The Consensus Operator for Combining Beliefs

Audun Jøsang [1]

Distributed Systems Technology Centre
Brisbane, Qld 4001, Australia, email: ajosang@dstc.edu.au

Abstract

The consensus operator provides a method for combining possibly conflicting beliefs within the Dempster-Shafer belief theory, and represents an alternative to the traditional Dempster's rule. This paper describes how the consensus operator can be applied to dogmatic conflicting opinions, i.e. when the degree of conflict is very high. It overcomes shortcomings of Dempster's rule and other operators that have been proposed for combining possibly conflicting beliefs.

Key words: Dempster's rule, belief, conflict, consensus operator, subjective logic

1 Introduction

Ever since the publication of Shafer's book *A Mathematical Theory of Evidence* [1] there has been continuous controversy around the so-called Dempster's rule. The purpose of Dempster's rule is to combine two conflicting beliefs into a single belief that reflects the two conflicting beliefs in a fair and equal way.

Dempster's rule has been criticised mainly because highly conflicting beliefs tend to produce counterintuitive results. This has been formulated in the form of examples by Zadeh [2] and Cohen [3] among others. The problem with Dempster's rule is due to its normalisation which redistributes conflicting belief masses to non-conflicting beliefs, and thereby tends to eliminate any conflicting characteristics in the resulting belief mass distribution. An alternative called the non-normalised Dempster's rule proposed by Smets [4] avoids this particular problem by allocating all conflicting belief masses to the empty set. Smets explains this by arguing that

[1] The work reported in this paper has been funded in part by the Co-operative Research Centre for Enterprise Distributed Systems Technology (DSTC) through the Australian Federal Government's CRC Programme (Department of Industry, Science & Resources).

Preprint of article published in Artificial Intelligence Journal, Vol.141/1-2 (2002), p.157-170

the presence of highly conflicting beliefs indicates that some possible event must have been overlooked (the open world assumption) and therefore is missing in the frame of discernment. The idea is that conflicting belief masses should be allocated to this missing (empty) event. Smets has also proposed to interpret the amount of belief mass allocated to the empty set as a measure of conflict between separate beliefs [5].

In this paper we describe an alternative rule for combining conflicting belief functions called the *consensus operator*. The consensus operator forms part of *subjective logic* which is described in [6]. Our consensus operator is different from Dempster's rule but has the same purpose; namely of combining possibly conflicting beliefs. The definition of the consensus operator in [6] and earlier publications does not cover combination of conflicting dogmatic beliefs, i.e. highly conflicting beliefs. This paper extends the definition of the consensus operator to also cover such cases. A comparison between the consensus operator and the two variants of Dempster's rule is provided in the form of examples.

Subjective logic is a framework for artificial reasoning with uncertain beliefs which for example can be applied to legal reasoning [7] and authentication in computer networks [8]. In subjective logic, beliefs must be expressed on binary frames of discernment, and coarsening is necessary if the original frame of discernment is larger than binary. Section 2 describes some basic elements from the Dempster-Shafer theory as well as some new concepts related to coarsening. Section 3 describes the opinion metric which is the binary belief representation used in subjective logic. Section 4 describes the consensus operator which operates on opinions and section 5 provides a comparison between the consensus operator and the two variants of Dempster's rule. A discussion of our results is provided in section 6.

2 Representing Uncertain Beliefs

The first step in applying the Dempster-Shafer belief model [1] is to define a set of possible states of a given system, called the *frame of discernment* denoted by Θ.

The powerset of Θ, denoted by 2^Θ, contains all possible unions of the sets in Θ including Θ itself. Elementary sets in a frame of discernment Θ will be called atomic sets because they do not contain subsets. It is assumed that only one atomic set can be true at any one time. If a set is assumed to be true, then all supersets are considered true as well.

An observer who believes that one or several sets in the powerset of Θ might be true can assign belief masses to these sets. Belief mass on an atomic set $x \in 2^\Theta$ is interpreted as the belief that the set in question is true. Belief mass on a non-atomic set $x \in 2^\Theta$ is interpreted as the belief that one of the atomic sets it contains is true, but that the observer is uncertain about which of them is true. The following definition is central in the Dempster-Shafer theory.

Definition 1 (Belief Mass Assignment) *Let Θ be a frame of discernment. If with each subset $x \in 2^\Theta$ a number $m_\Theta(x)$ is associated such that:*

1. $m_\Theta(x) \geq 0$,
2. $m_\Theta(\emptyset) = 0$,
3. $\sum_{x \in 2^\Theta} m_\Theta(x) = 1$.

then m_Θ is called a belief mass assignment [2] *on Θ, or BMA for short. For each subset $x \in 2^\Theta$, the number $m_\Theta(x)$ is called the* belief mass [3] *of x.*

A belief mass $m_\Theta(x)$ expresses the belief assigned to the set x and does not express any belief in subsets of x in particular. A BMA is called *dogmatic* if $m_\Theta(\Theta) = 0$ (see [5] p.277) because the total amount of belief mass has been committed.

In contrast to belief mass, the *belief* in a set must be interpreted as an observer's total belief that a particular set is true. The next definition from the Dempster-Shafer theory will make it clear that belief in x not only depends on belief mass assigned to x but also on belief mass assigned to subsets of x.

