Strategies for Combining Conflicting Dogmatic Beliefs*

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Abstract – *The combination of possibly conflicting beliefs and evidence forms an important part of various disciplines of artificial reasoning. In everyday discourse dogmatic beliefs are expressed by observers when they have a strong and rigid opinion about a subject of interest. Such beliefs can be expressed and formalised within the Demspter-Shafer belief theory. This paper describes and compares methods for combining dogmatic or highly conflicting beliefs within this framework.*

Keywords: Probability, Belief theory, Subjective logic, Dogmatic belief, Conflict, Dempster's rule

1 Introduction

In the belief theory community there has been continuous controversy around the so-called Dempster's rule since the publication of Shafer's book *A Mathematical Theory of Evidence* [17] . The purpose of Dempster's rule is to combine two beliefs into a single belief that reflects the two possibly conflicting beliefs in a fair and equal way. Dempster's rule has been criticised mainly because highly conflicting beliefs tend to produce counterintuitive results. This drawback has been pointed out by several authors [21, 1] including Lotfi Zadeh[21] who provided a very simple example which will be referred to as Zadeh's example below.

Alternatives to Dempster's rule have been proposed by several authors [20, 7, 18, 16, 3, 14, 11]. These rules express different behaviours with respect to the beliefs to be combined but have the same basic goal : Manage the conflict when combining beliefs. This shows that the problem of conflict management is of major importance. A recent theoretical contribution is the Dezert-Smarandache theory [4] where beliefs are defined on a so-called *hyper-power set* of a frame of discernment which breaks with the classic assumption that elements in a frame of discernment must be mutually exclusive, and thereby provides a new interpretation and framework for managing belief conflicts.

In this paper, we describe various proposals for combining beliefs and how they handle cases when the beliefs are conflicting and dogmatic. Section 2 presents a background on Dempster-Shafer (DS) theory of evidence and Section 3 describes Dempster's rule of combination as well as its unnormalised and disjunctive versions. The subsequent sections introduce alternatives to Dempster's rule i.e. the *Weighted Operator* (Section 4), the *minC Combination* (Section 5), and the *Consensus Operator* (Section 6). In Section 7, we compare the performance of these operators on some simple examples of combining conflicting beliefs. Finally, a discussion of the proposed combination rules is provided in Section 8 followed by the conclusion.

2 Fundamentals of D-S Theory

In this section several concepts of the Dempster-Shafer theory of evidence [17] are recalled in order to introduce notations used throughout the paper. Let $\Theta = \{\theta_k, k = 1, \cdots, K\}$ denote a finite set of exhaustive and exclusive possible values for a variable y of interest[1]. *A basic belief assignment* (bba) m on Θ is defined as a function from 2^Θ to $[0, 1]$ satisfying:

$$\sum_{A \subseteq \Theta} m(A) = 1 \ . \qquad (1)$$

Values of a bba are called *basic belief masses* (bbms). A bba m such that $m(\emptyset) = 0$ is said to be normal. This condition was originally imposed by Shafer, but it may be relaxed if one accepts the open-world assumption stating that the set Θ might not be complete and y might take its value outside

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[1] Some authors have proposed to allow non-exhaustive[18] or non-exclusive[4] sets.

Θ. Each subset $A \subseteq \Theta$ such as $m(A) > 0$ is called a focal element of m. A bba m can be equivalently represented by a non additive measure: a belief function bel: $2^{\Theta} \rightarrow [0, 1]$, defined as

$$bel(A) \triangleq \sum_{\emptyset \neq B \subseteq A} m(B) \qquad \forall\, A \subseteq \Theta\,. \tag{2}$$

The quantity $bel(A)$ can be interpreted as a measure of one's belief that hypothesis A is true. Note that functions m and bel are in one-to-one correspondence [17] and can be seen as two facets of the same piece of information. If all the focal elements are singletons (i.e. one-element subsets of Θ) then we speak about *Bayesian belief functions*. If all the focal elements are nestable (i.e. linearly ordered by inclusion) then we speak about *consonant belief functions*. A *dogmatic belief function* is defined by Smets as a belief function for which $m(\Theta) = 0$. Let us note, that trivially, every Bayesian belief function is dogmatic.

As already mentioned, this paper focuses on the case where the beliefs are highly conflicting and dogmatic. In the following sections we describe some rules and compare their performance when applied to such beliefs.

