The multiplicites of zeros of $\zeta(s)$ and its values over short intervals

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Professor A.A. Karatsuba (1937 – 2008), photo from Oberwolfach (1993), taken by Yoichi Motohashi

Let $r = m(\rho)$ $(\geqslant 1)$ denote the multiplicity of the complex zero $\rho = \beta + i\gamma$ of the Riemann zeta-function $\zeta(s)$. A zero ρ is simple if $m(\rho) = 1$. One may assume $\frac{1}{2} \leqslant \beta < 1, \gamma > 0$ since $\zeta(s) \neq 0$ for $\Re s \geqslant 1$, $\overline{\zeta(s)} = \zeta(\bar{s})$ and $\zeta(s) = \chi(s)\zeta(1-s)$, $\chi(s) = \frac{\Gamma(\frac{1}{2}(1-s))}{\Gamma(\frac{1}{2}s)}\pi^{s-1/2}$ (the **functional equation**).

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Namely the simplicity of zeros and the RH seem to be two statements independent of one another. Both could be **true**, or **false**, or **one true** and the **other one false**.

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He also says that the universality of $\zeta(s)$ (S.M. Voronin, 1975) should include the last conjecture, but that all these "are merely surmises".

Zeta zeros with large multiplicities, statistically speaking, are rare. A. Fujii (1981) proved that

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It seems plausible that, for any given $j \geqslant 2$, almost all zeros are simple, namely

$$N_j(T) = o\big(N(T)\big) \qquad (T \to \infty).$$

It follows when $j \to \infty$, but, in general, this is not known yet.

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Here $N_s(T)$ denotes the number of simple zeta-zeros of the form $\rho = 1/2 + i\gamma$, which are counted by $N(T)$.

There is a connection between multiplicities of zeta-zeros and the integral of $\zeta(s)$ over "very short intervals", namely lower bounds of the form

$$\int_{\delta}^{2\delta} |\zeta(\beta + i\gamma + i\alpha)|^k \, \mathrm{d}\alpha \geqslant \ell = \ell(\gamma, \delta, k) \quad (0 < \delta < \tfrac{1}{4}, k \in \mathbb{N}, \gamma \geqslant \gamma_0 > 0). \tag{1}$$

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For fixed β such that $\beta \geqslant \frac{1}{2}$, let $\mathcal{D}$ be the rectangle with vertices

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$$\frac{\zeta(\beta+i\gamma+i\alpha)}{(i\alpha)^r} = \frac{1}{2\pi i}\int_{\mathcal{D}} \Gamma(s-i\alpha)\frac{\zeta(s+\rho)}{s^r}\,\mathrm{d}s.$$

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Key fact: Since ρ is a zero of $\zeta(s)$ of multiplicity r, then the function $\zeta(s+\rho)s^{-r}$ is regular at $s=0$. Its only pole in $\mathcal{D}$ is $s=i\alpha$.

This gives

$$\zeta(\beta + i\gamma + i\alpha) \ll \alpha^r \left(\gamma(\beta - \tfrac{1}{4})^{-r} + 2^{-r}\right) \ll \alpha^r \gamma(\beta - \tfrac{1}{4})^{-r}.$$

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THEOREM 1. *If* $\beta \geqslant \frac{1}{2}$, $\gamma > \gamma_0 > 0$, $0 < \delta < 1/8$, $k \in \mathbb{N}$, *then with the notation introduced above we have*

$$m(\beta + i\gamma) = r \leqslant \frac{1}{\log\left(\frac{1}{8\delta}\right)} \left(\log\gamma - \frac{1}{k}\log\ell + O(1)\right) + O(1). \tag{2}$$

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We would like to **let** $\delta \to 0+$ in (2) and obtain

$$(3) \qquad m(\beta + i\gamma) \;=\; o(\log\gamma) \qquad (\beta \geqslant \tfrac{1}{2},\ \gamma \to \infty),$$

which is not yet known **unconditionally** in the **general case**, namely for the whole range $\frac{1}{2} \leqslant \beta < 1$.

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Obtaining (3) from (2) (or in any other way!) **seems very difficult**.

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H.L. Montgomery (1977) proved (on RH) that at least $2/3$ of the zeros ρ are simple, while H.M. Bui and D.R. Heath-Brown (2013) improved (also on RH) the constant $2/3$ to $19/27 = 0.\overline{703}$.

The classical Riemann–von Mangoldt formula says that

$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right).$$

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Here $N(T)$ is the number of zeta zeros with $0 < \gamma \leqslant T$ and we have $S(T) = \frac{1}{\pi}\arg\zeta(\frac{1}{2} + iT)$. This function regulates the **finer behavior** of $\zeta(s)$. The term $O(1/T)$ is a smooth function of T.