Definition 2 (Belief Function) *Let Θ be a frame of discernment, and let m_Θ be a BMA on Θ. Then the* belief function *corresponding with m_Θ is the function $b : 2^\Theta \rightarrow [0, 1]$ defined by:*

$$b(x) = \sum_{y \subseteq x} m_\Theta(y), \quad x, y \in 2^\Theta \ .$$

Similarly to belief, an observer's *disbelief* must be interpreted as the total belief that a set is **not** true. The following definition is ours.

Definition 3 (Disbelief Function) *Let Θ be a frame of discernment, and let m_Θ be a BMA on Θ. Then the* disbelief function *corresponding with m_Θ is the function $d : 2^\Theta \rightarrow [0, 1]$ defined by:*

$$d(x) = \sum_{y \cap x = \emptyset} m_\Theta(y), \quad x, y \in 2^\Theta \ .$$

The disbelief in x is equal to the belief in $\overline{x}$, and corresponds to the *doubt* of x in Shafer's book. However, we choose to use the term 'disbelief' because we feel that

[2] Called *basic probability assignment* in [1]
[3] Called *basic probability number* in [1]

for example the case when it is certain that a set is false can better be described by 'total disbelief' than by 'total doubt'. Our next definition expresses uncertainty regarding a given set as the sum of belief masses on supersets or on partly overlapping sets of x.

Definition 4 (Uncertainty Function) *Let Θ be a frame of discernment, and let m_Θ be a BMA on Θ. Then the* uncertainty function *corresponding with m_Θ is the function $u: 2^\Theta \rightarrow [0,1]$ defined by:*

$$u(x) = \sum_{\substack{y \cap x \neq \emptyset \\ y \not\subseteq x}} m_\Theta(y), \quad x, y \in 2^\Theta .$$

The sum of the belief, disbelief and uncertainty functions is equal to the sum of the belief masses in a BMA which according to Definition 1 is equal to 1. The following equality is therefore trivial to prove:

$$b(x) + d(x) + u(x) = 1 , \qquad x \in 2^\Theta,\ x \neq \emptyset. \tag{1}$$

For the purpose of deriving probability expectation values of sets in 2^Θ, we will show that knowing the relative number of atomic sets is also needed in addition to belief masses. For any particular set x the *atomicity* of x is the number of atomic sets it contains, denoted by $|x|$. If Θ is a frame of discernment, the atomicity of Θ is equal to the total number of atomic sets. Similarly, if $x, y \in 2^\Theta$ then the overlap between x and y relative to y can be expressed in terms of atomic sets. Our next definition captures this idea of relative atomicity:

Definition 5 (Relative Atomicity) *Let Θ be a frame of discernment and let $x, y \in 2^\Theta$. Then the* relative atomicity *of x to y is the function $a: 2^\Theta \rightarrow [0,1]$ defined by:*

$$a(x/y) = \frac{|x \cap y|}{|y|}, \quad x, y \in 2^\Theta .$$

It can be observed that $(x \cap y = \emptyset) \Rightarrow (a(x/y) = 0)$, and that $(y \subseteq x) \Rightarrow (a(x/y) = 1)$. In all other cases the relative atomicity will be a value between 0 and 1. The relative atomicity of an atomic set to its frame of discernment, denoted by $a(x/\Theta)$, can simply be written as $a(x)$. If nothing else is specified, the relative atomicity of a set then refers to the frame of discernment.

A frame of discernment with a corresponding BMA can be used to determine a probability expectation value for any given set. The greater the relative atomicity of a particular set the more the uncertainty function will contribute to the probability expectation value of that set.

Definition 6 (Probability Expectation) *Let Θ be a frame of discernment with BMA m_Θ, then the* probability expectation function *corresponding with m_Θ is the function $\mathrm{E}: 2^\Theta \to [0,1]$ defined by:*

$$\mathrm{E}(x) = \sum_y m_\Theta(y)\; a(x/y), \quad x, y \in 2^\Theta \; .$$

Definition 6 is equivalent to the pignistic probability justified by e.g. Smets & Kennes in [9], and corresponds to the principle of insufficient reason: a belief mass assigned to the union of n atomic sets is split equally among these n sets.

In order to simplify the representation of uncertain beliefs for particular sets we will define a *focused frame of discernment* which will always be binary, i.e. it will only contain (focus on) one particular set and its complement. The focused frame of discernment and the corresponding BMA will for the set in focus produce the same belief, disbelief and uncertainty functions as the original frame of discernment and BMA. The definitions of the focused frame of discernment and the focused BMA are given below.

