

Modeling of time with metamaterials

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Metamaterials have already been used to model various exotic optical spaces. Here we demonstrate that mapping of monochromatic extraordinary light distribution in a hyperbolic metamaterial along some spatial direction may model the flow of time. This idea is illustrated in experiments performed with plasmonic hyperbolic metamaterials. The appearance of the statistical arrow of time is examined in an experimental scenario that emulates a big-bang-like event. © 2011 Optical Society of America

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The nature of time has been a major subject of science, philosophy, and religion. Our everyday experiences tell us that time has a direction. On the other hand, most laws of physics appear to be symmetric with respect to time reversal. A few exceptions include the second law of thermodynamics, which states that entropy must increase over time, and the cosmological arrow of time, which points away from the big bang. While it is generally believed that the statistical and the cosmological arrows of time are connected, we cannot replay the big bang and prove this relationship in the experiment. Fortunately, it appears that electromagnetic metamaterials may provide us with interesting tools to better understand this relationship.

Recent advances in electromagnetic metamaterials have already demonstrated highly unusual curvilinear optical spaces [1–3]. While nothing affects the flow of time in metamaterials, very recently it was demonstrated that a spatial coordinate may become timelike in a hyperbolic metamaterial [4]. To better understand this effect, let us start with a nonmagnetic uniaxial anisotropic material with dielectric permittivities $\epsilon_x = \epsilon_y = \epsilon_1$ and $\epsilon_z = \epsilon_2$, and assume that this behavior holds in some frequency range around $\omega = \omega_0$. Any electromagnetic field propagating in this material can be expressed as a sum of the ordinary and extraordinary contributions, each of these being a sum of an arbitrary number of plane waves polarized in the ordinary (\vec{E} perpendicular to the optical axis) and extraordinary (\vec{E} parallel to the plane defined by the k vector of the wave and the optical axis) directions. Let us define our scalar extraordinary wave function as $\varphi = E_z$ so that the ordinary portion of the electromagnetic field does not contribute to φ . This definition will be extremely important for the discussion of our results below, since it turns out that ordinary and extraordinary photons do not experience the same effective metric. Since metamaterials generally exhibit high dispersion, let us work in the frequency domain and write the macroscopic Maxwell equations as

$$\frac{\omega^2}{c^2} \vec{D}_\omega = \vec{\nabla} \times \vec{\nabla} \times \vec{E}_\omega, \quad \vec{D}_\omega = \vec{\epsilon}_\omega \vec{E}_\omega. \quad (1)$$

Equation (1) results in the following wave equation for φ_ω if ϵ_1 and ϵ_2 are kept constant inside the metamaterial:

$$-\frac{\omega^2}{c^2} \varphi_\omega = \frac{\partial^2 \varphi_\omega}{\epsilon_1 \partial z^2} + \frac{1}{\epsilon_2} \left(\frac{\partial^2 \varphi_\omega}{\partial x^2} + \frac{\partial^2 \varphi_\omega}{\partial y^2} \right). \quad (2)$$

While in ordinary crystalline anisotropic media both ϵ_1 and ϵ_2 are positive, this is not necessarily the case in metamaterials. In hyperbolic metamaterials [5], ϵ_1 and ϵ_2 have opposite signs. These metamaterials are typically composed of multilayer metal–dielectric or metal wire array structures, as shown in Fig. 1. Optical properties of such metamaterials are quite extraordinary. For example, there is no usual diffraction limit in a hyperbolic metamaterial [6,7]. Let us consider the case of constant $\epsilon_1 > 0$ and $\epsilon_2 < 0$ and assume that this behavior holds in some frequency range around $\omega = \omega_0$. Let us assume that the metamaterial is illuminated by a coherent cw laser field at frequency ω_0 , and we study spatial distribution of the extraordinary field φ_ω at this frequency. Under these assumptions, Eq. (3) can be rewritten in the form of a three-dimensional (3D) Klein–Gordon equation describing a massive scalar φ_ω field:

$$-\frac{\partial^2 \varphi_\omega}{\epsilon_1 \partial z^2} + \frac{1}{|\epsilon_2|} \left(\frac{\partial^2 \varphi_\omega}{\partial x^2} + \frac{\partial^2 \varphi_\omega}{\partial y^2} \right) = \frac{\omega_0^2}{c^2} \varphi_\omega = \frac{m^{*2} c^2}{\hbar^2} \varphi_\omega. \quad (3)$$

in which the spatial coordinate $z = \tau$ behaves as a timelike variable. Therefore, Eq. (3) describes world lines of massive particles that propagate in a flat $(2 + 1)$ Minkowski spacetime [Fig. 2(d)]. When a metamaterial is built and illuminated with a coherent extraordinary cw laser beam, the stationary pattern of light propagation inside the metamaterial represents a complete history of a toy $(2 + 1)$ -dimensional spacetime populated with particles of mass m^* . This history is written as a collection of particle world lines along the timelike z coordinate.

The world lines of particles described by Eq. (3) are straight lines. These straight world lines are easy to observe in the experiment, as demonstrated in Fig. 2. These experiments were performed using polymethyl methacrylate (PMMA)-based plasmonic hyperbolic metamaterials described in detail in [6]. Rigorous theoretical description of these metamaterials was developed in [8]. While simple demonstration of a single straight world line presented in Fig. 2 is almost trivial, the

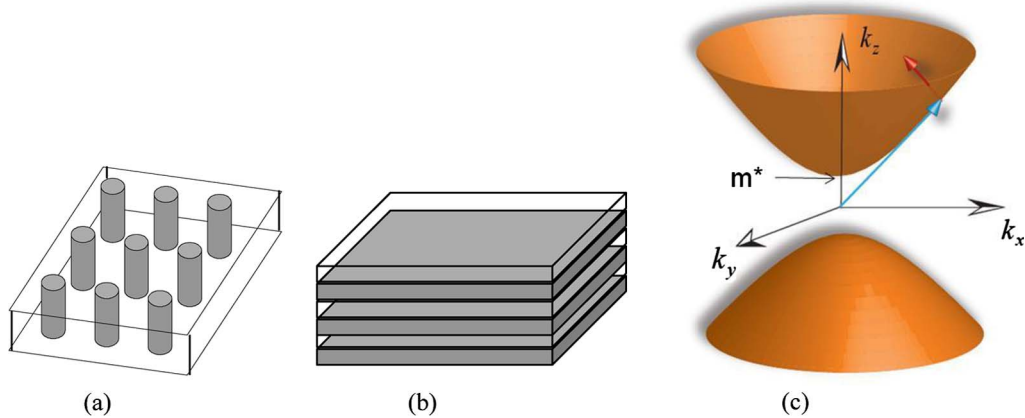


Fig. 1. (Color online) Schematic views of (a) wired and (b) layered hyperbolic metamaterials. (c) Hyperbolic dispersion relation is illustrated as a surface of constant frequency in k space. When the z coordinate is timelike, k_z behaves as effective energy.

situation may become much more interesting if we allow adiabatic variations of ε_1 and ε_2 inside the metamaterial. The Klein–Gordon equation for a massive particle in a gravitational field can be written as [9]

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left(g^{ik} \sqrt{-g} \frac{\partial \varphi}{\partial x^k} \right) = \frac{m^2 c^2}{\hbar^2} \varphi. \quad (4)$$

If ε_1 and ε_2 are allowed to vary, Eq. (3) will remain approximately valid if $\partial \varepsilon / \partial x^i \ll k_i$. Comparison of Eqs. (3) and (4)

