Image rotation, Wigner rotation, and the fractional Fourier transform

Adolf W. Lohmann

Angewandte Optik, Universität Erlangen-Nürnberg, Staudtstrasse 7/B2, 8520 Erlangen, Germany

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In this study the degree $p = 1$ is assigned to the ordinary Fourier transform. The fractional Fourier transform, for example with degree $P = 1/2$, performs an ordinary Fourier transform if applied twice in a row. Ozaktas and Mendlovic ["Fourier transforms of fractional order and their optical implementation," Opt. Commun. (to be published)] introduced the fractional Fourier transform into optics on the basis of the fact that a piece of graded-index (GRIN) fiber of proper length will perform a Fourier transform. Cutting that piece of GRIN fiber into shorter pieces corresponds to splitting the ordinary Fourier transform into fractional transforms. I approach the subject of fractional Fourier transforms in two other ways. First, I point out the algorithmic isomorphism among image rotation, rotation of the Wigner distribution function, and fractional Fourier transforming. Second, I propose two optical setups that are able to perform a fractional Fourier transform.

1. SURVEY AND MOTIVATION

The title of this study immediately provokes three questions:

1. Why are there actually *three* titles?
2. What is a fractional Fourier transform?
3. What does this transform have to do with optics?

First, a preliminary answer to the third question: the (ordinary) Fourier transform is of such central significance to physical optics and to optical information processing that everything that is somehow related to Fourier mathematics is likely to be important as well in the realm of optics. More-specific recommendations for the use of the fractional Fourier transform were made by Mendlovic and Ozaktas, the inventors of the fractional Fourier transform.[1] There exists some prior history, which will be mentioned in the final comments, when the context can be better appreciated.

Now a preliminary answer to the second question. Mendlovic and Ozaktas[1] proposed the following gedankenexperiment: suppose that a piece of graded-index fiber has the proper length L that is required for performing a Fourier transform of the coherent input image. Now imagine that the graded-index fiber is cut into pieces. A piece of length PL ($P < 1$) will do to the input image what Mendlovic and Ozaktas[1] call a fractional Fourier transform. Putting two pieces of lengths P_1L and P_2L together corresponds apparently to a fractional transform of degree $P_1 + P_2 = P$.

Mendlovic and Ozaktas proved that their definition satisfies this law of additivity and a few other laws as well. For example, since $P_1 + P_2 = P_2 + P_1$, the fractional Fourier transform is also commutative if the definition is based on this gedankenexperiment.

Now to the first question, which will lead us to an alternative approach for defining and implementing a fractional Fourier transform. Every image $u(x, y)$ or signal $u(t)$ can be described indirectly and uniquely by a Wigner distribution function, which will be reintroduced briefly later in the paper. The Wigner distribution function (WDF) undergoes certain changes if something happens to the signal [from now on called $u(x)$]. For example, propagation in free space means a horizontal shearing of the WDF, and passage through a lens corresponds to a vertical shearing of the WDF. A Fraunhofer diffraction (i.e., an ordinary Fourier transform) lets the WDF rotate by 90°. Hence it is plausible to define a fractional Fourier transform as what happens to the signal $u(x)$ while the WDF is rotated by an angle of $\phi = P\pi/2$. The P is the fractional degree. Notice that two consecutive rotations obey $\phi_1 + \phi_2 = \phi_{\mathrm{TOTAL}}$ and $\phi_1 + \phi_2 = \phi_2 + \phi_1$. Hence our definition is inherently additive and commutative.

This definition may be plausible. But it raises several further questions:

4. If a rotation of the WDF is accepted as the primary (but indirect) definition of a fractional Fourier transform, how then is the corresponding operator for $u(x)$ itself defined?
5. How can the WDF rotation be subdivided into other well-known WDF operations?
6. How can one implement experimentally the fractional Fourier transform?

Question 5 is relevant, since a direct rotation of the WDF is problematic, as we shall see later on.

Having raised these questions, we now attempt to answer them according to the following plan. In Section 2 we recall what is relevant about the WDF in our context. That enables us to answer in Section 3 the fourth question about the fractional Fourier operator. The mathematical result is not entirely satisfactory, since it does not lend itself to a straightforward experimental implementation. A special problem would arise if the signal were a two-dimensional image, which corresponds to a four-dimensional WDF. A useful contribution for the improvement of our WDF approach is the development of a new image rotator (Section 4). That image rotator can be

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described by an algorithm that is applicable also to the rotation of the WDF in a manner that provides a satisfactory answer to the fifth question (Section 5). From there on it is straightforward to devise two simple optical setups for performing a fractional Fourier transform (Section 6). I conclude with some comments on how this project relates to others.

