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## PSEUDODIFFERENTIAL ARITHMETIC AND THE RIEMANN HYPOTHESIS: REMINDERS

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**ABSTRACT.** The present preprint completes the arXiv: 2202.11652 preprint, entitled “Pseudodifferential arithmetic and the Riemann hypothesis”, devoted to a proof of the conjecture. The first 4 pages of that preprint make up a set of necessary reminders, given in a very concise way: we give here a self-contained, fully developed, version of this part.

### 1. INTRODUCTION

The paper is based on the consideration of the distribution

$$\mathfrak{T}_\infty(x, \xi) = \sum_{|j|+|k| \neq 0} a((j, k)) \delta(x - j) \delta(\xi - k) \quad (1.1)$$

in the plane, with  $(j, k) = \text{g.c.d.}(j, k)$  and  $a(r) = \prod_{p|r} (1 - p)$  for  $r = 1, 2, \dots$ . There is a collection  $(\mathfrak{E}_\nu)_{\nu \neq \pm 1}$  of so-called Eisenstein distributions,  $\mathfrak{E}_\nu$  homogeneous of degree  $-1 - \nu$ , such that, as an analytic functional,

$$\mathfrak{T}_\infty = \frac{1}{2i\pi} \int_{\text{Re } \nu = c > 1} \frac{\mathfrak{E}_{-\nu}}{\zeta(\nu)} d\nu = 12 \delta_0 + \sum_{\zeta(\rho)=0} \text{Res}_{\nu=\rho} \left( \frac{\mathfrak{E}_{-\nu}}{\zeta(\nu)} \right), \quad (1.2)$$

where  $\delta_0$  is the unit mass at the origin of  $\mathbb{R}^2$ . This decomposition immediately points towards a possible link between  $\mathfrak{T}_\infty$  and the zeros of the Riemann zeta function: only, to obtain a full benefit of this distribution, one must appeal to pseudodifferential analysis, more precisely to the Weyl symbolic calculus of operators. This is the rule  $\Psi$  that associates to any distribution  $\mathfrak{S} \in \mathcal{S}'(\mathbb{R}^2)$  the linear operator  $\Psi(\mathfrak{S})$  from  $\mathcal{S}(\mathbb{R})$  to  $\mathcal{S}'(\mathbb{R})$  weakly defined by the equation

$$(\Psi(\mathfrak{S}) u)(x) = \frac{1}{2} \int_{\mathbb{R}^2} \mathfrak{S} \left( \frac{x+y}{2}, \xi \right) e^{i\pi(x-y)\xi} u(y) dy d\xi. \quad (1.3)$$

It defines pseudodifferential analysis, which has been for more than half a century one of the main tools in the study of partial differential equations. The methods used in the present context do not intersect the usual ones

(and nothing would be gained here from previous acquaintance with the subject) and may call for the denomination of “pseudodifferential arithmetic” (Sections 6 and 7).

Eisenstein distributions will be described in detail in Section 3. They make up just a part of the domain of automorphic distribution theory, which relates to the classical one of modular form theory but is more precise. If one defines the Euler operator  $2i\pi\mathcal{E} = 1 + x\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial \xi}$  in the plane, the operator  $\pi^2\mathcal{E}^2$  transfers under some map (the dual Radon transformation, in an arithmetic context) to the operator  $\Delta - \frac{1}{4}$ , where  $\Delta$  is the automorphic Laplacian. While this is crucial in other applications [7], it is another feature of automorphic distribution theory that will be essential here: the way it can cooperate with the Weyl symbolic calculus.

We shall make use especially of the collection of rescaling operators  $Q^{2i\pi\mathcal{E}}$ , with

$$(Q^{2i\pi\mathcal{E}}\mathfrak{S})(x, \xi) = Q\mathfrak{S}(Qx, Q\xi), \quad Q > 0. \quad (1.4)$$

In [6], attention was brought to the hermitian form  $(w | \Psi(Q^{2i\pi\mathcal{E}}\mathfrak{T}_\infty) w)$ . The following criterion for R.H. was obtained: that, given  $\beta > 2$  and any function  $w \in C^\infty(\mathbb{R})$  supported in  $[0, \beta]$ , one should have

$$(w | \Psi(Q^{2i\pi\mathcal{E}}\mathfrak{T}_\infty) w) = O\left(Q^{\frac{1}{2}+\varepsilon}\right) \quad (1.5)$$

as  $Q \rightarrow \infty$  through squarefree integral values. We shall reprove this criterion in Section 5.

The next step consists in transforming the left-hand side of (1.5) into a finite-dimensional hermitian form. Given a positive integer  $N$ , one sets

$$\mathfrak{T}_N(x, \xi) = \sum_{j, k \in \mathbb{Z}} a((j, k, N)) \delta(x - j) \delta(\xi - k). \quad (1.6)$$

The distribution  $\mathfrak{T}_\infty$  is obtained as the limit, as  $N \nearrow \infty$  (a notation meant to express that  $N$  will be divisible by any given squarefree number, from a certain point on), of the distribution  $\mathfrak{T}_N^\times$  obtained from  $\mathfrak{T}_N$  by dropping the term corresponding to  $j = k = 0$ . If  $Q$  is squarefree, if the functions  $v, u \in C^\infty(\mathbb{R})$  are supported in  $[0, \beta]$ , finally if  $N = RQ$  is a squarefree integer divisible by all primes less than  $\beta Q$ , one has

$$(v | \Psi(Q^{2i\pi\mathcal{E}}\mathfrak{T}_\infty) u) = (v | \Psi(Q^{2i\pi\mathcal{E}}\mathfrak{T}_N) u). \quad (1.7)$$

Now, the hermitian form on the right-hand side is amenable to an algebraic-arithmetic version. Indeed, transferring functions in  $\mathcal{S}(\mathbb{R})$  to functions on  $\mathbb{Z}/(2N^2)\mathbb{Z}$  under the linear map  $\theta_N$  defined by the equation

$$(\theta_N u)(n) = \sum_{\substack{n_1 \in \mathbb{Z} \\ n_1 \equiv n \pmod{2N^2}}} u\left(\frac{n_1}{N}\right), \quad n \pmod{2N^2}, \quad (1.8)$$

one obtains the identity

$$(v \mid \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_N) u) = \sum_{m, n \pmod{2N^2}} c_{R,Q}(m, n) \overline{\theta_N v(m)} (\theta_N u)(n). \quad (1.9)$$

The coefficients  $c_{R,Q}(m, n)$  are fully explicit, and the symmetric matrix defining this hermitian form has a Eulerian structure.

If, under the isomorphism  $\mathbb{Z}/(2N^2)\mathbb{Z} \sim \mathbb{Z}/R^2\mathbb{Z} \times \mathbb{Z}/(2Q^2)\mathbb{Z}$ ,  $n$  identifies with a pair  $(n', n'')$ , let us denote as  $\check{n}$  the class that identifies with the pair  $(n', -n'')$ . Set, if  $u \in \mathcal{S}(\mathbb{R})$  and  $n \in \mathbb{Z}/(2N^2)\mathbb{Z}$ ,  $(\Lambda_{R,Q} \theta_N u)(n) = (\theta_N u)(\check{n})$ . There exists a transformation  $\Lambda_{R,Q}^\sharp$  of  $L^2(\mathbb{R})$  (a sum of Heisenberg unitaries) such that, for any  $u \in \mathcal{S}(\mathbb{R})$ , the transfer formula  $\Lambda_{R,Q} \theta_N u = \theta_N \Lambda_{R,Q}^\sharp u$  will hold. One has then the identity

$$(v \mid \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_N) u) = (v \mid \Psi(\mathfrak{T}_N) \Lambda_{R,Q}^\sharp u), \quad (1.10)$$

which substitutes for the action of the rescaling operator  $Q^{2i\pi\mathcal{E}}$  on  $\mathfrak{T}_N$  that of the reflection  $\Lambda_{R,Q}$  (or  $\Lambda_{R,Q}^\sharp$ ) on  $u$ . Under support constraints on  $v, u$ , this identity makes it possible to compute the hermitian form (1.10) exactly. This leads to a proof of the estimate making an application of the criterion (1.5), rather of a modification of it in which  $v$  is replaced by a  $Q$ -dependent function  $v_Q$ , possible.

The present paper is totally self-contained, and all that was already obtained in [6], though with a different normalization of the Weyl symbolic calculus, will be proved again in full.

## 2. THE WEYL CALCULUS OF OPERATORS

In space-momentum coordinates, the Weyl calculus, or pseudodifferential calculus, depends on one free parameter with the dimension of action,

called Planck's constant. In pure mathematics, even the more so when pseudodifferential analysis is applied to arithmetic, Planck's constant becomes a pure number: there is no question that the good such constant in "pseudodifferential arithmetic" is 2, as especially put into evidence [6, Chapter 6] in the pseudodifferential calculus of operators with automorphic symbols. To avoid encumbering the text with unnecessary subscripts, we shall denote as  $\Psi$  the rule denoted as  $\text{Op}_2$  in [6, (2.1.1)], to wit the rule that attaches to any distribution  $\mathfrak{S} \in \mathcal{S}'(\mathbb{R}^2)$  the linear operator  $\Psi(\mathfrak{S})$  from  $\mathcal{S}(\mathbb{R})$  to  $\mathcal{S}'(\mathbb{R})$  weakly defined as

$$(\Psi(\mathfrak{S})u)(x) = \frac{1}{2} \int_{\mathbb{R}^2} \mathfrak{S}\left(\frac{x+y}{2}, \xi\right) e^{i\pi(x-y)\xi} u(y) d\xi dy, \quad (2.1)$$

truly a superposition of integrals (integrate with respect to  $y$  first). The operator  $\Psi(\mathfrak{S})$  is called the operator with symbol  $\mathfrak{S}$ . Its integral kernel is the function

$$K(x, y) = (\mathcal{F}_2^{-1}\mathfrak{S})\left(\frac{x+y}{2}, \frac{x-y}{2}\right), \quad (2.2)$$

where  $\mathcal{F}_2^{-1}$  denotes the inverse Fourier transformation with respect to the second variable.