3 Dempster's Rule

Let now assume that we have two pieces of evidence expressed by m_1 and m_2 representing two distinct items concerning the truth value of A. These two bba's can be aggregated with the conjunctive operator $\textcircled{\scriptsize$\cap$}$, yielding to a unique belief function corresponding to bba $m_{\textcircled{\scriptsize$\cap$}}$ defined as:

$$(m_1 \textcircled{\scriptsize$\cap$} m_2)(A) \triangleq \sum_{B \cap C = A} m_1(B) m_2(C) \quad \forall\, A \subseteq \Theta. \tag{3}$$

This rule is sometimes referred to as the (unnormalised) Dempster's rule of combination. If necessary, the normality assumption $m(\emptyset) = 0$ may be recovered by dividing each mass by a normalisation coefficient. The resulting operator which is knows as Dempster's rule denoted by $m_{\oplus}$ is defined as:

$$(m_1 \oplus m_2)(A) \triangleq \frac{(m_1 \textcircled{\scriptsize$\cap$} m_2)(A)}{1 - m(\emptyset)} \quad \forall\, \emptyset \neq A \subseteq \Theta \tag{4}$$

where the quantity $m(\emptyset)$ is called the degree of conflict between m_1 and m_2 and can be computed using:

$$m(\emptyset) = (m_1 \textcircled{\scriptsize$\cap$} m_2)(\emptyset) = \sum_{B \cap C = \emptyset} m_1(B) m_2(C)\,. \tag{5}$$

The use of Dempster's rule is possible only if m_1 and m_2 are not totally conflicting, i.e., if there exist two focal elements B and C of m_1 and m_2 satisfying $B \cap C \neq \emptyset$. This rule verifies some interesting properties (associativity, commutativity, non-idempotence) and its use has been justified theoretically by several authors [19, 13, 5] according to specific axioms. For algebraic analysis of the Dempster's rule on binary frame of discernment see e.g. [8, 9].

The normalisation in Dempster's rule redistributes conflicting belief masses to non-conflicting ones, and thereby tends to eliminate any conflicting characteristics in the resulting belief mass distribution. The non-normalised Dempster's rule avoids this particular problem by allocating all conflicting belief masses to the empty set. Smets explains this by arguing that the presence of highly conflicting beliefs indicates that some possible event must have been overlooked (the open world assumption) and therefore is missing in the frame of discernment. The idea is that conflicting belief masses should be allocated to this missing (empty) event. Smets has also proposed to interpret the amount of belief mass allocated to the empty set as a measure of conflict between separate beliefs [2].

Another approach on how to eliminate conflicts from the Dempster's rule is to replace $\cap$ by $\cup$ in Eq.(3) which produces the *Disjunctive (or Dual Dempster's) Rule* [6] defined below.

$$(m_1 \textcircled{\scriptsize$\cup$} m_2)(A) \triangleq \sum_{B \cup C = A} m_1(B) m_2(C) \quad \forall\, A \subseteq \Theta. \tag{6}$$

The interpretation of (conjunctive) Dempster's rule is that both beliefs to be combined are assumed to be are correct, while at least one of them (and not necessarily both) is assumed to be correct in the case of the Disjunctive Rule.

4 The Weighted Operator

The weighted operator [14] has been created to overcome the sensibility problem of Dempster's rule which produces unexpected results when evidence conflicts. The idea of the weighted operator is to distribute the conflicting belief mass $m(\emptyset)$ on some subsets of Θ according to additional knowledge. More precisely, a part of the mass $m(\emptyset)$ is assigned to a subset $A \subseteq \Theta$ according to a weighting factor denoted w. This weighting factor can be a function of the considered subset A and belief functions $\mathbf{m} = \{m_j, j = 1, \cdots, J\}$ which are involved in the combination and have caused the conflict. This idea is formalised in the following definition of the Weighted Operator.

Definition 1 (Weighted Operator) *Let* $\mathbf{m} = \{m_j, j = 1, \cdots, J\}$ *be the set of belief functions defined on* Θ *to be combined. The combination of the belief functions* $\mathbf{m}$ *with the weighted operator, denoted* $\textcircled{\scriptsize W}$, *is defined as:*

$$\begin{aligned} m_{\textcircled{\scriptsize W}}(\emptyset) &\triangleq w(\emptyset, \mathbf{m}).m(\emptyset) & (7)\\ m_{\textcircled{\scriptsize W}}(A) &\triangleq m_{\textcircled{\scriptsize$\cap$}}(A) + w(A, \mathbf{m}).m(\emptyset) \ \ \forall\, A \neq \emptyset. & (8) \end{aligned}$$

[2]Note that conflicting beliefs can exist inside of so called non-consonant beliefs (non-consonant belief functions , see [17]). This conflict is an accomplice to conflict between separate beliefs when allocating positive belief mass to the empty set.

