Subjective Evidential Reasoning*

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Abstract

This paper describes a framework for combining and assessing subjective evidence from different sources. The approach is based on the Dempster-Shafer belief theory, but instead of using Dempster's rule we introduce a new rule called the consensus operator that is based on statistical inference. We show how this framework can be applied to subjective evidential reasoning.

1 Introduction

Several alternative calculi and logics which take uncertainty and ignorance into consideration have been proposed and quite successfully applied to practical problems where conclusions have to be made based on insufficient evidence (see for example Hunter 1996 [1] or Motro & Smets 1997 [2] for an analysis of some uncertainty logics and calculi). Although including uncertainty in the belief model is a significant step forward, it only goes half the way in realising the real nature of human beliefs. It is also necessary to take into account that beliefs always are held by individuals and that beliefs for this reason are fundamentally subjective.

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We describe two operators called *discounting*[1] and *consensus* that operate on subjective uncertain beliefs, and show how they can be used for assessing evidence from different sources. This work builds on Demspter-Shafer belief theory [3]. Our consensus operator is different from Dempster's rule but has the same purpose; namely of combining uncertain beliefs, and a comparison is provided at the end of the paper. Our discounting operator is similar to the Shaferian discounting operator, and can be used to model recommendation of beliefs. The operators form part of *Subjective Logic* which is described in Jøsang 2001 [4]. These operators can for example be applied to legal reasoning [5] and authentication in computer networks[6].

2 Representing Uncertain Beliefs

The first step in applying the Dempster-Shafer belief model [3] is to define a set of possible situations, which is called the *frame of discernment*, denoted by Θ and which delimits a set of possible states of a given system.

The powerset of Θ, denoted by 2^{Θ}, contains all possible unions of the states in Θ including Θ itself. Elementary states in a frame of discernment Θ will be called atomic states because they do not contain substates. It is assumed that only one atomic state can be true at any one time. If a state is assumed to be true, then all superstates are considered true as well.

An observer who believes that one or more states in the powerset of Θ might be true can assign belief mass to these states. Belief mass on an atomic

[1] Called *recommendation* in some earlier publications

state $x \in 2^\Theta$ is interpreted as the belief that the state in question is true. Belief mass on a non-atomic state $x \in 2^\Theta$ is interpreted as the belief that one of the atomic states it contains is true, but that the observer is uncertain about which of them is true. The following definition is central in the Dempster-Shafer theory.

Definition 1 (Belief Mass Assignment) *Let Θ be a frame of discernment. If with each substate $x \in 2^\Theta$ a number $m_\Theta(x)$ is associated such that:*

$$\begin{array}{ll} \mathbf{1}. & m_\Theta(x) \geq 0 \\ \mathbf{2}. & m_\Theta(\emptyset) = 0 \\ \mathbf{3}. & \sum_{x \in 2^\Theta} m_\Theta(x) = 1 \end{array}$$

then m_Θ is called a belief mass assignment[2] *on Θ, or BMA for short. For each substate $x \in 2^\Theta$, $m_\Theta(x)$ is called the* belief mass[3] *of x.*

A belief mass $m_\Theta(x)$ expresses the belief assigned to the state x and does not express any belief in substates of x in particular.

In contrast to belief mass, the *belief* in a state must be interpreted as an observer's total belief that a particular state is true. The next definition from the Dempster-Shafer theory will make it clear that belief in x not only depends on belief mass assigned to x but also on belief mass assigned to substates of x.

Definition 2 (Belief Function) *Let Θ be a frame of discernment, and let m_Θ be a BMA on Θ. Then the* belief function *corresponding with m_Θ is the function $b: 2^\Theta \rightarrow [0,1]$ defined by:*

$$b(x) = \sum_{y \subseteq x} m_\Theta(y), \quad x, y \in 2^\Theta .$$

Similarly to belief, an observer's *disbelief* must be interpreted as the total belief that a state is **not** true. The following definition is ours.