Admittedly, this introduction is somewhat involved. In order to proceed more quickly later on, I present once more an outline of this project in different terms: I start by calling the ordinary Fourier transform $F^{(1)}$ a transform of degree $P = 1$. A degree of 2 means that the Fourier transform is applied twice in a row: $F^{(2)} = F^{(1)}F^{(1)}$. So far, I have introduced only a slightly different nomenclature for a well-known procedure. However, new territory is entered if the degree P is no longer an integer but, for example, 1/2, 1/3, and so on. How can we define $F^{(P)}$?

We do this in two isomorphic transitions. The first transition is based on the following equivalence: a Fourier transform of a signal $u(x)$ corresponds to a 90° rotation of the Wigner distribution. Naturally, we want $F^{(1/2)}$ to correspond to a 45° rotation and $F^{(2/3)}$ to a 60° rotation of the WDF, and so on. This raises the question of what kind of experiment upon $u(x)$ would rotate the associated WDF accordingly. A preliminary answer will be found, but it is not quite satisfactory. Again, an isomorphic transition helps us. A special way of rotating a two-dimensional image consists of three steps, for which the three corresponding operations in Wigner space are known. Now we have gotten everything together for defining and implementing a fractional Fourier transform. As a by-product, we found a new method for rotating an image.

2. ABOUT THE WIGNER DISTRIBUTION FUNCTION

The WDF is defined as[2–4]

$$W(x,\nu) = \int u(x + x'/2)u^*(x - x'/2)\exp(-2\pi i x'\nu)\mathrm{d}x'. \tag{2.1}$$

We insert the Fourier representation of the signal $u(x)$:

$$u(x) = \int \tilde{u}(\nu)\exp(2\pi i\nu x)\mathrm{d}\nu, \tag{2.2}$$

$$W(x,\nu) = \int \tilde{u}(\nu + \nu'/2)\tilde{u}^*(\nu - \nu'/2)\exp(2\pi i\nu' x)\mathrm{d}\nu'. \tag{2.3}$$

The projections of the WDF have well-known physical meanings:

$$\int W\mathrm{d}\nu = |u(x)|^2, \tag{2.4}$$

$$\int W\mathrm{d}x = |\tilde{u}(\nu)|^2, \tag{2.5}$$

$$\iint W\mathrm{d}\nu\mathrm{d}x = E_{\mathrm{TOTAL}}. \tag{2.6}$$

The inversion from the WDF to the signal is unique, apart from a constant phase factor,

$$\int W(x,\nu)\exp(4\pi i\nu x)\mathrm{d}\nu = u(2x)u^*(0), \tag{2.7}$$

$$\int W(x,\nu)\exp(-4\pi i\nu x)\mathrm{d}x = \tilde{u}(2\nu)\tilde{u}^*(0). \tag{2.8}$$

If $u(0)$ happens to be very small or even zero we search for the maximum of $|u(x)|^2$ at $x = x_M$ and modify Eq. (2.7) accordingly:

$$u(2x - x_M) = \int W\exp[4\pi i\nu(x - x_M)]\mathrm{d}\nu/u^*(x_M). \tag{2.9}$$

The variable ν, the so-called spatial frequency, has the dimension (1/length). Instead, we introduce the coordinate of the Fourier plane of a typical optical setup:

$$\lambda f_1\nu = \xi. \tag{2.10}$$

λ is the wavelength and f_1 an arbitrary focal length, which we consider to be fixed from now on. Constant factors such as λf_1 are dropped into the following definitions.

$$u_F(\xi) = \tilde{u}_0(\xi/\lambda f_1) = \int u_0(x)\exp(-2\pi i x\xi/\lambda f_1)\mathrm{d}x, \tag{2.11}$$

$$W_0(x,\xi) = \int u_0(x + x'/2)u_0^*(x - x'/2) \times \exp(-2\pi i x'\xi/\lambda f_1)\mathrm{d}x', \tag{2.12}$$

$$W_0 = \int u_F(\xi + \xi'/2)u_F^*(\xi - \xi'/2)\exp(2\pi i x\xi'/\lambda f_1)\mathrm{d}\xi'. \tag{2.13}$$

Now we consider the consequences for the WDF if the signal is modified.

$$u_0(x) \to u_F(x),$$

then

$$W_0(x,\xi) \to W_F(x,\xi) = W_0(-\xi, x). \tag{2.14}$$

Notice that this can be interpreted as a 90° rotation of the WDF in the (x,ξ) plane. The associated physical process is a Fraunhofer diffraction.