Definition 7 (Focused Frame of Discernment) *Let Θ be a frame of discernment and let $x \in 2^\Theta$. The frame of discernment denoted by $\widetilde{\Theta}^x$ containing only x and $\overline{x}$, where $\overline{x}$ is the complement of x in Θ is then called a focused frame of discernment with focus on x.*

Definition 8 (Focused Belief Mass Assignment) *Let Θ be a frame of discernment with BMA m_Θ where $b(x)$, $d(x)$ and $u(x)$ are the belief, disbelief and uncertainty functions of x in 2^Θ, and let $a(x)$ be the real relative atomicity of x in Θ. Let $\widetilde{\Theta}^x$ be the focused frame of discernment with focus on x. The corresponding focused BMA $m_{\widetilde{\Theta}^x}$ and relative atomicity $a_{\widetilde{\Theta}^x}(x)$ on $\widetilde{\Theta}^x$ is defined according to:*

$$\begin{cases} m_{\widetilde{\Theta}^x}(x) &= b(x) \\ m_{\widetilde{\Theta}^x}(\overline{x}) &= d(x) \\ m_{\widetilde{\Theta}^x}(\widetilde{\Theta}^x) &= u(x) \end{cases} \qquad \begin{cases} a_{\widetilde{\Theta}^x}(x) = \frac{\mathrm{E}(x)-b(x)}{u(x)} & \textit{for } u(x) \neq 0 \\ a_{\widetilde{\Theta}^x}(x) = a(x) & \textit{for } u(x) = 0\;. \end{cases} \tag{2}$$

When the original frame of discernment Θ contains more than 2 atomic sets, the relative atomicity of x in the focused frame of discernment $\widetilde{\Theta}^x$ is in general different from $\frac{1}{2}$ although $\widetilde{\Theta}^x$ per definition contains exactly two sets. The focused relative atomicity of x in $\widetilde{\Theta}^x$ is defined so that the probability expectation value of x is equal in Θ and $\widetilde{\Theta}^x$, and the expression for $a_{\widetilde{\Theta}^x}(x)$ can be determined by using Definition 6. A focused relative atomicity represents the weighted average of relative atomicities of x to all other sets in function of their uncertainty belief mass. Working with focused BMAs makes it possible to represent the belief function of any set in 2^Θ using a binary frame of discernment, making the notation very compact.

3 The Opinion Space

For purpose of having a simple and intuitive representation of uncertain beliefs we will define a 3-dimensional metric called *opinion* but which will contain a 4th redundant parameter in order to allow a simple and compact definition of the consensus operator. It is assumed that all beliefs are held by individuals and the notation will therefore include belief ownership. Let for example agent A express his or her beliefs about the truth of set x in some frame of discernment. We will denote A's belief, disbelief, uncertainty and relative atomicity functions as b_x^A, d_x^A, u_x^A and a_x^A respectively, where the superscript indicates belief ownership and the subscript indicates the belief target.

Definition 9 (Opinion Metric) *Let Θ be a binary frame of discernment containing sets x and $\overline{x}$, and let m_Θ be the BMA on Θ held by A where b_x^A, d_x^A and u_x^A represent A's belief, disbelief and uncertainty functions on x in 2^Θ respectively, and let a_x^A represent the relative atomicity of x in Θ. Then A's* opinion *about x, denoted by ω_x^A, is the tuple:*

$$\omega_x^A = (b_x^A,\ d_x^A,\ u_x^A,\ a_x^A)\ .$$

The three coordinates (b, d, u) are dependent through Eq.(1) so that one is redundant. As such they represent nothing more than the traditional *Bel* (Belief) and *Pl* (Plausibility) pair of Shaferian belief theory, where *Bel* $= b$ and *Pl* $= b + u$. However, using (*Bel*, *Pl*) instead of $(b,\ d,\ u)$ would have produced unnecessary complexity in the definition of the consensus operator below. Eq.(1) defines a triangle that can be used to graphically illustrate opinions as shown in Fig.1.

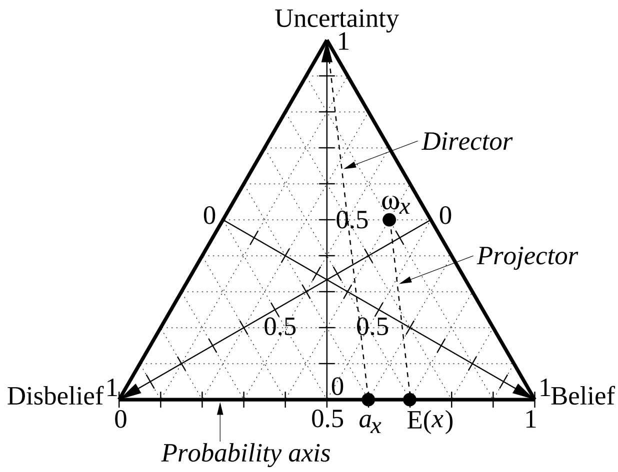

Fig. 1. Opinion triangle with ω_x as example

As an example the position of the opinion $\omega_x = (0.40,\ 0.10,\ 0.50,\ 0.60)$ is indicated as a point in the triangle. The horizontal base line between the belief and disbelief corners is called the *probability axis*. As shown in the figure, the probability expectation value $\mathrm{E}(x) = 0.7$ and the relative atomicity $a(x) = 0.60$ can be graphically represented as points on the probability axis. The line joining the top corner of the triangle and the relative atomicity point is called the *director*. The *projector* is parallel to the director and passes through the opinion point ω_x. Its intersection with the probability axis defines the probability expectation value which otherwise can be computed by the formula of Definition 6. Opinions situated on the probability axis are called *dogmatic opinions*, representing traditional probabilities without uncertainty. The distance between an opinion point and the probability axis can be interpreted as the degree of uncertainty. Opinions situated in the left or right corner, i.e. with either $b = 1$ or $d = 1$ are called *absolute opinions*, corresponding to TRUE or FALSE states in binary logic.

