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Agreement of fractional Fourier optics with the Huygens–Fresnel principle

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Abstract

According to the Huygens–Fresnel principle, the electric field transfer by diffraction from an emitter to a receiver can be split into two diffraction phenomena: from the emitter to an arbitrary intermediate surface and from this surface to the receiver. Expressing field transfers by the usual Fresnel-type integrals complies with this principle. Since a diffraction phenomenon can be mathematically represented by a fractional order Fourier transform, the Huygens–Fresnel principle should accompany the composition of fractional transforms. We show that diffraction representations by fractional Fourier transforms that have been proposed by some authors are not in conformity with the Huygens–Fresnel principle. We also explain how another representation preserves the Huygens–Fresnel principle and should provide a deeper physical insight in the mathematical expression of a diffraction phenomenon, in the framework of a scalar theory.

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1. Introduction

The Huygens–Fresnel principle plays a basic role in electromagnetic wave propagation and it is expected that every appropriate mathematical representation of light propagation by diffraction should be in accordance with it. In particular, this should hold true in fractional Fourier optics, a theory based on the use of fractional order Fourier transforms [1,2] in the scalar theory of diffraction [3] or light propagation [4].

In the framework of a scalar diffraction theory, the Huygens–Fresnel principle states that the (electric) field amplitude transfer from an emitter $\mathscr{A}$ to a receiver $\mathscr{B}$ can split into the field amplitude transfer from $\mathscr{A}$ to $\mathscr{C}$ followed by the field transfer from $\mathscr{C}$ to $\mathscr{B}$, where $\mathscr{C}$ is an arbitrary surface located between $\mathscr{A}$ and $\mathscr{B}$. Generally both emitter $\mathscr{A}$ and receiver $\mathscr{B}$ may be aerial surfaces, on which field amplitudes are defined.

We denote the fractional Fourier transform of order α by $\mathscr{F}_\alpha$ (α is a complex number). In fractional Fourier optics, $\mathscr{F}_\alpha$ is associated with the field amplitude transfer by diffraction by a given distance, after choosing appropriate α in ways that will be discussed later on.

The present paper deals with the consistency of the Huygens–Fresnel principle with the composition law of fractional Fourier transforms and investigates whether associating a fractional order Fourier transform with a diffraction phenomenon preserves this agreement. Let $\mathscr{F}_\alpha$, $\mathscr{F}_{\alpha_1}$ and $\mathscr{F}_{\alpha_2}$ be associated with the field transfers from $\mathscr{A}$ to $\mathscr{B}$, from $\mathscr{A}$ to $\mathscr{C}$ and from $\mathscr{C}$ to $\mathscr{B}$, respectively. Fractional Fourier transforms compose according to

$$\mathscr{F}_{\alpha_2} \circ \mathscr{F}_{\alpha_1} = \mathscr{F}_{\alpha_1+\alpha_2}, \tag{1}$$

and agreement with the Huygens–Fresnel principle demands that $\alpha = \alpha_1 + \alpha_2$, a relation that will be used as a guide in our analysis.

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Different ways of associating fractional Fourier transforms with diffraction phenomena have been proposed by several authors. We shall briefly consider some of them [5–9] and show that they are not compatible with the Huygens–Fresnel principle. On the other hand, we have proposed our own associating method [3,10,11] and in perfecting it, great attention has been paid to conformity with the Huygens–Fresnel principle. Indeed, we shall show below that our method preserves the Huygens–Fresnel principle.

In earlier papers [3,10], we have proposed a way of composing two fractional Fourier transforms in order to comply with the Huygens–Fresnel principle. Given an emitter $\mathscr{A}$ and two receivers $\mathscr{B}_1$ and $\mathscr{B}_2$ we can express the field transfers from $\mathscr{A}$ to $\mathscr{B}_1$ and from $\mathscr{A}$ to $\mathscr{B}_2$ by fractional Fourier transforms whose orders are α_1 and α_2. In these publications, we obtained a condition for the field transfer from $\mathscr{B}_1$ to $\mathscr{B}_2$ to be represented by a fractional Fourier transform whose order is $\beta = \alpha_2 - \alpha_1$. However, doing so was restrictive in the sense that the transfer from $\mathscr{B}_1$ to $\mathscr{B}_2$ was described with parameters and scaled variables referring to the emitter radius of curvature: β was defined with respect to $\mathscr{A}$. There is a need for a better accordance with the Huygens–Fresnel principle: all transfers should be represented in an identical form, explicitly, β should be related to $\mathscr{B}_1$ and $\mathscr{B}_2$ as α_1 is related to $\mathscr{A}$ and $\mathscr{B}_1$, and as α_2 is related to $\mathscr{A}$ and $\mathscr{B}_2$.