Next we consider the passage through a thin lens, from a plane immediately before the lens to a plane immediately behind it.

$$u_0(x) \to u_0(x)\exp(-i\pi x^2/\lambda f) = u_L(x). \tag{2.15}$$

The focal length f may be different from f_1.

$$f = f_1/Q, \tag{2.16}$$

$$W_0(x,\xi) \to W_0(x,\xi + Qx) = W_L(x,\xi). \tag{2.17}$$

The WDF underwent a shearing in the ξ direction.

Now we investigate what happens if the light propagates in free space over a distance Z. This process can be described best in the spatial-frequency domain of the signal:

$$\tilde{u}_0(\nu) \to \tilde{u}_0(\nu)\exp(-i\pi\lambda z\nu^2) = \tilde{u}_Z(\nu). \tag{2.18}$$

We relate the propagation length Z to the standard focal length f_1, and we rewrite relation (2.18) with the new variable ξ:

$$Z = Rf_1, \tag{2.19}$$

$$u_F(\xi) \to u_F(\xi)\exp(-i\pi R\xi^2/\lambda f_1) = u_Z(\xi), \tag{2.20}$$

$$W_0(x, \xi) \to W_Z(x, \xi) = W_0(x - R\xi, \xi). \tag{2.21}$$

Apparently, propagation in free space causes an x shearing of the WDF.

In Wigner space the three operations Fourier transform, passage through a lens, and free-space propagation do not change the values of W. Only the coordinates (x, ξ) are modified by affine transformations, as expressed in relations (2.22) and (2.23).

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} x' \\ \xi' \end{bmatrix}, \tag{2.22}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}; \quad \begin{bmatrix} 1 & 0 \\ Q & 1 \end{bmatrix}; \quad \begin{bmatrix} 1 & -R \\ 0 & 1 \end{bmatrix}. \tag{2.23}$$

3. DEFINITION OF THE FRACTIONAL FOURIER TRANSFORM

In accordance with Eqs. (2.2) and (2.11), we introduce the following notation for the standard Fourier transform:

$$F^{(1)}[u_0(x)] = \tilde{u}_0(\xi/\lambda f_1) = u_F(\xi). \tag{3.1}$$

Now we generalize by introducing a fractional degree P, which may be a noninteger.

$$F^{(P)}[u_0(x)] = u_P. \tag{3.2}$$

For the special case $P = 1$ we know that the associated WDF is rotated by 90° [relation (2.14)]:

$$W_0(x, \xi) \to W_0(-\xi, x) = W_1(x, \xi). \tag{3.3}$$

For defining a fractional Fourier transform with a continuous degree P we specify the coordinate transformation (2.22) of the WDF as

$$\begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} x\cos(\phi) - \xi\sin(\phi) \\ \xi\cos(\phi) + x\sin(\phi) \end{bmatrix}. \tag{3.4}$$

The fractional degree is then interpreted as an angle:

$$\phi = P\pi/2. \tag{3.5}$$

In other words, we have defined the fractional Fourier transform [Eq. (3.2)] indirectly by what happens to the associated WDF, i.e., a rotation.

$$\begin{aligned} W_0(x, \xi) &\to W_P(x, \xi) \\ &= W_0(x\cos\phi - \xi\sin\phi, \xi\cos\phi + x\sin\phi). \end{aligned} \tag{3.6}$$

This definition can be executed by the following three-step algorithm:

$$u_0(x) \to W_0(x, \xi), \tag{3.7}$$

$$W_0(x, \xi) \to W_P(x, \xi), \tag{3.8}$$

$$W_P(x, \xi) \to u_P(x) = F^{(P)}[u_0(x)]. \tag{3.9}$$

All three steps can be performed as optical experiments. The first step converts a one-dimensional signal into a two-dimensional Wigner display.[4] The second step is merely a rotation of the Wigner display. The third step [Eq. (2.7)] is a one-dimensional Fourier transform, which can be executed by a setup containing a spherical and a cylindrical lens. All three steps can be combined into one optical setup, which, however, would be somewhat involved. Hence we will look for better solutions in Section 6 below. The algorithm [relations (3.7)–(3.9)] can be compacted into one equation that is somewhat clumsy. It has the undesirable property of being bilinear in $u(x)$:

$$\begin{aligned} u_P(2x) = &\iint u_0(x\cos\phi - \xi\sin\phi + x'/2) \\ &\times u_0^*(\xi\cos\phi + x\sin\phi - x'/2) \\ &\times \exp(..)\mathrm{d}x'\mathrm{d}\xi, \\ \exp(..) = &\exp\{(2\pi i/\lambda f_1)[2\xi x - x'(\xi\cos\phi + x\sin\phi)]\}. \end{aligned} \tag{3.10}$$