In the definition of the weighted operator $\oplus$, the first term, $m_{\textcircled{\cap}}$, corresponds to the conjunctive rule of combination. The second one is the part of the conflicting mass assigned to each subset A and added to the conjunctive term. The symbol $\oplus$ has been chosen to highlight these two aspects. Weighting factors $w(A, \mathbf{m}) \in [0, 1]$ are coefficients which depend on each subset $A \subseteq \Theta$ and on the belief functions $\mathbf{m}$ to be combined. They must be constrained by:

$$\sum_{A \subseteq \Theta} w(A, \mathbf{m}) = 1 \tag{9}$$

so as to respect the property that the sum of mass functions must be equal to 1 (cf. Eq.(1)). In order to completely define this operator, we need additional information to choose the values of w which allow to have a particular behaviour of the operator.

This generic framework allows Dempster's rule of combination and other proposed by Smets [18], Yager [20] and Dubois and Prade [7] to be rewritten. For each operator, we only have to define the weighting factors $w(A, \mathbf{m})$ associated to each subset $A \subseteq \Theta$. For example, the unnormalised Dempster's rule is no more than the weighted operator with $w(A, \mathbf{m}) = 0$ for all $A \subseteq \Theta \backslash \{\emptyset\}$ and $w(\emptyset, \mathbf{m}) = 1$. This is the open world assumed by Smets. Yager [20] assumes that the frame of discernment Θ is exhaustive but its idea consists in assigning the conflicting mass $m(\emptyset)$ to the whole set Θ. According to the weighted operator previously presented, it is easy to reformulate the Yager's idea in setting $w(\Theta, \mathbf{m}) = 1$ and $w(\emptyset, \mathbf{m}) = 0$. According to the choice of weights w, we can define a family of weighted operators. Another operator of this family is the proportionalised combination which has been proposed by Daniel in [2].

In [14], another example has been proposed by the authors in the case of a pattern recognition problem. It consists in optimising the weighting factors values according to some criteria minimisation. This application gives interesting performance when compared to Dempster's rule. Unfortunately, the operator previously defined is not associative as remarked by Haenni [15]. However, an n-ary version of the operator exists, and combining n input bbas simultaneously can be a practical substitute for associativity in many real world applications.

According to the definition 1, we define a weighted average in computing the values of weighting factors w from $\mathbf{m}$ using a statistical average. This leads to the definition of the Weighted Average Operator.

Definition 2 (Weighted Average Operator) *Let m_a the belief function resulting from the weighted average combination of belief functions $\mathbf{m}$ using the operator $\oplus$ and weighting factors w^a. It is defined as:*

$$m_a = \underset{w^a}{\oplus} \mathbf{m} \tag{10}$$

with w^a computed $\forall A \subseteq \Theta \setminus \{\emptyset\}$ as:

$$w^a(A, \mathbf{m}) = \frac{1}{J} \sum_{j=1}^{J} m_j(A) \tag{11}$$

and $w^a(\emptyset, \mathbf{m}) = 0$.

It is easy to see from the definitions 1 and 2 that the weighted average operator has two different behaviours according to the value of $m(\emptyset)$. This operator, part of the family of weighted operators, can be used for combining dogmatic and conflicting beliefs.

5 The minC Combination

The minC combination (the minimal contradiction/conflict combination) is a generalisation of the unnormalised Dempster's rule. $m(\emptyset)$ is not considered as an argument for new unknown elements of the frame of discernment, $m(\emptyset)$ is considered as a conflict[3] arising by conjunctive combination. To handle it a system of different types of conflicts is considered with respect to basic belief masses which produce it.

We distinguish conflicts according to the sets to which the original bbms were assigned by m_j. Let us assume different singletons A, B, C (i.e. single-element subsets of Θ). Conflict is denoted by "$\times$" so that e.g. $A \times B$ means conflict between beliefs in A and B.

There is only one possible type of conflict on belief functions defined on binary frame of discernment, and in that case the minC combination after proportionalisation coincides with the conjunctive rule. In the case of an n-ary frame of discernment we distinguish different types of conflicts, e.g. $A \times B$, $A \times (B \cup C)$, $A \times B \times C$, if $m_i(A), m_j(B) > 0$, $m_i(A), m_j(B \cup C) > 0$, $m_i(A), m_j(B), m_k(C) > 0$ etc. The very important role play so called *potential conflicts*, e.g. $(A \cup B) \times (B \cup C)$ which is not a conflict in the case of combination of two beliefs ($(A \cup B) \cap (B \cup C) = B$), but it can cause a conflict in a later combination with another belief, e.g. real conflict $(A \cup B) \times (B \cup C) \times (A \cup C)$ because there is $(A \cup B) \cap (B \cup C) \cap (A \cup C) = \emptyset$. In order not to have an infinite number of different conflicts, the conflicts are divided into classes of equivalence which are called *types of conflicts*, e.g. $A \times B \sim B \times A \sim A \times B \times B \times B \times A \times A \times A$, etc. For more detail see [3].