If one defines the Wigner function  $\text{Wig}(v, u)$  of two functions in  $\mathcal{S}(\mathbb{R})$  as the function in  $\mathcal{S}(\mathbb{R}^2)$  such that

$$\text{Wig}(v, u)(x, \xi) = \int_{-\infty}^{\infty} \bar{v}(x+t) u(x-t) e^{2i\pi\xi t} dt, \quad (2.3)$$

one has

$$(v | \Psi(\mathfrak{S})u) = \langle \mathfrak{S}, \text{Wig}(v, u) \rangle, \quad (2.4)$$

with  $(v|u) = \int_{-\infty}^{\infty} \bar{v}(x) u(x) dx$ , while straight brackets refer to the bilinear operation of testing a distribution on a function. Note that  $\text{Wig}(v, u)(x, \xi) = 0$  unless  $2x$  lies in the algebraic sum of the supports of  $v$  and  $u$ . The function  $\text{Wig}(v, u)$  has exactly the same role, in connection to the Weyl calculus  $\Psi = \text{Op}_2$ , as the function  $W(v, u)$  [6, (2.1.3)] in connection to the rule  $\text{Op}_1$  used in that reference and denoted as  $\text{Op}$  there.

Another useful property of the calculus  $\Psi$  is expressed by the following two equivalent identities, obtained with the help of elementary manipulations of the Fourier transformation or with that of (2.2),

$$\Psi(\mathcal{F}^{\text{symp}}\mathfrak{S})w = \Psi(\mathfrak{S})\check{w}, \quad \mathcal{F}^{\text{symp}}\text{Wig}(v, u) = \text{Wig}(v, \check{u}), \quad (2.5)$$

where  $\check{w}(x) = w(-x)$  and the symplectic Fourier transformation in  $\mathbb{R}^2$  is defined in  $\mathcal{S}(\mathbb{R}^2)$  or  $\mathcal{S}'(\mathbb{R}^2)$  by the equation

$$(\mathcal{F}^{\text{symp}} \mathfrak{S})(x, \xi) = \int_{\mathbb{R}^2} \mathfrak{S}(y, \eta) e^{2i\pi(x\eta - y\xi)} dy d\eta. \quad (2.6)$$

Introduce the Euler operator  $2i\pi\mathcal{E} = 1 + x\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial \xi}$  and, for  $t > 0$ , the operator  $t^{2i\pi\mathcal{E}}$  such that  $(t^{2i\pi\mathcal{E}} \mathfrak{S})(x, \xi) = t \mathfrak{S}(tx, t\xi)$ . Denoting as  $U[2]$  the unitary rescaling operator such that  $(U[2]u)(x) = 2^{\frac{1}{4}}u(x\sqrt{2})$ , one can connect the Weyl calculi with Planck constants 1 and 2 by the equation [6, (2.1.14)]

$$U[2] \Psi(\mathfrak{S}) U[2]^{-1} = \text{Op}_1(\text{Resc } \mathfrak{S}), \quad (2.7)$$

with  $\text{Resc} = 2^{-\frac{1}{2} + i\pi\mathcal{E}}$  or  $(\text{Resc } \mathfrak{S})(x, \xi) = \mathfrak{S}(2^{\frac{1}{2}}x, 2^{\frac{1}{2}}\xi)$ . This would enable us not to redo, in Section 6 and in Proposition 7.1, the proofs already written with another normalization in [6]: but, for self-containedness and simplicity (specializing certain parameters), we shall rewrite all proofs dependent on the symbolic calculus.

The choice of the rule  $\Psi$  makes it possible to avoid splitting into cases, according to the parity of the integers present there, the developments of pseudodifferential arithmetic in Section 6.

### 3. EISENSTEIN DISTRIBUTIONS

The objects in the present section are taken from the more detailed versions in [6, Section 2.2] or, especially, [5, Section 1.1]. Automorphic distributions are distributions in the Schwartz space  $\mathcal{S}'(\mathbb{R}^2)$ , invariant under the linear changes of coordinates associated to matrices in  $SL(2, \mathbb{Z})$ . It is the theory of automorphic and modular distributions, developed over a 20-year span, that led to the definition of the basic distribution  $\mathfrak{T}_\infty$  (4.3), and to that of Eisenstein distributions. It connects [5, Section 2.1] to the classical theory of automorphic functions and modular forms of the non-holomorphic type in the hyperbolic half-plane through a version of the dual Radon transformation dealing with  $SL(2, \mathbb{Z})$ -invariant objects. Our initial interest in developing these notions was stimulated [2, section 18] by the Lax-Phillips scattering theory [1]. Then, our two aims – to wit, obtaining a pseudodifferential calculus of operators with automorphic symbols [3], [5], and approaching the Riemann hypothesis – were finally realized: our proof of the Ramanujan-Petersson conjecture for Maass forms [7] was more of a

surprise.

**Definition 3.1.** If  $\nu \in \mathbb{C}$ ,  $\operatorname{Re} \nu > 1$ , the Eisenstein distribution  $\mathfrak{E}_{-\nu}$  is defined by the equation, valid for every  $h \in \mathcal{S}(\mathbb{R}^2)$ ,

$$\langle \mathfrak{E}_{-\nu}, h \rangle = \sum_{|j|+|k| \neq 0} \int_0^\infty t^\nu h(jt, kt) dt = \frac{1}{2} \sum_{|j|+|k| \neq 0} \int_{-\infty}^\infty |t|^\nu h(jt, kt) dt. \quad (3.1)$$

The second equation is convenient when taking Fourier transforms with respect to  $t$ . It is immediate that the series of integrals converges if  $\operatorname{Re} \nu > 1$ , in which case  $\mathfrak{E}_{-\nu}$  is well defined as a tempered distribution. Obviously, it is  $SL(2, \mathbb{Z})$ -invariant as a distribution, i.e., an automorphic distribution. It is homogeneous of degree  $-1 + \nu$ , i.e.,  $(2i\pi\mathcal{E}) \mathfrak{E}_{-\nu} = \nu \mathfrak{E}_{-\nu}$ . Its name stems from its relation with the classical notion of non-holomorphic Eisenstein series, as made explicit in [6, p.93]: it is, however, a more precise concept.

**Proposition 3.2.** [5, p.13] *As a tempered distribution,  $\mathfrak{E}_{-\nu}$  extends as a meromorphic function of  $\nu \in \mathbb{C}$ , whose only poles are the points  $\nu = \pm 1$ : these poles are simple, and the residues of  $\mathfrak{E}_\nu$  there are*

$$\operatorname{Res}_{\nu=1} \mathfrak{E}_{-\nu} = 1 \quad \text{and} \quad \operatorname{Res}_{\nu=-1} \mathfrak{E}_{-\nu} = -\delta_0, \quad (3.2)$$

*the unit mass at the origin of  $\mathbb{R}^2$ . Recalling the definition (2.6) of the symplectic Fourier transformation  $\mathcal{F}^{\text{symp}}$ , one has, for  $\nu \neq \pm 1$ ,  $\mathcal{F}^{\text{symp}} \mathfrak{E}_{-\nu} = \mathfrak{E}_\nu$ .*

*Proof.* We reproduce the proof given in [5, p.13-14] not only for self-containedness, but also because we shall need more precise results later. Denote as  $(\mathfrak{E}_{-\nu})_{\text{princ}}$  (*resp.*  $(\mathfrak{E}_{-\nu})_{\text{res}}$ ) the distribution defined in the same way as  $\mathfrak{E}_{-\nu}$ , except for the fact that in the integral (3.1), the interval of integration  $(0, \infty)$  is replaced by the interval  $(0, 1)$  (*resp.*  $(1, \infty)$ ), and observe that the distribution  $(\mathfrak{E}_{-\nu})_{\text{res}}$  extends as an entire function of  $\nu$ . As a consequence of Poisson's

formula, one has when  $\operatorname{Re} \nu > 1$  the identity

$$\begin{aligned} \int_1^\infty t^{-\nu} \sum_{(j,k) \in \mathbb{Z}^2} (\mathcal{F}^{\text{symp}} h)(tk, tj) dt &= \int_1^\infty t^{-\nu} \sum_{(j,k) \in \mathbb{Z}^2} t^{-2} h(t^{-1}j, t^{-1}k) dt \\ &= \int_0^1 t^\nu \sum_{(j,k) \in \mathbb{Z}^2} h(tj, tk) dt, \end{aligned} \quad (3.3)$$

from which one obtains that

$$\langle \mathcal{F}^{\text{symp}}(\mathfrak{E}_\nu)_{\text{res}}, h \rangle = \langle (\mathfrak{E}_{-\nu})_{\text{princ}}, h \rangle + \frac{h(0, 0)}{1 + \nu} + \frac{(\mathcal{F}^{\text{symp}} h)(0, 0)}{1 - \nu}. \quad (3.4)$$

From this identity, one finds the meromorphic continuation of the function  $\nu \mapsto \mathfrak{E}_\nu$ , including the residues at the two poles, as well as the fact that  $\mathfrak{E}_\nu$  and  $\mathfrak{E}_{-\nu}$  are the images of each other under  $\mathcal{F}^{\text{symp}}$ .  $\square$

Note that if  $h = \text{Wig}(v, u)$ , one has

$$h(0, 0) = (v \mid \check{u}), \quad (\mathcal{F}^{\text{symp}} h)(0, 0) = (v \mid u). \quad (3.5)$$

If dealing with pairs of functions  $v, u \in \mathcal{S}(\mathbb{R})$  such that the supports of  $v$  and  $u$  are disjoint, and so are the supports of  $v$  and  $\check{u}$ , one has, using (2.5), and without any condition on  $\nu$ ,

$$\langle \mathfrak{E}_{-\nu}, \text{Wig}(v, u) \rangle = \langle (\mathfrak{E}_{-\nu})_{\text{res}}, \text{Wig}(v, u) \rangle + \langle (\mathfrak{E}_\nu)_{\text{res}}, \text{Wig}(v, \check{u}) \rangle \quad (3.6)$$

**Lemma 3.3.** ([3, p.22], [5, p.15]) *One has if  $\nu \neq \pm 1$  the Fourier expansion*

$$\mathfrak{E}_{-\nu}(x, \xi) = \zeta(\nu) |\xi|^{\nu-1} + \zeta(1 + \nu) |x|^\nu \delta(\xi) + \sum_{r \neq 0} \sigma_{-\nu}(r) |\xi|^{\nu-1} \exp\left(2i\pi \frac{rx}{\xi}\right) \quad (3.7)$$

where  $\sigma_{-\nu}(r) = \sum_{1 \leq d \mid r} d^{-\nu}$ : the first two terms must be grouped when  $\nu = 0$ .