The bilinearity can be implemented with classical optics by letting the light pass twice through the signal mask $u_0(x)$. Nevertheless, neither the algorithm nor the suggested experiment is very elegant. But we did arrive at a working definition for the fractional Fourier transform. Two simple experiments for implementing our definition will evolve in Section 6. However, two preliminary steps are needed: image rotation (Section 4) and Wigner rotation (Section 5).

4. IMAGE ROTATION BASED ON IMAGE SHEARING

Rotation of an image can be described best in polar coordinates:

$$I(r, \theta) \to I(r, \theta + \phi). \tag{4.1}$$

Usually the process of rotation is reduced to two successive flips [relations (4.2) and (4.3)], because flips are easy to implement optically.

$$\theta_0 \to \theta_0 + 2(\alpha - \theta_0) = -\theta_0 + 2\alpha, \tag{4.2}$$

$$\theta_1 \to \theta_1 + 2(\beta - \theta_1) = \theta_0 + 2(\beta - \alpha). \tag{4.3}$$

Apparently a rotation by $2(\beta - \alpha)$ has been accomplished. This method is commonly implemented with Dove prisms. It would be the preferred method if the WDF existed in two dimensions. Here I propose another way for rotating an image. This alternative method can be extended to four dimensions. It consists of three successive shearing operations (as illustrated in Fig. 1):

$$(x_0, y_0) \to (x_0 - Ay_0, y_0) = (x_1, y_1), \tag{4.4}$$

$$(x_1, y_1) \to (x_1, y_1 + Bx_1) = (x_2, y_2), \tag{4.4'}$$

$$(x_2, y_2) \to (x_2 - Cy_2, y_2) = (x_3, y_3). \tag{4.4''}$$

These three steps can be expressed together as

$$x_3 = x_0(1 - BC) - y_0(A + C - ABC), \tag{4.5}$$

$$y_3 = y_0(1 - AB) + x_0B. \tag{4.5'}$$

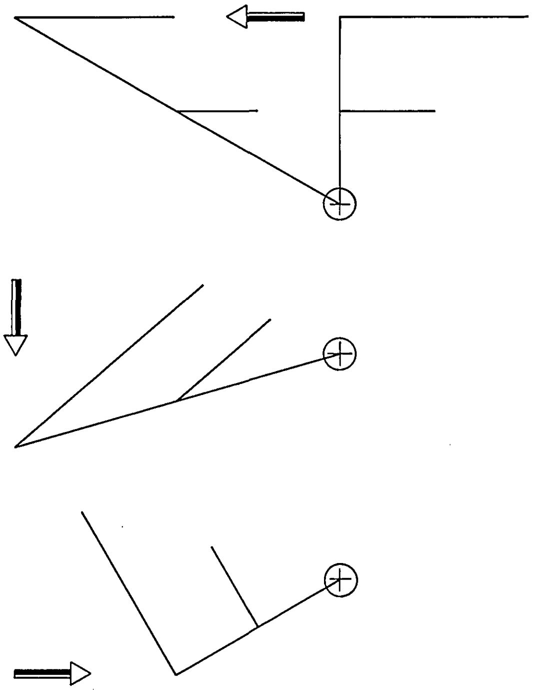

Fig. 1. Image rotation, generated by three shearing processes (top to bottom): to the left, down, to the right.

Equation (4.5) describes a coordinate rotation from (x_0, y_0) to (x_3, y_3) if the coefficients satisfy

$$B = \sin(\phi), \qquad A = C = \tan(\phi/2). \tag{4.6}$$

The process of relation (4.4) consisted of x shear, y shear, and again x shear. An equivalent result can be achieved by the following three steps: y shear, x shear, and y shear.

$$(x_0, y_0) \to (x_0, y_0 + ax_0) = (x_1, y_1), \tag{4.7}$$

$$(x_1, y_1) \to (x_1 - by_1, y_1) = (x_2, y_2), \tag{4.7'}$$

$$(x_2, y_2) \to (x_2, y_2 + cx_2) = (x_3, y_3). \tag{4.7''}$$

It follows that

$$x_3 = x_0(1 - ab) - by_0, \tag{4.8}$$

$$y_3 = y_0(1 - bc) + x_0(a + c - abc). \tag{4.8'}$$

The conditions for this to constitute a rotation are

$$b = \sin(\phi), \qquad a = c = \tan(\phi/2). \tag{4.9}$$

Shearing of two-dimensional images can be implemented optically.[5] We will not pursue this approach to image rotation since we need to know only Eqs. (4.6) and (4.9) in what follows.

