.12}$$

then (2.9) also follows from (2.11) (or from (2.8)). Thus if (2.12) holds for some zero  $\rho$ , then (2.9) shows that  $\rho$  lies to the left of the sharpest known zero-free region for  $\zeta(s)$  implied by (2.3).

The condition (2.12) appears significant in another context. Namely if one assumes that the zeros near the line  $\sigma = 1$  are isolated from one another, then one can improve the known zero-free region of  $\zeta(s)$  (see Ch. 6 of [1]) that

$$\beta \le 1 - C(\log \gamma)^{-2/3} (\log \log \gamma)^{-1/3} \qquad (\zeta(\beta + i\gamma) = 0, \, \gamma \ge \gamma_0 > 0).$$

This was done by N. Levinson [3], who proved the following result: If for some  $\delta > 0$  and sufficiently large constant  $T_0 > 0$  the zeros  $\rho$  which lie in  $\beta > 1 - \delta$ ,  $|\gamma| > T_0$  are all isolated in the sense that there is no zero of  $\zeta(s)$  other than  $\rho = \beta + i\gamma$  in the rectangle  $(s = \sigma + it)$ 

$$1 - \delta < \sigma < 1, \quad |t - \gamma| < 2\delta, \tag{2.13}$$

then there are no zeros of  $\zeta(s)$  in the region

$$\sigma > 1 - \frac{C}{\log \log t} \qquad (|t| > T_0) \tag{2.14}$$

for suitable C > 0. Although isolated, these zeros need not be simple.

By going through Levinson's proof and making the appropriate modifications one can obtain an upper bound for  $\beta$  depending on  $m(\beta + i\gamma)$ , and which shows that under he condition (2.12) the zero-free region (2.14) can be improved. This is

THEOREM 4. Suppose that for some  $\delta > 0$  and sufficiently large constant  $T_0 > 0$  the zeros  $\rho$  which lie in  $\beta > 1 - \delta$ ,  $|\gamma| > T_0$  are all isolated in the sense that there is no zero of  $\zeta(s)$  other than  $\rho = \beta + i\gamma$  in the rectangle (2.13). If  $r = m(\beta + i\gamma)$  for such a zero, then

$$\beta \le 1 - \frac{r^{\frac{1}{2\log\log\gamma}}}{C\log\log\gamma}. (2.15)$$

**Proof of Theorem 4.** The theorem actually shows that  $r = m(\beta + i\gamma)$  cannot be larger than  $(C \log \log \gamma)^{\log \log \gamma}$ , for otherwise (2.15) would yield that  $\beta < 1 - \delta$ , which is impossible. One can rewrite (2.15) as

$$m(\beta + i\gamma) \leq (C(1-\beta)\log\log\gamma)^{2\log\log\gamma}$$

which is the upper bound for  $m(\beta + i\gamma)$  under the "isolation hypothesis". To prove the theorem, let  $(\Lambda(n))$  is the von Mangoldt function

$$F_m(s) := (-1)^m \frac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}} \left( \frac{\zeta'(s)}{\zeta(s)} \right) = \sum_{n=1}^{\infty} \Lambda(n) n^{-s} \log^{m-1} n \quad (\sigma = \text{Re } s > 1; \ m \ge 1).$$

We use (see (1.43) of [1]; here exceptionally in the next two formulas  $\gamma$  is Euler's constant and not the imaginary part of a zeta-zero)

$$\frac{\zeta'(s)}{\zeta(s)} = \log(2\pi) - 1 - \frac{1}{2}\gamma - \frac{1}{2}\frac{\Gamma'(\frac{s}{2} + 1)}{\Gamma(\frac{s}{2} + 1)} + \sum_{\rho} \left(\frac{1}{s - \rho} + \frac{1}{\rho}\right),$$