Definition 3 (Disbelief Function) *Let Θ be a frame of discernment, and let m_Θ be a BMA on Θ. Then the* disbelief function *corresponding with m_Θ is the function $d: 2^\Theta \rightarrow [0,1]$ defined by:*

$$d(x) = \sum_{y \cap x = \emptyset} m_\Theta(y), \quad x, y \in 2^\Theta .$$

[2] Called *basic probability assignment* in Shafer 1976[3]
[3] Called *basic probability number* in Shafer 1976[3]

The disbelief of x corresponds to the *doubt* of x in Shafer's book. However, we choose to use the term 'disbelief' because we feel that for example the case when it is certain that a state is false can better be described by 'total disbelief' than by 'total doubt'. Our next definition expresses uncertainty regarding a given state as the sum of belief masses on superstates or on partly overlapping states.

Definition 4 (Uncertainty Function) *Let Θ be a frame of discernment, and let m_Θ be a BMA on Θ. The* uncertainty function *corresponding with m_Θ is the function $u: 2^\Theta \rightarrow [0,1]$ defined by:*

$$u(x) = \sum_{\substack{y \cap x \neq \emptyset \\ y \not\subseteq x}} m_\Theta(y), \quad x, y \in 2^\Theta .$$

A BMA with zero belief mass assigned to Θ is called a dogmatic BMA. The sum of the belief, disbelief and uncertainty functions is equal to the sum of the belief masses in a BMA which according to Def.1 sums up to 1. The following equality is therefore trivial to prove:

$$b(x) + d(x) + u(x) = 1 , \quad \text{for } x \neq \emptyset . \qquad (1)$$

For the purpose of deriving probability expectation values of sets in 2^Θ, the relative number of atomic sets is also needed in addition to belief masses. For any particular state x the *atomicity* of x is the number of atomic states it contains, denoted by $|x|$. If Θ is a frame of discernment, the atomicity of Θ is equal to the total number of atomic states it contains. Similarly, if $x, y \in 2^\Theta$ then the overlap between x and y relative to y can be expressed in terms of atomic states. Our next definition captures this idea of relative atomicity:

Definition 5 (Relative Atomicity) *Let Θ be a frame of discernment and let $x, y \in 2^\Theta$. Then the* relative atomicity *of x to y is the function $a: 2^\Theta \rightarrow [0,1]$ defined by:*

$$a(x/y) = \frac{|x \cap y|}{|y|}, \quad x, y \in 2^\Theta .$$

It can be observed that $x \cap y = \emptyset \;\Rightarrow\; a(x/y) = 0$, and that $y \subseteq x \;\Rightarrow\; a(x/y) = 1$. In all other cases it will have a value between 0 and 1.

The relative atomicity of an atomic state to its frame of discernment, denoted by $a(x/\Theta)$, can simply be written as $a(x)$, i.e. it refers to the frame of discernment by default. The relative atomicity function can be used to determine a probability expectation value for any given state.

Definition 6 (Probability Expectation) *Let Θ be a frame of discernment with BMA m_Θ, then the* probability expectation function *corresponding with m_Θ is the function $\mathrm{E} : 2^\Theta \rightarrow [0,1]$ defined by:*

$$\mathrm{E}(x) = \sum_y m_\Theta(y)\ a(x/y), \quad x, y \in 2^\Theta\ . \quad (2)$$

Def.6 is equivalent to pignistic probability described in e.g. Smets and Kennes 1994 [7], and is based on the principle of insufficient reason: a belief mass assigned to the union of n atomic states is split equally among these n states.

In order to simplify the representation of uncertain beliefs for particular states we will define a *focused frame of discernment* and a *focused BMA*. The focused frame of discernment and the corresponding BMA will for a given state produce the same belief, disbelief and uncertainty functions as the original frame of discernment and BMA.