We shall investigate how our diffraction representation by fractional Fourier transforms [3,10,11] preserves the Huygens–Fresnel principle in a general context: given an emitter $\mathscr{A}$ and a receiver $\mathscr{B}$ at a distance D, we shall look for an intermediate surface $\mathscr{C}$ located at an arbitrary distance D_1 from $\mathscr{A}$, such that $\alpha = \alpha_1 + \alpha_2$, where $\mathscr{F}_\alpha$ will be associated with the field transfer from $\mathscr{A}$ to $\mathscr{B}$, $\mathscr{F}_{\alpha_1}$ with the field transfer from $\mathscr{A}$ to $\mathscr{C}$ and $\mathscr{F}_{\alpha_2}$ with the field transfer from $\mathscr{C}$ to $\mathscr{B}$. Definitions of α_1, α_2 and α will be equivalent for every transfer (in other words: α will be defined with respect to $\mathscr{A}$ and $\mathscr{B}$ as α_1 will be defined with respect to $\mathscr{A}$ and $\mathscr{C}$, and α_2 to $\mathscr{C}$ and $\mathscr{B}$).

2. The Huygens–Fresnel principle and an integral expression of a diffraction phenomenon

2.1. Integral expression of a diffraction phenomenon

As done in earlier papers [3,10,11], we use spherical emitters and receivers in order to include quadratic phase factors that appear in integral expressions of diffraction phenomena. (Spherical emitters and receivers are only parts of spheres.)

Let $\mathscr{A}$ be a spherical emitter with a radius of curvature R_A and $\mathscr{B}$ a spherical receiver with radius R_B, located at a distance D (taken from vertex to vertex, see Fig. 1). The emitter is monochromatic (wavelength λ). We use vectorial spatial variables $\boldsymbol{r}$ on $\mathscr{A}$ and $\boldsymbol{s}$ on $\mathscr{B}$. With orthogonal coordinates x and y on $\mathscr{A}$ we have $\boldsymbol{r} = (x,y)$, $r = \|\boldsymbol{r}\| = (x^2 + y^2)^{1/2}$ and we denote $\mathrm{d}\boldsymbol{r} = \mathrm{d}x\,\mathrm{d}y$. The field transfer from $\mathscr{A}$ to $\mathscr{B}$ can be written

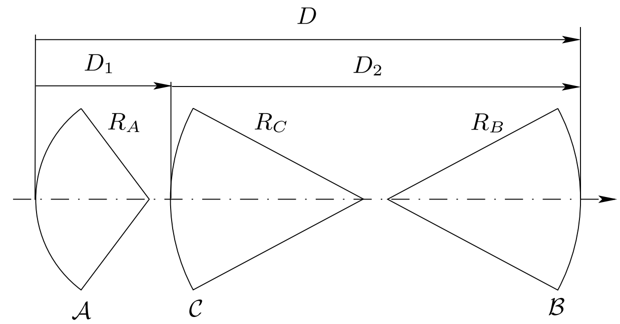

Fig. 1. According to the Huygens–Fresnel principle, the field transfer from the spherical emitter $\mathscr{A}$ to the spherical receiver $\mathscr{B}$ can be split into the field transfer from $\mathscr{A}$ to $\mathscr{C}$ followed by the field transfer from $\mathscr{C}$ to $\mathscr{B}$, where $\mathscr{C}$ is an intermediate sphere.

$$U_\mathrm{B}(\boldsymbol{s}) = \frac{\mathrm{i}}{\lambda D}\exp\left[-\frac{\mathrm{i}\pi}{\lambda}\left(\frac{1}{R_\mathrm{B}}+\frac{1}{D}\right)s^2\right] \times \int_{\mathbb{R}^2}\exp\left[-\frac{\mathrm{i}\pi}{\lambda}\left(\frac{1}{D}-\frac{1}{R_\mathrm{A}}\right)r^2\right]\exp\left[\frac{2\mathrm{i}\pi}{\lambda D}\boldsymbol{s}\cdot\boldsymbol{r}\right]U_\mathrm{A}(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}, \tag{2}$$

where U_A is the field amplitude on $\mathscr{A}$ and U_B on $\mathscr{B}$.