*Proof.* Isolating the term for which  $k = 0$  in (3.1), we write if  $\operatorname{Re} \nu > 1$ , after a change of variable,

$$\begin{aligned} \langle \mathfrak{E}_{-\nu}, h \rangle &= \zeta(1 + \nu) \int_{-\infty}^{\infty} |t|^{\nu} h(t, 0) dt + \frac{1}{2} \sum_{j \in \mathbb{Z}, k \neq 0} \int_{-\infty}^{\infty} |t|^{\nu} h(jt, kt) dt \\ &= \zeta(1 + \nu) \int_{-\infty}^{\infty} |t|^{\nu} h(t, 0) dt + \frac{1}{2} \sum_{j \in \mathbb{Z}, k \neq 0} \int_{-\infty}^{\infty} |t|^{\nu-1} (\mathcal{F}_1^{-1} h) \left( \frac{j}{t}, kt \right) dt, \end{aligned} \quad (3.8)$$

where we have used Poisson's formula at the end and denoted as  $\mathcal{F}_1^{-1} h$  the inverse Fourier transform of  $h$  with respect to the first variable. Isolating now the term such that  $j = 0$ , we obtain

$$\begin{aligned} \frac{1}{2} \sum_{j \in \mathbb{Z}, k \neq 0} \int_{-\infty}^{\infty} |t|^{\nu-1} (\mathcal{F}_1^{-1} h) \left( \frac{j}{t}, kt \right) dt &= \zeta(\nu) \int_{-\infty}^{\infty} |t|^{\nu-1} (\mathcal{F}_1^{-1} h)(0, t) dt \\ &\quad + \frac{1}{2} \sum_{jk \neq 0} \int_{-\infty}^{\infty} |t|^{\nu-1} (\mathcal{F}_1^{-1} h) \left( \frac{j}{t}, kt \right) dt, \end{aligned} \quad (3.9)$$

from which the main part of the lemma follows if  $\operatorname{Re} \nu > 1$  after we have made the change of variable  $t \mapsto \frac{t}{k}$  in the main term. The continuation of the identity uses also the fact that the product  $\zeta(\nu) |t|^{\nu-1}$  is regular at  $\nu = -2, -4, \dots$  and the product  $\zeta(1 + \nu) |t|^{\nu}$  is regular at  $\nu = -3, -5, \dots$ , thanks to the trivial zeros of zeta. That the sum  $\zeta(\nu) |\xi|^{\nu-1} + \zeta(1 + \nu) |x|^{\nu} \delta(\xi)$  is regular at  $\nu = 0$  follows from the facts that  $\zeta(0) = -\frac{1}{2}$  and that the residue at  $\nu = 0$  of the distribution  $|\xi|^{\nu-1} = \frac{1}{\nu} \frac{d}{d\xi} (|x|^{\nu} \operatorname{sign} \xi)$  is  $\frac{d}{d\xi} \operatorname{sign} \xi = 2\delta(\xi)$ .  $\square$

Decompositions into homogeneous components of functions or distributions in the plane will be ever-present. Any function  $h \in \mathcal{S}(\mathbb{R}^2)$  can be decomposed in  $\mathbb{R}^2 \setminus \{0\}$  into homogeneous components according to the equations, in which  $c > -1$ ,

$$h = \frac{1}{i} \int_{\operatorname{Re} \nu = c} h_{\nu} d\nu, \quad h_{\nu}(x, \xi) = \frac{1}{2\pi} \int_0^{\infty} t^{\nu} h(tx, t\xi) dt. \quad (3.10)$$

Indeed, the integral defining  $h_{\nu}(x, \xi)$  is convergent for  $|x| + |\xi| \neq 0, \operatorname{Re} \nu > -1$ , and the function  $h_{\nu}$  so defined is  $C^{\infty}$  in  $\mathbb{R}^2 \setminus \{0\}$  and homogeneous of degree  $-1 - \nu$ ; it is also analytic with respect to  $\nu$ . Using twice the integration by parts associated to Euler's equation  $-(1 + \nu) h_{\nu} = \left( x \frac{\partial}{\partial x} + \xi \frac{\partial}{\partial \xi} \right) h_{\nu}(x, \xi)$ , one sees that the integral  $\frac{1}{i} \int_{\operatorname{Re} \nu = c} h_{\nu}(x, \xi) d\nu$  is convergent for  $c > -1$ : its



value does not depend on  $c$ . Taking  $c = 0$  and setting  $t = e^{2\pi\tau}$ , one has for  $|x| + |\xi| \neq 0$

$$h_{i\lambda}(x, \xi) = \int_{-\infty}^{\infty} e^{2i\pi\tau\lambda} \cdot e^{2\pi\tau} h(e^{2\pi\tau}x, e^{2\pi\tau}\xi) d\tau, \quad (3.11)$$

and the Fourier inversion formula shows that  $\int_{-\infty}^{\infty} h_{i\lambda}(x, \xi) d\lambda = h(x, \xi)$ : this proves (3.10).

As a consequence, some automorphic distributions of interest (not all: Hecke distributions are needed too in general [7]) can be decomposed into Eisenstein distributions. A basic one is the “Dirac comb”

$$\begin{aligned} \mathfrak{D}(x, \xi) &= 2\pi \sum_{|j|+|k| \neq 0} \delta(x-j) \delta(\xi-k) \\ &= 2\pi [\mathcal{D}ir(x, \xi) - \delta(x)\delta(\xi)] \end{aligned} \quad (3.12)$$

if, as found convenient in some algebraic calculations, one introduces also the “complete” Dirac comb  $\mathcal{D}ir(x, \xi) = \sum_{j,k \in \mathbb{Z}} \delta(x-j)\delta(\xi-k)$ .

Noting the inequality  $|\int_0^\infty t^\nu h(tx, t\xi) dt| \leq C(|x| + |\xi|)^{-\operatorname{Re} \nu - 1}$ , one obtains if  $h \in \mathcal{S}(\mathbb{R}^2)$  and  $c > 1$ , pairing (3.12) with (3.10), the identity

$$\langle \mathfrak{D}, h \rangle = \frac{1}{i} \sum_{|j|+|k| \neq 0} \int_{\operatorname{Re} \nu = c} d\nu \int_0^\infty t^\nu h(tj, tk) dt. \quad (3.13)$$

Now, assuming that  $h$  is even as is possible since both  $\mathfrak{D}$  and  $\mathfrak{E}_{-\nu}$  are, one can rewrite (3.1) as

$$\langle \mathfrak{E}_{-\nu}, h \rangle = \sum_{|m|+|n| \neq 0} \int_0^\infty t^{-\nu} h(mt, nt) dt, \quad c > 1. \quad (3.14)$$

It follows that, for  $c > 1$  [5, p.14],

$$\mathfrak{D} = \frac{1}{i} \int_{\operatorname{Re} \nu = c} \mathfrak{E}_{-\nu} d\nu = 2\pi + \frac{1}{i} \int_{\operatorname{Re} \nu = 0} \mathfrak{E}_{-\nu} d\nu, \quad (3.15)$$

the second equation being a consequence of the first in view of (3.2). Using the second equation (3.2), one could also write

$$\mathcal{D}ir = \frac{1}{2i\pi} \int_{\operatorname{Re} \nu = -c} \mathfrak{E}_{-\nu} d\nu, \quad c > 1. \quad (3.16)$$

In the reverse direction, Eisenstein distributions can be obtained from the Dirac comb, making use of the resolvent of the Euler operator defined [4, p.188] as

$$\left[ (2i\pi\mathcal{E} + \nu)^{-1} h \right] (x, \xi) = \begin{cases} \int_0^1 t^\nu h(tx, t\xi) dt, & \operatorname{Re} \nu > 0, \\ -\int_1^\infty t^\nu h(tx, t\xi) dt, & \operatorname{Re} \nu < 0. \end{cases} \quad (3.17)$$

That the two maps define bounded endomorphisms of  $L^2(\mathbb{R}^2)$  [4, (5.1.5)] is actually just Hardy's inequality. The resolvent  $(2i\pi\mathcal{E} + \nu)^{-1}$  does not preserve the space  $\mathcal{S}(\mathbb{R}^2)$  but, if  $h \in \mathcal{S}(\mathbb{R}^2)$ , the two expressions

$$\begin{aligned} \frac{1}{2\pi} \langle \mathfrak{D}, (2i\pi\mathcal{E} + \nu)^{-1} h \rangle &= \sum_{|j|+|k| \neq 0} \int_0^1 t^\nu h(tj, tk) dt = \langle (\mathfrak{E}_{-\nu})_{\text{res}}, h \rangle, \\ \frac{1}{2\pi} \langle \mathfrak{D}, (2i\pi\mathcal{E} - \nu)^{-1} h \rangle &= - \sum_{|j|+|k| \neq 0} \int_1^\infty t^{-\nu} h(tj, tk) dt = \langle (\mathfrak{E}_\nu)_{\text{res}}, h \rangle \end{aligned} \quad (3.18)$$

are convergent if  $c = \operatorname{Re} \nu > 1$ . This is obvious so far as the second one is concerned; for the first, choosing  $A > c + 1$ , we bound  $h(tj, tk)$  by  $C [1 + t(|j| + |k|)]^{-A}$  and make a change of variable.

Integral superpositions of Eisenstein distributions, such as the one in (3.15), are to be interpreted in the weak sense in  $\mathcal{S}'(\mathbb{R}^2)$ , i.e., they make sense when tested on arbitrary functions in  $\mathcal{S}(\mathbb{R}^2)$ . Of course, pole-chasing is essential when changing contours of integration. But no difficulty concerning the integrability with respect to  $\operatorname{Im} \nu$  on the line ever occurs, because of the identities

$$(a - \nu)^A \langle \mathfrak{E}_{-\nu}, W \rangle = \langle (a - 2i\pi\mathcal{E})^A \mathfrak{E}_{-\nu}, W \rangle = \langle \mathfrak{E}_{-\nu}, (a + 2i\pi\mathcal{E})^A W \rangle, \quad (3.19)$$

in which  $A = 0, 1, \dots$  may be chosen arbitrarily large and  $a$  is arbitrary.

**Lemma 3.4.** *Let  $\nu \in \mathbb{C}$ ,  $\nu \neq \pm 1$ , and let  $v, u \in C^\infty(\mathbb{R})$  be two functions, with disjoint supports contained in  $[a, b]$  with  $a \geq 0$  and  $b^2 - a^2 < 8$ . One has*

$$(v \mid \Psi(\mathfrak{E}_{-\nu}) u) = 2 \int_0^\infty t^{\nu-1} \bar{v}(t + t^{-1}) u(t - t^{-1}) dt. \quad (3.20)$$

*Proof.* Let us use the expansion (3.7), but only after we have substituted the pair  $(\xi, -x)$  for  $(x, \xi)$ , which does not change  $\mathfrak{E}_{-\nu}(x, \xi)$  for any  $\nu \neq \pm 1$

because the Eisenstein distribution  $\mathfrak{E}_{-\nu}$  is automorphic: hence,

$$\mathfrak{E}_{-\nu}(x, \xi) = \zeta(\nu) |x|^{\nu-1} + \zeta(1+\nu) \delta(x) |\xi|^\nu + \sum_{r \neq 0} \sigma_{-\nu}(r) |x|^{\nu-1} \exp\left(-2i\pi \frac{r\xi}{x}\right). \quad (3.21)$$

We immediately observe that, under the given support assumptions regarding  $v$  and  $u$ , the first two terms of this expansion will not contribute to  $(v \mid \Psi(\mathfrak{E}_{-\nu})u)$ : the first because the supports of  $v$  and  $u$  do not intersect, the second because those of  $v$  and  $\overset{\vee}{u} = u$  do not. Only the contribution of the series for  $k \neq 0$ , to be designated as  $\mathfrak{E}_{-\nu}^{\text{trunc}}(x, \xi)$ , remains to be examined.