The minC combination is commutative and associative. It overcomes some disadvantages of both versions of Dempster's rule (normalised and unnormalised). On the other hand, this theoretically nice combining rule has a computational complexity increasing with the size of the frame of discernment.

[3]The term "contradiction" is used in [3], while we use "conflict" here in order to have a uniform terminology.

Due to the fact that belief masses are assigned also to elements represented as types of conflicts the result of the minC combination is a generalised belief function. To obtain a belief function according to Eq.(2) we have to apply proportionalisation of bbms assigned to conflicts. Note that such a proportionalisation does not keep associativity. Hence we have to always keep the generalised version to be prepared for later combination with other beliefs.

Several variants of proportionalisation is discussed in [3]. We use the following one in this text. Belief mass assigned to a conflict is proportionally distributed among all focal elements which are contained in the conflict.

E.g. we present now how to proportionalize $m_0(A \times (B \cup C))$: let us assume $m_0(A \times (B \cup C)) = 0.24$ and $m_0(A) = 0.1$, $m_0(B) = 0.1$, $m_0(C) = 0.2$, $m_0(A \cup B) = 0.0$, $m_0(A \cup C) = 0.0$, $m_0(B \cup C) = 0.1$, $m_0(A \cup B \cup C) = 0.1$. It should be noted that the sum of the stated bbms $\neq 1$ because the other positive bbms are assigned to the another element of Θ (if $|\Theta| > 3$) and/or to another types of conflict ($A \times B,\ B \times (A \cup C), \ldots$). All elements corresponding to A, B, and C are contained in $A \times (B \cup C)$, and thus we distribute $m_0(A \times (B \cup C))$ proportionally among all nonempty subsets of $A \cup B \cup C$. We add $\frac{m_0(A)}{0.6} m_0(A \times (B \cup C)) = \frac{0.24}{6} = 0.04$ to $m_0(A)$; in general $\frac{m_0(X)}{\sum_{Y \subseteq A \cup B \cup C} m_0(Y)} m_0(A \times (B \cup C))$ to $m_0(X)$, for any $X \subseteq A \cup B \cup C$.

6 The Consensus Operator

6.1 The Opinion Space

The consensus operator [10, 11] is not defined for general frames of discernment, but only on binary frames of discernment. If the original frame of discernment is larger than binary it is possible to derive a binary frame of discernment containing any element A and its complement $\overline{A}$ through simple or normal coarsening[12]. After normal coarsening, the relative atomicity of A is equal to the relative cardinality of A in the original frame of discernment.

An *opinion* basically consists of a bba on a (coarsened) binary Θ with an additional *relative atomicity* parameter that enables the computation of the probability expectation value (or pignistic belief) of an opinion.

Definition 3 (Opinion) *Let Θ be a (coarsened) binary frame of discernment containing sets A and $\overline{A}$, and let m^X be the (coarsened) bba on Θ held by X. Let $b^X_A = m^X(A)$, $d^X_A = m^X(\overline{A})$ and $u^X_A = m^X(\Theta)$*[4] *be called the* belief, disbelief *and* uncertainty *components respectively, and let a^X_A represent the relative atomicity of A. Then X's* opinion *about A, denoted by ω^X_A, is the ordered tuple:*

$$\omega^X_A \triangleq (b^X_A,\ d^X_A,\ u^X_A,\ a^X_A)\ .$$

[4]Note that $u = 1 - b - d$ corresponds to vagueness in Hájek-Valdés approach [8, 9].