We introduce the function h such that

$$h(\boldsymbol{s},\boldsymbol{r}) = \frac{\mathrm{i}}{\lambda D}\exp\left[-\frac{\mathrm{i}\pi}{\lambda}\left(\frac{\|\boldsymbol{s}-\boldsymbol{r}\|^2}{D}+\frac{s^2}{R_\mathrm{B}}-\frac{r^2}{R_\mathrm{A}}\right)\right], \tag{3}$$

and Eq. (2) can be written

$$U_\mathrm{B}(\boldsymbol{s}) = \int_{\mathbb{R}^2} h(\boldsymbol{s},\boldsymbol{r})U_\mathrm{A}(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}. \tag{4}$$

Planar emitters and receivers are spherical emitters or receivers with infinite curvature radii: if $\mathscr{A}$ and $\mathscr{B}$ are planes, we set $1/R_\mathrm{A}$ and $1/R_\mathrm{B}$ to 0 in Eq. (2) and obtain the diffraction formula between two planes. Then Eq. (4) reduces to a convolution product: $U_\mathrm{B} = h * U_\mathrm{A}$.

2.2. Agreement with the Huygens–Fresnel principle

We consider an emitter $\mathscr{A}$ and a receiver $\mathscr{B}$ at a distance D. Let $\mathscr{C}$ be an intermediate sphere, with a radius of curvature R_C, located between $\mathscr{A}$ and $\mathscr{B}$ (Fig. 1). The distance from $\mathscr{A}$ to $\mathscr{C}$ is D_1 and the distance from $\mathscr{C}$ to $\mathscr{B}$ is D_2. We have $D = D_1 + D_2$. We take coordinates $\boldsymbol{r}'$ on $\mathscr{C}$. (More generally $\mathscr{C}$ can be located before $\mathscr{A}$ (it is a virtual receiver) or after $\mathscr{B}$. Then D_1 or D_2 may be negative.) The field transfer from $\mathscr{A}$ to $\mathscr{C}$ is expressed as

$$U_\mathrm{C}(\boldsymbol{r}') = \int_{\mathbb{R}^2} h_1(\boldsymbol{r}',\boldsymbol{r})U_\mathrm{A}(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}, \tag{5}$$

where

$$h_1(\boldsymbol{r}',\boldsymbol{r}) = \frac{\mathrm{i}}{\lambda D_1}\exp\left[-\frac{\mathrm{i}\pi}{\lambda}\left(\frac{\|\boldsymbol{r}'-\boldsymbol{r}\|^2}{D_1}+\frac{r'^2}{R_\mathrm{C}}-\frac{r^2}{R_\mathrm{A}}\right)\right]. \tag{6}$$

The field transfer from $\mathscr{C}$ to $\mathscr{B}$ is expressed as

$$U_{\mathrm{B}}(\boldsymbol{s}) = \int_{\mathbb{R}^2} h_2(\boldsymbol{s},\boldsymbol{r}')U_{\mathrm{C}}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}', \tag{7}$$

where

$$h_2(\boldsymbol{s},\boldsymbol{r}') = \frac{\mathrm{i}}{\lambda D_2}\exp\left[-\frac{\mathrm{i}\pi}{\lambda}\left(\frac{\|\boldsymbol{s}-\boldsymbol{r}'\|^2}{D_2}+\frac{s^2}{R_{\mathrm{B}}}-\frac{r'^2}{R_{\mathrm{C}}}\right)\right]. \tag{8}$$

The Huygens–Fresnel principle states that the field transfer from $\mathscr{A}$ to $\mathscr{B}$ can be seen as the composition of the field transfer from $\mathscr{A}$ to $\mathscr{C}$ and the field transfer from $\mathscr{C}$ to $\mathscr{B}$. Thus we must have

$$\int_{\mathbb{R}^2} h(\boldsymbol{s},\boldsymbol{r})U_{\mathrm{A}}(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r} = U_{\mathrm{B}}(\boldsymbol{s}) = \int_{\mathbb{R}^2} h_2(\boldsymbol{s},\boldsymbol{r}')\int_{\mathbb{R}^2} h_1(\boldsymbol{r}',\boldsymbol{r})U_{\mathrm{A}}(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'. \tag{9}$$

Indeed, we prove in Appendix A that Eq. (9) holds true if h, h_1 and h_2 are given by Eqs. (3), (6) and (8). We conclude that the integral expression of a diffraction phenomenon, as given by Eq. (4), is in accordance with the Huygens–Fresnel principle. The result holds true if $\mathscr{A}$ and $\mathscr{B}$ are planes.

