Journal of Physics A: Mathematical and General

A simple model of Aharonov - Berry's superoscillations

To cite this article: Wang Qiao 1996 *J. Phys. A: Math. Gen.* **29** 2257

J. Phys. A: Math. Gen. **29** (1996) 2257–2258. Printed in the UK

A simple model of Aharonov–Berry's superoscillations

Wang Qiao
Department of Mathematics, Wuhan University, Wuhan, 430072, People's Republic of China
and
Information Science Center, Peking University, Beijing, 100871, People's Republic of China

Received 29 January 1996

Abstract. A simple method of constructing superoscillatory functions is given. It is based on a zero-shifting principle of functions in Paley–Wiener space.

There exist functions that oscillate like $\cos x$ but they can do so arbitrarily faster than $\cos x$ in an interval with arbitrary length. This strange phenomenon, called superoscillation, is firstly described by Aharonov (cf [1]). Using the asymptotic of integrals, Berry gave a clever way to construct a large class of superoscillation functions ([1]). This fact has many interesting applications in quantum mechanics such as [2] and [3]. In this paper we will give a zero-shifting principle which implies that we can get arbitrary superoscillation from any band-limited function. Any physical signal has finite energy, so we consider the function $f(x) \in L^2(R)$ with $\widehat{f}(\xi) = 0$ for $|\xi| \geqslant \pi$, where $\widehat{f}$ means Fourier–Laplace transform. These functions form a Paley–Wiener space W_π. The following theorem is classical (cf [4]).

Theorem (Paley–Wiener). Suppose $\xi \in R$. If $\widehat{g}(\xi) \in L^2([-\pi, \pi])$ then $\widehat{g}(\xi) = 0$ for $|\xi| \geqslant \pi$ if and only if there is a $f(x) \in L^2(R)$, $f(z)$ is entire and $f(z) = O(e^{\pi|z|})$ such that $\widehat{f}(\xi) = \widehat{g}(\xi)$.

Now we can give

Theorem (Zero-shifting principle). If $f(x) \in W_\pi$ is continuous with c zero, then for any real number s, $g(x) = \frac{x-s}{x-c} f(x)$ is also in W_π.

Proof. $g(z) = \frac{z-s}{z-c} f(z)$ is of $O(e^{\pi|z|})$ and $g(x) \in L^2(R)$ since $g(z) \sim f(z)$ for $|z| \to \infty$.

Using this theorem we can get arbitrary superoscillation functions from arbitrary given band-limited functions very easily. For example, consider

$$f(x) = \frac{\sin \pi x}{\pi x} = \prod_{n=1}^{\infty} (1 - \frac{x^2}{n^2})$$

which oscillates like $\cos \pi x$. Slightly modifying it, we get

$$g(x) = \prod_{n=1}^{N} (1 - \frac{k^2 x^2}{n^2}) \prod_{n=N+1}^{\infty} (1 - \frac{x^2}{n^2}) \qquad k > 1$$

which oscillates like $\cos kx$ during $[-N/k, N/k]$ but globally like $\cos x$.

0305-4470/96/092257+02$19.50

Acknowledgments

The author would like to thank Professor M Berry for his help. He also thanks the referees of this paper for their careful revision.

References

[1] Berry M V 1994 Faster than Fourier *Fundamental Problems in Quantum Theory* ed J A Anandan and J Safko (Singapore: World Scientific)
[2] Aharonov Y *et al* 1990 *Phys. Rev. Lett.* **64** 2965–8
[3] Berry M V 1994 *J.Phys. A: Math. Gen.* **27** L391–8
[4] Zygmund 1959 *Trigonometric Series* (Cambridge: Cambridge University Press)