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+z} \right),$$

(2.13) and  $N(T+1) - N(T) \ll \log T$  to obtain

$$F_m(s) = -\sum_{\rho; |\gamma - t| \le 1} (s - \rho)^{-m} + O(\log|t|) \quad (s = \sigma + it \in \mathbb{C}, \ m \ge 2), \tag{2.16}$$

where the O-constants throughout the proof are uniform in m. Therefore the isolation hypothesis yields

$$F_m(\sigma + i\gamma) = -\frac{r}{(\sigma - \beta)^m} + O\left(\frac{\log \gamma}{\delta^m}\right) \qquad (r = m(\beta + i\gamma); m \ge 2). \tag{2.17}$$

In the rectangle  $1-\delta < \sigma < 1, |t-2\gamma| < \delta$ , there is at most one zero of  $\zeta(s)$  by the isolation hypothesis. Consider the case when there is such a zero, say  $\beta_1 + i\gamma_1$ ; if there is no such zero (2.20) will hold with  $r_1 = 0$ . Then in place of (2.17) one obtains from (2.16)

$$F_m(\sigma + 2i\gamma) = -\frac{r_1}{(\sigma - \beta_1 + i(2\gamma - \gamma_1))^m} + O\left(\frac{\log \gamma}{\delta^m}\right) \qquad (r_1 = m(\beta_1 + i\gamma_1); m \ge 2)$$
(2.18)

and, for  $\sigma > 1$ , one also has

$$F_m(\sigma) = \frac{1}{(\sigma - 1)^m} + O(1) \qquad (m \ge 2).$$
 (2.19)

From the definition of  $F_m(s)$  and the classical inequality

$$3 + 4\cos\theta + \cos 2\theta = 2(1 + \cos\theta)^2 \ge 0 \qquad (\theta \in \mathbb{R})$$

it follows, for  $\sigma > 1$ , that

$$\Re \{3F_m(\sigma) + 4F_m(\sigma + it) + F_m(\sigma + 2it)\} \ge 0,$$

which in view of (2.17)–(2.19) yields, for  $\sigma > 1$  and  $m \ge 2$ ,

$$\frac{3}{(\sigma-1)^m} - \frac{4r}{(\sigma-\beta)^m} - \Re \left(\frac{r_1}{(\sigma_1-\beta_1+i(2\gamma-\gamma_1))^m}\right) \ge -\frac{C\log\gamma}{\delta^m}.$$
 (2.20)

Now one chooses

$$M = [\log \log \gamma] + 1, \quad \sigma = 1 + 100M(1 - \beta),$$

and the chief feature of Levinson's method is that m in (2.20) can be chosen in such a way that  $m \asymp M$  and that  $\text{Re}(\cdots)$  is non-negative. As shown in detail in [3], there are two cases: in the first case  $m = M (\geq 2)$  suffices for the proof, and in the second case  $M \leq m < 4M$  will hold. Since the analysis is identical with the one needed in our proof, it will be omitted here. With the above choices of M and  $\sigma$  one obtains from (2.20) in the case m = M (the other case is quite analogous)

$$(100M(1-\beta))^{-M} \left\{ 3 - 4r \left( \frac{100M}{1 + 100M} \right)^M \right\} \ge -\frac{C \log \gamma}{\delta^M}. \tag{2.21}$$

But we have

$$3 - 4r \left(\frac{100M}{1 + 100M}\right)^{M} = 3 - 4r \left(1 + \frac{1}{100M}\right)^{-M} < 3 - 4re^{-\frac{1}{100}}$$
$$< 3 - \frac{7r}{2} \le -\frac{r}{2}.$$

Consequently (2.21) yields

$$\frac{r\delta^M}{2C\log\gamma} \le (100M(1-\beta))^M,$$

which easily leads to (2.15). The choice of M is made so that  $(\log \gamma)^{1/M}$  is bounded.