3. Fractional Fourier transform representation and the Huygens–Fresnel principle

3.1. Remarks on the problem under consideration

The fractional Fourier transform of order α of the two-dimensional function f is defined by [1,2]

$$\mathscr{F}_\alpha[f](\boldsymbol{\sigma}) = \frac{\mathrm{i}e^{-\mathrm{i}\alpha}}{\sin\alpha}\exp[-\mathrm{i}\pi\sigma^2\cot\alpha]\int_{\mathbb{R}^2}\exp[-\mathrm{i}\pi\rho^2\cot\alpha] \times\exp\left[\frac{2\mathrm{i}\pi\boldsymbol{\sigma}\cdot\boldsymbol{\rho}}{\sin\alpha}\right]f(\boldsymbol{\rho})\,\mathrm{d}\boldsymbol{\rho}. \tag{10}$$

Eq. (10) is very similar to Eq. (2), so that it is possible to express Eq. (2) as a fractional order Fourier transform [3,10,11]. In so doing, we may ask if the result remains compatible with the Huygens–Fresnel principle. The problem is not trivial: one could believe that since the diffraction integrals are in accordance with the Huygens–Fresnel principle, expressing these integrals as fractional Fourier transforms would necessarily preserve their compatibility with the principle. This is not true. The reason is that several methods have been proposed for choosing a fractional Fourier transform representation of a diffraction phenomenon or light propagation through optical elements. Of course fractional Fourier transform definitions slightly different from Eq. (10) are sometimes used, but they do not make any difference for the purpose of this paper, as far as they comply with the fractional transform composition rule on which our analysis is based. Indeed, the above mentioned methods differ both in the way they choose the order of the transform associated with a diffraction phenomenon at a given distance of observation and in the way they define scaled variables that allow us to write Eq. (2) in the form of Eq. (10). In other words: all proposed methods are not equivalent to each other. The fact that diffraction is expressed by integrals complying with the Huygens–Fresnel principle does not ensure that fractional Fourier transform representations do.

As explained in Section 1, our guide in deciding whether a fractional Fourier transform representation of diffraction is in accordance with the Huygens–Fresnel principle consists in inspecting whether the equation $\alpha = \alpha_1 + \alpha_2$ holds true, where $\mathscr{F}_\alpha$, $\mathscr{F}_{\alpha_1}$ and $\mathscr{F}_{\alpha_2}$ are associated with the field transfers from $\mathscr{A}$ to $\mathscr{B}$, from $\mathscr{A}$ to $\mathscr{C}$ and from $\mathscr{C}$ to $\mathscr{B}$, respectively (see Fig. 1).

3.2. Some associations that do not preserve the Huygens–Fresnel principle

The main point in associating a fractional Fourier transform with a diffraction phenomenon is finding the order α associated to the propagation distance D (besides providing appropriate scaled variables). Some authors [5,6,8,9] propose to choose α such that

$$\tan a\alpha = bD, \tag{11}$$

where a and b are constant (or their choice reduces to Eq. (11) in the end). It should be clear that such a choice is not consistent with the Huygens–Fresnel principle: it leads to $\tan a\alpha_1 = bD_1$ and $\tan a\alpha_2 = bD_2$ and since in general

$$\tan(a\alpha_1 + a\alpha_2) \neq \tan a\alpha_1 + \tan a\alpha_2, \tag{12}$$

we cannot obtain $\alpha = \alpha_1 + \alpha_2$, although we have $D = D_1 + D_2$. We note that although it does not preserve the Huygens–Fresnel principle, the choice of Eq. (11) may be useful in managing some numerical simulations [9]. Nevertheless it is not acceptable from a physical point of view, since it violates the Huygens–Fresnel principle.

We remark that some authors [4,7] manage with two ways of associating a fractional Fourier transform with light propagation: one way reduces to Eq. (11) and the other [7] is close to our method that will be explained later on.

We now comment some results obtained in cascading optical systems [12,13]: the fractional Fourier transform associated with a plane-to-plane propagation phenomenon splits into two fractional transforms if one sets lenses where the composition should take place. Various optical setups can be used for this purpose [14,15]. By so doing, the authors introduce suitable quadratic phase factors and can play on transform orders and scaled variables. We interpret the use of lenses as implicitly recognizing that the ordinary composition of fractional transforms cannot be obtained so easily when associating these transforms with light propagation by diffraction. We also remark that the proposed solution does not provide agreement with the Huygens–Fresnel principle in the sense that the transform order of the cascaded system depends on the transform orders associated with the two successive setups:

transform orders are not identically defined and the variable scaling process has to be adapted for each transfer [13].