5. ROTATION OF THE WIGNER DISTRIBUTION FUNCTION

We know how to synthesize image rotation by image shearing. Now we want to synthesize the rotation of the WDF by shearing the WDF three times properly. What is wanted is

$$\begin{aligned} W_0(x, \xi) &\to W_P(x, \xi) \\ &= W_0(x \cos\phi - \xi \sin\phi, \xi \cos\phi + x \sin\phi), \end{aligned} \tag{5.1}$$

with

$$\phi = P\pi/2. \tag{5.2}$$

We know specifically [from Eq. (2.22) and relation (2.18)] that

$$W_Z(x, \xi) = W_0(x - R\xi, \xi), \tag{5.3}$$

$$W_L(x, \xi) = W_0(x, \xi + Qx), \tag{5.4}$$

$$Z = Rf_1; \qquad f = f_1/Q. \tag{5.5}$$

There exist essentially two ways to synthesize a WDF rotation, which we will call type I (RQR) and type II (QRQ):

$$\text{I:} \quad R = \tan(\phi/2), \qquad Q = \sin(\phi), \tag{5.6}$$

$$\text{II:} \quad Q = \tan(\phi/2), \qquad R = \sin(\phi). \tag{5.7}$$

Type I is isomorphic with the image rotation of Eqs. (4.6), and type II is related to Eqs. (4.9).

Before we discuss the experimental implementation of the two approaches, I want to point out that our strategy can be generalized easily to two-dimensional signals, i.e., images $u(x, y)$, whose WDFs are four dimensional.

$$\begin{aligned} W(x, y, \xi, \eta) &= \iint u(x + x'/2, y + y'/2) \\ &\quad \times u^*(-, -)\exp[..]dx'dy', \\ [..] &= -(2\pi i/\lambda f_1)(x'\xi + y'\eta). \end{aligned} \tag{5.8}$$

Propagation of $u(x, y)$ in free space in the Z direction is described by multiplication of $\tilde{u}(\xi, \eta)$, with

$$\exp[-(i\pi/\lambda f_1)R(\xi^2 + \eta^2)]. \tag{5.9}$$

Transition through a lens means that $u(x, y)$ is multiplied by

$$\exp[-(i\pi/\lambda f_1)Q(x^2 + y^2)]. \tag{5.10}$$

The associated WDF shearing processes are

$$W_Z = W_0(x - R\xi, y - R\eta, \xi, \eta), \tag{5.11}$$

$$W_L = W_0(x, y, \xi + Qx, \eta + Qy). \tag{5.12}$$

The 4×4 matrix, which rotates the coordinates (x, ξ, y, η), is

$$\begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 & 0 \\ \sin(\phi) & \cos(\phi) & 0 & 0 \\ 0 & 0 & \cos(\phi) & -\sin(\phi) \\ 0 & 0 & \sin(\phi) & \cos(\phi) \end{bmatrix}. \tag{5.13}$$

This particular rotation matrix is not the most general one in four-dimensional space. But it is appropriate, if the fractional Fourier operator is supposed to have the same fractional degrees for both x and y:

$$P_x = P_y. \tag{5.14}$$

A generalization would be possible, if so desired. For example, an anamorphic or an astigmatic lens would lead to $Q_x \neq Q_y$. And a birefringent medium would be characterized by $R_x \neq R_y$. The inversion from $W(x, y, \xi, \eta)$ to $u(x, y)$

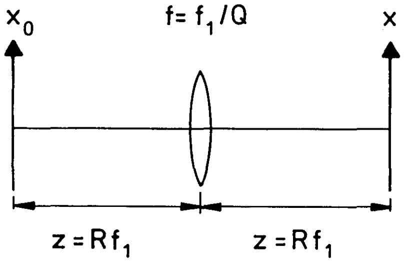

Fig. 2. Setup (type I) for performing a fractional Fourier transform. Parameters R and Q determine the degree P and the angle $\phi = P\pi/2$. The signals are two dimensional; the lens is spherical.

