Simplifying Jermann and Quadrini [2012]

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This note contains two sections. In the first section, we derive the equilibrium characterization of the model developed by Jermann and Quadrini 2012. In the second section, we present many simplified version of [Jermann and Quadrini, 2012]. All these models are intended to be solved the method proposed by Feng et al. [2011].

1 The Jermann and Quadrini [2012] Model

1.1 Firm's Problem

$$V(\mathbf{s}_t; k_t, b_t) = \max_{d_t, n_t, k_{t+1}, b_{t+1}} \{d_t + \mathbb{E} m_{t+1} V(\mathbf{s_{t+1}}; k_{t+1}, b_{t+1})\}$$

subject to

$$\begin{aligned}
(1-\delta)k_t + F(z_t, k_t, n_t) + \frac{b_{t+1}}{R_t} &= w_t n_t + b_t + \varphi(d_t) + k_{t+1} \\
\xi_t \left(k_{t+1} - \frac{b_{t+1}}{1+r_t}\right) &\geq F(z_t, k_t, n_t) \\
n_t &\geq 0 \\
k_{t+1} &\geq 0
\end{aligned}$$

The FOCs are given by:

$$\begin{aligned}
\lambda_t &= \frac{1}{\varphi_d(d_t)} \\
\lambda_t F_n(z_t, k_t, n_t) &= \lambda_t w_t + \mu_t F_n(z_t, k_t, n_t) \\
\lambda_t - \mu_t \xi_t &= \mathbb{E}_t m_{t+1} \{\lambda_{t+1} [1 - \delta + F_k(z_{t+1}, k_{t+1}, n_{t+1})] - \mu_{t+1} F_k(z_{t+1}, k_{t+1}, n_{t+1})\} \\
\frac{\lambda_t}{R_t} - \frac{\mu_t \xi_t}{1+r_t} &= \mathbb{E}_t m_{t+1} \lambda_{t+1} \\
0 &= \mu_t \left\{F(z_t, k_t, n_t) - \xi_t \left(k_{t+1} - \frac{b_{t+1}}{1+r_t}\right)\right\} \\
(1-\delta)k_t + F(z_t, k_t, n_t) + \frac{b_{t+1}}{R_t} &= b_t + \varphi(d_t) + k_{t+1} + w_t n_t \\
\mu_t &\geq 0 \\
\xi_t (k_{t+1} - \frac{b_{t+1}}{1+r_t}) &\geq F(z_t, k_t, n_t)
\end{aligned}$$

1.2 Household's Problem

$$\max_{\{n_t, c_t, b_{t+1}, s_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t, n_t)$$

subject to

$$w_t n_t + b_t + s_t\left(d_t + p_t\right) = \frac{b_{t+1}}{1+r_t} + s_{t+1}p_t + c_t + T_t.$$

The FOCs are given by:

$$\begin{aligned}
w_t &= \frac{-U_n(c_t,n_t)}{U_c(c_t,n_t)} \\
U_c(c_t,n_t)\frac{1}{1+r_t} &= \beta\mathbb{E}_t U_c(c_{t+1},n_{t+1}) \\
p_t U_c(c_t,n_t) &= \beta\mathbb{E}_t(d_{t+1}+p_{t+1})U_c\left(c_{t+1},n_{t+1}\right)
\end{aligned}$$

1.3 Government

To complete the model, we specify where the tax benefit on interest payment comes from:

$$\begin{aligned}
T_t &= \frac{b_{t+1}}{R_t} - \frac{b_{t+1}}{1+r_t} \\
R_t &= 1 + r_t(1-\tau)
\end{aligned}$$

1.4 Market Clearing

1.4.1 Financial Market

The share supply (s_t) is fixed at one unit, i.e. there's only one tree in this economy. Firms issue positive(negative) amount of bonds (b_t) to borrow(lend) from(to) households.

1.4.2 Goods Market

Combining all three budget constraints we have the goods market clearing condition:

$$F(z_t,k_t,n_t) = c_t + k_{t+1} - (1-\delta)k_t + \underbrace{\varphi(d_t) - d_t}_{AdjustmentCost}.$$

We can see that the output has three final uses: consumption, investment, and adjustment.

2 The Simplified Model #1 (No Labor Supply)

This version of the model is just a normal RBC model augmented by the enforcement constraint. Also firms own capital, but are not allow to issue bonds. One simplification is that the household supply labor inelastically.