4. A fractional Fourier transform representation of diffraction in accordance with the Huygens–Fresnel principle

4.1. Diffraction representation

In order to express Eq. (2) as a fractional order Fourier transform we proceed as follows. Let L be such that

$$L=\frac{(D+R_{\mathrm{B}})(R_{\mathrm{A}}-D)}{D(D-R_{\mathrm{A}}+R_{\mathrm{B}})}. \tag{13}$$

If $L \geqslant 0$, the field transfer from $\mathscr{A}$ to $\mathscr{B}$ is said to be a real order transfer [11]. If $L<0$, we have a complex order transfer. For the sake of simplicity, in this paper we consider only real order transfers and define α by

$$\cot^2\alpha=\frac{(D+R_{\mathrm{B}})(R_{\mathrm{A}}-D)}{D(D-R_{\mathrm{A}}+R_{\mathrm{B}})}, \quad -\pi<\alpha<\pi, \quad \alpha D \geqslant 0, \tag{14}$$

the sign of $\cot\alpha$ being that of $R_{\mathrm{A}}D(R_{\mathrm{A}}-D)$. We introduce an auxiliary parameter ε such that

$$\varepsilon^2=\frac{D(R_{\mathrm{B}}+D)}{(R_{\mathrm{A}}-D)(D-R_{\mathrm{A}}+R_{\mathrm{B}})}. \tag{15}$$

For a real transfer obviously $\varepsilon^2 \geqslant 0$, so that ε is a real number and we choose it such that $\varepsilon R_{\mathrm{A}}>0$. We define scaled variables on $\mathscr{A}$ and $\mathscr{B}$ by

$$\boldsymbol{\rho}=\frac{\boldsymbol{r}}{\sqrt{\lambda\varepsilon R_{\mathrm{A}}}}, \tag{16}$$

$$\boldsymbol{\sigma}=\frac{\boldsymbol{s}}{\sqrt{\lambda\varepsilon R_{\mathrm{A}}}}(\cos\alpha+\varepsilon\sin\alpha), \tag{17}$$

and we introduce scaled field amplitudes on $\mathscr{A}$ and $\mathscr{B}$ such that

$$V_{\mathrm{A}}(\boldsymbol{\rho})=U_{\mathrm{A}}\left(\sqrt{\lambda\varepsilon R_{\mathrm{A}}}\boldsymbol{\rho}\right), \tag{18}$$

$$V_{\mathrm{B}}(\boldsymbol{\sigma})=U_{\mathrm{B}}\left(\frac{\sqrt{\lambda\varepsilon R_{\mathrm{A}}}\boldsymbol{\sigma}}{\cos\alpha+\varepsilon\sin\alpha}\right). \tag{19}$$

Eq. (2) then becomes

$$V_{\mathrm{B}}(\boldsymbol{\sigma})=\mathrm{e}^{\mathrm{i}\alpha}(\cos\alpha+\varepsilon\sin\alpha)\mathscr{F}_{\alpha}[V_{\mathrm{A}}](\boldsymbol{\sigma}), \tag{20}$$

which means that the scaled field amplitude on $\mathscr{B}$ is the fractional Fourier transform of order α of the scaled field amplitude on $\mathscr{A}$.

4.2. Accordance with the Huygens–Fresnel principle

We consider the intermediate sphere $\mathscr{C}$ again (see Section 2.2). (For language convenience we still call $\mathscr{C}$ an intermediate surface, although it can be a virtual surface, located before the emitter or after the receiver.) The field transfer from $\mathscr{A}$ to $\mathscr{C}$ can be expressed as a fractional Fourier transform whose order α_1 is defined as α in Eq. (14) by replacing D with D_1 and R_{B} with R_{C}. The parameter ε becomes ε_1. Scaled variables $\boldsymbol{\rho}_1$ on $\mathscr{A}$ and $\boldsymbol{\sigma}_1$ on $\mathscr{C}$ are such that

$$\boldsymbol{\rho}_1=\frac{\boldsymbol{r}}{\sqrt{\lambda\varepsilon_1 R_{\mathrm{A}}}}, \quad \boldsymbol{\sigma}_1=\frac{\boldsymbol{r}'}{\sqrt{\lambda\varepsilon_1 R_{\mathrm{A}}}}(\cos\alpha_1+\varepsilon_1\sin\alpha_1). \tag{21}$$

The scaled field amplitudes $V_{\mathrm{A}}^{(1)}$ on $\mathscr{A}$ and $V_{\mathrm{C}}^{(1)}$ on $\mathscr{C}$ are

$$V_{\mathrm{A}}^{(1)}(\boldsymbol{\rho}_1)=U_{\mathrm{A}}\left(\sqrt{\lambda\varepsilon_1 R_{\mathrm{A}}}\boldsymbol{\rho}_1\right), \tag{22}$$

$$V_{\mathrm{C}}^{(1)}(\boldsymbol{\sigma}_1)=U_{\mathrm{C}}\left(\frac{\sqrt{\lambda\varepsilon_1 R_{\mathrm{A}}}\boldsymbol{\sigma}_1}{\cos\alpha_1+\varepsilon_1\sin\alpha_1}\right). \tag{23}$$

