2. Properties of the Jensen function $\Phi(t)$

A large number of properties can be associated with the Jensen functions $\phi_J(t)$ (1.6.1) or $\Phi(t)$ (1.6.3) appearing in the Fourier cosine transform representation of the Riemann entire function $\xi(s)$. It is convenient to list them here with references to previous publications and to related sections in this document. Conjectures are indicated by a question mark.

2.1. Definition.

$$\Phi(t) := \frac{\phi_J(2t)}{4} := \sum_{n=1}^{\infty} \left(2n^4\pi^2 e^{9t} - 3n^2\pi e^{5t}\right) \exp(-n^2\pi e^{4t}).$$

The rescaling $t \mapsto 2t$ is the norm in current publications on $\xi(s)$ (§1.2, §1.6). This leaves most properties invariant while avoiding distracting fractions, and we will use $\Phi(t)$ exclusively. Another simplification comes from defining an auxiliary variable which appears implicitely in [175, 97, 60, 161, 51] :

$$\Phi(t) := e^t \sum_{n=1}^{\infty} \left(2n^4u^2 - 3n^2u\right) e^{-n^2u}, \qquad (u = \pi e^{4t}).$$

2.2. Theta function. $16\,\Phi(t) = (D^2-1)\{e^t\theta(e^{4t})\}$, where $\theta(x) := \sum_{-\infty}^{\infty} e^{-n^2\pi x}$.
From Pólya–Jensen [174, p. 10–11], from §1.6, or simply, with $y = e^t$ and $D := d/dt$,

$$(D+1)\cdot(D-1)\cdot y\,\theta(y^4) = \left(y^{-1}Dy\right)\cdot\left(yDy^{-1}\right)\cdot y\,\theta(y^4) = 8\left(2y^9\theta''(y^4) + 3y^5\theta'(y^4)\right).$$

This yields the even parity property in §2.10 from the transformation equation $\theta(1/x) = \sqrt{x}\theta(x)$ (1.2.2) for the Jacobi theta function (1.2.1).

2.3. Analyticity. $\Phi(z)$ is analytic in the horizontal band $B = \{z \in \mathbb{C} \mid |\Im z| < \frac{\pi}{8}\}$.
The series $\sum_{n=1}^{\infty} e^{-n^2u}$ is analytic for $\Re u > 0$ from [174, p. 12], or from §3.2.

2.4. Singularities. $\Phi(z)$ has essential singularities on the boundary of B.
The Jacobi theta function (1.2.1) satisfies $\theta(i+h) = 2\theta(4h) - \theta(h)$ which, after the transformation (1.2.2), is asymptotic to $\exp(-1/h)/\sqrt{h}$ as $h \to 0$ [174, p. 13]. Moreover, all the derivatives of $\theta(x)$ also tend to zero as $x \to i$ [94], [139, p. 219].

2.5. Fourier transform. $\Xi(x/2) = 8\int_0^{\infty} \Phi(t)\cos(xt)\,dt$ for $x \in \mathbb{C}$.
Reformulation by Jensen of Riemann's integral representation (1.1.8) for $\Xi(t)$ (§1.6).

2.6. Fourier transform. $\Phi(t) = 1/(4\pi)\int_0^{\infty} \Xi(x/2)\cos(tx)dx$ for $t \in B$.
By inversion of the cosine Fourier transform in §2.5 or from [210, p. 36].

2.7. Limits. $\lim_{t\to\infty} \Phi^{(n)}(t)\exp(4(\pi-\epsilon)t) = 0$ for all $\epsilon > 0$ and all $n = 0, 1, 2, \ldots$
From Jensen [174, p. 11].