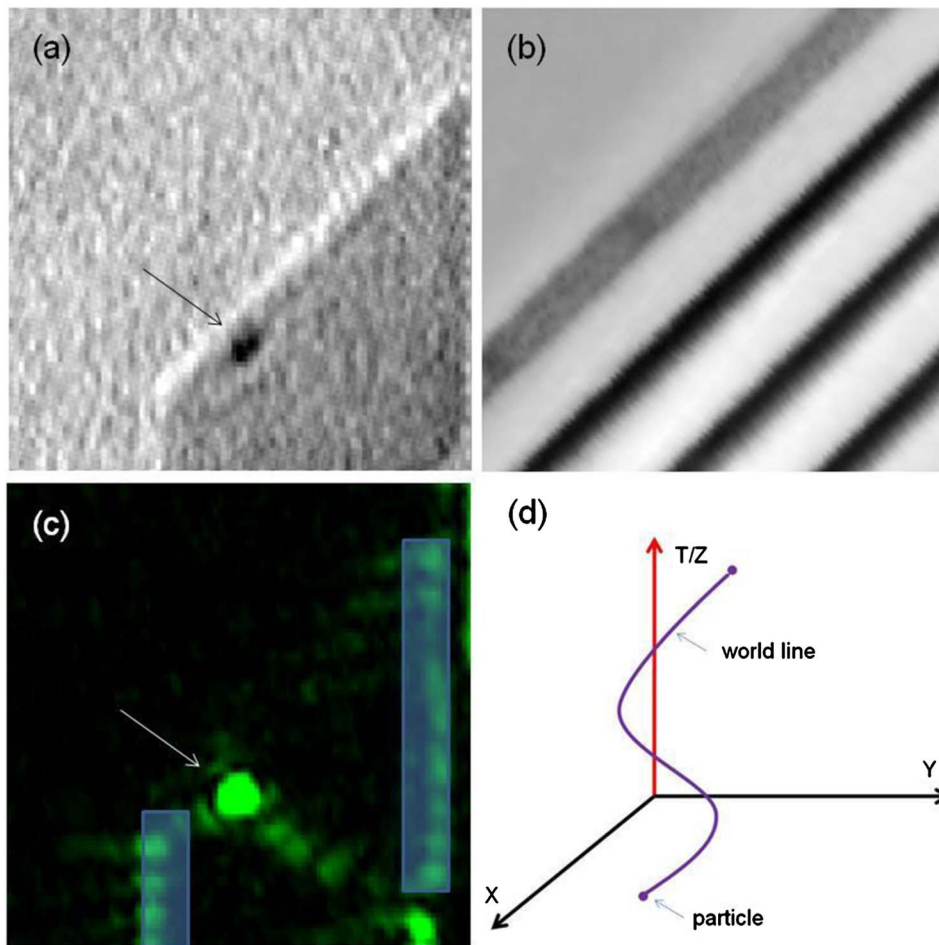


Fig. 2. (Color online) Experimental demonstration of straight world lines in a hyperbolic metamaterial: (a) Image of the plasmonic hyperbolic metamaterial obtained using an optical microscope under white-light illumination. The defect used as a plasmon source is shown by an arrow. (b) Atomic force microscope (AFM) image of the metamaterial shows stripes of PMMA formed on the gold film surface using electron-beam lithography. (c) Straight plasmonic ray, or world line, is emitted from the defect under illumination with 532 nm laser light. The ray direction is indicated by the arrow. For the sake of clarity, light scattering by the edges of the PMMA pattern is partially blocked by semitransparent rectangles. (d) Schematic view of a particle world line in a $(2+1)$ -dimensional Minkowski spacetime.