One has

$$(\mathcal{F}_2^{-1} \mathfrak{E}_{-\nu}^{\text{trunc}})(x, z) = \sum_{r \neq 0} \sigma_{-\nu}(r) |x|^{\nu-1} \delta\left(z - \frac{r}{x}\right). \quad (3.22)$$

Still using (2.2), the integral kernel of the operator  $\Psi(\mathfrak{S}_r)$ , with

$$\mathfrak{S}_r(x, \xi) := |x|^{\nu-1} \exp\left(-2i\pi \frac{r\xi}{x}\right) = \mathfrak{T}_r\left(x, \frac{\xi}{x}\right), \quad (3.23)$$

is

$$\begin{aligned} K_r(x, y) &= (\mathcal{F}_2^{-1} \mathfrak{S}_r)\left(\frac{x+y}{2}, \frac{x-y}{2}\right) = \left|\frac{x+y}{2}\right| (\mathcal{F}_2^{-1} \mathfrak{T}_r)\left(\frac{x+y}{2}, \frac{x^2-y^2}{4}\right) \\ &= \left|\frac{x+y}{2}\right|^\nu \delta\left(\frac{x^2-y^2}{4} - r\right) = \left|\frac{x+y}{2}\right|^{\nu-1} \delta\left(\frac{x-y}{2} - \frac{2r}{x+y}\right). \end{aligned} \quad (3.24)$$

If  $v(x)u(y) \neq 0$ , one has  $0 < x^2 - y^2 < 8$  so that  $r = \frac{x^2-y^2}{4} = 1$ . Making in the integral  $\int_{\mathbb{R}^2} K(x, y) \overline{v}(x) u(y) dx dy$  the change of variable which amounts to taking  $\frac{x+y}{2}$  and  $x-y$  as new variables, one obtains (3.20).  $\square$

The following lemma will be used later, in the proof of Theorem 5.2.

**Lemma 3.5.** *Given  $\nu \in \mathbb{C}$  and  $\beta > 2$ , there exists a pair  $v, u$  of  $C^\infty$  functions supported in  $[0, \beta]$ , with disjoint supports, such that*

$$(v \mid \Psi(\mathfrak{E}_{-\nu})u) \neq 0. \quad (3.25)$$

*Proof.* Assuming without loss of generality that  $\beta < 2^{\frac{3}{2}}$ , one uses (3.20), with  $u$  supported in  $[0, \sqrt{\beta^2 - 4}]$  and  $v$  supported in  $[2, \beta]$ . Given  $y_0 \in ]0, \sqrt{\beta^2 - 4}[$ , one can choose for  $v(x)$  a function approaching  $\delta(x - \sqrt{y_0^2 + 4})$ : then, the limit of the integral cannot be zero if  $u(y_0) \neq 0$ .  $\square$

#### 4. SOME DISTRIBUTIONS OF ARITHMETIC INTEREST

Set for  $j \neq 0$

$$a(j) = \prod_{p|j} (1 - p), \quad (4.1)$$

where  $p$ , in the role of defining the range of the subscript in a product, is always tacitly assumed to be prime. The distribution

$$\mathfrak{T}_N(x, \xi) = \sum_{j, k \in \mathbb{Z}} a((j, k, N)) \delta(x - j) \delta(\xi - k), \quad (4.2)$$

where the notation  $(j, k, N)$  refers to the g.c.d. of the three numbers, depends only on the “squarefree version” of  $N$ , defined as  $N_\bullet = \prod_{p|N} p$ . We also denote as  $\mathfrak{T}_N^\times$  the distribution obtained from  $\mathfrak{T}_N$  by discarding the term  $a(N) \delta(x) \delta(\xi)$ , in other words by limiting the summation to all pairs of integers  $j, k$  such that  $|j| + |k| \neq 0$ . As  $N \nearrow \infty$ , a notation meant to convey that  $N \rightarrow \infty$  in such a way that any given finite set of primes constitutes eventually a set of divisors of  $N$ , the distribution  $\mathfrak{T}_N^\times$  converges in the space  $\mathcal{S}'(\mathbb{R}^2)$  towards the distribution

$$\mathfrak{T}_\infty(x, \xi) = \sum_{|j| + |k| \neq 0} a((j, k)) \delta(x - j) \delta(\xi - k). \quad (4.3)$$

Indeed, if  $q$  denotes the least prime not dividing  $N$ , one has

$$\mathfrak{T}_\infty(x, \xi) - \mathfrak{T}_N^\times(x, \xi) = \sum_{|j| + |k| \neq 0} [a((j, k)) - a((j, k, N))] \delta(x - j) \delta(\xi - k), \quad (4.4)$$

where the sum may be restricted to the set of pairs  $j, k$  such that at least one prime factor of  $(j, k)$  does not divide  $N$ , which implies  $(j, k) \geq q$ . This convergence is very slow since, as a consequence of the theorem of prime numbers, in the case when  $N = \prod_{p < q} p$ ,  $q$  is roughly of the size of  $\log N$  only.

**Lemma 4.1.** [6, Lemma 3.1.1] *For any squarefree integer  $N \geq 1$ , defining*

$$\zeta_N(s) = \prod_{p|N} (1 - p^{-s})^{-1}, \quad \text{so that } \frac{1}{\zeta_N(s)} = \sum_{1 \leq T|N} \mu(T) T^{-s}, \quad (4.5)$$

where  $\mu$  is the Möbius indicator function, one has

$$\mathfrak{T}_N^\times = \frac{1}{2i\pi} \int_{\operatorname{Re} \nu = c} \frac{1}{\zeta_N(\nu)} \mathfrak{E}_{-\nu} d\nu, \quad c > 1. \quad (4.6)$$

Then,

$$\mathfrak{T}_\infty = \frac{1}{2i\pi} \int_{\operatorname{Re} \nu = c} \frac{\mathfrak{E}_{-\nu}}{\zeta(\nu)} d\nu, \quad c > 1. \quad (4.7)$$

*Proof.* Using the equation  $T^{-x} \frac{d}{dx} \delta(x - j) = \delta\left(\frac{x}{T} - j\right) = T \delta(x - Tj)$ , one has with  $\mathfrak{D}$  as introduced in (3.21)

$$\begin{aligned} \frac{1}{2\pi} \prod_{p|N} (1 - p^{-2i\pi\mathcal{E}}) \mathfrak{D}(x, \xi) &= \sum_{T|N_\bullet} \mu(T) T^{-2i\pi\mathcal{E}} \sum_{|j|+|k| \neq 0} \delta(x - j) \delta(\xi - k) \\ &= \sum_{T|N_\bullet} \mu(T) T \sum_{|j|+|k| \neq 0} \delta(x - Tj) \delta(\xi - Tk) \\ &= \sum_{T|N_\bullet} \mu(T) T \sum_{\substack{|j|+|k| \neq 0 \\ j \equiv k \equiv 0 \pmod T}} \delta(x - j) \delta(\xi - k) \\ &= \sum_{\substack{|j|+|k| \neq 0 \\ T|(N_\bullet, j, k)}} \mu(T) T \delta(x - j) \delta(\xi - k). \end{aligned} \quad (4.8)$$

Since  $\sum_{T|(N_\bullet, j, k)} \mu(T) T = \prod_{p|(N, j, k)} (1 - p) = a((N, j, k))$ , one obtains

$$\frac{1}{2\pi} \prod_{p|N} (1 - p^{-2i\pi\mathcal{E}}) \mathfrak{D}(x, \xi) = \mathfrak{T}_N^\times(x, \xi). \quad (4.9)$$

One has

$$\begin{aligned} [\mathfrak{T}_N - \mathfrak{T}_N^\times](x, \xi) &= a(N) \delta(x) \delta(\xi) \\ &= \delta(x) \delta(\xi) \prod_{p|N} (1 - p) = \prod_{p|N} (1 - p^{-2i\pi\mathcal{E}}) (\delta(x) \delta(\xi)), \end{aligned} \quad (4.10)$$

so that

$$\mathfrak{T}_N = \prod_{p|N} (1 - p^{-2i\pi\mathcal{E}}) \mathcal{D}ir. \quad (4.11)$$

Combining (4.9) with (3.15) and with  $(2i\pi\mathcal{E})\mathfrak{E}_{-\nu} = \nu \mathfrak{E}_{-\nu}$ , one obtains if  $c > 1$

$$\begin{aligned}\mathfrak{T}_N^\times &= \prod_{p|N} (1 - p^{-2i\pi\mathcal{E}}) \left[ \frac{1}{2i\pi} \int_{\operatorname{Re} \nu = c} \mathfrak{E}_{-\nu} d\nu \right] \\ &= \frac{1}{2i\pi} \int_{\operatorname{Re} \nu = c} \mathfrak{E}_{-\nu} \prod_{p|N} (1 - p^{-\nu}) d\nu,\end{aligned}\tag{4.12}$$

which is just (4.6). Equation (4.7) follows as well, taking the limit as  $N \nearrow \infty$ . Recall what was said immediately between (3.15) and (3.19) about integral superpositions of Eisenstein distributions such as the ones in (3.15) and (4.7). □

*Remark.* In (4.7), introducing a sum of residues over all zeros of zeta with a real part above some large negative number, one can replace the line  $\operatorname{Re} \nu = c$  with  $c > 1$  by a line  $\operatorname{Re} \nu = c'$  with  $c'$  very negative: one cannot go further in the distribution sense. But [6, Theor. 3.2.2, Theor. 3.2.4], one can get rid of the integral if one agrees to interpret the identity in the sense of a certain analytic functional. Then, all zeros of zeta, non-trivial and trivial alike, enter the formula.