2.8. Asymptotics. $\Phi(t) = O\left(\exp(9|t| - \pi e^{4|t|})\right)$ for $|t| \to \infty$.
From Pólya [172, p. 305].

2.9. Approximation. $\Phi(t) \approx 16\pi^2\cosh(9t)\exp(-2\pi\cosh 4t)$ for $t \to \infty$.
From Pólya [172, p. 305].

2.10. Parity. $\Phi(z) = \Phi(-z)$ for $z \in B$.
Already known by Hurwitz in 1899 [174, footnote, p. 11] and "not hard (although somewhat tedious)" [200, p. 498] to derive from the transformation equation (1.2.2) of the Jacobi function $\theta(x)$ (1.2.1) – see §1.6. Pólya made this a corollary of the Hardy theorem in §1.7 on the infinite number of real roots of $\Xi(t)$ ([173, p. 99], [174, p. 15]).

2.11. Parity. $\Phi^{(k)}(0) = 0$ for $k = 1, 3, 5, \ldots$
Equivalent, with the analyticity property of §2.3, to the even parity property in §2.10.

2.12. **Maclaurin coefficients.** $\Phi^{(2k)}(0)/\theta(1)$ is a rational polynomial in $\Omega := \pi^2\theta(1)^8/2 \approx 9.57$. From §2.2 and an expression found by Romik [190, 191],

$$(-8)^k \frac{\theta^{(k)}(1)}{\theta(1)} = \sum_{j=0}^{[k/2]} \frac{(2k)!}{(k-2j)!} \frac{(4\Omega)^j}{(4j)!} d_j, \qquad (d_j \text{ integer } = 1, 1, -1, 51, 849, -26199, \ldots).$$

2.13. **Positivity.** $\Phi(t) > 0$ for $t \in \mathbb{R}$.
Obvious from the series definition above [174, p. 12], or from the positivity of the factor $f_0(u)$ of $\Phi(t)$ represented with S-series (§2.46, §4.6, §5.1.1). See §2.27 for a more precise inequality.

2.14. **Monotonicity.** $\Phi'(t) < 0$ for $t > 0$.
From Wintner [221], rediscovered by Spira [200], with many more proofs (§3.6, §3.7, §3.12, §4.7.3, §5.1.6) based on the positivity for $u > \pi$ of the factor $f_1(u)$ of $-\Phi'(t)$ (§2.47). Corollary of (2.17) since $\Phi'(t)/\Phi(t)$ is then decreasing. See §2.29 and §2.30 for more precise inequalities.

2.15. **Concavity1.** $\Phi(t)$ is strictly concave for $|t| < t_2$, where $t_2 \approx 0.12$.
Equivalent to the negativity of $\Phi''(t)$ on the interval $0 \le t < t_2$ in §2.37.

2.16. **Convexity1.** $\Phi(t)$ is strictly convex for $|t| > t_2$, where $t_2 \approx 0.12$.
Equivalent to the positivity of $\Phi''(t)$ on the interval $t > t_2$ in §2.37.

2.17. **Concavity2.** $\Phi(t)$ is strictly log-concave for $t \in \mathbb{R}$.
From Csordas [58] as a corollary of the log-concavity of $\Phi(\sqrt{t})$ for $t > 0$ (§2.19), from an independent proof of Coffey and Csordas [51], from the log-concavity of the factor $f_0(u)$ of $\Phi(t)$ (§3.7), from the upper inequality for $\Phi''(t)$ in §2.32, or from $W(\Phi(t), \Phi'(t)) < 0$ (§2.24). Implies $\Phi'(t) < 0$ (§2.14).

2.18. **Concavity3.** $(\Phi^{(k+1)}(t))^2 - \Phi^{(k)}(t)\Phi^{(k+2)}(t) > 0$ for $t \in \mathbb{R}$ and $k \ge 0$?
From Coffey and Csordas [51, Conjecture 2.5, p. 5]. This inequality cannot be satisfied for $t = 0$ and $k = 8$ due to the two extra zeros of $\Phi^{(9)}(t)$ from the Spira observation in §2.44.

2.19. **Concavity4.** $\Phi(\sqrt{t})$ is strictly log-concave for $t > 0$.
From Csordas and Varga [62, p. 197], and follows from the inequality in §2.33 as shown in §4.12.