The belief, disbelief and uncertainty components of an opinion represent exactly the same as a bba, so the following equality holds:

$$b_A + d_A + u_A = 1\ , \qquad A \in 2^\Theta \setminus \emptyset. \tag{12}$$

Opinions have an equivalent represention as beta probability density functions (pdf) denoted by $\mathrm{beta}\,(\alpha, \beta)$ through the following bijective mapping:

$$\begin{aligned} &(b_A, d_A, u_A, a_A) \longleftrightarrow \\ &\mathrm{beta}\left(\tfrac{2b_A}{u_A} + 2a_A,\ \ \tfrac{2d_A}{u_A} + 2(1 - a_A)\right). \end{aligned} \tag{13}$$

This means for example that an opinion with $u_A = 1$ and $a_A = 0.5$ which maps to $\mathrm{beta}\,(1, 1)$ is equivalent to a uniform pdf. It also means that a dogmatic opinion with $u_A = 0$ which maps to $\mathrm{beta}\,(b_A \eta,\ d_A \eta)$ where $\eta \to \infty$ is equivalent to a spike pdf with infinitesimal width and infinite height. Dogmatic opinions can thus be interpreted as being based on an infinite amount of evidence.

6.2 The Consensus Operator

The Consensus Operator defined below is derived from the combination of two beta pdfs. More precisely, the two input opinions are mapped to beta pdfs according to Eq.(13) and combined, and the resulting beta pdf mapped back to the opinion space again, as described in [10]. The Consensus Operator can thus be interpreted as the statistical combination of two beta pdfs.

Definition 4 (Consensus Operator)
Let $\omega^X_A = (b^X_A, d^X_A, u^X_A, a^X_A)$ and $\omega^Y_A = (b^Y_A, d^Y_A, u^Y_A, a^Y_A)$ be opinions respectively held by agents X and Y about the same element A, and let $\kappa = u^X_A + u^Y_A - u^X_A u^Y_A$. When $u^X_A, u^Y_A \to 0$, the relative dogmatism between ω^X_A and ω^Y_A is defined by $\gamma^{X/Y}_A$ so that $\gamma^{X/Y}_A = u^Y_A / u^X_A$. Let $\omega^{X,Y}_A = (b^{X,Y}_A, d^{X,Y}_A, u^{X,Y}_A, a^{X,Y}_A)$ be the opinion such that:

for $\kappa \neq 0$:

$$\begin{cases} b^{X,Y}_A = (b^X_A u^Y_A + b^Y_A u^X_A)/\kappa \\ d^{X,Y}_A = (d^X_A u^Y_A + d^Y_A u^X_A)/\kappa \\ u^{X,Y}_A = (u^X_A u^Y_A)/\kappa \\ a^{X,Y}_A = \frac{a^X_A u^Y_A + a^Y_A u^X_A - (a^X_A + a^Y_A) u^X_A u^Y_A}{u^X_A + u^Y_A - 2u^X_A u^Y_A} \end{cases},$$

for $\kappa = 0$:

$$\begin{cases} b^{X,Y}_A = (\gamma^{X/Y}_A\, b^X_A + b^Y_A)/(\gamma^{X/Y}_A + 1) \\ d^{X,Y}_A = (\gamma^{X/Y}_A\, d^X_A + d^Y_A)/(\gamma^{X/Y}_A + 1) \\ u^{X,Y}_A = 0 \\ a^{X,Y}_A = (\gamma^{X/Y}_A\, a^X_A + a^Y_A)/(\gamma^{X/Y}_A + 1)\ . \end{cases}$$

Then $\omega^{X,Y}_A$ is called the consensus opinion between ω^X_A and ω^Y_A, representing an imaginary agent $[X, Y]$'s opinion about A, as if that agent represented both X and Y.

fe	m_1	m_2	DR	UDR	DDR	WAO	minC	CO
A	0.9	0.0	0.4737	0.09	0.00	0.4545	0.4737	0.9
B	0.1	0.9	0.5263	0.10	0.09	0.5050	0.5263	0.1
Θ	0.0	0.1	0.0000	0.00	0.81	0.0405	0.0000	0.0
$\emptyset$	0.0	0.0	0.0000	0.81	0.00	0.0000	0.0000	0.0

Table 1: Results of combining a dogmatic belief with a non-dogmatic belief.

The consensus operator is commutative, associative and non-idempotent. Associativity in case $\kappa = 0$ is a special case which is explained in Section 6.3 below.

In case of two totally uncertain opinions (i.e. $u = 1$) it is required that the observers agree on the relative atomicity so that the consensus relative atomicity for example can be defined as $a_A^{X,Y} = a_A^X$.

6.3 Associativity i.c.o. Dogmatic Beliefs

In case of dogmatic opinions the associativity of the consensus operator does not emerge directly from Def.4. In fact, the consensus operator is only defined and associative for dogmatic opinions if a relative dogmatism can be determined for each pair of opinions to be combined.