Definition 7 (Focused Frame Of Discernment) *Let Θ be a frame of discernment and let $x \in 2^\Theta$. The frame of discernment denoted by $\widetilde{\Theta}^x$ containing only x and $\overline{x}$, where $\overline{x}$ is the complement of x in Θ is then called a focused frame of discernment with focus on x.*

Definition 8 (Focused Belief Mass Assignment) *Let Θ be a frame of discernment with BMA m_Θ and let $b(x)$, $d(x)$ and $u(x)$ be the belief, disbelief and uncertainty functions of x in 2^Θ. Let $\widetilde{\Theta}^x$ be the the focused frame of discernment with focus on x. The corresponding focused BMA $m_{\widetilde{\Theta}^x}$ and relative atomicity $a_{\widetilde{\Theta}^x}(x)$ on $\widetilde{\Theta}^x$ is defined by:*

$$\begin{aligned} m_{\widetilde{\Theta}^x}(x) &= b(x)\ , \\ m_{\widetilde{\Theta}^x}(\overline{x}) &= d(x)\ , \\ m_{\widetilde{\Theta}^x}(\widetilde{\Theta}^x) &= u(x)\ , \\ a_{\widetilde{\Theta}^x}(x) &= [\mathrm{E}(x) - b(x)]/u(x)\ . \end{aligned} \quad (3)$$

A focused relative atomicity represents the weighted average of relative atomicities of x to all other states in function of their uncertainty belief mass. Working with focused BMAs makes it possible to represent the belief functions relative to x using a binary frame of discernment, making the notation compact.

3 The Opinion Space

We define a 3-dimensional belief metric called *opinion* containing a 4th redundant component. For compactness of notation the belief, disbelief, uncertainty and relative atomicity functions will be denoted as b_x, d_x, u_x and a_x respectively.

Definition 9 (Opinion) *Let Θ be a frame of discernment with 2 atomic states x and $\overline{x}$, and let m_Θ be a BMA on Θ where b_x, d_x, u_x, and a_x are the belief, disbelief, uncertainty and relative atomicity functions on x in 2^Θ respectively. Then the* opinion *about x, denoted by ω_x, is the tuple:*

$$\omega_x \equiv (b_x,\ d_x,\ u_x,\ a_x)\ . \quad (4)$$

Because the three coordinates (b, d, u) are dependent through Eq.(1) they represent nothing more than the traditional (*Belief*, *Plausibility*) pair of Shaferian belief theory. It is useful to keep all three coordinates in order to obtain simple operator expressions. Eq.(1) defines a triangle that can be used to graphically illustrate opinions as shown in Fig.1.

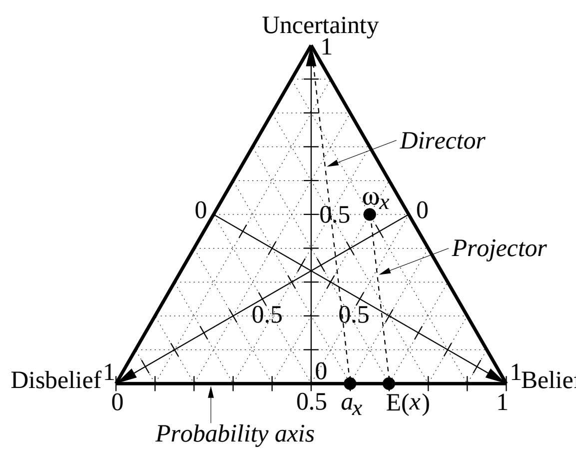

Figure 1: Opinion triangle with ω_x as example

As an example the position of the opinion $\omega_x = (0.40,\ 0.10,\ 0.50,\ 0.60)$ is indicated as a point in the triangle. Also shown are the probability expectation value and the relative atomicity. The horizontal bottom line between the belief and disbelief corners in Fig.1 is called the *probability axis*. The relative atomicity can be graphically represented as a point on the probability axis. The line joining the top corner of the triangle and the relative atomicity point becomes the *director*. In Fig.1 $a(x) = 0.60$ is represented as a point, and the dashed line pointing at it represents the director.