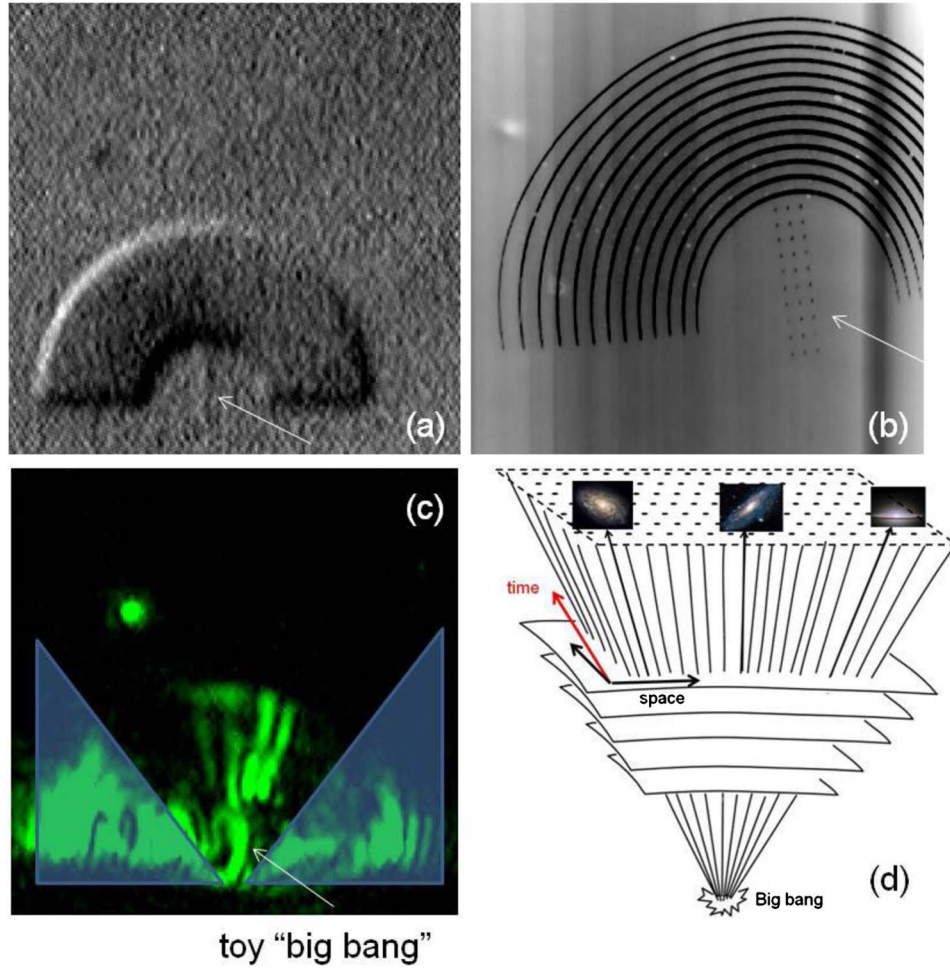


Fig. 3. (Color online) Experimental demonstration of world line behavior in an "expanding universe" using a plasmonic hyperbolic metamaterial. (a) Optical and (b) AFM images of the plasmonic hyperbolic metamaterial based on PMMA stripes on gold. The defect used as a plasmon source is shown by an arrow. (c) Plasmonic rays or world lines increase their spatial separation as a function of a timelike radial coordinate. The point (or moment) $r = \tau = 0$ corresponds to a toy big bang. For the sake of clarity, light scattering by the edges of the PMMA pattern is partially blocked by semitransparent triangles. (d) Schematic view of world lines behavior near the big bang.

demonstrates that world lines of massive particles in some well-known curvilinear spacetimes can indeed be emulated by using hyperbolic metamaterials in the case of slowly varying ϵ_1 and ϵ_2 . For example, let us consider an experimental situation in which we allow slow adiabatic variation of ϵ_2 as a function of z , while ϵ_1 is kept constant. According to Eqs. (3) and (4), this situation corresponds to cosmological expansion of the $(2+1)$ -dimensional universe as a function of timelike variable z . Therefore, spatial separation of particle world lines must increase with increasing z [see Fig. 3(d)]. Some complications within such a model may arise due to loss and spatial dispersion in practical hyperbolic metamaterials. On the other hand, losses may be compensated if a gain medium is used as a dielectric component of the metamaterial, while effects of spatial dispersion are widely believed to affect the behavior of real spacetime at the Planck scale. Thus, difficulties presented to our model by losses and spatial dispersion are not insurmountable.