4 The Consensus Operator

The consensus opinion of two possibly conflicting argument opinions is an opinion that reflects both argument opinions in a fair and equal way, i.e. when two observers have beliefs about the truth of x resulting from distinct pieces of evidence about x, the consensus operator produces a consensus belief that combines the two separate beliefs into one. If for example a process can produce two outcomes x and $\overline{x}$, and A and B have observed the process over two different time intervals so that they have formed two independent opinions about the likelihood of x to occur, then the consensus opinion is the belief about x to occur which a single agent would have had after having observed the process during both periods.

Definition 10 (Consensus Operator)
Let $\omega_x^A = (b_x^A, d_x^A, u_x^A, a_x^A)$ and $\omega_x^B = (b_x^B, d_x^B, u_x^B, a_x^B)$ be opinions respectively held by agents A and B about the same state x, and let $\kappa = u_x^A + u_x^B - u_x^A u_x^B$. When $u_x^A, u_x^B \to 0$, the relative dogmatism between ω_x^A and ω_x^B is defined by γ so that $\gamma = u_x^B / u_x^A$. Let $\omega_x^{A,B} = (b_x^{A,B}, d_x^{A,B}, u_x^{A,B}, a_x^{A,B})$ be the opinion such that:

for $\kappa \neq 0$:	*for $\kappa = 0$:*
$\mathbf{1}.\ b_x^{A,B} = (b_x^A u_x^B + b_x^B u_x^A)/\kappa$	$\mathbf{1}.\ b_x^{A,B} = \frac{\gamma b_x^A + b_x^B}{\gamma + 1}$
$\mathbf{2}.\ d_x^{A,B} = (d_x^A u_x^B + d_x^B u_x^A)/\kappa$	$\mathbf{2}.\ d_x^{A,B} = \frac{\gamma d_x^A + d_x^B}{\gamma + 1}$
$\mathbf{3}.\ u_x^{A,B} = (u_x^A u_x^B)/\kappa$	$\mathbf{3}.\ u_x^{A,B} = 0$
$\mathbf{4}.\ a_x^{A,B} = \frac{a_x^A u_x^B + a_x^B u_x^A - (a_x^A + a_x^B) u_x^A u_x^B}{u_x^A + u_x^B - 2 u_x^A u_x^B}$,	$\mathbf{4}.\ a_x^{A,B} = \frac{\gamma a_x^A + a_x^B}{\gamma + 1}$.

Then $\omega_x^{A,B}$ is called the consensus opinion between ω_x^A and ω_x^B, representing an imaginary agent $[A, B]$'s opinion about x, as if that agent represented both A and B. By using the symbol '$\oplus$' to designate this operator, we define $\omega_x^{A,B} \equiv \omega_x^A \oplus \omega_x^B$.

It is easy to prove that the consensus operator is both commutative and associative which means that the order in which opinions are combined has no importance. It can also be shown that the consensus opinion satisfies Eq.(1), i.e. that $b_x^{A,B} + d_x^{A,B} + u_x^{A,B} = 1$. Opinion independence must be assumed, which for example translates into not allowing an agent's opinion to be counted more than once, and also that that the argument opinions must be based on distinct pieces of evidence.

Briefly said, the consensus operator is obtained by mapping beta-probability density functions to the opinion space. It can be shown that posteriori probabilities of binary events can be represented by the beta-pdf (see e.g. [10] p.298). The beta-family of density functions is a continuous family of functions indexed by the two parameters α and β. The parameters of beta-distributions, which for example can represent the number of observations of events, can be combined by simple addition, and thus a way of combining evidence emerges. We refer the reader to [6] for a detailed description of how the consensus operator can be derived from the combination of beta-distributions.

The consensus of two totally uncertain opinions results in a new totally uncertain opinion, although the relative atomicity is not well defined in that case. Two observers would normally agree on the relative atomicity, and in case of two totally uncertain opinions we require that they do so, so that the consensus relative atomicity for example can be defined as $a_x^{A,B} = a_x^A$.

In [6] it is incorrectly stated that the consensus operator can not be applied to two dogmatic opinions, i.e. when $\kappa = 0$. The definition above rectifies this so that dogmatic opinions can be combined. This result is obtained by computing the limits of $(b_x^{A,B}, d_x^{A,B}, u_x^{A,B}, a_x^{A,B})$ as $u_x^A, u_x^B \to 0$ using the relative dogmatism between A and B defined by $\gamma = u_x^B / u_x^A$. This result makes the consensus operator more general than Dempster's rule because the latter excludes the combination of totally conflicting beliefs.