The following reduction of  $\mathfrak{T}_\infty$  to  $\mathfrak{T}_N$  is immediate, and fundamental for our purpose. Assume that  $v$  and  $u \in C^\infty(\mathbb{R})$  are supported in  $[0, \beta]$ . Then, given a squarefree integer  $N = RQ$  (with  $R, Q$  integers) divisible by all primes  $< \beta Q$ , one has

$$(v \mid \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_\infty) u) = (v \mid \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_N) u).\tag{4.13}$$

Indeed, using (2.4) and noting that the transpose of  $2i\pi\mathcal{E}$  is  $-2i\pi\mathcal{E}$ , one can write the right-hand side as

$$(v \mid \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_N) u) = Q^{-1} \sum_{|j|+|k| \neq 0} a((j, k, N)) \operatorname{Wig}(v, u) \left( \frac{j}{Q}, \frac{k}{Q} \right).\tag{4.14}$$

In view of the observation that follows (2.4), one has  $0 < \frac{j}{Q} < \beta$  for all nonzero terms on the right-hand side, in which case all prime divisors of  $j$  divide  $N$ .

We shall also need the distribution

$$\mathfrak{T}_{\frac{\infty}{2}}(x, \xi) = \sum_{|j|+|k| \neq 0} a\left(j, k, \frac{\infty}{2}\right) \delta(x - j) \delta(\xi - k),\tag{4.15}$$

where  $a((j, k, \frac{\infty}{2}))$  is the product of all factors  $1 - p$  with  $p$  prime  $\neq 2$  dividing  $(j, k)$ . One has

$$\mathfrak{T}_{\frac{\infty}{2}}(x, \xi) = \frac{1}{2i\pi} \int_{\operatorname{Re} \nu = c} (1 - 2^{-\nu})^{-1} \frac{\mathfrak{E}_{-\nu}(x, \xi)}{\zeta(\nu)} d\nu, \quad c > 1. \quad (4.16)$$

In [6, Prop. 3.4.2 and 3.4.3], it was proved (with some differences essentially due to the present change of  $\operatorname{Op}_1$  to  $\Psi$ ) that, if for some  $\beta > 2$  and every function  $w \in C^\infty(\mathbb{R})$  supported in  $[0, \beta]$ , one has

$$(w \mid \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_\infty) w) = O\left(Q^{\frac{1}{2} + \varepsilon}\right) \quad (4.17)$$

as  $Q \rightarrow \infty$  through squarefree integral values, the Riemann hypothesis follows. Except for the change of the pair  $w, w$  to a pair  $v, u$  with specific support properties, this is the criterion to be discussed in the next section.

## 5. A CRITERION FOR THE RIEMANN HYPOTHESIS

**Lemma 5.1.** *Let  $\rho_0 \in \mathbb{C}$  and consider a product  $h(\nu) f(s - \nu)$ , where the function  $f = f(z)$ , defined and meromorphic near the point  $z = 1$ , has a simple pole at this point, and the function  $h$ , defined and meromorphic near  $\rho_0$ , has at that point a pole of order  $\ell \geq 1$ . Then, the function  $s \mapsto \operatorname{Res}_{\nu=\rho_0} [h(\nu) f(s - \nu)]$  has at  $s = 1 + \rho_0$  a pole of order  $\ell$ .*

*Proof.* If  $h(\nu) = \sum_{j=1}^{\ell} a_j (\nu - \rho_0)^{-j} + O(1)$  as  $\nu \rightarrow \rho_0$ , one has for  $s$  close to  $\rho_0$  but distinct from this point

$$\operatorname{Res}_{\nu=\rho_0} [h(\nu) f(s - \nu)] = \sum_{j=1}^{\ell} (-1)^{j-1} a_j \frac{f^{(j-1)}(s - \rho_0)}{(j-1)!}, \quad (5.1)$$

and the function  $s \mapsto f^{(j-1)}(s - \rho_0)$  has at  $s = 1 + \rho_0$  a pole of order  $j$ .  $\square$

**Theorem 5.2.** *Assume that, for some  $\beta \in ]2, 2^{\frac{3}{2}}[$  and for every given pair  $v, u$  of  $C^\infty$  functions with  $v$  supported in  $[2, \beta]$  and  $u$  supported in  $[0, \sqrt{\beta^2 - 4}]$ , and for every  $\varepsilon > 0$ , there exists  $C > 0$  such that, for every squarefree odd integer  $Q \geq 1$ , one has*

$$(v \mid \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_{\frac{\infty}{2}}) u) \leq C Q^{\frac{1}{2} + \varepsilon}. \quad (5.2)$$

*Then, the Riemann hypothesis follows.*

*Proof.* Consider the function

$$F(s) = \sum_{Q \in \text{Sq}^{\text{odd}}} Q^{-s} \left( v \mid \Psi \left( Q^{2i\pi\mathcal{E}} \mathfrak{T}_{\frac{\infty}{2}} \right) u \right), \quad (5.3)$$

where we denote as  $\text{Sq}^{\text{odd}}$  the set of squarefree odd integers. In view of the assumption, it is analytic for  $\text{Re } s > \frac{3}{2}$ . For  $\text{Re } s > 2$ , one can compute it in a different way, starting from

$$\begin{aligned} \sum_{Q \in \text{Sq}^{\text{odd}}} Q^{-\theta} &= \prod_{\substack{q \text{ prime} \\ q \neq 2}} (1 + q^{-\theta}) \\ &= (1 + 2^{-\theta})^{-1} \prod_q \frac{1 - q^{-2\theta}}{1 - q^{-\theta}} = (1 + 2^{-\theta})^{-1} \frac{\zeta(\theta)}{\zeta(2\theta)}. \end{aligned} \quad (5.4)$$

One has  $(v \mid \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{E}_{-\nu}) u) = Q^\nu \langle \mathfrak{E}_{-\nu}, \text{Wig}(v, u) \rangle$  according to (2.4) and the homogeneity of  $\mathfrak{E}_{-\nu}$ . Using (4.15), (4.16) and (5.4), one obtains if  $\text{Re } s > 2$ , noting that the denominator  $\zeta(\nu)$  takes care of the pole of  $\mathfrak{E}_{-\nu}$  at  $\nu = 1$ , which makes it possible to replace the line  $\text{Re } \nu = c$ ,  $c > 1$  by the line  $\text{Re } \nu = 1$ ,

$$\begin{aligned} F(s) &= \frac{1}{2i\pi} \int_{\text{Re } \nu=1} (1 - 2^{-\nu})^{-1} (1 + 2^{-s+\nu})^{-1} \\ &\quad \frac{\zeta(s-\nu)}{\zeta(2(s-\nu))} \langle \mathfrak{E}_{-\nu}, \text{Wig}(v, u) \rangle \frac{d\nu}{\zeta(\nu)}. \end{aligned} \quad (5.5)$$

Note that the integrability at infinity is taken care of by (3.19).

Assume that such a zero exists: one may assume that the real part of  $\rho_0$  is the largest one among those of all zeros of zeta (if any other should exist !) with the same imaginary part. Choose  $\alpha > 0$  such that  $\frac{1}{2} < \text{Re } \rho_0 - \alpha$ . Assuming that  $\text{Re } s > 2$ , change the line  $\text{Re } \nu = 1$  to a simple contour  $\gamma$  on the left of the initial line, enclosing the point  $\rho_0$  but no other point  $\rho$  with  $\zeta(\rho) = 0$ , coinciding with the line  $\text{Re } \nu = 1$  for  $|\text{Im } \nu|$  large, and such that  $\text{Re } \nu > \text{Re } \rho_0 - \alpha$  for  $\nu \in \gamma$ . Let  $\Omega$  be the relatively open part of the half-plane  $\text{Re } \nu \leq 1$  enclosed by  $\gamma$  and the line  $\text{Re } \nu = 1$ . Let  $\mathcal{D}$  be the domain consisting of the numbers  $s$  such that  $s - 1 \in \Omega$  or  $\text{Re } s > 2$ . Taking for the boundary of  $\Omega$  a narrow rectangle, one can manage so that, for  $\nu \in \gamma$  but  $\text{Re } \nu \neq 1$ , and  $s \in 1 + \Omega$ , one will have  $|\text{Im } (\nu - s)| < \frac{\pi}{\log 2}$ : this will ensure that  $1 + 2^{-s+\nu} \neq 0$ .



Still assuming that  $\operatorname{Re} s > 2$ , one obtains the equation

$$\begin{aligned} F(s) &= \frac{1}{2i\pi} \int_{\gamma} [(1 - 2^{-\nu})(1 + 2^{-s+\nu})]^{-1} \frac{\zeta(s - \nu)}{\zeta(2(s - \nu))} \langle \mathfrak{E}_{-\nu}, \operatorname{Wig}(v, u) \rangle \frac{d\nu}{\zeta(\nu)} \\ &\quad + \operatorname{Res}_{\nu=\rho_0} [h(\nu)f(s - \nu)], \end{aligned} \quad (5.6)$$

with

$$\begin{aligned} h(\nu) &= (1 - 2^{-\nu})^{-1} \times \frac{\langle \mathfrak{E}_{-\nu}, \operatorname{Wig}(v, u) \rangle}{\zeta(\nu)}, \\ f(s - \nu) &= (1 + 2^{-s+\nu})^{-1} \times \frac{\zeta(s - \nu)}{\zeta(2(s - \nu))}. \end{aligned} \quad (5.7)$$

Let us insist that this formula has been obtained under the assumption that  $\operatorname{Re} s > 2$ . We shall benefit presently from a continuation of this formula for values of  $s$  such that  $s - 1 \in \Omega$ . We remark now that the integral term in (5.6) is holomorphic in the domain  $\mathcal{D}$ . This is immediate when  $\operatorname{Re} s > 2$  since  $\operatorname{Re} \nu \leq 1$  on  $\gamma$ , so that  $\operatorname{Re}(s - \nu) > 1$ . When  $s - 1 \in \Omega$  and  $\nu \in \gamma$ , that  $\zeta(2(s - \nu)) \neq 0$  follows from the inequalities  $\operatorname{Re}(s - \nu) \geq \operatorname{Re} \rho_0 - \alpha > \frac{1}{2}$ , since  $\operatorname{Re} s > 1 + (\operatorname{Re} \rho_0 - \alpha)$  and  $\operatorname{Re} \nu \leq 1$ . On the other hand, one has  $s - \nu \neq 1$  since the conditions  $s - 1 \in \Omega$  and  $s - 1 \in \gamma$  are incompatible.

Since  $F(s)$  is analytic for  $\operatorname{Re} s > \frac{3}{2}$ , it follows that the residue present in (5.6) extends as an analytic function of  $s$  in  $\mathcal{D}$ . But an application of Lemmas 5.1 and 3.5 shows that this residue is singular at  $s = 1 + \rho_0$  for some choice of the pair  $v, u$ .