While experimental demonstration of the expanding universe with 3D metamaterials would require sophisticated nanofabrication, experimental demonstration of this concept using plasmonic hyperbolic metamaterials is much simpler. Let us start by rewriting Eq. (1) in cylindrical coordinates

(r, θ, z) . Let us assume that ϵ_θ , ϵ_z , and ϵ_r change very slowly as a function of coordinates so that their derivatives can be neglected. In addition, let us assume that $\epsilon_\theta = \epsilon_z > 0$, $\epsilon_r < 0$, and redefine our scalar extraordinary wave function as $\varphi = E_r$. Under these approximations, we obtain

$$-\frac{\partial^2 E_r}{\epsilon_\theta \partial r^2} + \frac{1}{|\epsilon_r|} \left(\frac{\partial^2 E_r}{\partial z^2} + \frac{\partial^2 E_r}{r^2 \partial \theta^2} \right) - \frac{\partial E_r}{\epsilon_\theta r \partial r} + \frac{E_r}{\epsilon_\theta r^2} + \frac{\partial E_\theta}{\epsilon_r r^2 \partial \theta} = -\frac{\partial^2 E_r}{\partial t^2}. \quad (5)$$

At large enough r , extraordinary rays become well defined, the last three terms on the left side of Eq. (5) can be neglected, and the new scalar wave equation can be written [5] as

$$-\frac{\partial^2 \varphi_\omega}{\epsilon_\theta \partial r^2} + \frac{1}{|\epsilon_r|} \left(\frac{\partial^2 \varphi_\omega}{\partial z^2} + \frac{\partial^2 \varphi_\omega}{r^2 \partial \theta^2} \right) = \frac{\omega_0^2}{c^2} \varphi_\omega = \frac{m^* c^2}{\hbar^2} \varphi_\omega, \quad (6)$$

where $r = \tau$ becomes a timelike variable [note that the local metric experienced by the so-defined extraordinary field coincides with the metric in Eq. (3) at large enough r , while ordinary rays do not experience this metric].

In the case of plasmonic hyperbolic metamaterial, Eq. (6) can be simplified even further:

$$-\frac{\partial^2 \varphi_\omega}{\varepsilon_\theta \partial r^2} + \frac{1}{|\varepsilon_r| r^2} \frac{\partial^2 \varphi_\omega}{\partial \theta^2} = \frac{m^{*2} c^2}{\hbar^2} \varphi_\omega. \quad (7)$$

Now it is apparent that the cosmological expansion as a function of $r = \tau$ may be realized with constant $\varepsilon_\theta > 0$ and $\varepsilon_r < 0$, which validates our original assumptions. Experimental demonstration of the world line behavior in an expanding universe as a function of timelike $r = \tau$ is presented in Fig. 3. This experiment is similar to the experimental demonstration of the plasmonic hyperlens [6]. Plasmon rays are launched into the hyperbolic metamaterial near the $r = 0$ point via the central phase-matching structure marked with an arrow in Figs. 3(a) and 3(b). Similar to the world line behavior near the big bang, which is shown in Fig. 3(d), plasmonic rays or

world lines indeed increase their spatial separation as a function of timelike radial coordinate $r = \tau$. The point (or moment) $r = \tau = 0$ corresponds to a moment of the toy big bang. Comparison of Fig. 3(c) with Fig. 3(d) indicates that the cosmological arrow of time is well defined and points from $r = 0$ outward.

Let us examine if this experimental model may help us illustrate the relationship between the statistical and the cosmological arrows of time. As a first step, let us define the operational notion of entropy within the scope of our model. Let us follow the classical particle-in-a-box approach. Let us define our boxes as circular sectors with an angular width equal to the width of an individual plasmonic ray or world line in Fig. 3(c), and number these boxes from 1 to m . As a result, field intensity $I(r)$ at a given radius r can be interpreted as a distribution of identical particles of mass m^* among these boxes. Therefore, entropy can be written as