Let $\omega_A^{X_1}, \omega_A^{X_2}$ and $\omega_A^{X_3}$ be conflicting dogmatic opinions about A held by the observers X_1, X_2 and X_3 respectively. If e.g. $\gamma_A^{X_1/X_2}$ and $\gamma_A^{X_2/X_3}$ are known, then a consensus opinion can be determined because it is possible to derive the relative dogmatism between any pair of opinions. For example $\gamma_A^{X_1/X_3} = \gamma_A^{X_1/X_2} \cdot \gamma_A^{X_2/X_3}$. However, the consensus opinion can not be determined if e.g. only $\gamma_A^{X_1/X_2}$ is known. In case it is difficult to determine specific relative dogmatisms it is natural to always use the default relative dogmatism $\gamma = 1$ in which case it is only required to store the past number of argument bba's that have already been combined. The following algorithm[5] describes how three or more dogmatic opinions can be combined.

1. Choose the order in which the n opinions are to be combined: $\omega_A^{X_1}, ..., \omega_A^{X_n}$.

2. Define the relative dogmatism between $\omega_A^{X_1}$ and each other opinion on the form $\gamma_A^{X_1/X_i}$ where $i \in \{2, .., n\}$. The default value is $\gamma_A^{X_1/X_i} = 1$.

3. Compute the consensus opinions $\omega_A^{X_1,...,X_i}$ stepwise for each $i \in \{2, ..., n\}$ where the relative dogmatism between each opinion pair $\omega_A^{X_1,...,X_{i-1}}$ and $\omega_A^{X_i}$ is $\gamma_A^{X_1,...,X_{i-1}/X_i} = \gamma_A^{X_1/X_i}\left(1 + \sum_{j=2}^{i-1}\left(\frac{1}{\gamma_A^{X_1/X_j}}\right)\right)$ for $i \in \{3, ..., n\}$. The relative dogmatism $\gamma_A^{X_1/X_2}$ was already defined in the previous step.

[5]Erratum in the printed version, corrected here.

7 Comparison of Combination Rules

In this section, we present several examples in order to compare the performance of the different rules described in the previous sections. The following notation is used: DR : Dempster's Rule, UDR : the Unnormalised Dempster's Rule, DDR : the Disjunctive (Dempster's) Rule, WAO : the Weighted Average Operator, n-WAO : the n-ary version of WAO, minC : the minimal Contradiction rule and CO : the Consensus Operator. The abbreviation "fe" is used to denote "focal element".

7.1 Example 1 : One Dogmatic Belief

Let m_1 and m_2 represent two distinct pieces of evidence about the states in $\Theta = \{A, B\}$. In this example, we suppose that we want to combine a dogmatic belief function m_1 with a non-dogmatic one m_2. Table 1 presents these two bba's with the results obtained by the previously presented operators. Because the frame Θ is binary, minC combination after proportionalisation coincides with DR. For CO, the dogmatic bba totally overrules the non-dogmatic bba, whereas for WAO and minC the results are close statistical average.

7.2 Example 2 : Zadeh's Example

Let m_1 and m_2 represent two distinct testimonies about the guilt of three suspects $\{A, B, C\} = \Theta$ in a murder case (see Table 2). Because of the symmetry between the two testimonies the default relative dogmatisms $\gamma_A^{m_1/m_2}, \gamma_B^{m_1/m_2}, \gamma_C^{m_1/m_2} = 1$ are used. It can be seen that the weighted average and the consensus operators produce equal results which in fact is the statistical average of m_1 and m_2. The results in the column for the minC rule in Table 2 are produced by proportionalisation of bbms allocated to conflicts, and are quite similar to those of WAO and CO. The generalised bba is: $m(A) = 0.0$, $m(B) = 0.0001$, $m(C) = 0.0$, $m(A \times B) = 0.0099$, $m(A \times C) = 0.9801$ and $m(B \times C) = 0.0099$. The unnormalised Dempster's rule indicates, in line with the open world assumption, that neither of the suspects is guilty.

7.3 Example 3 : Zadeh's Example Modified

By introducing a small amount of uncertainty in the witnesses testimonies (see Table 3), the weighted average and

fe	m_1	m_2	DR	UDR	DDR	WAO	minC	CO
A	0.99	0.00	0.00	0.0000	0.0000	0.495	0.4905	0.495
B	0.01	0.01	1.00	0.0001	0.0001	0.010	0.0199	0.010
C	0.00	0.99	0.00	0.0000	0.0000	0.495	0.4905	0.495
$A \cup B$	0.00	0.00	0.00	0.0000	0.0099	0.000	0.0000	0.000
$A \cup C$	0.00	0.00	0.00	0.0000	0.9801	0.000	0.0000	0.000
$B \cup C$	0.00	0.00	0.00	0.0000	0.0099	0.000	0.0000	0.000
Θ	0.00	0.00	0.00	0.0000	0.0000	0.000	0.0000	0.000
$\emptyset$	0.00	0.00	0.00	0.9999	0.0000	0.000	0.0000	0.000