We obtain

$$V_{\mathrm{C}}^{(1)}(\boldsymbol{\sigma}_1)=\mathrm{e}^{\mathrm{i}\alpha_1}(\cos\alpha_1+\varepsilon_1\sin\alpha_1)\mathscr{F}_{\alpha_1}[V_{\mathrm{A}}^{(1)}](\boldsymbol{\sigma}_1), \tag{24}$$

which is equivalent to Eq. (20).

The field transfer from $\mathscr{C}$ to $\mathscr{B}$ can also be represented by a fractional Fourier transform. We change R_{A} into R_{C} and D into D_2 in Eqs. (14) and (15) and obtain parameters α_2 and ε_2. Scaled variables on $\mathscr{C}$ and $\mathscr{B}$ are

$$\boldsymbol{\rho}_2=\frac{\boldsymbol{r}'}{\sqrt{\lambda\varepsilon_2 R_{\mathrm{C}}}}, \quad \boldsymbol{\sigma}_2=\frac{\boldsymbol{s}}{\sqrt{\lambda\varepsilon_2 R_{\mathrm{C}}}}(\cos\alpha_2+\varepsilon_2\sin\alpha_2). \tag{25}$$

The scaled field amplitudes are

$$V_{\mathrm{C}}^{(2)}(\boldsymbol{\rho}_2)=U_{\mathrm{C}}\left(\sqrt{\lambda\varepsilon_2 R_{\mathrm{C}}}\boldsymbol{\rho}_2\right), \tag{26}$$

$$V_{\mathrm{B}}^{(2)}(\boldsymbol{\sigma}_2)=U_{\mathrm{B}}\left(\frac{\sqrt{\lambda\varepsilon_2 R_{\mathrm{C}}}\boldsymbol{\sigma}_2}{\cos\alpha_2+\varepsilon_2\sin\alpha_2}\right), \tag{27}$$

so that the field transfer from $\mathscr{C}$ to $\mathscr{B}$ becomes

$$V_{\mathrm{B}}^{(2)}(\boldsymbol{\sigma}_2)=\mathrm{e}^{\mathrm{i}\alpha_2}(\cos\alpha_2+\varepsilon_2\sin\alpha_2)\mathscr{F}_{\alpha_2}[V_{\mathrm{C}}^{(2)}](\boldsymbol{\sigma}_2). \tag{28}$$

We remark that $\mathscr{F}_{\alpha_1}$ and $\mathscr{F}_{\alpha_2}$ are defined with respect to $\mathscr{A}$ and $\mathscr{C}$ and to $\mathscr{C}$ and $\mathscr{B}$ as $\mathscr{F}_{\alpha}$ is defined with respect to $\mathscr{A}$ and $\mathscr{B}$.

According to the Huygens–Fresnel principle the field transfer from $\mathscr{A}$ to $\mathscr{B}$ can be decomposed into two successive transfers: from $\mathscr{A}$ to $\mathscr{C}$ and from $\mathscr{C}$ to $\mathscr{B}$. Expressed in terms of fractional Fourier transforms this means that we should compose $\mathscr{F}_{\alpha_1}$ and $\mathscr{F}_{\alpha_2}$ and obtain $\mathscr{F}_{\alpha}$ with $\alpha=\alpha_1+\alpha_2$. We shall prove that this happens for a suitable choice of R_{C}.

We split the analysis in six steps. Proofs are given in Appendix A.

Step i: To compose the two transfers associated with $\mathscr{F}_{\alpha_1}$ and $\mathscr{F}_{\alpha_2}$ and obtain the transfer associated with $\mathscr{F}_{\alpha}$ as a result, we should have

$$\cos\alpha+\varepsilon\sin\alpha=(\cos\alpha_1+\varepsilon_1\sin\alpha_1)(\cos\alpha_2+\varepsilon_2\sin\alpha_2). \tag{29}$$

We have (see Appendix A for a proof)

$$(\cos\alpha+\varepsilon\sin\alpha)^2=\frac{R_{\mathrm{A}}(R_{\mathrm{B}}+D)}{R_{\mathrm{B}}(R_{\mathrm{A}}-D)}. \tag{30}$$