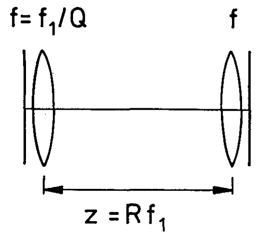

Fig. 3. Setup (type II) for performing a fractional Fourier transform.

is straightforward:

$$\iint W \exp[(4\pi i/\lambda f_1)(x\xi + y\eta)]\mathrm{d}\xi\mathrm{d}\eta = u(2x, 2y)u^*(0, 0). \tag{5.15}$$

6. TWO OPTICAL SYSTEMS FOR PERFORMING THE FRACTIONAL FOURIER TRANSFORM

By now we know how to shear the WDF [Eqs. (5.3)–(5.5)] and how to synthesize a WDF rotation in three shearing steps [Eqs. (5.6) and (5.7)]. The optical processes, which correspond to WDF shearing, are simply free-space propagation over a distance $Z = Rf_1$ and passage through a lens with a focal length $f = f_1/Q$. System I (RQR) is shown in Fig. 2 and is defined in Eq. (5.6), with the parameters Q and R established in Eq. (5.7).

We now want to derive explicitly the fractional Fourier operator $F^{(P)}$ by analyzing the wave propagation through system II (Fig. 3). The complex amplitude $u_0(x_0)$ is the input of the fractional Fourier transform:

$$Z = 0 - : u_0(x_0), \tag{6.1}$$

$$Z = 0 + : u_0(x_0)\exp[-(i\pi/\lambda f_1)Qx_0^2] = u_1(x_0), \tag{6.2}$$

$$Z = Rf_1 - 0 : \tilde{u}_1(\xi)\exp[-(i\pi/\lambda f_1)R\xi^2] = \tilde{u}_2(\xi), \tag{6.3}$$

$$Z = Rf_1 + 0 : u_2 \exp[-(i\pi/\lambda f_1)Qx^2] = u_P(x). \tag{6.4}$$

The minus and the plus associated with the Z coordinates refer to before the lens and behind the lens, respectively. Equations (6.2) and (6.3) are connected by a Fourier transform:

$$\tilde{u}_1(\xi) = \int u_1(x_0)\exp[-(2\pi i/\lambda f_1)x_0\xi]\mathrm{d}x_0. \tag{6.5}$$

Similarly, between Eqs. (6.3) and (6.4),

$$u_2(x) = \int \tilde{u}_2(\xi)\exp[(2\pi i/\lambda f_1)x\xi]\mathrm{d}\xi. \tag{6.6}$$

By tying these equations together and by using Eq. (5.7), we obtain

$$\begin{aligned} u_P(x) &= F^{(P)}[u_0(x_0)] \\ &= \int u_0(x_0)\exp[(i\pi/\lambda f_1 \tan\phi)(x_0^2 + x^2)] \\ &\quad \times \exp[-(2\pi i/\lambda f_1 \sin\phi)xx_0]\mathrm{d}x_0. \end{aligned} \tag{6.7}$$

The angle ϕ and the fractional degree P are connected by

$$\phi = P\pi/2. \tag{6.8}$$

Equation (6.7) holds for system I (RQR) also.

It is valid to ask whether the explicit form of the fractional Fourier transform [Eq. (6.7)] is indeed equivalent to the earlier indirect result [Eq. (3.10)]. The equivalence can be shown by computing the WDF of the explicit transform integral. The result is a rotated WDF, as in relation (3.6).

7. FINAL COMMENTS

After finishing this project, my colleagues and I realized that our results are not quite as new as we thought. The first such realization was that we ourselves had actually discovered a generalized Fourier transform with noninteger degree $P = 4/3$. That result[6] was a special case from a study on self-Fourier objects.[7] A much more significant piece of history was brought to our attention by a colleague from the computer science department. In 1980 Namias published a paper, "The fractional order Fourier transform and its application to quantum mechanics,"[8] which has been extended more recently by McBride and Kerr.[9] Their results seem to be compatible with ours when similar questions are addressed, but the motivation and the conceptual applications are quite different. Dickinson and Steiglitz defined the fractional version of the discrete Fourier transform.[10] Our results are compatible with theirs but go beyond as a result of a different context. Dickinson and Steiglitz' motivation is the optimal separation of two signals, for example for cryptography with the parameter P as the key.

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