2.1 Firm's Problem

$$V(\mathbf{s}_t;k_t) = \max_{d_t,n_t,k_{t+1},b_{t+1}} \left\{d_t + \mathbb{E}m_{t+1}V\left(\mathbf{s_{t+1}};k_{t+1}\right)\right\}$$

subject to

$$\begin{aligned}
(1-\delta)k_t + F(z_t,k_t,n_t) &= w_t n_t + d_t + k_{t+1} \\
\xi_t\left(k_{t+1}\right) &\geq F(z_t,k_t,n_t) \\
n_t &\geq 0 \\
k_{t+1} &\geq 0
\end{aligned}$$

The FOCs are given by (not updated yet

$$
\begin{aligned}
(1-\mu_t)F_n(z_t,k_t,n_t) &= w_t \\
1-\mu_t\xi_t &= \mathbb{E}_t m_{t+1}[1-\delta+(1-\mu_{t+1})F_k\left(z_{t+1},k_{t+1},n_{t+1}\right)] \\
0 &= \mu_t[F(z_t,k_t,n_t)-\xi_t k_{t+1}] \\
(1-\delta)k_t+F(z_t,k_t,n_t) &= d_t+k_{t+1}+w_t n_t \\
\mu_t &\geq 0 \\
\xi_t k_{t+1} &\geq F(z_t,k_t,n_t)
\end{aligned}
$$

2.2 Household's Problem

$$
\max_{\{n_t,c_t,b_{t+1},s_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t)
$$

subject to

$$
w_t + s_t\left(d_t+p_t\right) = s_{t+1}p_t + c_t.
$$

The FOCs are given by:

$$
p_t U_c(c_t,n_t) = \beta\mathbb{E}_t(d_{t+1}+p_{t+1})U_c\left(c_{t+1},n_{t+1}\right)
$$

2.3 Market Clearing

2.3.1 Financial Market

The share supply (s_t) is fixed at one unit, i.e. there's only one tree in this economy. Firms issue positive(negative) amount of bonds (b_t) to borrow(lend) from(to) households.

2.3.2 Goods Market

Combining all three budget constraints we have the goods market clearing condition:

$$
F(z_t,k_t,n_t) = c_t + k_{t+1} - (1-\delta)k_t.
$$

2.3.3 Labor Market

Since the household doesn't derive disutility from supplying labor we know $n_t = 1$ in equilibrium.

2.4 Feng et al. [2011] Representation

For computation purposes, we assume Cobb-Douglous production function and log-linear utility. Then we can collect all equilibrium conditions and rewrite them in an alternative form:

$$
\begin{aligned}
(1-\delta)k_t + z_t k_t^{\theta} &= c_t + k_{t+1} \\
\frac{1}{c_t}(1-\mu_t\xi_t) &= \beta\mathbb{E}_t m_{k,t+1} \\
m_{k,t} &= \frac{1}{c_t}[1-\delta+(1-\mu_t)z_t\theta k_t^{\theta-1}] \\
\xi_t\left(k_{t+1}\right) &\geq z_t k_t^{\theta} \\
0 &= \mu_t[F(z_t,k_t)-\xi_t k_{t+1}] \\
\mu_t &\geq 0 \\
w_t &= (1-\mu_t)(1-\theta)z_t k_t^{\theta} \\
w_t + d_t &= c_t
\end{aligned}
$$

2.5 Numerical Results

In the numerical practice, the ξ is fixed at the steady-state level $\bar{\xi}$ and z_t is i.i.d around 1.

2.5.1 Parameters

$$\begin{aligned}\beta &= 0.9825\\ \theta &= 0.36\\ \delta &= 0.0025\\ \overline{\xi} &= 0.1640\end{aligned}$$

2.5.2 Results

- The steady state must be that the enforcement constraint is not binding, because if you assume the enforcement constraint is binding in steady state you get negative Lagrange Multiplier μ_{ss}.
- Around the steady state $(0.9 \times k_{ss},\ 1.1 \times k_{ss})$ the enforcement constraint is not binding. Behaves just like a RBC.
- Once we expand the range capital $(0.5 \times k_{ss},\ 2 \times k_{ss})$, the region where capital is low and TFP is high (high MPK) doesn't admit any equilibrium (Figure 1 on page 6).
- For example, let $k = 13.9276$, $z = 1.2$, $m = 0.00001$. If you guess the constraint is binding, consumption is -2.2784. If you guess the enforcement is not binding, you get $\xi k_{t+1} < z_t k_t^\theta$. Change m to 100000, you get the same result. So no value of m can survive here.