In order to understand the meaning of the relative dogmatism γ, it is useful to consider a process with possible outcomes $\{x, \overline{x}\}$ that produces γ times as many x as $\overline{x}$. For example when throwing a fair dice and some mechanism makes sure that A only observes the outcome of 'six' and B only observes the outcome of 'one', 'two', 'three', 'four' and 'five', then A will think the dice only produces 'six' and B will think that the dice never produces 'six'. After infinitely many observations A and B will have the conflicting dogmatic opinions $\omega^A_{\text{`six'}} = (1.0,\ 0.0,\ 0.0,\ \frac{1}{6})$ and $\omega^B_{\text{`six'}} = (0.0,\ 1.0,\ 0.0,\ \frac{1}{6})$ respectively. On the average B observes 5 times more events than A so that B remains 5 times more dogmatic than A as $u^A_{\text{`six'}}, u^B_{\text{`six'}} \to 0$, meaning that the relative dogmatism between A and B is $\gamma = 1/5$. By combining their opinions according to the case where $\kappa = 0$ in Definition 10 and inserting the value of γ, the combined opinion about obtaining a 'six' with the dice can be computed as $\omega^{A,B}_{\text{`six'}} = (\frac{1}{6},\ \frac{5}{6},\ 0,\ \frac{1}{6})$, which is exactly what one would expect.

In the last example, the relative dogmatism was finite and non-zero, but it is also possible to imagine extreme relative dogmatisms, e.g. $\gamma = \infty$ or $\gamma = \varepsilon$ where ε is an infinitesimal (i.e. close to zero). This is related to the concept of epsilon belief functions which has been applied to default reasoning by Benferhat *et al.* in [11]. Epsilon belief functions are opinions with $b, d, u \in \{0,\ \varepsilon,\ 1-\varepsilon\}$, i.e. opinions situated close to a corner of the triangle in Fig.1. Without going into details it can be shown that some properties of extreme relative dogmatisms seem suitable for default reasoning. For example, when the relative dogmatism between A and B is infinite ($\gamma^{A/B} = \infty$) the consensus opinion is equal to A's argument opinion ($\omega_x^{A,B} = \omega_x^A$), and when the relative dogmatism is infinitesimal ($\gamma^{A/B} = \varepsilon$) the consensus opinion is equal to B's argument opinion ($\omega_x^{A,B} = \omega_x^B$). However, with three agents A, B, and C where $\gamma^{A/B} = \varepsilon_1$ and $\gamma^{A/C} = \varepsilon_2$ the consensus opinion $\omega_x^{A,B,C}$ is non-conclusive as long as the relationship between ε_1 and ε_2 is unknown.

5 Comparing the Consensus Operator with Dempster's Rule

This section describes three examples that compare Dempster's rule, the non-normalised Dempster's rule and the consensus operator. The definition of Dempster's rule and the non-normalised rule is given below. In order to distinguish between the consensus operator and Dempster's rule, the latter will be denoted by $\oplus'$.

Definition 11 *Let Θ be a frame of discernment, and let m_Θ^A and m_Θ^B be BMAs on Θ. Then $m_\Theta^A \oplus' m_\Theta^B$ is a function $m_\Theta^A \oplus' m_\Theta^B :\ 2^\Theta \rightarrow [0,1]$ such that:*

1. $m_\Theta^A \oplus' m_\Theta^B(\emptyset) = \sum_{y \cap z = \emptyset} m_\Theta^A(y) \cdot m_\Theta^B(z) - K$, *and*
2. $m_\Theta^A \oplus' m_\Theta^B(x) = \frac{\sum_{y \cap z = x} m_\Theta^A(y) \cdot m_\Theta^B(z)}{1-K}$, *for all $x \neq \emptyset$*

where $K = \sum_{y \cap z = \emptyset} m_\Theta^A(y) \cdot m_\Theta^B(z)$ and $K \neq 1$ in Dempster's rule, and where $K = 0$ in the non-normalised version.

5.1 Example 1: Dogmatic Conflicting Beliefs

We will start with the well known example that Zadeh [2] used for the purpose of criticising Dempster's rule. Smets [4] used the same example in defence of the non-normalised version of Dempster's rule.

Suppose that we have a murder case with three suspects; Peter, Paul and Mary and two witnesses W_1 and W_2 who give highly conflicting testimonies. Table 1 gives the witnesses' belief masses in Zadeh's example and the resulting belief masses after applying Dempster's rule, the non-normalised rule and the consensus operator.

	W_1	W_2	Dempster's rule	Non-normalised Dempster's rule	Consensus operator
Peter	0.99	0.00	0.00	0.0000	0.495
Paul	0.01	0.01	1.00	0.0001	0.010
Mary	0.00	0.99	0.00	0.0000	0.495
Θ	0.00	0.00	0.00	0.0000	0.000
$\emptyset$	0.00	0.00	0.00	0.9999	0.000

Table 1
Comparison of operators in Zadeh's example

Because the frame of discernment in Zadeh's example is ternary, a focused binary frame of discernment must be derived in order to apply the consensus operator. The focused opinions are:

$$\begin{aligned}
\omega^{W_1}_{\text{Peter}} &= (0.99,\ 0.01,\ 0.00,\ \tfrac{1}{3})\ , & \omega^{W_2}_{\text{Peter}} &= (0.00,\ 1.00,\ 0.00,\ \tfrac{1}{3})\ ,\\
\omega^{W_1}_{\text{Paul}} &= (0.01,\ 0.99,\ 0.00,\ \tfrac{1}{3})\ , & \omega^{W_2}_{\text{Paul}} &= (0.01,\ 0.99,\ 0.00,\ \tfrac{1}{3})\ ,\\
\omega^{W_1}_{\text{Mary}} &= (0.00,\ 1.00,\ 0.00,\ \tfrac{1}{3})\ , & \omega^{W_2}_{\text{Mary}} &= (0.99,\ 0.01,\ 0.00,\ \tfrac{1}{3})\ .
\end{aligned}$$