The *projector* is parallel to the director and passes through the opinion point ω_x. Its intersection with the probability axis defines the probability expectation value which otherwise can be computed by the formula of Def.6. The position of the probability expectation $\mathrm{E}(x) = 0.70$ is shown. For simplicity the probability expectation of an opinion ω_x can be denoted according to $\mathrm{E}(\omega_x) \equiv \mathrm{E}(x)$.

Opinions situated on the probability axis are called *dogmatic opinions*, representing traditional probabilities without uncertainty. The distance between an opinion point and the probability axis can be interpreted as the degree of uncertainty. Opinions situated in the left or right corner, i.e. with either $b = 1$ or $d = 1$ are called *absolute opinions*. They represent situations where it is absolutely certain that a state is either true or false, and correspond to TRUE or FALSE proposition in binary logic.

Opinions can be ordered according to probability expectation value, but additional criteria are needed in case of equal probability expectation values. The following definition can be used to determine the order of opinions:

Definition 10 (Ordering of Opinions) *Let ω_x and ω_y be two opinions. They can be ordered according to the following criteria by priority:*

1. *The opinion with the greatest probability expectation is the greatest opinion.*
2. *The opinion with the least uncertainty is the greatest opinion.*
3. *The opinion with the least relative atomicity is the greatest opinion.*

The first criterion is self evident. The second criterion is less so, but it is supported by experimental findings described by Ellsberg 1961 [8]. The third criterion is more an intuitive guess and so is the priority between the second and third criterion, and before these assumptions can be supported by evidence from practical experiments we invite the readers to judge whether they agree.

4 Evidential Operators

So far we have described the elements of a frame of discernment as states. In practice states can be verbally described as propositions; if for example Θ consists of possible colours of a ball when drawn from an urn with red and black balls, and x designates the state when the colour drawn from the urn is red then it can be interpreted as the verbal proposition x: "*I will draw a red ball*".

Opinions are considered to be held by individuals, and will therefore have an ownership assigned whenever relevant. In our notation, superscripts indicate ownership, and subscripts indicate the proposition to which the opinion applies. For example ω_x^A is an opinion held by agent A about the truth of proposition x.

In this section we describe the operators *discounting* and *consensus* which are suitable for subjective evidential reasoning. Other operators suitable for subjective logic reasoning are described in [4].

4.1 Discounting

Assume two agents A and B where A has an opinion about B in the form of the proposition: "*B is knowledgeable and will tell the truth*". In addition B has an opinion about a proposition x. Agent A can then form an opinion about x by discounting B's opinion about x with A's opinion about B. There is no such thing as physical belief discounting, and discounting of opinions therefore lends itself to different interpretations. The main difficulty lies with describing the effect of A disbelieving that B will give a good advice. This we will interpret as if A thinks that B is uncertain about the truth value of x so that A also is uncertain about the truth value of x no matter what B's actual advice is. Our next definition captures this idea.

Definition 11 (Discounting)
Let $\omega_B^A = (b_B^A, d_B^A, u_B^A, a_B^A)$ be agent A's opinion about agent B as a recommender, and let x be a proposition where $\omega_x^B = (b_x^B, d_x^B, u_x^B, a_x^B)$ is B's opinion about x expressed in a recommendation to A. Let $\omega_x^{A:B} = (b_x^{A:B}, d_x^{A:B}, u_x^{A:B}, a_x^{A:B})$ be the opinion such that:

$$\begin{array}{ll} \mathbf{1}. & b_x^{A:B} = b_B^A b_x^B \\ \mathbf{2}. & d_x^{A:B} = b_B^A d_x^B \\ \mathbf{3}. & u_x^{A:B} = d_B^A + u_B^A + b_B^A u_x^B \\ \mathbf{4}. & a_x^{A:B} = a_x^B \ , \end{array}$$

then $\omega_x^{A:B}$ is called the discounting of ω_x^B by ω_B^A expressing A's opinion about x as a result of B's advice to A. By using the symbol '$\otimes$' to designate this operator, we define $\omega_x^{A:B} \equiv \omega_B^A \otimes \omega_x^B$.