2.5.3 An Example Where No Equilibrium Exists

To further illustrate the last point in previous subsection, an informal proof is sketched here. Let $k_0 = 13.9276$, $z_0 = 1.2$. If an equibrium starts here, exactly one of the following two cases is true:

1. Enforcement constraint is binding.
2. Enforcement constraint is not binding.

In both cases, the current output y_0 is given by $z_0 \times k_0^\theta = 3.0973$, and the current marginal product of capital MPK_0 is 0.224.

Case 1: Binding Now we know $k_1 = \frac{y_0}{\xi} = 18.9550$. The budget constraint immediately implies that the consumption $c_0 = z_0 \times k_0^\theta + (1-\delta)k_0 - k_1 = -2.2784$. It's a contradiction.

Case 2: Not Binding Now the budget constraint implies that k_1 is within [0,16.6707) as long as $c_0 > 0$. Now the LHS of the enforcement constraint is $\bar{\xi} k_1 \in [0, 2.734)$, which is always smaller than the RHS of the enforcement constraint ($z_0 k_0^\theta = 3.0973$). This contradicts the assumption that the enforcement constraint is not binding.

Appendix

A The Simplified Model #2 (Added Labor Supply)

This version of the model is just a normal RBC model augmented by the enforcement constraint. Also firms own capital, but are not allow to issue bonds.

A.1 Firm's Problem

$$V(\mathbf{s}_t; k_t) = \max_{d_t, n_t, k_{t+1}, b_{t+1}} \{d_t + \mathbb{E} m_{t+1} V(\mathbf{s_{t+1}}; k_{t+1})\}$$

subject to

$$\begin{aligned}
(1-\delta)k_t + F(z_t, k_t, n_t) &= w_t n_t + d_t + k_{t+1} \\
\xi_t (k_{t+1}) &\geq F(z_t, k_t, n_t) \\
n_t &\geq 0 \\
k_{t+1} &\geq 0
\end{aligned}$$

The FOCs are given by (not updated yet

$$\begin{aligned}
(1-\mu_t) F_n(z_t, k_t, n_t) &= w_t \\
1 - \mu_t \xi_t &= \mathbb{E}_t m_{t+1}[1 - \delta + (1-\mu_{t+1}) F_k(z_{t+1}, k_{t+1}, n_{t+1})] \\
0 &= \mu_t [F(z_t, k_t, n_t) - \xi_t k_{t+1}] \\
(1-\delta)k_t + F(z_t, k_t, n_t) &= d_t + k_{t+1} + w_t n_t \\
\mu_t &\geq 0 \\
\xi_t k_{t+1} &\geq F(z_t, k_t, n_t)
\end{aligned}$$

A.2 Household's Problem

$$\max_{\{n_t, c_t, b_{t+1}, s_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t, n_t)$$

subject to

$$w_t n_t + s_t (d_t + p_t) = s_{t+1} p_t + c_t.$$

The FOCs are given by:

$$\begin{aligned}
w_t &= \frac{-U_n(c_t, n_t)}{U_c(c_t, n_t)} \\
p_t U_c(c_t, n_t) &= \beta \mathbb{E}_t (d_{t+1} + p_{t+1}) U_c(c_{t+1}, n_{t+1})
\end{aligned}$$

A.3 Feng et al. [2011] Representation

For computation purposes, we assume Cobb-Douglous production function and log-linear utility. Then we can collect all equilibrium conditions and rewrite them in an alternative form:

$$\begin{aligned}
(1-\delta)k_t + z_t k_t^{\theta} n_t^{1-\theta} &= w_t n_t + d_t + k_{t+1} \\
\frac{1}{c_t}(1 - \mu_t \xi_t) &= \beta \mathbb{E}_t m_{k,t+1} \\
m_{k,t} &= \frac{1}{c_t}[1 - \delta + (1-\mu_t) z_t \theta k_t^{\theta-1} n_t^{1-\theta}] \\
\xi_t (k_{t+1}) &\geq z_t k_t^{\theta} n_t^{1-\theta} \\
0 &= \mu_t [F(z_t, k_t, n_t) - \xi_t k_{t+1}] \\
\mu_t &\geq 0 \\
w_t &= \frac{\alpha c_t}{1 - n_t} \\
w_t &= (1-\mu_t) F_n(z_t, k_t, n_t) \\
w_t n_t + d_t &= c_t
\end{aligned}$$

References

Zhigang Feng, Jianjun Miao, Adrian Peralta-Alva, and Manuel Santos. Numerical simulation of nonoptimal dynamic equilibrium models. *Federal Reserve Bank of St. Louis Working Paper No*, 2011.