We also remark that there is a possibility to bound  $m(\beta + i\gamma)$ , provided one has a good lower bound of the form

$$\int_{\delta}^{2\delta} |\zeta(\beta + i\gamma + i\alpha)|^k d\alpha \ge \ell = \ell(\gamma, \delta, k) \qquad (0 < \delta < \frac{1}{4}, \ \gamma \ge \gamma_0 > 0)$$
 (2.22)

for k = 1, 2. Namely for  $\beta \ge \frac{1}{2}$  let  $\mathcal{D}$  be the rectangle with vertices  $\frac{1}{4} - \beta \pm i \log^2 \gamma, 2 \pm i \log^2 \gamma, \zeta(\rho) = 0, \rho = \beta + i\gamma, \gamma \ge \gamma_0 > 0$ , and let  $0 < \alpha \le 1$ . Then by the residue theorem

$$\frac{\zeta(\beta + i\gamma + i\alpha)}{(i\alpha)^r} = \frac{1}{2\pi i} \int_{\mathcal{D}} \Gamma(s - i\alpha) \frac{\zeta(s + \rho)}{s^r} \, \mathrm{d}s.$$

This gives

$$\zeta(\beta + i\gamma + i\alpha) \ll \alpha^r \left(\gamma(\beta - \frac{1}{4})^{-r} + 2^{-r}\right) \ll \alpha^r \gamma(\beta - \frac{1}{4})^{-r},$$

and consequently, if  $\delta$  is a constant satisfying  $0 < \delta < \frac{1}{4}$ , we have

$$\int_{\delta}^{2\delta} |\zeta(\beta + i\gamma + i\alpha)|^k d\alpha \ll \gamma^k (\beta - \frac{1}{4})^{-rk} \int_{\delta}^{2\delta} \alpha^{rk} d\alpha \ll \delta^{rk} \gamma^k.$$

Therefore by using (2.22) and taking logarithms we obtain

$$m(\beta + i\gamma) = r \le \frac{1}{\log\left(\frac{1}{\delta}\right)} \left(\log\gamma - \frac{1}{k}\log\ell + O(1)\right). \tag{2.23}$$

Hence (2.23) shows that the upper bound for  $m(\beta + i\gamma)$  can be made to depend on  $\ell$  in (2.22). We would like to let  $\delta \to 0+$  in (2.23) and obtain (1.2). However, by using the argument on top of p. 219 of [6] and the first inequality on p. 230, it follows that in (2.22) one can take  $\ell = \delta \gamma^{-A/\delta}$  or even  $\delta \gamma^{A \log \delta}$  for some absolute A > 0. These bounds, unfortunately, are too weak to yield (1.2).

## 3. The zero-density counting function

A problem related to the estimation of  $m(\beta+i\gamma)$  is to estimate  $N^{(r)}(\sigma,T)$ , the number of zeros  $\rho=\beta+i\gamma$  of  $\zeta(s)$  such that  $m(\rho)\geq r, \beta\geq\sigma\,(\geq\frac{1}{2})$  and  $|\gamma|\leq T$ . Note that  $N^{(1)}(\sigma,T)\equiv N(\sigma,T)$ , where as usual  $N(\sigma,T)$  denotes the number of zeros  $\rho$  such that  $\beta\geq\sigma$  and  $|\gamma|\leq T$ , counted with their multiplicities. To deal with this problem we define, for x,c>0 and integral  $r\geq0$ ,

$$f_r(x) := \frac{1}{2\pi i} \int_{(c)} \Gamma(s) x^{-s} s^{-r} ds,$$
 (3.1)

where as usual  $\int_{(c)} = \lim_{T\to\infty} \int_{c-iT}^{c+iT}$ . We have  $f_0(x) = e^{-x}$  by the Mellin inversion formula of the gamma-integral, and since the integral in (3.1) is absolutely convergent we obtain

$$f'_r(x) = -\frac{1}{2\pi i} \int_{(c)} \Gamma(s) x^{-s-1} s^{1-r} \, \mathrm{d}s = -x^{-1} f_{r-1}(x) \qquad (r \ge 1). \tag{3.2}$$

For  $c \ge 1$  we have, by (3.1) and trivial estimation,  $f_r(x) \ll_c x^{-c}$ , hence by integration of (3.2) we obtain

$$f_r(x) = \int_x^\infty f_{r-1}(t) \frac{\mathrm{d}t}{t} \qquad (r \ge 1). \tag{3.3}$$