2.20. **Concavity5.** $-W(\Phi(\sqrt{t}), (\Phi(\sqrt{t}))')$ is strictly log-concave for $t > 0$?
Conjecture of Csordas and Dimitrov [59, Problem 3.3], validated in §5.25.

2.21. **Convexity2.** $\Phi(\sqrt{t})$ is strictly convex for $t > 0$.
From Csordas [58], and proven in §3.13.4.

2.22. **Concavity6.** $(\log \Phi(t))''' < 0$ for $t > 0$.
From Newman [161, Theorem 1], proven in §4.13 and in §5.20; sufficient for the Wronskian property in §2.25 as shown in [54, Theorem 1].

2.23. **Concavity7.** $(\log \Phi(t))^{iv} < 0$ for $t \ge 0$.
From Newman [161, Proposition 5] for $0 < t < 1/40$, from §5.21 for $t \ge 0$, and sufficient for the concavity property in §2.22.

2.24. **Wronskian1.** $W(\Phi(t), \Phi'(t)) < 0$ for $t \in \mathbb{R}$.
Equivalent to the log-concavity property of $\Phi(t)$ in §2.17 as shown in §4.11.1, and proven in §4.11.

2.25. **Wronskian2.** $W(t\Phi(t), \Phi'(t)) < 0$ for $t > 0$.
From Matiyasevich [153], and equivalent to the log-concavity of $\Phi(\sqrt{t})$ in §2.19 as shown in §4.12.1. Proven in §4.12 and in §5.22.

2.26. **Wronskian3.** $W(t\Phi(t), \Phi'(t), t\Phi''(t)) < 0$ for $t > 0$?
Conjecture.

2.27. **Inequality0.** If $u = \pi e^{4t}$ and $t \geq 0$, then

$$(2u^2 - 3u)e^{t-u} < \Phi(t) < (2u^2 - 3u)e^{t-u} + 32u^2 e^{t-4u}.$$

A more precise version of the positivity property in §2.13, equivalent to the inequalities in §2.46 for the factor $f_0(u)$ of $\Phi(t)$.

2.28. **Inequality0m.** If $u = \pi e^{4t}$, $m \geq 1$ and $t \geq 0$, then

$$2 - \frac{3}{m^2 u} < \frac{\Phi_m(t)}{m^4 u^2 e^{t-m^2 u}} < 2 - \frac{3 - 10^{-2}}{m^2 u}, \qquad (\Phi_m(t) = \sum_{n=m}^{\infty} \left(2n^4 u^2 - 3n^2 u\right) e^{t-n^2 u}).$$

From Haviland with $m = 2$ [97, eq. 8], and proven for all m in §3.8.

2.29. **Inequality1.** If $u = \pi e^{4t}$ and $t > 0$, then

$$4\,\Phi(t) < -\Phi'(t)/(u - \pi) < 4\,\Phi(t) + (8\pi - 18)ue^{t-u}.$$

A more precise version of the monotonicity property of §2.14, equivalent to the inequalities in §2.47 for the factor $f_1(u)$ of $-\Phi'(t)$.

2.30. **Inequality1a.** If $u = \pi e^{4t}$ and $t > 0$, then for some positive constants λ_j with values between 3 and 10 given explicitely as polynomials in π in §3.12.7,

$$-2\frac{\lambda_5}{u^5} - \frac{\lambda_4}{u^4} < \frac{\Phi'(t)}{(u-\pi)\Phi(t)} + 4 + \frac{\lambda_1}{u} + \frac{\lambda_2}{u^2} + \frac{\lambda_3}{u^3} < 0, \qquad (u > \pi).$$

This is a different version of the inequalities of §2.29, equivalent to the inequalities in §2.48 for the factor $f_1(u)$ of $-\Phi'(t)$.

2.31. **Inequality1m.** If $u = \pi e^{4t}$, $m \geq 1$ and $t \geq 0$, then

$$-\frac{1}{m^4 u^2} < \frac{\Phi_m'(t)}{m^6 u^3 e^{t-m^2 u}} + q_1(m^2 u) < 0, \qquad (q_1(u) = 8 - \frac{30}{u} + \frac{15}{u^2}, \Phi_m \text{ as in §2.28}).$$

From [60, lemma 3.3] and [51, Proposition 2.1] with $m = 2$, and proven for all m in §3.9.