The discounting function defined by Shafer[3] uses a *discounting rate* that can be denote by c, where the belief mass on each state in 2^Θ except the belief mass on Θ itself is multiplied by $(1-c)$. By setting $(1-c) = b_B^A$ our definition becomes equivalent to Shafer's definition.

It is easy to prove that $\otimes$ is associative but not commutative. This means that in case of a chain the discounting of opinions can start in either end, but that the order of opinions is significant. In a chain with more than one advisor, opinion independence must be assumed, which for example translates into not allowing the same entity to appear more than once in a chain.

4.2 Consensus

The consensus of two possibly conflicting opinions is an opinion that reflects both opinions in a fair and equal way, i.e. when two observers have beliefs about the truth of x resulting from distinct pieces of evidence about x, the consensus operator produces a consensus belief that combines the two separate beliefs into one. If for example a process can produce two outcomes x and $\overline{x}$, and A and B have observed the process over two different time intervals so that they have formed two independent opinions about the likelihood of x to occur, then the consensus opinion is the belief about x to occur which a single agent would have had after having observed the process during both periods. See Jøsang 2001 [4] for a formal justification of the consensus operator.

Definition 12 (Consensus)
Let $\omega_x^A = (b_x^A, d_x^A, u_x^A, a_x^A)$ and $\omega_x^B = (b_x^B, d_x^B, u_x^B, a_x^B)$ be A's and B's opinions about the same proposition x. When $u_x^A, u_x^B \to 0$, the relative dogmatism between ω_x^A and ω_x^B is defined by γ so that $\gamma = u_x^B / u_x^A$. Let $\kappa = u_x^A + u_x^B - u_x^A u_x^B$. The opinion $\omega_x^{A,B} = (b_x^{A,B}, d_x^{A,B}, u_x^{A,B}, a_x^{A,B})$ defined by:

for $\kappa \neq 0$:

$$\begin{array}{ll} \mathbf{1}. & b_x^{A,B} = (b_x^A u_x^B + b_x^B u_x^A)/\kappa \\ \mathbf{2}. & d_x^{A,B} = (d_x^A u_x^B + d_x^B u_x^A)/\kappa \\ \mathbf{3}. & u_x^{A,B} = (u_x^A u_x^B)/\kappa \\ \mathbf{4}. & a_x^{A,B} = \frac{a_x^B u_x^A + a_x^A u_x^B - (a_x^A + a_x^B) u_x^A u_x^B}{u_x^A + u_x^B - 2 u_x^A u_x^B} \end{array}$$

for $\kappa = 0$:

$$\begin{array}{ll} \mathbf{1}. & b_x^{A,B} = \frac{\gamma b_x^A + b_x^B}{\gamma + 1} \\ \mathbf{2}. & d_x^{A,B} = \frac{\gamma d_x^A + d_x^B}{\gamma + 1} \\ \mathbf{3}. & u_x^{A,B} = 0 \\ \mathbf{4}. & a_x^{A,B} = \frac{\gamma a_x^A + a_x^B}{\gamma + 1} \end{array}$$

is then called the consensus between ω_x^A and ω_x^B, representing an imaginary agent $[A, B]$'s opinion about x, as if she represented both A and B. By using the symbol '$\oplus$' to designate this operator, we define $\omega_x^{A,B} \equiv \omega_x^A \oplus \omega_x^B$.

It is easy to prove that the consensus operator is both commutative and associative. Opinion independence must be assumed. The consensus of two totally uncertain opinions results in a new totally uncertain opinion, although the relative atomicity is not well defined in that case.