We have reached a contradiction. □

## 6. PSEUDODIFFERENTIAL ARITHMETIC

As noted in (4.13), one can substitute for the analysis of the hermitian form  $(v \mid \Psi(Q^{2i\pi\mathcal{E}}\mathfrak{T}_{\frac{\infty}{2}})u)$  that of  $(v \mid \Psi(Q^{2i\pi\mathcal{E}}\mathfrak{T}_N)u)$ , under the assumption that  $v, u$  are supported in  $[0, \beta]$ , provided that  $N$  is a squarefree odd integer divisible by all odd primes  $< \beta Q$ . The new hermitian form should be amenable to an algebraic treatment. In this section, we make no support assumptions on  $v, u$ , just taking them in  $\mathcal{S}(\mathbb{R})$ .

We consider operators of the kind  $\mathcal{A} = \Psi(Q^{2i\pi\mathcal{E}}\mathfrak{S})$  with

$$\mathfrak{S}(x, \xi) = \sum_{j, k \in \mathbb{Z}} b(j, k) \delta(x - j) \delta(\xi - k), \quad (6.1)$$

under the following assumptions: that  $N$  is an odd squarefree integer decomposing as the product  $N = QR$  of two positive integers, and that  $b$  satisfies the periodicity conditions

$$b(j, k) = b(j + N, k) = b(j, k + N). \quad (6.2)$$

A special case consists of course of the symbol  $\mathfrak{S} = \mathfrak{T}_N$ . The aim is to transform the hermitian form associated to the operator  $\Psi(Q^{2i\pi\mathcal{E}}\mathfrak{T}_N)$  to an arithmetic version.

**Proposition 6.1.** *Define the operator  $\mathcal{B}$  by the identity*

$$(\mathcal{B}u)(Qx) = Q^{-1} [\Psi(Q^{2i\pi\mathcal{E}}\mathfrak{S})(y \mapsto u(Qy))](x), \quad (6.3)$$

*in other words*

$$(\mathcal{B}u)(x) = \frac{1}{2Q^2} \int_{\mathbb{R}^2} \mathfrak{S}\left(\frac{x+y}{2}, \xi\right) u(y) \exp\left(\frac{i\pi}{Q^2}(x-y)\xi\right) dy d\xi. \quad (6.4)$$

*Setting  $v_Q(x) = v(Qx)$  and  $u_Q(x) = u(Qx)$ , one has the identity*

$$(v | \mathcal{B}u) = (v_Q | \mathcal{A}u_Q). \quad (6.5)$$

*Proof.* Starting from

$$(\mathcal{B}u)(Qx) = \frac{1}{2} \int_{\mathbb{R}^2} \mathfrak{S}\left(\frac{Q(x+y)}{2}, Q\xi\right) e^{i\pi(x-y)\xi} u(Qy) dy d\xi, \quad (6.6)$$

one obtains (6.4). From (6.3),

$$\begin{aligned} (v | \mathcal{B}u) &= Q \int_{-\infty}^{\infty} \bar{v}(Qx) (\mathcal{B}u)(Qx) dx = Q \int_{-\infty}^{\infty} \bar{v}(Qx) \cdot (Q^{-1} \mathcal{A}u_Q)(x) dx \\ &= \int_{-\infty}^{\infty} \bar{v}(Qx) \cdot (\mathcal{A}u_Q)(x) dx = (v_Q | \mathcal{A}u_Q). \end{aligned} \quad (6.7)$$

□

**Lemma 6.2.** *Set, for  $u \in \mathcal{S}(\mathbb{R})$ ,*

$$(\theta_N u)(n) = \sum_{\ell \in \mathbb{Z}} u\left(\frac{n}{N} + 2\ell N\right), \quad n \bmod 2N^2. \quad (6.8)$$

On the other hand, set

$$(\kappa u)(n) = \sum_{\ell_1 \in \mathbb{Z}} u\left(\frac{n}{R} + 2QN\ell_1\right), \quad n \in \mathbb{Z}/(2N^2)\mathbb{Z}. \quad (6.9)$$

One has

$$(\kappa u)(n) = (\theta_N u_Q)(n). \quad (6.10)$$

*Proof.* It is immediate. □

The following proposition reproduces [6, Prop. 4.1.2], with the simplification brought by the fact, made possible by the choice of the Weyl calculus  $\Psi$  rather than  $\text{Op}_1$ , that we may take  $N$  odd.

**Proposition 6.3.** *With  $N = RQ$  and  $b(j, k)$  satisfying the properties in the beginning of this section, define the function*

$$f(j, s) = \frac{1}{N} \sum_{k \bmod N} b(j, k) \exp\left(\frac{2i\pi ks}{N}\right), \quad j, s \in \mathbb{Z}/N\mathbb{Z}. \quad (6.11)$$

Set, noting that the condition  $m - n \equiv 0 \bmod 2Q$  implies that  $m + n$  too is even,

$$c_{R,Q}(\mathfrak{S}; m, n) = \text{char}(m+n \equiv 0 \bmod R, m-n \equiv 0 \bmod 2Q) f\left(\frac{m+n}{2R}, \frac{m-n}{2Q}\right). \quad (6.12)$$

Then, one has

$$(v \mid \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{S}) u) = \sum_{m, n \in \mathbb{Z}/(2N^2)\mathbb{Z}} c_{R,Q}(\mathfrak{S}; m, n) \overline{\theta_N v(m)} (\theta_N u)(n). \quad (6.13)$$

*Proof.* There is no restriction here on the supports of  $v, u$ , and one can replace these two functions by  $v_Q, u_Q$ . In view of (6.5) and (6.10), the identity (6.13) is equivalent to

$$(v \mid \mathcal{B}u) = \sum_{m, n \in \mathbb{Z}/(2N^2)\mathbb{Z}} c_{R,Q}(\mathfrak{S}; m, n) \overline{\kappa v(m)} (\kappa u)(n). \quad (6.14)$$

From (6.4), one has

$$\begin{aligned}
(v | \mathcal{B}u) &= \frac{1}{2Q^2} \int_{-\infty}^{\infty} \bar{v}(x) dx \int_{\mathbb{R}^2} \mathfrak{S}\left(\frac{x+y}{2}, \xi\right) u(y) \exp\left(\frac{i\pi}{Q^2}(x-y)\xi\right) dy d\xi \\
&= \frac{1}{Q^2} \int_{-\infty}^{\infty} \bar{v}(x) dx \int_{\mathbb{R}^2} \mathfrak{S}(y, \xi) u(2y-x) \exp\left(\frac{2i\pi}{Q^2}(x-y)\xi\right) dy d\xi \\
&= \frac{1}{Q^2} \int_{-\infty}^{\infty} \bar{v}(x) dx \sum_{j,k \in \mathbb{Z}} b(j, k) u(2j-x) \exp\left(\frac{2i\pi}{Q^2}(x-j)k\right).
\end{aligned} \tag{6.15}$$

Since  $b(j, k) = b(j, k + N)$ , one replaces  $k$  by  $k + N\ell$ , the new  $k$  lying in the interval  $[0, N - 1]$  of integers. One has (Poisson's formula)

$$\sum_{\ell \in \mathbb{Z}} \exp\left(\frac{2i\pi}{Q^2}(x-j)\ell N\right) = \sum_{\ell \in \mathbb{Z}} \exp\left(\frac{2i\pi}{Q}(x-j)\ell R\right) = \frac{Q}{R} \sum_{\ell \in \mathbb{Z}} \delta\left(x-j-\frac{\ell Q}{R}\right), \tag{6.16}$$

and

$$\begin{aligned}
&(\mathcal{B}u)(x) \\
&= \frac{1}{N} \sum_{\substack{j \in \mathbb{Z} \\ 0 \leq k < N}} b(j, k) \sum_{\ell \in \mathbb{Z}} u\left(j - \frac{\ell Q}{R}\right) \exp\left(\frac{2i\pi(x-j)k}{Q^2}\right) \delta\left(x-j-\frac{\ell Q}{R}\right) \\
&= \sum_{m \in \mathbb{Z}} t_m \delta\left(x - \frac{m}{R}\right), \tag{6.17}
\end{aligned}$$

with  $m = Rj + \ell Q$  and  $t_m$  to be made explicit: we shall drop the summation with respect to  $\ell$  for the benefit of a summation with respect to  $m$ . Since, when  $x = j + \frac{\ell Q}{R} = \frac{m}{R}$ , one has  $\frac{x-j}{Q^2} = \frac{\ell}{N} = \frac{m-Rj}{NQ}$  and  $j - \frac{\ell Q}{R} = 2j - x =$

$2j - \frac{m}{R}$ , one has

$$\begin{aligned}
t_m &= \frac{1}{N} \sum_{\substack{j \in \mathbb{Z} \\ 0 \leq k < N}} b(j, k) \operatorname{char}(m \equiv Rj \bmod Q) u \left( 2j - \frac{m}{R} \right) \exp \left( \frac{2i\pi k(m - Rj)}{NQ} \right) \\
&= \frac{1}{N} \sum_{\substack{0 \leq j < QN \\ 0 \leq k < N}} b(j, k) \operatorname{char}(m \equiv Rj \bmod Q) \\
&\quad \sum_{\ell_1 \in \mathbb{Z}} u \left( 2(j + \ell_1 QN) - \frac{m}{R} \right) \exp \left( \frac{2i\pi k(m - Rj)}{QN} \right). \quad (6.18)
\end{aligned}$$

Recalling the definition (6.9) of  $\kappa v$ , one obtains

$$\begin{aligned}
t_m &= \frac{1}{N} \sum_{\substack{0 \leq j < QN \\ 0 \leq k < N}} b(j, k) \operatorname{char}(m \equiv Rj \bmod Q) \\
&\quad (\kappa u)(2Rj - m) \exp \left( \frac{2i\pi k(m - Rj)}{QN} \right). \quad (6.19)
\end{aligned}$$

The function  $\kappa v$  is  $(2N^2)$ -periodic, so one can replace the subscript  $0 \leq j < RQ^2$  by  $j \bmod RQ^2$ . Using (6.17), we obtain

$$\begin{aligned}
(v | \mathcal{B}u) &= \frac{1}{N} \sum_{j \bmod RQ^2} \sum_{0 \leq k < N} b(j, k) \sum_{\substack{m_1 \in \mathbb{Z} \\ m_1 \equiv Rj \bmod Q}} \\
&\quad \bar{v} \left( \frac{m_1}{R} \right) (\kappa u)(2Rj - m_1) \exp \left( \frac{2i\pi k(m_1 - Rj)}{QN} \right). \quad (6.20)
\end{aligned}$$

The change of  $m$  to  $m_1$  is just a change of notation.