Table 2: Zadeh's example.

fe	m_1	m_2	DR	UDR	DDR	WAO	minC	CO
A	0.98	0.00	0.4900	0.0098	0.0000	0.4900	0.4995	0.492
B	0.01	0.01	0.0150	0.0003	0.0001	0.0101	0.0009	0.010
C	0.00	0.98	0.4900	0.0098	0.0000	0.4900	0.4995	0.492
$A \cup B$	0.00	0.00	0.0000	0.0000	0.0098	0.000	0.0000	0.000
$A \cup C$	0.00	0.00	0.0000	0.0000	0.9801	0.000	0.0000	0.000
$B \cup C$	0.00	0.00	0.0000	0.0000	0.0098	0.000	0.0000	0.000
Θ	0.01	0.01	0.0050	0.0001	0.0001	0.0099	0.0001	0.005
$\emptyset$	0.00	0.00	0.0000	0.9800	0.0000	0.0000	0.0000	0.000

Table 3: The modified Zadeh's example.

the consensus operators produce different but still very similar results. The unnormalised Dempster's rule still indicates that new suspects must be found. The results in the column of the minC rule in Table 3 are produced by proportionalisation of bbms allocated by minC to conflicts, and are quite similar to those of WAO and CO. The generalised bba is: $m(A) = 0.0098$, $m(B) = 0.0003$, $m(C) = 0.0098$, $m(A \times B) = 0.0098$, $m(A \times C) = 0.9604$ and $m(B \times C) = 0.0098$.

7.4 Example 4 : TRUE and FALSE

The most extreme case of conflicting dogmatic belief is the case of combining TRUE and FALSE, i.e. when the two bba's assign all belief to two distinct non-overlapping sets in Θ. Results obtained by the different operators are illustrated in Table 4. Note that the Dempster's rule is not applicable because there doesn't exist two focal elements A and B of m_1 and m_2 satisfying $A \cap B \neq \emptyset$. This is the only case where minC after proportionalisation does not coincide with DR on binary frames of discernment.

In this special case, several authors propose to use the disjunctive rule of combination as expressed in Eq.(6). For this example, the disjunctive combination of m_1 and m_2 leads to the belief function $m(A \cup B) = 1$ which in the binary case is the vacuous belief function $m(\Theta) = 1$. This result can be seen as cautious opinion. However, the major drawback of this rule concerns the non-specificity of the resulting belief function which tends to dramatically increase in case of highly conflicting beliefs.

The WA and CO operators both produce the statistical average between m_1 and m_2. The results in the column for minC in Table 4 are produced by proportionalisation of the bbms allocated by minC to conflicts, and the results are different from those of WAO and CO. The generalised bba is $m(A) = 0.0$, $m(B) = 0.0$, $m(\Theta) = 0.0$ and $m(A \times B) = 1$.

7.5 Example 5 : Three Dogmatic Beliefs

Let m_1, m_2 and m_3 represent three distinct pieces of evidence about the states in $\Theta = \{A, B\}$. In case the combination rule is not associative the result will not be unique. All the results are shown in table 5. In fact, the bba's can be combined in three different ways according to the orders which are defined as follows: 1: $m_1 \circledcirc (m_2 \circledcirc m_3)$, 2: $m_2 \circledcirc (m_1 \circledcirc m_3)$, 3: $m_3 \circledcirc (m_1 \circledcirc m_2)$. In this table, the abbreviation "o" is used to denote "order".

As mentioned earlier, the WAO is not associative and results depend on the order of the belief functions to be combined. If we suppose that the three belief functions are simultaneously available the n-ary WAO version produces the results shown in the column for n-WAO.