Urban Jermann and Vincenzo Quadrini. Macroeconomic effects of financial shocks. *The American Economic Review*, 102(1):238–271, 2012.

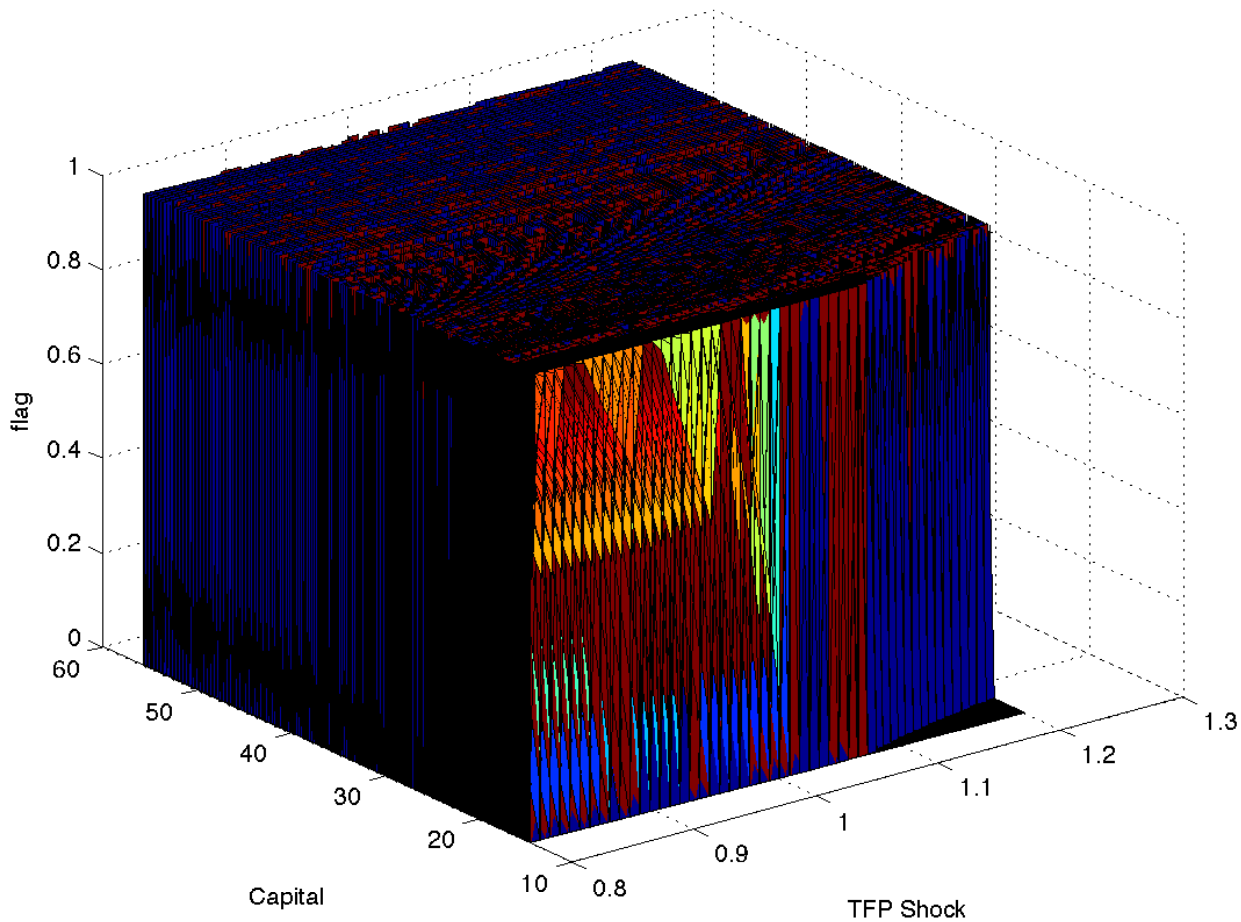

Figure 1: The Portion of m_k survive