Fixing  $k$ , we trade the set of pairs  $m_1, j$  with  $m_1 \in \mathbb{Z}, j \bmod RQ^2, m_1 \equiv Rj \bmod Q$  for the set of pairs  $m, n \in (\mathbb{Z}/(2N^2)\mathbb{Z}) \times (\mathbb{Z}/(2N^2)\mathbb{Z})$ , where  $m$  is the class mod  $2N^2$  of  $m_1$  and  $n$  is the class mod  $2N^2$  of  $2Rj - m_1$ . Of necessity,  $m + n \equiv 0 \bmod 2R$  and  $m - n \equiv 2(m - Rj) \equiv 0 \bmod 2Q$ . Conversely, given a pair of classes  $m, n \bmod 2N^2$  satisfying these conditions, the equation  $2Rj - m = n$  uniquely determines  $j \bmod \frac{2N^2}{2R} = RQ^2$ , as it should.

The sum  $\sum_{m_1 \equiv m \pmod{2N^2}} v\left(\frac{m_1}{R}\right)$  is just  $(\kappa v)(m)$ , and we have obtained the identity

$$(v | \mathcal{B}u) = \sum_{m, n \pmod{2N^2}} c_{R,Q}(\mathfrak{S}; m, n) \overline{(\kappa v)(m)} (\kappa u)(n), \quad (6.21)$$

provided we define

$$c_{R,Q}(\mathfrak{S}; m, n) = \frac{1}{N} \text{char}(m + n \equiv 0 \pmod{R}, m - n \equiv 0 \pmod{2Q}) \sum_{k \pmod{N}} b\left(\frac{m+n}{2R}, k\right) \exp\left(\frac{2i\pi k}{N} \frac{m-n}{2Q}\right), \quad (6.22)$$

which is just the way indicated in (6.11), (6.12).  $\square$

## 7. THE $(R, Q)$ -DECOMPOSITION OF THE MAIN HERMITIAN FORM

We assume that  $N = RQ$  is a squarefree odd integer and we now make the coefficients  $c_{R,Q}(\mathfrak{T}_N; m, n)$  fully explicit. The following reproduces [6, Prop. 4.2.8] with the necessary modifications necessitated by the choice of the symbolic calculus  $\Psi$ . Note that this choice saved us much irritation with the prime 2.

**Proposition 7.1.** *One has*

$$c_{R,Q}(\mathfrak{T}_N; m, n) = \text{char}(m + n \equiv 0 \pmod{R}) \text{char}(m - n \equiv 0 \pmod{2Q}) \times \mu\left(\left(\frac{m+n}{2R}, N\right)\right) \text{char}\left(N = \left(\frac{m+n}{2R}, N\right) \left(\frac{m-n}{2Q}, N\right)\right). \quad (7.1)$$

*Proof.* We shall give two proofs of this formula, fundamental for our aim. One has the identity, most certainly found many times,

$$S(N, k) := \sum_{\substack{r \pmod{N} \\ (r, N) = 1}} \exp\left(-\frac{2i\pi rk}{N}\right) = \mu(N) a((k, N)). \quad (7.2)$$

For completeness' sake, we reproduce the proof given in [6, (4.2.53)]. Set  $N = pN_p$  for every  $p|N$  and, choosing integral coefficients  $\alpha_p$  such that  $\sum_{p|N} \alpha_p N_p = 1$ , write  $\frac{1}{N} = \sum_{p|N} \frac{\alpha_p}{p}$ . The coefficient  $\alpha_p$ , which is unique

$\bmod p$ , is not divisible by  $p$ . Under the identification of  $\mathbb{Z}/N\mathbb{Z}$  with  $\prod_{p|N} \mathbb{Z}/p\mathbb{Z}$ , one writes  $r = (r^p)_{p|N}$ . Then,

$$\exp\left(-2i\pi \frac{kr}{N}\right) = \prod_{p|N} \exp\left(-2i\pi kr \frac{\alpha_p}{p}\right) = \prod_{p|N} \exp\left(-2i\pi k \frac{\alpha_p r^p}{p}\right) \quad (7.3)$$

and

$$S(N; k) = \prod_{p|N} \sum_{r^p \in (\mathbb{Z}/p\mathbb{Z})^\times} \exp\left(-2i\pi k \sum \frac{\alpha_p r^p}{p}\right). \quad (7.4)$$

Since  $p \nmid \alpha_p$ , the value of the  $p$ th factor is  $p-1$  if  $p|k$ ,  $-1$  if  $p \nmid k$ . Hence,

$$S(N; k) = \prod_{p|N} [(-1)(1 - p \operatorname{char}(p|k))] = \mu(N) a((k, N)). \quad (7.5)$$

Then,

$$\mu((j, N)) a((j, k, N)) = \mu((j, N)) a((k, (j, N))) = \sum_{\substack{r \bmod (j, N) \\ (r, (j, N)) = 1}} \exp\left(-\frac{2i\pi rk}{(j, N)}\right). \quad (7.6)$$

With  $b(j, k) = a((j, k, N))$ , one has

$$\begin{aligned} f(j, s) &= \frac{1}{N} \sum_{k \bmod N} a((j, k, N)) \exp\left(\frac{2i\pi ks}{N}\right) \\ &= \frac{\mu((j, N))}{N} \sum_{\substack{r \bmod (j, N) \\ (r, j, N) = 1}} \sum_{k \bmod N} \exp\left(\frac{2i\pi k}{N} \left(s - \frac{rN}{(j, N)}\right)\right). \end{aligned} \quad (7.7)$$

The inside sum coincides with  $N \operatorname{char}\left(s \equiv \frac{rN}{(j, N)} \bmod N\right)$ . This requires first that  $s \equiv 0 \bmod \frac{N}{(j, N)}$ , in other words  $js \equiv 0 \bmod N$ , and we set  $s = \frac{Ns'}{(j, N)}$ . Then,

$$\begin{aligned} N \operatorname{char}\left(s - \frac{rN}{(j, N)} \equiv 0 \bmod N\right) &= N \operatorname{char}\left(\frac{N}{(j, N)} (s' - r) \equiv 0 \bmod N\right) \\ &= N \operatorname{char}(s' - r \equiv 0 \bmod (j, N)). \end{aligned} \quad (7.8)$$

Since

$$\sum_{\substack{r \bmod (j, N) \\ (r, j, N) = 1}} N \operatorname{char}(s' - r \equiv 0 \bmod (j, N)) = N \operatorname{char}((s', (j, N)) = 1), \quad (7.9)$$

one obtains

$$\begin{aligned} f(j, s) &= \mu((j, N)) \operatorname{char}(js \equiv 0 \bmod N) \times \operatorname{char}\left[\left(\frac{(j, N)s}{N}, (j, N)\right) = 1\right] \\ &= \mu((j, N)) \operatorname{char}(js \equiv 0 \bmod N) \times \operatorname{char}[(j, N)s, (j, N)N = N]. \end{aligned} \quad (7.10)$$

The condition  $((j, N)s, (j, N)N) = N$ , or  $(j, N)(s, N) = N$ , implies the condition  $js \equiv 0 \bmod N$ . Hence,

$$f(j, s) = \mu((j, N)) \operatorname{char}[(j, N), (s, N)) = N]. \quad (7.11)$$

The equation (7.1) follows then from (6.12).  $\square$

In view of the importance of the formula (7.1), let us give a short alternative proof of it, based on using Proposition 6.3 “in reverse”. So that (6.12) should be satisfied with  $c_{R,Q}(\mathfrak{I}_N; m, n)$  as it appears in (7.1), we must take

$$f(j, s) = \operatorname{char}(j, s \in \mathbb{Z}) \mu((j, N)) \operatorname{char}(N = (j, N)(s, N)). \quad (7.12)$$

A function  $b(j, k)$  leading to (6.11) can then be obtained by an inversion of that formula, to wit

$$b(j, k) = \sum_{s \bmod N} f(j, s) \exp\left(-\frac{2i\pi ks}{N}\right). \quad (7.13)$$

If  $N = N_1 N_2$ , one writes, choosing  $a, d$  such that  $aN_1 + dN_2 = 1$ ,  $\frac{1}{N} = \frac{d}{N_1} + \frac{a}{N_2}$ , so that

$$\exp\left(-\frac{2i\pi ks}{N}\right) = \exp\left(-\frac{2i\pi dks}{N_1}\right) \exp\left(-\frac{2i\pi aks}{N_2}\right) \quad (7.14)$$

This leads to the equation (with an obvious notation)  $b(j, k) = b_{N_1}(j, k) b_{N_2}(j, k)$ , and the proof may be reduced to the case when  $N$  is a prime  $p$ . Then, in the formula (7.13), the term obtained for  $s \equiv 0$  is  $\operatorname{char}(j \not\equiv 0 \bmod p)$ , and the sum of terms obtained for  $s \not\equiv 0$  is  $-\operatorname{char}(j \equiv 0 \bmod p) \times [p \operatorname{char}(k \equiv$



$0 \bmod p) - 1]$ . Overall, the  $p$ -factor under investigation is thus

$$\text{char}(j \not\equiv 0) + \text{char}(j \equiv 0) [1 - p \text{char}(k \equiv 0)] = 1 - p \text{char}(j \equiv k \equiv 0). \quad (7.15)$$

This concludes this verification.

## 8. REFLECTIONS

**Lemma 8.1.** *Let  $N$  be a squarefree odd integer, and let  $v, u$  be two functions in  $\mathcal{S}(\mathbb{R})$ . One has for every squarefree odd integer  $N$  the identity*

$$(v \mid \Psi(\mathfrak{T}_N) u) = 2 \sum_{T|N} \mu(T) \sum_{j,k \in \mathbb{Z}} \bar{v} \left( Tj + \frac{k}{T} \right) u \left( Tj - \frac{k}{T} \right). \quad (8.1)$$

*Proof.* Together with the operator  $2i\pi\mathcal{E}$ , let us introduce the operator  $2i\pi\mathcal{E}^\natural = r \frac{\partial}{\partial r} - s \frac{\partial}{\partial s}$  when the coordinates  $(r, s)$  are used on  $\mathbb{R}^2$ . One has  $\mathcal{F}_2^{-1}[(2i\pi\mathcal{E})\mathfrak{S}] = (2i\pi\mathcal{E}^\natural)\mathcal{F}_2^{-1}\mathfrak{S}$  for every tempered distribution  $\mathfrak{S}$ .