The result in the column for the minC rule in Table 5 is obtained after proportionalisation of bbms allocated by minC to conflicts. The generalised bba is $m(A) = 0.126$, $m(B) = 0.024$, $m(\Theta) = 0.0$ and $m(A \times B) = 0.850$.

fe	m_1	m_2	DR	UDR	DDR	WAO	minC	CO
A	1.0	0.0	N.A.	0.0	0.0	0.5	0.3333	0.5
B	0.0	1.0	N.A.	0.0	0.0	0.5	0.3333	0.5
Θ	0.0	0.0	N.A.	0.0	1.0	0.0	0.3333	0.0
$\emptyset$	0.0	0.0	N.A.	1.0	0.0	0.0	0.0000	0.0

Table 4: Results of combining TRUE and FALSE.

fe	m_1	m_2	m_3	DR	UDR	DDR		WAO		n-WAO	minC	CO
o							1	2	3			
A	0.9	0.7	0.2	0.840	0.126	0.63	0.749	0.710	0.589	0.636	0.840	0.6
B	0.1	0.3	0.8	0.160	0.124	0.03	0.251	0.290	0.411	0.364	0.160	0.4
Θ	0.0	0.0	0.0	0.000	0.000	0.34	0.000	0.000	0.000	0.000	0.000	0.0
$\emptyset$	0.0	0.0	0.0	0.000	0.850	0.00	0.000	0.000	0.000	0.000	0.000	0.0

Table 5: Results of combining three dogmatic bba's.

8 Discussion

Three important characteristics for discussing and comparing the operators and their results are:

1. **Associativity.** From a theoretical point of view associativity is always desirable. This property generally allows to simply rewrite formulae and can be useful to preserve several other properties. From a practical point of view, n-ary versions of operators can be used. This means that in case n bba's are to be combined, the n input bba's must be available simultaneously and the result is computed with an n-ary operator. Theoretically, n-ary versions of operators can't be replaced associativity but under some assumptions they can be easily implemented for real world applications.

2. **Simplicity.** This can refer to various aspects of using a particular combination rule. It might for example mean that the rule is simple to implement in software or that it does not require storing old input arguments or intermediate results for future use.

3. **Interpretation and assumptions.** An operator should always have a meaningful interpretation and provide assumptions for when the operator can be applied. Without this, any operator becomes a purely mathematical concept deprived of any real world meaning. The previous examples show that the operators produce different results, and each result should be justifiable through the operator's interpretation.

DR which is associative and simple, assumes that both input beliefs are correct, which seems to be incompatible with the combination of highly conflicting beliefs. DR therefore produces counterintuitive results with increasing degree of conflict, and it is not even applicable in the case of combining TRUE and FALSE.

UDR which is also associative and simple, assumes that the frame of discernment is not exhaustive (open world), which makes it unsuitable in case the frame can be well defined. As a result of the open world assumption all conflicting belief is contained in the bbm $m(\emptyset)$. Instead of producing some kind of compromise belief, UDR assumes that all conflicting belief terms are wrong simply because they are conflicting. There is a problem of increasing non-specificity, especially in the case of vacuous belief function as one of its arguments.

DDR is associative and simple, and moreover has no difficulties with conflicts. There is a problem of increasing non-specificity especially in the case of vacuous belief function as one of its arguments. In this extreme case, DDR will overrule any non-vacuous belief.

WAO is simple but not associative. The n-ary version of the rule can be used as a substitute for associativity, but it increases the complexity by requiring that all argument bba's must be stored. WAO makes no assumptions about the correctness of the input beliefs. WAO amplifies consonant belief terms, and converges towards statistical average when the degree of conflict is high.

minC is associative, produces intuitive results but has high computational complexity in case the frame of discernment contains many elements. It commutes with refinement/coarsening of frame of discernment on the generalised level, but unfortunately we do not know how to reconstruct n-nary belief from the corresponding set of binary ones in order to exploit the relative simplicity of the operator on binary frames of discernment. It still needs further investigation before the possibility of practical application.

CO is simple and associative and makes no assumptions about the correctness of the input beliefs. CO amplifies consonant belief terms, and converge towards statistical average of conflicting beliefs with equal levels of uncertainty.

Less uncertain beliefs have more weight, and CO interprets dogmatic beliefs to have infinite weight so that they always overrule any non-dogmatic beliefs. The combination of two dogmatic beliefs requires the additional relative dogmatism parameter, which represents an increase in complexity.

9 Conclusion

In this paper we have focused on the problem of combining highly conflicting and dogmatic beliefs within the belief functions theory, where dogmatic beliefs are defined according to Smets as having $m(\Theta) = 0$. There is no single philosophical definition of dogmatism in general, and there is no philosophical interpretation that corresponds to Smets' mathematical definition of dogmatism in particular. The Weighted Average Operator, the minimum Contradiction Combination, and the Consensus Operator provide alternatives to the traditional operators. By not making any assumptions about missing elements in the frame of discernment or about the input beliefs being correct or wrong these new operators produce results that are much more convincing than those of the traditional operators. Each of the new operators have advantages and disadvantages, and the differences in their computational results must be seen in the light of the interpretations of belief and dogmatism which can be associated with each operator.

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