The above opinions are all dogmatic, and the case where $\kappa = 0$ in Definition 10 must be invoked. Because of the symmetry between W_1 and W_2 we determine the relative dogmatism between W_1 and W_2 to be $\gamma = 1$. The consensus opinion values and their corresponding probability expectation values can then be computed as:

$$\begin{aligned}
\omega^{W_1,W_2}_{\text{Peter}} &= (0.495,\ 0.505,\ 0.000,\ \tfrac{1}{3})\ , & \mathrm{E}(\omega^{W_1,W_2}_{\text{Peter}}) &= 0.495\ ,\\
\omega^{W_1,W_2}_{\text{Paul}} &= (0.010,\ 0.990,\ 0.000,\ \tfrac{1}{3})\ , & \mathrm{E}(\omega^{W_1,W_2}_{\text{Paul}}) &= 0.010\ ,\\
\omega^{W_1,W_2}_{\text{Mary}} &= (0.495,\ 0.505,\ 0.000,\ \tfrac{1}{3})\ , & \mathrm{E}(\omega^{W_1,W_2}_{\text{Mary}}) &= 0.495\ .
\end{aligned}$$

The column for the consensus operator in Table 1 is obtained by taking the 'belief' coordinate from the consensus opinions above. Dempster's rule selects the least suspected by both witnesses as the guilty. The non-normalised version acquits all the suspects and indicates that the guilty has to be someone else. This is explained by Smets [4] with the so-called open world interpretation of the frame of discernment. In [5] Smets also proposed to interpret $m(\emptyset)$ (= 0.9999 in this case) as a measure of the degree of conflict between the argument beliefs.

The consensus operator respects conflicting beliefs by giving the average of beliefs to Peter and Mary, whereas the non-conflicting beliefs on Paul is kept unaltered. This result is consistent with classical estimation theory (see e.g. comments to Smets [4] p.278 by M.R.B.Clarke) which is based on taking the average of probability estimates when all estimates have equal weight.

5.2 Example 2: Conflicting Beliefs with Uncertainty

In the following example uncertainty is introduced by allocating some belief to the set $\Theta = \{\text{Peter, Paul, Mary}\}$. Table 2 gives the modified BMAs and the results of applying the rules.

	W_1	W_2	Dempster's rule	Non-normalised Dempster's rule	Consensus operator
Peter	0.98	0.00	0.490	0.0098	0.492
Paul	0.01	0.01	0.015	0.0003	0.010
Mary	0.00	0.98	0.490	0.0098	0.492
Θ	0.01	0.01	0.005	0.0001	0.005
$\emptyset$	0.00	0.00	0.000	0.9800	0.000

Table 2
Comparison of operators after introducing uncertainty in Zadeh's example

The frame of discernment in this modified example is again a ternary, and a focused binary frame of discernment must be derived in order to apply the consensus operator. The focused opinions are given below:

$$\begin{aligned}
\omega^{W_1}_{\text{Peter}} &= (0.98,\ 0.01,\ 0.01,\ \tfrac{1}{3})\ , & \omega^{W_2}_{\text{Peter}} &= (0.00,\ 0.99,\ 0.01,\ \tfrac{1}{3})\ ,\\
\omega^{W_1}_{\text{Paul}} &= (0.01,\ 0.98,\ 0.01,\ \tfrac{1}{3})\ , & \omega^{W_2}_{\text{Paul}} &= (0.01,\ 0.98,\ 0.01,\ \tfrac{1}{3})\ ,\\
\omega^{W_1}_{\text{Mary}} &= (0.00,\ 0.99,\ 0.01,\ \tfrac{1}{3})\ , & \omega^{W_2}_{\text{Mary}} &= (0.98,\ 0.01,\ 0.01,\ \tfrac{1}{3})\ .
\end{aligned}$$

The consensus opinion values and their corresponding probability expectation values are:

$$\begin{aligned}
\omega^{W_1,W_2}_{\text{Peter}} &= (0.492,\ 0.503,\ 0.005,\ \tfrac{1}{3})\ , & \mathrm{E}(\omega^{W_1,W_2}_{\text{Peter}}) &= 0.494\ ,\\
\omega^{W_1,W_2}_{\text{Paul}} &= (0.010,\ 0.985,\ 0.005,\ \tfrac{1}{3})\ , & \mathrm{E}(\omega^{W_1,W_2}_{\text{Paul}}) &= 0.012\ ,\\
\omega^{W_1,W_2}_{\text{Mary}} &= (0.492,\ 0.503,\ 0.005,\ \tfrac{1}{3})\ , & \mathrm{E}(\omega^{W_1,W_2}_{\text{Mary}}) &= 0.494\ .
\end{aligned}$$

The column for the consensus operator in Table 2 is obtained by taking the 'belief' coordinate from the consensus opinions above. When uncertainty is introduced, Dempster's rule corresponds well with intuitive human judgement. The non-normalised Dempster's rule however still indicates that none of the suspects are guilty and that new suspects must be found, or alternatively that the degree of conflict is still high, despite introducing uncertainty.