2.32. **Inequality2.** If $u = \pi e^{4t}$ and $t \geq 0$, then

$$-32u^3 e^{t-u} < \Phi''(t) + 8(u - \pi)\Phi'(t) + 16(u - \pi)^2 \Phi(t) < 0.$$

These follow from the inequalities in §2.49 for the factor $f_2(u)$ of $\Phi''(t)$. The right-hand side inequality is sufficient for the log-concavity property of $\Phi(t)$ in §2.17, and more precise versions are provided in §2.33 and §4.11.7.

2.33. **Inequality2a.** If $u = \pi e^{4t}$ and $t > 0$, then

$$\Phi''(t) + 8(u - \pi)\Phi'(t) + 16(u - \pi)^2 \Phi(t) < \frac{\Phi'(t)}{t}.$$

Sufficient for the log-concavity property of $\Phi(\sqrt{t})$ in §2.19 , and proven in §4.12, §5.22.

2.34. **Inequality2m.** If $u = \pi e^{4t}$, $m \geq 1$ and $t \geq 0$, then

$$0 < \frac{\Phi_m''(t)}{m^8 u^4 e^{t-m^2 u}} - q_2(m^2 u) < \frac{11}{m^6 u^3}, \qquad (q_2(u) = 32 - \frac{224}{u} + \frac{330}{u^2} - \frac{75}{u^3}, \Phi_m \text{ as in §2.28}).$$

From [62, lemma 3.1] and [51, Proposition 2.1] with $m = 2$, and proven in §3.10.

2.35. **Zeros0.** $\Phi(t)$ has no real zero and is positive as $t \to \infty$.
From the positivity property, the inequality for $\Phi(t)$ in §2.27, or §4.6.

2.36. **Zeros1.** $\Phi'(t)$ has one simple zero for $t \in \mathbb{R}$ and is negative for $t > 0$.
From the parity property in §2.10 and the monotonicity property in §2.14, the inequalities for $-\Phi'(t)$ in §2.29, or §4.7.3. The zero is $t = 0$.

2.37. **Zeros2.** $\Phi''(t)$ has two simple zeros for $t \in \mathbb{R}$ and is positive as $t \to \infty$.
From the properties of the factor $f_2(u)$ of $\Phi''(t)$ in §2.49. The zeros are $t \approx \pm 0.12$.

2.38. **Zeros3.** $\Phi^{(3)}(t)$ has three simple zeros for $t \in \mathbb{R}$ and is negative as $t \to \infty$.
From the property in §2.50. The zeros are $t = 0$ and $t \approx \pm 0.20$.

2.39. **Zeros4.** $\Phi^{(4)}(t)$ has four simple zeros for $t \in \mathbb{R}$ and is positive as $t \to \infty$.
From the property in §2.51. The zeros are $t \approx \pm 0.10$ and $t \approx \pm 0.27$.

2.40. **Zeros5.** $\Phi^{(5)}(t)$ has five simple zeros for $t \in \mathbb{R}$ and is negative as $t \to \infty$.
From the property in §2.52. The zeros are $t = 0$, $t \approx \pm 0.17$, and $t \approx \pm 0.32$.

2.41. **Zeros6.** $\Phi^{(6)}(t)$ has six simple zeros for $t \in \mathbb{R}$ and is positive as $t \to \infty$.
From the property in §2.53. The zeros are $t \approx \pm 0.095$, $t \approx \pm 0.24$, and $t \approx \pm 0.37$.

2.42. **Zeros7.** $\Phi^{(7)}(t)$ has seven simple zeros for $t \in \mathbb{R}$ and is negative as $t \to \infty$.
From the property in §2.54. The zeros are $t = 0$, $t \approx \pm 0.17$, $t \approx \pm 0.29$, and $t \approx \pm 0.41$.