By induction on r we see from (3.2) and (3.3) that  $f_r(x)$  is positive and monotonically decreasing in  $(0, \infty)$  (and  $\lim_{x\to 0+} f_r(x) = +\infty$  for  $r \ge 1$ ). From (3.2) and (3.3) we obtain, using repeatedly integration by parts,

$$(3.4) f_r(x) = f_{r-1}(t) \log \frac{t}{x} \Big|_x^{\infty} - \int_x^{\infty} f'_{r-1}(t) \log \frac{t}{x} dt$$

$$= \int_x^{\infty} f_{r-2}(t) \log \frac{t}{x} \frac{dt}{t} = \frac{1}{2!} \int_x^{\infty} f_{r-3}(t) \log^2 \frac{t}{x} \frac{dt}{t} = \dots$$

$$= \frac{1}{r!} \int_x^{\infty} \log^r (\frac{t}{x}) e^{-t} dt \qquad (r \ge 1).$$

10 Aleksandar Ivić

From (3.3) we obtain, by induction on r,

$$f_r(x) \le x^{-r}e^{-x} \quad (x > 0, r = 0, 1, 2, \dots).$$
 (3.5)

From (3.4) we have

$$f_r(x) \ge \frac{1}{r!} \int_1^\infty \log^r \left(\frac{t}{x}\right) e^{-t} \, dt \ge \frac{\log^r \left(\frac{1}{x}\right)}{r!} \int_1^\infty e^{-t} \, dt = \frac{\log^r \left(\frac{1}{x}\right)}{r!e} \quad (0 < x \le 1, r \ge 1).$$
(3.6)

Set  $(\mu(n))$  is the Möbius function

$$M_X(s) = \sum_{n \le X} \mu(n) n^{-s} \quad (X \ge 2), \qquad a(n) = \sum_{d \mid n, d \le X} \mu(d),$$

and note that a(1)=1, a(n)=0 for  $2\leq n\leq X$  and  $|a(n)|\leq d(n)$ , the number of divisors of n. Let  $r=R+1=m(\rho)=m(\beta+i\gamma)\to\infty, \beta>\frac{1}{2}, \gamma\geq\gamma_0>0$ , and suppose that for some constant B>0

$$1 \ll X \le Y, \quad Y^{1-\beta} \ll \gamma^B. \tag{3.7}$$

Then by (3.1) and absolute convergence we obtain, since  $\Gamma(s+1) = s\Gamma(s)$ ,

$$\sum_{n=1}^{\infty} a(n) n^{-\rho} f_R(\frac{n}{Y}) = \sum_{n=1}^{\infty} a(n) n^{-\rho} \cdot \frac{1}{2\pi i} \int_{(2)} \left(\frac{Y}{n}\right)^s \Gamma(s) s^{-R} \, \mathrm{d}s$$

$$= \frac{1}{2\pi i} \int_{(2)} \zeta(\rho + s) M_X(\rho + s) Y^s \Gamma(s + 1) s^{-r} \, \mathrm{d}s.$$
(3.8)

Note that by (3.5) we have, as  $Y \to \infty$ , uniformly in R

$$\sum_{n=1}^{\infty} a(n)n^{-\rho} f_R(\frac{n}{Y}) = a(1)f_R(\frac{1}{Y}) + \sum_{n>X} a(n)n^{-\rho} f_R(\frac{n}{Y})$$

$$= f_R(\frac{1}{Y}) + \sum_{X < n \le 2Y \log Y} a(n)n^{-\rho} f_R(\frac{n}{Y}) + O\left(\sum_{n>2Y \log Y} d(n)n^{-\beta} e^{-n/Y}\right)$$

$$= f_R(\frac{1}{Y}) + \sum_{X < n \le 2Y \log Y} a(n)n^{-\rho} f_R(\frac{n}{Y}) + o(1).$$
(3.9)