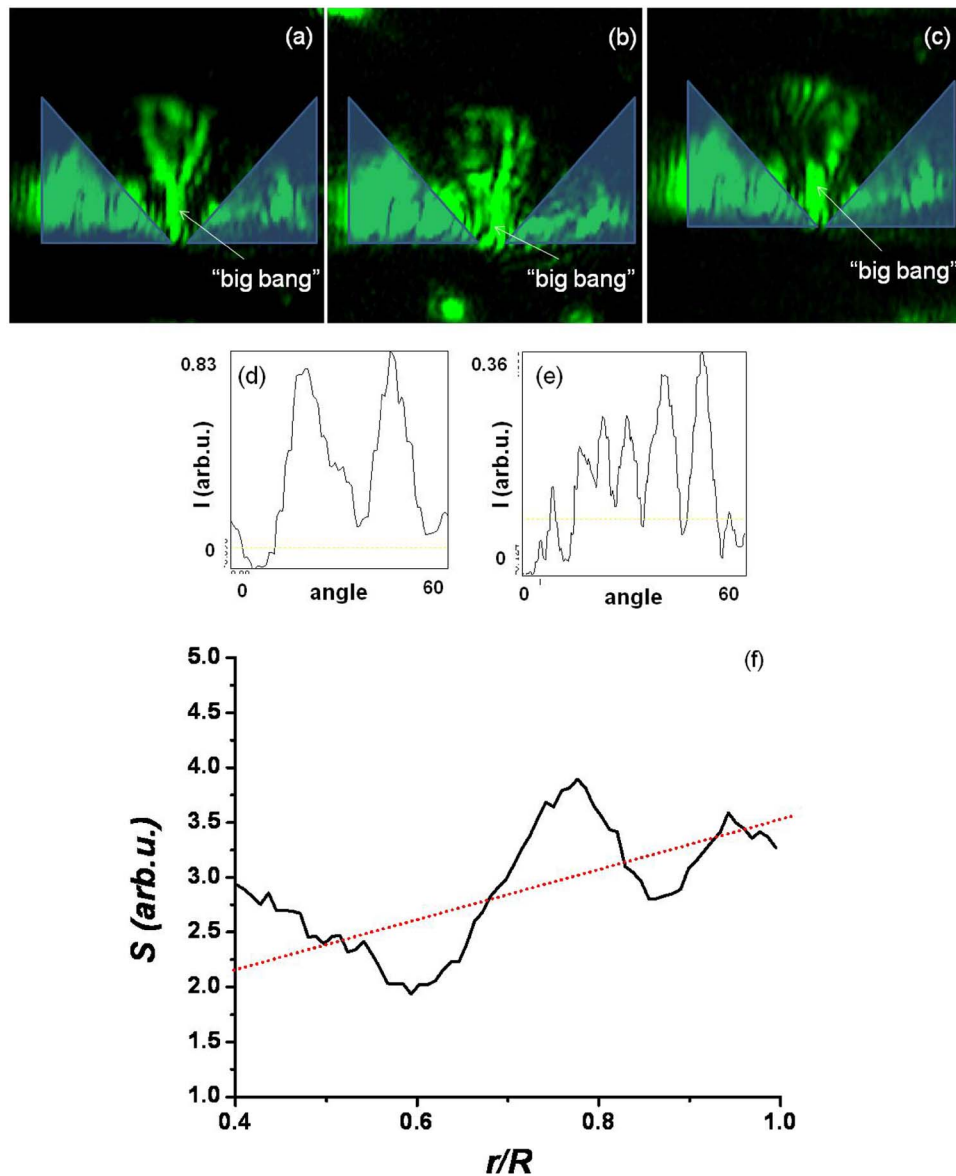


Fig. 4. (Color online) (a) (b), (c) Effect of various degrees of disorder on field distribution inside the metamaterial structure shown in Fig. 3. The cross section of image (c) near the launch point (d) and near the outer rim of the structure (e) demonstrates the increase of entropy. (f) Entropy [as defined by Eqs. (8) and (9)] plotted as a function of a timelike radial coordinate for the experiment presented in (c) demonstrates that statistical and cosmological arrows of time do coincide in our experiments.