The consensus operator corresponds well with human judgement and gives almost the same result as Dempster's rule, but not exactly. Note that the values resulting from the consensus operator have been rounded off after the third decimal.

The belief masses resulting from Dempster's rule in Table 2 add up to 1. The 'belief' parameters of the consensus opinions resulting from the consensus operator do not add up to 1 because they are actually taken from 3 different focused frames of discernment, but the following holds:

$$\mathrm{E}(\omega_{\text{Peter}}^{W_1,W_2}) + \mathrm{E}(\omega_{\text{Paul}}^{W_1,W_2}) + \mathrm{E}(\omega_{\text{Mary}}^{W_1,W_2}) = 1 \ .$$

5.3 *Example 3: Harmonious Beliefs*

The previous example seemed to indicate that Dempster's rule and the consensus operator give very similar results in the presence of uncertainty. However, this is not always the case as illustrated by the following example. Let two witnesses W_1 and W_2 have equal beliefs about the truth of x. The agents' BMAs and the results of applying the rules are give in Table 3.

	W_1	W_2	Dempster's rule	Non-normalised Dempster's rule	Consensus operator
x	0.90	0.90	0.99	0.99	0.947
$\overline{x}$	0.00	0.00	0.00	0.00	0.000
Θ	0.10	0.10	0.01	0.01	0.053
$\emptyset$	0.00	0.00	0.00	0.00	0.000

Table 3
Comparison of operators i.c.o. equal beliefs

The consensus opinion about x and the corresponding probability expectation value are:

$$\omega_x^{W_1,W_2} = (0.947,\ 0.000,\ 0.053,\ 0.500)\ , \quad \mathrm{E}(\omega_x^{W_1,W_2}) = 0.974\ .$$

It is difficult to give an intuitive judgement of these results. It can be observed that Dempster's rule and the non-normalised version produce equal results because the witnesses' BMAs are non-conflicting. The two variants of Dempster's rule amplify the combined belief twice as much as the consensus operator and this difference needs an explanation. The consensus operator produces results that are consistent with statistical analysis (see [6]) and in the absence of other criteria for intuitive or formal judgement, this constitutes a strong argument in favour of the consensus operator.

6 Discussion and Conclusion

In addition to the three belief combination rules analysed here, numerous others have been presented in the literature, e.g. the rule proposed by Yager [12] that transfers conflicting belief mass $m^A_\Theta(x) \oplus m^B_\Theta(y)$ to Θ whenever $x \cap y = \emptyset$, and the rule proposed by Dubois & Prade [13] that transfers conflicting belief mass $m^A_\Theta(x) \oplus m^B_\Theta(y)$ to $x \cup y$ whenever $x \cap y = \emptyset$. These rules are commutative, but unfortunately they are not associative, which seems counterintuitive. Assuming that beliefs from different sources should be treated in the same way, why should the result depend on the order in which they are combined? After analysing the rules of Dempster, Smets, Yager, Dubois & Prade as well as simple statistical average, Murphy [14] rejects the rules of Yager and Dubois & Prade for their lack of associativity, and concludes that Dempster's rule performs best for its convergence properties, accompanied by statistical average to warn of possible errors when the degree of conflict is high. Our consensus operator seems to combine both the desirable convergence properties of Dempster's rule when the degree of conflict is low, and the natural average of beliefs when the degree of conflict is high. As mentioned in Lefévre *et al.* [15], Dempster's rule and it's non-normalised version require that all belief sources are reliable, whereas Yager's and Dubois & Prade's rules require that at least one of the belief sources is reliable for the result to be meaningful. The consensus operator does not make any assumption about reliability of the belief sources, but does of course not escape the 'garbage in, garbage out' principle.

An argument that could be used against our consensus operator, is that it does not give any indication of possible belief conflict. Indeed, by looking at the result only, it does not tell whether the original beliefs were in harmony or in conflict, and it would have been nice if it did. A possible way to incorporate the degree of conflict is to add an extra 'conflict' parameter. This could for example be the belief mass assigned to $\emptyset$ in Smets' rule, which in the opinion notation can be defined as $c^{A,B}_x = b^A_x d^B_x + b^B_x d^A_x$ where $c^{A,B}_x \in [0,1]$. The consensus opinion with conflict parameter would then be expressed as $\omega^{A,B}_x = (b^{A,B}_x, d^{A,B}_x, u^{A,B}_x, a^{A,B}_x, c^{A,B}_x)$. The conflict parameter would only be relevant for combined belief, and not for original beliefs. A default value $c = -1$ could for example indicate original belief, because a default value $c = 0$ could be misunderstood as indicating that a belief comes from combined harmonious beliefs, even though it is an original belief.

Opinions can be derived by coarsening any frame of discernment and BMA through the focusing process, where focusing on different states produces different opinions. In this context it is in general not meaningful to relate belief, disbelief and uncertainty functions from opinions that focus on different states even though the opinions are derived from the same frame of discernment and belief mass assignment. The only way to relate such opinions is through the probability expectation value $\mathrm{E}(\omega_x)$ (which can also be written as $\mathrm{E}(x)$), and this leads to interesting results. The proof of the following theorem can be found in [6].