From (4.11) and Poisson's formula, one obtains

$$\mathcal{F}_2^{-1}\mathfrak{T}_N = \prod_{p|N} \left( 1 - p^{-2i\pi\mathcal{E}^\natural} \right) \mathcal{F}_2^{-1}\mathcal{D}ir = \sum_{T|N} \mu(T) T^{-2i\pi\mathcal{E}^\natural} \mathcal{D}ir, \quad (8.2)$$

explicitly

$$(\mathcal{F}_2^{-1}\mathfrak{T}_N)(r, s) = \sum_{T|N} \mu(T) \sum_{j,k \in \mathbb{Z}} \delta \left( \frac{r}{T} - j \right) \delta(Ts - k). \quad (8.3)$$

The integral kernel of the operator  $\Psi(\mathfrak{T}_N)$  is

$$\begin{aligned} K(x, y) &= (\mathcal{F}_2^{-1}\mathfrak{T}_N) \left( \frac{x+y}{2}, \frac{x-y}{2} \right) \\ &= \sum_{T|N} \mu(T) \sum_{j,k \in \mathbb{Z}} \delta \left( \frac{x+y}{2T} - j \right) \delta \left( \frac{T(x-y)}{2} - k \right) \\ &= 2 \sum_{T|N} \mu(T) \sum_{j,k \in \mathbb{Z}} \delta \left( x - Tj - \frac{k}{T} \right) \delta \left( y - Tj + \frac{k}{T} \right). \end{aligned} \quad (8.4)$$

The equation (8.1) follows. □

For a given  $N$ , the  $(j, k)$ -series on the right-hand side of (8.1) is absolutely convergent. One can consider also its limit as  $N \nearrow \infty$  if dropping the terms in which  $j = 0$ : it suffices, with some  $A > 3$ , to write

$$\begin{aligned} \left| v \left( Tj + \frac{k}{T} \right) u \left( Tj - \frac{k}{T} \right) \right| &\leq C (1 + |Tj|)^{-A} \left( 1 + \frac{|k|}{T} \right)^{-A} \\ &\leq C (1 + T)^{-2} (1 + |Tj|)^{-A+2} \left( 1 + \frac{|k|}{T} \right)^{-A}. \end{aligned} \quad (8.5)$$

We have thus recovered the fact that  $\mathfrak{T}_N^\times$  (but not  $\mathfrak{T}_N$ ) has a limit in  $\mathcal{S}'(\mathbb{R}^2)$  as  $N \nearrow \infty$ .

The sole difficulty in the search for the estimate that would yield R.H. consists in the dependence on  $Q$ : finding a  $O(Q^{1+\varepsilon})$  is easy, but we need a  $O(Q^{\frac{1}{2}+\varepsilon})$ . The following theorem exhibits the link between the operators with symbols  $\mathfrak{T}_N$  and  $Q^{2i\pi\varepsilon}\mathfrak{T}_N$ .

**Theorem 8.2.** *Let  $N = RQ$  be a squarefree odd integer. Given a complex-valued function  $\psi$  on  $\mathbb{Z}/(2N^2)\mathbb{Z}$ , identified with  $(\mathbb{Z}/R^2\mathbb{Z}) \times (\mathbb{Z}/(2Q^2)\mathbb{Z})$ , a product in which the coordinates are denoted as  $n', n''$ , define the function  $\Lambda_{R,Q}\psi$  by the equation*

$$(\Lambda_{R,Q}\psi)(n', n'') = \psi(n', -n''). \quad (8.6)$$

*We also denote as  $n \mapsto \check{n}$  the reflection  $(n', n'') \mapsto (n', -n'')$ . Introduce the automorphism  $\Lambda_{R,Q}^\#$  of  $\mathcal{S}(\mathbb{R})$  defined by the equation*

$$\Lambda_{R,Q}^\# = \frac{1}{Q^2} \sum_{0 \leq \sigma, \tau < Q^2} \exp \left[ \frac{2R}{Q} \left( \tau \frac{d}{dx} + i\pi\sigma x \right) \right], \quad (8.7)$$

*in other words*

$$(\Lambda_{R,Q}^\#)(x) = \frac{1}{Q^2} \sum_{0 \leq \sigma, \tau < Q^2} w \left( x + \frac{2R\tau}{Q} \right) \exp \left( 2i\pi \frac{\sigma(Nx + R^2\tau)}{Q^2} \right), \quad (8.8)$$

*so that, for every  $w$ ,*

$$\theta_N \Lambda_{R,Q}^\# w = \Lambda_{R,Q} \theta_N w. \quad (8.9)$$

*Then, for every pair of functions  $v, u \in \mathcal{S}(\mathbb{R})$ , one has*

$$(v \mid \Psi(Q^{2i\pi\varepsilon}\mathfrak{T}_N) u) = \mu(Q) (v \mid \Psi(\mathfrak{T}_N) \Lambda_{R,Q}^\# u). \quad (8.10)$$

*Proof.* The proof of (8.9) goes as follows. From (6.8), one has

$$\begin{aligned} & \left( \theta_N \Lambda_{R,Q}^\# w \right) (n) \\ &= \frac{1}{Q^2} \sum_{\ell \in \mathbb{Z}} \sum_{0 \leq \sigma, \tau < Q^2} w \left( \frac{n}{N} + \frac{2R\tau}{Q} + 2\ell N \right) \exp \left( \frac{2i\pi\sigma}{Q^2} (n + 2\ell N^2 + R^2\tau) \right). \end{aligned} \quad (8.11)$$

Summing with respect to  $\sigma$ , this is the same as

$$\sum_{\ell \in \mathbb{Z}} w \left( \frac{n + 2R^2\tau + 2\ell N^2}{N} \right), \quad (8.12)$$

where the integer  $\tau \in [0, Q^2[$  is characterized by the condition  $n + 2\ell N^2 + R^2\tau \equiv 0 \pmod{Q^2}$ , or  $n + R^2\tau \equiv 0 \pmod{Q^2}$ . Finally, as  $\ell \in \mathbb{Z}$ , the number  $n + 2R^2\tau + 2\ell N^2$  runs through the set of integers  $n_2$  such that  $n_2 \equiv -n \pmod{Q^2}$  and  $n_2 \equiv n \pmod{2R^2}$ , in other words (moving the class mod 2 from  $2R^2$  to  $2Q^2$  does not change anything) the set of numbers  $n_2$  such that  $n_2 \equiv \check{n} \pmod{2N^2}$ .

What Theorem 8.2 says is that (up to the factor  $\mu(Q)$ ), applying  $Q^{2i\pi\mathcal{E}}$  to  $\mathfrak{T}_N$  has the same effect as transforming the  $\theta_N$ -transform of the second function  $u$  of the pair under the reflection  $n \mapsto \check{n}$ . With another normalization of the operator calculus, this was given in [6, Cor.4.2.7]. The main part was given two different proofs, one in [6, Prop.4.2.3] and the other in [6, p.62-64]. Here is a shorter one.

Let us use the expression of  $c_{R,Q}(\mathfrak{T}_N; m, n)$  given in Proposition 7.1. A special case of it is

$$\begin{aligned} c_{N,1}(\mathfrak{T}_N; m, n) &= \text{char}(m + n \equiv 0 \pmod{2N}) \\ &\times \mu \left( \left( \frac{m+n}{2N}, N \right) \right) \text{char} \left( N = \left( \frac{m+n}{2N}, N \right) (m-n, N) \right). \end{aligned} \quad (8.13)$$

To prove Theorem 8.2, we need to show the identity

$$c_{R,Q}(\mathfrak{T}_N; m, n) = \mu(Q) c_{N,1}(\mathfrak{T}_N; m, \check{n}). \quad (8.14)$$

The condition  $N = \left(\frac{m+n}{2N}, N\right) (m - \check{n}, N)$  can be rewritten as

$$N = \left(\frac{m+n}{2R}, R\right) \left(\frac{m-n}{2Q}, Q\right) (m-n, R)(m+n, Q), \quad (8.15)$$

which is the same as  $N = \left(\frac{m+n}{2R}, N\right) \left(\frac{m-n}{2Q}, N\right)$ . It remains to be verified that, under this condition, one has  $\mu\left(\left(\frac{m+n}{R}, N\right)\right) = \mu(Q) \mu\left(\left(\frac{m+n}{N}, N\right)\right)$ , or

$$\mu\left(\left(\frac{m+n}{R}, R\right)\right) \mu((m+n, Q)) = \mu(Q) \mu\left(\left(\frac{m+n}{R}, R\right)\right) \mu\left(\left(\frac{m-n}{Q}, Q\right)\right), \quad (8.16)$$

or

$$\mu((m+n, Q)) = \mu(Q) \mu\left(\left(\frac{m-n}{Q}, Q\right)\right), \quad (8.17)$$

or finally (since  $Q$  is squarefree and  $\mu(\cdot) = \pm 1$ )

$$\mu((m+n, Q)) \mu\left(\left(\frac{m-n}{Q}, Q\right)\right) = \mu(Q). \quad (8.18)$$

The equation (8.15) implies  $Q = \left(\frac{m-n}{Q}, Q\right) (m+n, Q)$ , from which (8.18) follows.

□

This is the point at which we had arrived in the summary given in the first 4 pages of [8]. There is no need to reproduce here the main part of that preprint, devoted to a proof of R.H.

Let us stress that, in any attempt on R.H. related to the present one, the consideration of pseudodifferential arithmetic (which may have come as a surprise to some readers) really lies at the core of the question. Distributions somewhat similar to  $\mathfrak{T}_\infty$  but living on the line, with the factor  $\zeta(\nu)$  in the denominator of their decomposition into homogeneous components, certainly exist. Examples are [6, p.28-29], with  $c > 1$  in both cases,

$$\begin{aligned} \mathfrak{t}_\infty(x) &:= \sum_{k \neq 0} a(k) \delta(x - k) = \frac{1}{2i\pi} \int_{\operatorname{Re} \nu = c} \frac{\zeta(\nu + 1)}{\zeta(\nu)} |x|^\nu d\nu, \\ \mathfrak{t}_\infty^{\text{Möb}}(x) &:= \sum_{k \neq 0} \mu(k) \delta(x - k) = \frac{1}{2i\pi} \int_{\operatorname{Re} \nu = c} \frac{|x|^{\nu-1}}{\zeta(\nu)} d\nu. \end{aligned} \quad (8.19)$$

But the specificity of the support conditions in Theorem 5.2 seems to indicate that operator theory is really at stake here.

The reader may wonder whether the functional equation of the zeta function played any role in the proof, apart from dispensing us with the consideration of non-trivial zeros with a real part  $< \frac{1}{2}$ . It certainly did, in a somewhat unusual way: in the proof of Lemma 3.4, we used the fact that the Eisenstein distribution  $\mathfrak{E}_{-\nu}$  is automorphic. Now, if one uses [5, Theor.1.2.2] and [5, (1.1.53)], one sees that, granted that  $\mathcal{F}^{\text{symp}}\mathfrak{E}_{\nu} = \mathfrak{E}_{-\nu}$ , the functional equation of zeta is equivalent to the automorphy property of  $\mathfrak{E}_{\nu}$ : this gives the functional equation another significance, possibly preparing at the same time for similar questions, in relation to the  $L$ -functions attached to Hecke eigenforms.

## REFERENCES

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