In [4] it is incorrectly stated that the consensus operator can not be applied to two dogmatic opinions, i.e. when $\kappa = 0$. The definition above rectifies this so that dogmatic opinions can be combined. This result is obtained by computing the limits of $b_x^{A,B}$, $d_x^{A,B}$, $u_x^{A,B}$, and $a_x^{A,B}$, as $u_x^A, u_x^B \to 0$ using the relative dogmatism $\gamma = u_x^B / u_x^A$. This result makes the consensus operator even more general than Dempster's rule because the latter excludes the combination of two totally conflicting beliefs.

In order to understand the meaning of the relative dogmatism γ, it is useful to consider a process with possible outcomes $\{x, \overline{x}\}$ that produces γ times as many x as $\overline{x}$. For example when throwing a fair dice and some mechanism makes sure that A only observes the outcome of “six” and B only observes the outcome of “one”, “two”, “three”, “four” and “five”, then A will think the dice only produces “six” and B will think that the dice never produces “six”. After infinitely many observations A and B will have the conflicting dogmatic opinions $\omega^A_{\text{“six”}} = (1.0,\ 0.0,\ 0.0,\ \frac{1}{6})$ and $\omega^B_{\text{“six”}} = (0.0,\ 1.0,\ 0.0,\ \frac{1}{6})$ respectively. On the average B observes 5 times more events than A so that B remains 5 times more dogmatic than A as $u^A_{\text{“six”}}, u^B_{\text{“six”}} \rightarrow 0$, meaning that the relative dogmatism between A and B is $\gamma = 1/5$. By combining their opinions according to the case where $\kappa = 0$ in Def.12 and inserting the value of γ, we get a combined opinion about obtaining a “six” with the dice as $\omega^{A,B}_{\text{“six”}} = (\frac{1}{6},\ \frac{5}{6},\ 0,\ \frac{1}{6})$. We have assumed that A and B initially agreed on the relative atomicity of obtaining “six” with the dice.

4.3 Example: Assessment of Testimony from Witnesses

We consider a court case where 3 witnesses A, B and C are giving testimony to express their opinions about a verbal proposition x which has been made about the accused. We assume that the verbal proposition is either true or false, and let each witness express his or her opinion about the truth of the proposition as the opinions ω^A_x, ω^B_x and ω^C_x, to the courtroom. The judge J then has to determine his or her own opinion about x as a function of her trust ω^J_A: ’*Witness A is reliable*’, and similarly for B and C. This situation is illustrated in Fig.2 where the arrows denote trust or opinions about truth.

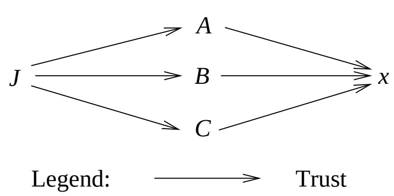

Figure 2: Trust in testimony from witnesses

The effect on the judge of each individual testimony can be computed using the discounting operator, so that for example A’s testimony causes the judge to have the opinion:

$$\omega^{J:A}_x = \omega^J_A \otimes \omega^A_x$$

about the truth of x. Assuming that the opinions resulting from each witness are independent, they can finally be combined using the consensus operator to produce the judge’s own opinion about x:

$$\omega^{J:(A,B,C)}_x = (\omega^J_A \otimes \omega^A_x) \oplus (\omega^J_B \otimes \omega^B_x) \oplus (\omega^J_C \otimes \omega^C_x) \tag{5}$$

4.4 Comparing the Consensus Operator with Dempster’s Rule

We will start with the well known example that Zadeh 1984 [9] used for the purpose of criticising Dempster’s rule. Smets 1988 [10] used the same example in defence of the non-normalised version of Dempster’s rule. The definition of Dempster’s rule and the no-normalised Dempster’s rule is given below.