Theorem 1 (Kolmogorov Axioms) *Given a frame of discernment* Θ *with a BMA* m_Θ*, the probability expectation function* E *with domain* 2^Θ *satisfies:*

1. $\mathrm{E}(x) \geq 0$ *for all* $x \in 2^\Theta$,
2. $\mathrm{E}(\Theta) = 1$,
3. *If* $x_1, x_2 \ldots \in 2^\Theta$ *are pairwise disjoint, then* $\mathrm{E}(\cup_{i=1}^{|2^\Theta|} x_i) = \sum_{i=1}^{|2^\Theta|} \mathrm{E}(x_i)$.

This shows that probability theory can be built on top of belief theory through the probability expectation value. As such belief functions should not be interpreted as probabilities, instead there is a surjective (onto) mapping from the belief space to the probability space. Belief and possibility functions have been interpreted as upper and lower probability bounds respectively (see e.g. Halpern & Fagin [16] and de Cooman & Ayles [17]). Belief functions can be useful for estimating probability values but not to set bounds, because the probability of a real event can never be determined with absolute certainty, and neither can upper and lower bounds to it. Our view is that probability always is a subjective notion, inasmuch as it is a 1-dimensional belief measure felt by a given person facing a given event. Objective, physical or real probability is a meaningless notion. This view is shared by e.g. de Finetti [18]. In the same way, an opinion as defined here, is a 3-dimensional belief measure felt by a given person facing a given event.

It has also been suggested to interpret belief functions as *evidence* (see e.g. Fagin and Halpern [16]). Belief can result from evidence in the form of observing an event or knowing internal properties of a system, or from more subjective and intangible experience. Statistical evidence can for example be translated into belief functions, as described in [6], and other types of evidence can be intuitively translated into belief functions, but belief and evidence are not the same. We prefer to leave belief functions as a distinct concept in its own right, and in general not try to interpret them as anything else.

The opinion metric described here provides a simple and compact notation for beliefs in the Shaferian belief model. We have presented an alternative to Dempster's rule which is consistent with probabilistic and statistical analysis, and which seems more suitable for combining highly conflicting beliefs as well as for combining harmonious beliefs, than Dempster's rule and its non-normalised version. The fact that a binary focused frame of discernment must be derived in order to apply the consensus operator puts no restriction on its applicability. The resulting beliefs for each event can still be compared and can form the basis for decision making.

References

[1] G. Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, 1976.

[2] L.A. Zadeh. Review of Shafer's A Mathematical Theory of Evidence. *AI Magazine*, 5:81–83, 1984.

[3] M.S. Cohen. An expert system framework for non-monotonic reasoning about probabilistic assumptions. In L.N. Kanal and J.F. Lemmer, editors, *Uncertainty in Artificial Intelligence 1*. North-Holland, 1986.

[4] Ph. Smets. Belief functions. In Ph. Smets et al., editors, *Non-Standard Logics for Automated Reasoning*, pages 253–286. Academic Press, 1988.

[5] Ph. Smets. The transferable belief model for quantified belief representation. In D.M. Gabbay and Ph. Smets, editors, *Handbook of Defeasible Reasoning and Uncertainty Management Systems, Vol.1*, pages 267–301. Kluwer, Doordrecht, 1998.

[6] A. Jøsang. A Logic for Uncertain Probabilities. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 9(3):279–311, June 2001.

[7] A. Jøsang and V.A. Bondi. Legal Reasoning with Subjective Logic. *Artificial Reasoning and Law*, 8(4):289–315, winter 2000.

[8] A. Jøsang. An Algebra for Assessing Trust in Certification Chains. In J. Kochmar, editor, *Proceedings of the Network and Distributed Systems Security Symposium (NDSS'99)*. The Internet Society, 1999.

[9] Ph. Smets and R. Kennes. The transferable belief model. *Artificial Intelligence*, 66:191–234, 1994.

[10] George Casella and Roger L. Berger. *Statistical Inference*. Duxbury Press, 1990.

[11] S. Benferhat, A. Saffiotti, and Ph. Smets. Belief Functions and Default Reasoning. *Artificial Intelligence*, 122(1–2):1–69, 2000.

[12] R. Yager. On the Dempster-Shafer framework and new combination rules. *Information Sciences*, 41:93–137, 1987.

[13] D. Dubois and H. Prade. Representation and combination of uncertainty with belief functions and possibility measures. *Computational Intelligence*, 4:244–264, 1988.

[14] Catherine K. Murphy. Combining belief functions when evidence conflicts. *Decision Support Systems*, 29:1–9, 2000.

[15] E. Lefèvre et al. A generic framework for resolving the conflict in the combination of belief structures. In *Proceedings of the 3rd International Conference on Information Fusion (FUSION'2000) Volume 1, Session MoD4*, pages 11–18, Paris, July 2000.

[16] Joseph Y. Halpern and Ronald Fagin. Two views of belief: Belief as generalised probability and belief as evidence. *Artificial Intelligence*, 54:275–317, 1992.

[17] G. de Cooman and D. Ayles. Supremum preserving upper probabilities. *Information Sciences*, 1998.

[18] Bruno de Finetti. The value of studying subjective evaluations of probability. In Carl-Axel Staël von Holstein, editor, *The concept of probability in psychological experiments*, pages 1–14, Dordrecht, Holland, 1974. D.Reidel Publishing Company.