Equations equivalent to Eq. (30) can be written for both α_1 and α_2, and Eq. (29) leads to

$$\frac{R_{\mathrm{A}}(R_{\mathrm{B}}+D)}{R_{\mathrm{B}}(R_{\mathrm{A}}-D)}=\frac{R_{\mathrm{A}}(R_{\mathrm{C}}+D_1)}{R_{\mathrm{C}}(R_{\mathrm{A}}-D_1)}\frac{R_{\mathrm{C}}(R_{\mathrm{B}}+D_2)}{R_{\mathrm{B}}(R_{\mathrm{C}}-D_2)}. \tag{31}$$

We solve Eq. (31) in R_C and we obtain

$$R_C = \frac{D_1(R_B + D_2)(R_A - D) + D_2(R_B + D)(R_A - D_1)}{D_1(R_A - D) + D_2(D + R_B)}. \tag{32}$$

We conclude that the intermediate sphere $\mathscr{C}$ is perfectly defined.

Step ii: We assume R_C to be defined by Eq. (32). Then (see Appendix A for a proof)

$$\varepsilon_1^2 = \frac{D_1(R_C + D_1)}{(R_A - D_1)(D_1 - R_A + R_C)} = \frac{D(R_B + D)}{(R_A - D)(D - R_A + R_B)} = \varepsilon^2. \tag{33}$$

Since ε is real, we conclude that ε_1 is real; moreover, since $\varepsilon_1 R_A > 0$ and $\varepsilon R_A > 0$, we obtain $\varepsilon_1 = \varepsilon$.

Step iii: We show (see Appendix A for a proof)

$$\varepsilon_2^2 = \left[\varepsilon \frac{D_1(R_A - D) + D_2(R_B + D)}{D(D + R_B)}\right]^2, \tag{34}$$

and we conclude that ε_2^2 is positive, so that the field transfer from $\mathscr{C}$ to $\mathscr{B}$ is a real order transfer.

Step iv: Composition of $\mathscr{F}_{\alpha_1}$ and $\mathscr{F}_{\alpha_2}$ (taken in this order) makes sense if the output scaled variables of $\mathscr{F}_{\alpha_1}$ (that is, $\boldsymbol{\sigma}_1$) are input scaled variables of $\mathscr{F}_{\alpha_2}$ (that is, $\boldsymbol{\rho}_2$). We have

$$\begin{aligned}\frac{(\cos\alpha_1 + \varepsilon_1 \sin\alpha_1)^2}{\varepsilon_1 R_A} &= \frac{R_C + D_1}{\varepsilon R_C(R_A - D_1)} \\ &= \frac{D(D + R_B)}{\varepsilon R_C[D_1(R_A - D) + D_2(D + R_B)]} = \frac{1}{\varepsilon_2 R_C}.\end{aligned} \tag{35}$$

We conclude that $\boldsymbol{\sigma}_1 = \boldsymbol{\rho}_2$, so that $\mathscr{F}_{\alpha_2} \circ \mathscr{F}_{\alpha_1}$ makes sense.

Step v: We have $\alpha = \alpha_1 + \alpha_2$ (see Appendix A for a proof).

Step vi: We have $\boldsymbol{\rho} = \boldsymbol{\rho}_1$ and $\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}$, so that $\mathscr{F}_{\alpha_2}[\mathscr{F}_{\alpha_1}[V_A]](\boldsymbol{\sigma}) = \mathscr{F}_{\alpha}[V_A](\boldsymbol{\sigma})$.

We come to the following conclusion. Let the transfer from $\mathscr{A}$ to $\mathscr{B}$ be represented by a fractional Fourier transform whose order is α, according to Section 4.1. At a distance D_1 from $\mathscr{A}$, stands a unique spherical surface $\mathscr{C}$ such that the field transfer from $\mathscr{A}$ to $\mathscr{C}$ is represented by $\mathscr{F}_{\alpha_1}$, the field transfer from $\mathscr{C}$ to $\mathscr{B}$ by $\mathscr{F}_{\alpha_2}$, and $\alpha = \alpha_1 + \alpha_2$. (Both α_1 and α_2 are chosen according to Section 4.1.) The radius of curvature of $\mathscr{C}$ is given by Eq. (32).

4.3. Remarks

(1) Let $\mathscr{A}$ be an emitter and $\mathscr{B}$ a receiver at a distance D. For every distance D_1 there exists an intermediate receiver $\mathscr{C}$ complying with the Huygens–Fresnel principle. If $D_1 = 0$, Eq. (32) gives $R_C = R_A$, and if $D_1 = D$, it gives $R_C = R_B$.