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If T is the ordinate of a zeta-zero, then one defines $S(T) = S(T + 0)$. Here $\arg\zeta(\frac{1}{2} + iT)$ is obtained by continuous variation along the segments joining the points $2, 2 + iT, \frac{1}{2} + iT$, starting with the value 0.

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One has the bounds

$$S(T) \ll \log T, \quad S(T) = o(\log T) \quad (\text{LH}), \quad S(T) \ll \frac{\log T}{\log\log T} \quad (\text{RH}).$$

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Recall that the bound $m(\beta + i\gamma) \ll \log\gamma$ has **not been improved yet** in the whole range $\frac{1}{2} \leqslant \beta < 1$.

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Here as usual $f(x) = \Omega_\pm(g(x))$ means that

$$\limsup_{x\to\infty} f(x)/g(x) > 0, \quad \liminf_{x\to\infty} f(x)/g(x) < 0,$$

and $g(x) > 0$ for $x \geqslant x_0 > 0$.

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The Ω–results are due to H.L. Montgomery (1977) and K.-M. Tsang (1986), respectively.

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Conjecture 1. There exists a positive function $\Delta = \Delta(T) \to 0$ as $T \to \infty$ such that, for some constant $A > 0$,

$$F(T,\Delta) \;\geqslant\; T^{-A}.$$

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These conjectures have not been proved unconditionally yet. Clearly Conjecture 3 implies Conjecture 2, which in turn implies Conjecture 1.

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Shao-Ji Feng (2004) proved that the LH implies Conjecture 1 with an arbitrary constant $A > 0$. Other relevant works are due to M.E. Changa, B. Kerr and M.A. Korolev.

THEOREM 2. *If Conjecture 1 holds, then*

$$m(\tfrac{1}{2} + i\gamma) \;=\; o(\log\gamma) \qquad (\gamma \to \infty). \tag{5}$$

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The connection between $F(T,\Delta)$ and integrals over short intervals is **easy**:

$$\int_\delta^{2\delta} |\zeta(\tfrac{1}{2}+i\gamma+i\alpha)|^k \,\mathrm{d}\alpha \;=\; \int_0^\delta |\zeta(\tfrac{1}{2}+i\gamma+i\delta+ix)|^k \,\mathrm{d}x \;\leqslant\; \delta F^k(\gamma+\delta,\delta),$$

where $k,\gamma>0$.

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Thus the Karatsuba conjectures have less stringent counterparts involving the above integral.

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which is valid unconditionally for

$$-1 \leqslant \sigma \leqslant 2, s \neq \rho, -\pi < \Im\log(s-\rho) \leqslant \pi,$$

where $\rho = \beta + i\gamma$ denotes complex zeros of $\zeta(s)$.

To get rid of the logarithms one uses (this is a consequence of the arithmetic-geometric means inequality)

$$\log\Big\{\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\Big\} \geqslant \frac{1}{b-a}\int_a^b \log f(t)\,\mathrm{d}t$$

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$$\begin{aligned} m(\beta + i\gamma) &\leqslant C + \frac{13.35\beta}{3(1-\beta)\log 6 + \beta \log 2}(1-\beta)^{3/2}\log\gamma \\ &+ \frac{7(3-2\beta)+\varepsilon}{9(1-\beta)\log 6 + 3\beta\log 2}\log\log\gamma. \end{aligned}$$

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Corollary 2. If $m(\beta + i\gamma) \geqslant 8\log\log\gamma$ for $5/6 \leqslant \beta < 1$ and $\gamma \geqslant \gamma_2$, then

$$\beta \leqslant 1 - \left(\frac{m(\beta + i\gamma)}{40\log\gamma}\right)^{2/3}.$$

Starting point of proof: Let $\beta \geqslant 5/6$, $r = m(\beta + i\gamma)$ and $\mathcal{E}$ be the rectangle with vertices $-2(1-\beta) \pm 2i\log^2\gamma$, $1 \pm 2i\log^2\gamma$. If X, with $0 < X \ll \gamma^C$, is a parameter which will be suitably chosen, then by the residue theorem we obtain

$$\frac{\zeta(1-\beta+\rho)}{(1-\beta)^r} = \frac{1}{2\pi i}\int_{\mathcal{E}} X^{s-1+\beta}\Gamma(s-1+\beta)\frac{\zeta(s+\rho)}{s^r}\,\mathrm{d}s \quad (\rho = \beta + i\gamma).$$

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To bound the zeta-factor we shall use the explicit inequality

$$|\zeta(\sigma+it)| \;\leqslant\; At^{B(1-\sigma)^{3/2}}\log^{2/3} t \qquad (t \geqslant 3, \quad \tfrac{1}{2} \leqslant \sigma \leqslant 1).$$

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To bound $\zeta(1+i\gamma)$ we also use another consequence of Vinogradov's method (zero-free region for $\zeta(s)$):

$$\zeta(1+it) \gg (\log|t|)^{-2/3}(\log\log|t|)^{-1/3}.$$

Thank you for your attention!