7.4.4. Pólya's counterexample of growth order two with parameter α in $(-1,1]$,

$$\begin{aligned}
F_{4,\alpha}(z) &:= e^{z^2/2}\,(\cosh z+\alpha)\\
&= \int_0^\infty \Psi_\alpha(t)\,\cosh zt\,dt, \quad \Big(\Psi_\alpha(t) = 2\frac{e^{-t^2/2}}{\sqrt{2\pi e}}\,\big[\cosh(t)+\alpha\sqrt{e}\big]\Big)\\
&= \sum_{k=0}^\infty N_{k,\alpha}\frac{z^{2k}}{(2k)!}, \qquad \Big(N_{k,\alpha} = \int_0^\infty \Psi_\alpha(t)t^{2k}\,dt\Big)\\
&= (\alpha+1)\,e^{z^2/2}\prod_{n=1}^\infty\left(1+\frac{z^2}{\delta_{n,\alpha}^2}\right), \qquad (\delta_{n,\alpha}>0, -1<\alpha\le 1).
\end{aligned}$$

Proof: The infinite product of $\cosh z+\alpha$ was obtained by Euler [79, p. 120] and can be derived simply from the infinite product of $\cosh z$. If $-1<\alpha\le 1$, let $\alpha=\cos\theta=\cosh i\theta$ where $0\le\theta<\pi$. Then, after "factorizing and rearranging a little" [213, p. 81], we get

$$\begin{aligned}
\frac{\cosh z+\alpha}{1+\alpha} &= \frac{2}{1+\alpha}\cosh\frac{z+i\theta}{2}\cosh\frac{z-i\theta}{2} = \prod_{n=1}^\infty\left(1+\frac{z^2}{((2n-1)\pi\pm\theta)^2}\right)\\
&= \prod_{n=-\infty}^\infty\left(1+\frac{z^2}{((2n-1)\pi+\theta)^2}\right).
\end{aligned}$$

For $\alpha=\cosh\theta, \theta\ge 0$, we can write, following Euler [79, p. 122],

$$\frac{\cosh z+\alpha}{1+\alpha} = \prod_{n=1}^\infty\left(1+\frac{z^2\pm 2\theta z}{((2n-1)\pi)^2+\theta^2}\right) = \prod_{n=1}^\infty\left(1+\frac{z^2}{((2n-1)\pi\pm i\theta)^2}\right).$$

In a first proof, Euler proceeded in fact directly with his quadratic factorization method, starting from his general formula of $n=2p+1$ *distinct* factors [3, p. 84]

$$X^{2n}-2X^nY^n\cos g+Y^{2n} = \prod_{k=-p}^{p}\left(X^2-2XY\cos\frac{2k\pi+g}{n}+Y^2\right).$$

This yields, with $g=\theta-\pi$, $n=2p+1$, $X=1+z/(2n)$, $Y=1-z/(2n)$,

$$\begin{aligned}
&\frac{\left(1+\frac{z}{2n}\right)^{2n}+2\left(1+\frac{z}{2n}\right)^n\left(1-\frac{z}{2n}\right)^n\cos\theta+\left(1-\frac{z}{2n}\right)^{2n}}{2(1+\cos\theta)}\\
&= \prod_{k=-p}^{p}\left(1+\frac{z^2}{z_k^2}\frac{1}{\gamma_{k,n}^2}\right) \qquad \left(z_k=(2k-1)\pi+\theta, \gamma_{k,n}=\frac{\tan(z_k/n)}{z_k/n}\right).
\end{aligned}$$

Letting $n=2p+1$ increase without bound,

$$\begin{aligned}
\frac{\cosh z+\cos\theta}{1+\cos\theta} &= \frac{e^z+2e^{z/2}e^{-z/2}\cos\theta+e^{-z}}{2(1+\cos\theta)} = \prod_{k=-\infty}^\infty\left(1+\frac{z^2}{((2k-1)\pi+\theta)^2}\right)\\
&= \prod_{k=1}^\infty\left(1+\frac{z^2}{((2k-1)\pi\pm\theta)^2}\right).
\end{aligned}$$

In this case, the z-polynomial of degree $2n$ is not a Jensen polynomial, but the convergence of the square of the roots of the scaled polynomials is still monotone in n.

We note that for the critical parameter value $\alpha=1$, *all* zeros of $G_{4,1}(w)=F_{4,1}(\sqrt{w})$ become simultaneously double zeros, and that the polynomials used by Euler have the same property :

$$\frac{\cosh z+1}{2} = \cosh^2\frac{z}{2} = \lim_{n\to\infty}\Big[g_n(\cosh;\frac{z}{2n})\Big]^2 = \lim_{n\to\infty}\left[\frac{\left(1+\frac{z}{2n}\right)^n+\left(1-\frac{z}{2n}\right)^n}{2}\right]^2.$$

[79] Leonhard Euler. Introduction à l'analyse infinitésimale I{II. Barois, Paris, 1797.
French translation of Introductio in Analysin Infinitorum, 1748.
[213] V.S. Varadarajan. Euler through time: a new look at old themes. American Mathematical Society, 2006.

If α increases towards 1, two real zeros of $G_{4,\alpha}(-w)$ converge along the real line from both sides towards the square of each odd multiple of π, and if α increases further from 1, two complex zeros diverge from both sides along a vertical line away from the square of each odd multiple of π.

Finally, we give the Maclaurin coefficients in terms of the parameter α :

$$N_{k,\alpha} = \alpha\,(2k-1)!! + \sum_{j=0}^{k} \binom{2k}{2j} (2j-1)!!,$$

where $(2k-1)!! = (2k)!/(k!2^k)$ is the product of the first k odd integers [2, 6.1.49]. Abramovitch+Stegun 1964