$$S = k \ln G = k \ln \frac{N!}{N_1! N_2! \dots N_m!}, \quad (8)$$

where N_m is the number of particles in box m , and N is the total number of particles [10]. Since the numbers of particles are very large, we may use the Stirling's approximation, $\ln N! \approx N \ln N$, and obtain an approximate expression

$$S \sim I \ln I - \sum I_m \ln I_m, \quad (9)$$

where $I = \sum I_m$. Equations (8) and (9) allow us to calculate entropy $S(r)$ at a given radius or moment of time $r = \tau$, and examine if the statistical arrow of time in our model coincides with the cosmological one. The possible sources of entropy in these model experiments are nonlinear optical interactions of plasmonic rays and random disorder of the samples. In the absence of both factors, entropy defined by Eqs. (8) and (9) should remain constant. Ideally, the best case scenario for these experiments would be to use a perfect defectless hyperbolic metamaterial structure, which is made using a highly nonlinear dielectric, such as photorefractive lithium niobate or barium titanate. Such an experimental geometry would present us with a good opportunity to study entropy increase due to particle interaction in a truly irreversible nonlinear optical scenario. Unfortunately, in our current experiments, we must rely only on the random disorder of the samples. This is analogous to experiments performed with a small number of molecules of an ideal gas, which is composed of perfectly elastic molecules. Entropy increase in such a gas is known to be reversible in principle. In a similar fashion, the effects of linear random scattering of light are known to be reversible using, for instance, an adaptive optics approach. Therefore, our experiments must be considered only as an indication of potentially interesting research direction. The summary of these experiments is presented in Fig. 4. In one of these experiments, shown in Fig. 4(c), a low entropy state created at the moment of a big bang by launching two plasmon rays into the structure [Fig. 4(d)] has been gradually scattered by random defects into a high entropy state [Fig. 4(e)]. This behavior is strikingly different from the almost ideal defectless case presented in Fig. 3(c), in which entropy stays almost constant. Figure 4(f) clearly demonstrates that entropy plotted as a function of timelike radial coordinates does grow with time in the presence of disorder. Thus, statistical and cosmological arrows of time do coincide in our model experiments.

Finally, let us ask a natural question if closed timelike curves (CTCs) are allowed in a 3D hyperbolic metamaterial. At first glance, this question is simple, and the answer should be "yes." If the angular coordinate θ is made timelike, while r and z coordinates are spacelike, we seem to have a clear case of CTC in which all the grandfather paradoxes are resolved by the cyclic boundary conditions. However, more detailed

analysis described below indicates that nature resists creation of even such trivial CTCs in a 3D metamaterial. In order to perform this analysis, we need to assume that ε_θ , ε_z , and ε_r change very slowly as a function of coordinates so that their derivatives can be neglected. In addition, let us assume that $\varepsilon_r = \varepsilon_z > 0$, $\varepsilon_\theta < 0$, and redefine our scalar extraordinary wave function as $\varphi = E_\theta$. Under these approximations, we obtain

$$-\frac{\partial^2 E_\theta}{\varepsilon_r r^2 \partial \theta^2} + \frac{1}{|\varepsilon_\theta|} \left(\frac{\partial^2 E_\theta}{\partial z^2} + \frac{\partial^2 E_\theta}{\partial r^2} \right) - \frac{2\partial E_r}{\varepsilon_\theta r^2 \partial \theta} = -\frac{\partial^2 E_\theta}{\partial t^2}. \quad (10)$$

At large enough r , extraordinary rays become well defined, the last term on the left side of Eq. (10) can be neglected, and the new scalar wave equation can be written as

$$-\frac{\partial^2 \varphi_\omega}{\varepsilon_r r^2 \partial \theta^2} + \frac{1}{|\varepsilon_\theta|} \left(\frac{\partial^2 \varphi_\omega}{\partial z^2} + \frac{\partial^2 \varphi_\omega}{\partial r^2} \right) = \frac{\omega_0^2}{c^2} \varphi_\omega = \frac{m^* c^2}{\hbar^2} \varphi_\omega, \quad (11)$$

where $\theta = \tau$ becomes a timelike variable [once again, note that the local metric experienced by the so-defined extraordinary field coincides with the metric in Eq. (3) at large enough r , while ordinary rays do not experience this metric]. An infinite number of CTCs defined by condition $r = \text{const}$ and $z = \text{const}$ do appear in such a toy spacetime, and an extraordinary ray may be sent over a world line, which is very close to a CTC. However, the only rays that can propagate along exact cyclic CTC trajectories are ordinary rays. These rays do not experience the effective metric defined by Eq. (11), and they do not perceive θ as a timelike variable.

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