2.43. **Zeros8.** $\Phi^{(8)}(t)$, has eigth simple zeros for $t \in \mathbb{R}$ and is positive as $t \to \infty$.
From the property in §2.55. The zeros are $t \approx \pm 0.11$, $t \approx \pm 0.23$, $t \approx \pm 0.33$, and $t \approx \pm 0.44$.

2.44. **Zeros9.** $\Phi^{(9)}(t), t \in \mathbb{R}$, has eleven simple zeros for $t \in \mathbb{R}$ and is negative as $t \to \infty$.
Observation of Spira [200, p. 500] based on "rough calculations", seen to be valid from the property in §2.56. The zeros are $t = 0$, $t \approx \pm 0.06$, $t \approx \pm 0.17$, $t \approx \pm 0.28$, $t \approx \pm 0.37$, and $t \approx \pm 0.47$.

2.45. **Zeros1–8.** $\Phi^{(k)}(t), t \in \mathbb{R}$, has k simple zeros for $k = 0(1)8$.
Observation of Spira [200, p. 500] based on "rough calculations", which is valid from the properties in §2.46–§2.55, or simply from the property in §2.55 – see §5.14.

2.46. **Factor0.** $\Phi(t) = e^{t-u}u^2 f_0(u), u = \pi e^{4t}$ where for $u \geq \pi$, $f_0(u)$ is positive, concave, log-concave, increasing from[2] $e^{\pi}\theta(1)/16/\pi^2(\Omega - 3) \approx 1.0473$ up to 2 as $u \to \infty$, and satisfies

$$2 - \frac{3}{u} < f_0(u) < 2 - \frac{3}{u} + 32e^{-3u}, \quad (u \geq \pi).$$

Proven in §3.4, and yielding the inequalities in §2.27.

2.47. **Factor1.** $-\Phi'(t) = e^{t-u}u^3 f_1(u), u = \pi e^{4t}$ where for $u \geq \pi$, $f_1(u)$ is concave, log-concave, increasing from 0 up to 8 as $u \to \infty$, has one simple zero at $u = \pi$, and satisfies

$$4f_0(u) < \frac{u\, f_1(u)}{u - \pi} < 4\, f_0(u) + \frac{8\pi - 18}{u}, \quad (u > \pi)$$

From §3.12.5, §3.12.6, with a second proof of the inequalities in §5.17, and yielding the inequalities in §2.29. See §2.48 for more precise inequalities.

2.48. **Factor1a.** $-\Phi'(t) = e^{t-u}u^3 f_1(u), u = \pi e^{4t}$ where $f_1(u)$ satisfies, for some positive constants λ_j with values between 3 and 10,

$$0 < \frac{uf_1(u)}{(u-\pi)f_0(u)} - 4 - \frac{\lambda_1}{u} - \frac{\lambda_2}{u^2} - \frac{\lambda_3}{u^3} < \frac{\lambda_4}{u^4} + 2\frac{\lambda_5}{u^5}, \qquad (u > \pi).$$

Equivalent to §2.30 and proven in §3.12.7, where the constants are given as polynomials in π.

2.49. **Factor2.** $\Phi''(t) = e^{t-u}u^4 f_2(u), u = \pi e^{4t}$ where for $u \geq \pi$, $f_2(u)$ is increasing up to 32 from $-e^{\pi}\theta(1)/16/\pi^4(\Omega^2 + 45\Omega - 30) \approx -7.949$ with a zero at $u \approx 5.05$, $uf_2(u)$is convex, and satisfies

$$0 < u^2 f_2(u) - 8(u-\pi)uf_1(u) + 16(u-\pi)^2 f_0(u) + 32u < 2(4\pi - 9)^2 + 48, \quad (u \geq \pi).$$

From §3.11, §4.11, or §5.18, and yielding the inequalities in §2.32. A more precise version of the inequalities is proven in §4.11.6.