Definition 13 *Let Θ be a frame of discernment, and let m^A_Θ and m^B_Θ be BMAs on Θ. Then $m^A_\Theta \oplus' m^B_\Theta$ is a function $m^A_\Theta \oplus' m^B_\Theta : 2^\Theta \rightarrow [0,1]$ such that for all $x \neq \emptyset$:*

$$m^A_\Theta \oplus' m^B_\Theta(\emptyset) = \sum_{y \cap z = \emptyset} m^A_\Theta(y) \cdot m^B_\Theta(z) - K \ ,$$

$$m^A_\Theta \oplus' m^B_\Theta(x) = \frac{\sum_{y \cap z = x} m^A_\Theta(y) \cdot m^B_\Theta(z)}{1 - K} \ ,$$

where $K = \sum_{y \cap z = \emptyset} m^A_\Theta(y) \cdot m^B_\Theta(z)$ and $K \neq 1$ in Dempster’s rule, and where $K = 0$ in the non-normalised version.

Suppose that we have a murder case with three suspects; Peter, Paul and Mary and two witnesses W_1 and W_2 who give highly conflicting testimonies. Table 1 gives the witnesses’ belief masses in Zadeh’s example and the resulting belief masses after applying Dempster’s rule, the non-normalised Dempster’s rule and the consensus operator.

Because the frame of discernment in Zadeh’s example is ternary a focused binary frame of discernment must be derived in order to apply the consensus operator. The resulting opinions are all dogmatic, and the case where $\kappa = 0$ in Def.12 must be invoked. Because of the symmetry between W_1 and W_2 we determine the relative dogmatism between W_1 and W_2 to be $\gamma = 1$.

	W_1	W_2	Dempster's rule	Non-norm. rule	Cons. operator
Peter	0.99	0.00	0.00	0.0000	0.495
Paul	0.01	0.01	1.00	0.0001	0.010
Mary	0.00	0.99	0.00	0.0000	0.495
Θ	0.00	0.00	0.00	0.0000	0.000
$\emptyset$	0.00	0.00	0.00	0.9999	0.000

Table 1: Comparison of operators in Zadeh's example

The column for the consensus operator is obtained by taking the 'belief' coordinate from the consensus opinions. The consensus opinion values and their corresponding probability expectation values are:

$$\omega^{W_1,W_2}_{\text{Peter}} = (0.495,\ 0.505,\ 0.000,\ \tfrac{1}{3})\ ,$$

$$\omega^{W_1,W_2}_{\text{Paul}} = (0.010,\ 0.990,\ 0.000,\ \tfrac{1}{3})\ ,$$

$$\omega^{W_1,W_2}_{\text{Mary}} = (0.495,\ 0.505,\ 0.000,\ \tfrac{1}{3})\ ,$$

$$\mathrm{E}(\omega^{W_1,W_2}_{\text{Peter}}) = 0.495\ ,$$

$$\mathrm{E}(\omega^{W_1,W_2}_{\text{Paul}}) = 0.010\ ,$$

$$\mathrm{E}(\omega^{W_1,W_2}_{\text{Mary}}) = 0.495\ .$$

Dempster's rule selects the least suspected by both witnesses as the guilty. The non-normalised version acquits all the suspects and indicates that somebody else is the guilty. This is explained by Smets 1988 [10] with the so-called open world interpretation of the frame of discernment.

Murphy 2000 [11] argues that Dempster's rule can only be considered safe for combining beliefs with low conflict and that statistical average is the best way to combine highly conflicting beliefs. The consensus operator gives precisely the average of beliefs to Peter and Mary, and leaves the non-conflicting beliefs on Paul unaltered. This result is further consistent with classical estimation theory (see e.g. comments to Smets 1988 [10] p.278 by M.R.B.Clarke) which is based on taking the average of probability estimates when all estimates have equal weight.

In the following example uncertainty is introduced by allocating some belief to the set $\Theta =$ {Peter, Paul, Mary}. Table 2 gives the modified BMAs and the results of applying the rules.