Now if  $\rho$  is counted by  $N^{(r)}(\sigma, T)$ , then (3.6), (3.8) and (3.9) give that either  $|\gamma| \leq \log^2 T$  or

$$1 \ll \left| \sum_{X < n \le 2Y \log Y} b(n) n^{-\rho} \right| \qquad \left( b(n) = a(n) \frac{f_R(\frac{n}{Y})}{f_R(\frac{1}{Y})} \right)$$

or

$$1 \ll \frac{R!}{\log^R Y} \int_{-\infty}^{\infty} |\zeta(\frac{1}{2} + it + i\gamma) M_X(\frac{1}{2} + it + i\gamma)| e^{-\pi|t|/2} Y^{1/2 - \beta} \frac{\mathrm{d}t}{|\frac{1}{2} - \beta + it|^r}.$$

From the properties of  $f_r(x)$  it follows that  $|b(n)| \leq a(n)$ , and the above bounds are analogous to (11.9) and (11.10) of [1], which appear as the starting point in the derivation of upper bounds for  $N(\sigma, T)$ . The difference is that, if  $r \geq 2$  is fixed, then in the analogue of (11.10) of [1] we shall have an additional factor  $\log^{-R} Y = \log^{1-r} Y$ . Hence eventually for  $N^{(r)}(\sigma, T)$  we shall obtain the same upper bound as we would for  $N(\sigma, T)$ , only it will be smaller by a factor of  $\log^A T$ ,  $A = A(r, \sigma) > 0$  (for  $r \geq 2$  fixed). In each specific upper bound for  $N^{(r)}(\sigma, T)$  this constant  $A = A(r, \sigma)$  can be explicitly evaluated. Actually one expects that, for  $\frac{1}{2} \leq \sigma < 1$  fixed,

$$\lim_{T \to \infty} \frac{N^{(r)}(\sigma, T)}{N(\sigma, T)} = 0 \qquad (r \ge 2)$$
(3.10)

will hold, although proving this is out of reach at present. Note that the function  $f_r(x)$ , defined by (3.1), is not the only kernel function which may be used to estimate  $N^{(r)}(\sigma, T)$ , but the use of other similar functions does not appear to yield sharper results. One could also use the above method to obtain a pointwise estimate for  $m(\beta + i\gamma)$  (e.g., by choosing  $X = 2Y \log Y$  in (3.8)), but the bound that will be obtained will not be better than (2.1).

One may compare the conjectural (3.10) with the known bound (see p. 246 of [6])

$$N_r(T) \ll N(T)e^{-C\sqrt{r}},\tag{3.11}$$

where N(T) denotes the number of zeros  $\rho$  with  $0 < \gamma \le T$  and  $N_r(T)$  the number of zeros  $\rho$  with  $0 < \gamma \le T$  and  $m(\beta + i\gamma) = r$ . The methods used in obtaining (3.11) involve moments of the function S(t+h) - S(t) and do not seem to be able to yield (3.10).

## REFERENCES

- [1] A. Ivić, The Riemann zeta-function, John Wiley and Sons, New York, 1985.
- [2] S. Lang, Complex Analysis (2nd ed.), Springer, GTM 103, New York etc., 1985.
- [3] N. Levinson, Zeros of the Riemann zeta-function near the 1-line, J. Math. Anal. Appl. 25(1969), 250-253.
- [4] H.L. Montgomery, The pair correlation of zeros of the zeta-function, Proc. Symp. Pure Math. 24, AMS, Providence R.I., 1973, 181-193.
- [5] H.L. Montgomery, Extreme values of the Riemann zeta-function, Comment. Math. Helv. **52**(1977), 511-518.
- [6] E.C. Titchmarsh, The theory of the Riemann zeta-function (2nd ed.), Oxford, Clarendon Press, 1986.
- [7] K.-M. Tsang, Some  $\Omega$ -theorems for the Riemann zeta-function, Acta Arithmetica 46(1986), 369-395.

Aleksandar Ivić, Katedra Matematike RGF-a Universiteta u Beogradu, Đušina 7 11000 Beograd, Serbia and Montenegro e-mail: aivic@matf.bg.ac.yu, ivic@rgf.bg.ac.yu