2.50. **Factor3.** $-\Phi^{(3)}(t) = e^{t-u}u^5 f_3(u), u = \pi e^{4t}$ where for $u \geq \pi$, $f_3(u)$ has two simple zeros at $u = \pi$ and $u \approx 7.08$, tends to 128 as $u \to \infty$, and $uf_3(u)$ is convex.
From §3.13, with other proofs of the convexity in §4.9 or §4.14.5, and of the decreasing property of $uf_3(u)$ on $(3, 4)$ in §5.8.

2.51. **Factor4.** $\Phi^{(4)}(t) = e^{t-u}u^6 f_4(u), u = \pi e^{4t}$ where $f_4(u)$ has two simple zeros for $u \geq \pi$ at $u \approx 4.63$ and $u \approx 9.21$, is positive on $[\pi, 4.6125]$, tends to 512 as $u \to \infty$ from $f_4(\pi) = e^{\pi}\theta(1)/16/\pi^4(51\Omega^3 + 191\Omega^2 + 3600\Omega - 1140) \approx 156.4$, and $uf_4(u)$ is convex for $u \geq 17/4$.
From §3.14, with other proofs of the convexity in §4.10.2 and of the positivity in §5.9.

[2]See §2.12, $\Omega := \pi^2\theta(1)^8/2 = \Gamma(\frac{1}{4})^8/(32\pi^4)$ from [190, p. 2].

2.52. **Factor5.** $-\Phi^{(5)}(t) = e^{t-u}u^7 f_5(u), u = \pi e^{4t}$ where $f_5(u)$ has three simple zeros for $u \geq \pi$, is positive on $(\pi, 25/4]$, tends to 2^{11} as $u \to \infty$, and $uf_5(u)$ is convex for $u \geq 23/4$.
Proven in §5.11.

2.53. **Factor6.** $\Phi^{(6)}(t) = e^{t-u}u^8 f_6(u), u = \pi e^{4t}$ where $f_6(u)$ has three simple zeros for $u \geq \pi$, is negative on $(3, 4)$ but positive on $(19/4, 31/4]$, tends to 2^{13} as $u \to \infty$, and $uf_6(u)$ is convex for $u \geq 31/4$.
Proven in §5.12.

2.54. **Factor7.** $-\Phi^{(7)}(t) = e^{t-u}u^9 f_7(u), u = \pi e^{4t}$ where $f_7(u)$ has four simple zeros for $u \geq \pi$, is negative on $(\pi, 19/4]$ but positive on $[25/4, 10]$, tends to 2^{15} as $u \to \infty$, and $uf_7(u)$ is convex for $u \geq 10$.
Proven in §5.13.

2.55. **Factor8.** $\Phi^{(8)}(t) = e^{t-u}u^{10} f_8(u), u = \pi e^{4t}$ where $f_8(u)$ has four simple zeros for $u \geq \pi$; it is positive on $[\pi, 31/8]$, decreasing on $[31/8, 5]$, negative on $[5, 6]$, increasing on $[6, 8]$, positive on $[8, 10]$, decreasing on $[9.5, 13]$; it tends to 2^{17} as $u \to \infty$, and $uf_8(u)$ is convex for $u \geq 12.5$.
Proven in §5.14, and implying that $f_k(u)$ has exactly k simple zeros for $u \in (0, \infty)$ and $0 \leq k \leq 8$.

2.56. **Factor9.** $-\Phi^{(9)}(t) = e^{t-u}u^{11} f_9(u), u = \pi e^{4t}$ where $f_9(u)$ has six simple zeros for $u \geq \pi$; it is decreasing on $[3, 13/4]$, negative on $(\pi, 15/4]$, increasing on $[15/4, 17/4]$, positive on $[17/4, 6]$, decreasing on $[6, 13/2]$, negative on $[13/2, 9]$, increasing on $[9, 10]$, positive on $[10, 14]$, and decreasing on $[12, 16]$; it tends to 2^{19} as $u \to \infty$, and $uf_9(u)$ is convex for $u \geq 15$.
Proven in §5.15.