	W_1	W_2	Dempster's rule	Non-norm. rule	Cons. operator
Peter	0.98	0.00	0.490	0.0098	0.492
Paul	0.01	0.01	0.015	0.0003	0.010
Mary	0.00	0.98	0.490	0.0098	0.492
Θ	0.01	0.01	0.005	0.0001	0.005
$\emptyset$	0.00	0.00	0.000	0.9800	0.000

Table 2: Comparison of operators after introducing uncertainty in Zadeh's example

The column for the consensus operator is obtained by taking the 'belief' coordinate from the consensus opinions. The consensus opinion values and their corresponding probability expectation values are:

$$\omega^{W_1,W_2}_{\text{Peter}} = (0.492,\ 0.503,\ 0.005,\ \tfrac{1}{3})\ ,$$

$$\omega^{W_1,W_2}_{\text{Paul}} = (0.010,\ 0.985,\ 0.005,\ \tfrac{1}{3})\ ,$$

$$\omega^{W_1,W_2}_{\text{Mary}} = (0.492,\ 0.503,\ 0.005,\ \tfrac{1}{3})\ ,$$

$$\mathrm{E}(\omega^{W_1,W_2}_{\text{Peter}}) = 0.494\ ,$$

$$\mathrm{E}(\omega^{W_1,W_2}_{\text{Paul}}) = 0.012\ ,$$

$$\mathrm{E}(\omega^{W_1,W_2}_{\text{Mary}}) = 0.494\ .$$

When uncertainty is introduced Dempster's rule corresponds well with intuitive human judgement. The non-normalised Dempster's rule however still indicates that none of the suspects are guilty and that new suspects must be found. Our consensus operator corresponds well with human judgement and gives almost the same result as Dempster's rule. Note that the values resulting from the consensus operator have been rounded off after the third decimal. The belief masses resulting from Dempster's rule in Table 2 add up to 1. The 'belief' parameters of the consensus opinions do not add up to 1 because they are actually taken from 3 different focused frames of discernment, but the following holds:

$$\mathrm{E}(\omega^{W_1,W_2}_{\text{Peter}}) + \mathrm{E}(\omega^{W_1,W_2}_{\text{Paul}}) + \mathrm{E}(\omega^{W_1,W_2}_{\text{Mary}}) = 1\ .$$

The above example indicates that Dempster's rule and the Consensus operator give similar results. However, this is not always the case as illustrated by the following example. Let two agents D and E have equal beliefs about the truth of a binary proposition x. The agents' BMAs and the results of applying the rules are give in Table 3.

	D	E	Demp-ster's rule	Non-norm. rule	Cons. oper-ator
x	0.90	0.90	0.99	0.99	0.947
$\overline{x}$	0.00	0.00	0.00	0.00	0.000
Θ	0.10	0.10	0.01	0.01	0.053
$\emptyset$	0.00	0.00	0.00	0.00	0.000

Table 3: Comparison of operators i.c.o. equal beliefs

The consensus opinion about x and the corresponding probability expectation value are:

$$\omega_x^{D,E} = (0.947,\ 0.000,\ 0.053,\ 0.500)\ ,$$

$$\mathrm{E}(\omega_x^{D,E}) = 0.974\ .$$

It is difficult to give an intuitive judgement of these results. It can be observed that Dempster's rule and the non-normalised version produce equal results because the witnesses' BMAs are non-conflicting. The two variants of Dempster's rule amplify the combined belief twice as much as the consensus operator and this difference needs an explanation. The consensus operator produces results that are consistent with statistical analysis [4] and in the absence of good criteria for intuitive judgement this constitutes a strong argument in favour of the consensus operator.

5 Conclusion

The Opinion metric described here provides a simple and compact notation for beliefs in the Shaferian belief model. We have defined an alternative to Dempster's rule which we believe corresponds more closely with human intuitive reasoning. We have also defined a discounting operator which combined with the consensus operator can be used for subjective evidential reasoning.

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