(2) The radius of curvature R_C is infinite for $D_1 = D_0$ with

$$D_0 = \frac{D(D + R_B)}{2D - R_A + R_B}. \tag{36}$$

We take D_1 as a variable and we associate a receiver $\mathscr{C}(D_1)$ with each value of D_1: its radius is given by Eq. (32). Receivers $\mathscr{C}(D_1)$ form a continuous family of spherical receivers. These receivers can be shown to be equiphase surfaces of the Gaussian beam generated by an optical resonator whose mirrors are $\mathscr{A}$ and $\mathscr{B}$. Eq. (36) is no more than the usual expression for the beam waist localization from a mirror ($D_1 = D_0$) [16].

(3) It may happen that $\mathscr{A}$ and $\mathscr{B}$ are concentric; then $D = R_A - R_B$. Since $D = D_1 + D_2$ we obtain $R_A - D_1 = R_B + D_2$. Eq. (32) gives

$$R_C = \frac{D_1(R_B + D_2)R_B + D_2 R_A(R_A - D_1)}{D_1 R_B + D_2 R_A} = R_A - D_1. \tag{37}$$

The relation $R_C = R_A - D_1$ means that $\mathscr{A}$ and $\mathscr{C}$ are concentric too. If $\mathscr{A}$ and $\mathscr{B}$ are planes, they can be considered as concentric spheres and we conclude that $\mathscr{C}$ is a plane. Indeed, for infinite R_A and R_B, Eq. (32) shows that R_C is infinite.

5. An example

We illustrate the former results by providing a practical example related to Gaussian beams and optical resonators. A Gaussian beam can be described as a series of spherical wave surfaces (not concentric) to which the beam is orthogonal. On each wave surface the field amplitude takes the form $\exp[-r^2/w^2]$ up to a multiplicative factor (including a phase factor that does not depend on $\boldsymbol{r}$ anymore). The parameter w ($w > 0$) defines the beam width on the wave surface (w is minimum on the beam waist). Let $\mathscr{A}$ and $\mathscr{B}$ be two spherical wave surfaces with curvature radii R_A and R_B and let D be the distance from $\mathscr{A}$ to $\mathscr{B}$. The beam width on $\mathscr{A}$ is w_A and it is w_B on $\mathscr{B}$. Surfaces $\mathscr{A}$ and $\mathscr{B}$ may be seen as mirrors of an optical resonator that generates a Gaussian beam. We only consider the fundamental Gaussian mode, but higher order modes can be handled in the same manner.

We use the method and notations of Section 4 to represent the field transfer from $\mathscr{A}$ to $\mathscr{B}$ as a fractional Fourier transform, and introduce appropriate scaled variables and field amplitudes. Since the field amplitudes on $\mathscr{A}$ and $\mathscr{B}$ are described by Gaussian functions, scaled field amplitudes are also Gaussian functions.

For every α, the fractional Fourier transform is such that [1]

$$\mathscr{F}_\alpha[e^{-\pi\rho^2}](\boldsymbol{\sigma}) = e^{-\pi\sigma^2}. \tag{38}$$

We remark that Eq. (38) only holds true for functions $\exp[-\pi\rho^2]$, we mean, not for $\exp[-\pi a\rho^2]$ ($a \neq 1$), since the fractional Fourier transform, unlike the standard Fourier transform, has no simple similarity property [1]. As a consequence we require the scaled field amplitudes on $\mathscr{A}$ and $\mathscr{B}$ to be $e^{-\pi\rho^2}$ and $e^{-\pi\sigma^2}$ (up to a multiplicative factor), so that Eq. (38) can be applied, as for example in Eq. (20).

Eqs. (15) and (16) show that the scaled field amplitude associated with $U_A(\boldsymbol{r}) = \exp[-r^2/w_A^2]$ is $V_A(\boldsymbol{\rho}) = \exp[-\pi\rho^2]$ if

$$w_A^4 = \frac{\lambda^2 R_A^2 D(D + R_B)}{\pi^2 (R_A - D)(D - R_A + R_B)}. \tag{39}$$

Eqs. (15), (17) and (30) show that the scaled field amplitude on $\mathscr{B}$ is $V_B(\boldsymbol{\sigma}) = \exp[-\pi\sigma^2]$ if

$$w_B^4 = \frac{\lambda^2 R_B^2 D(R_A - D)}{\pi^2 (D + R_B)(D - R_A + R_B)}. \tag{40}$$

Eqs. (39) and (40) are classical formulae of Gaussian beams or optical resonator theory [16,17]: they provide the fundamental mode widths on the mirrors of an optical resonator, given the distance between the two mirrors and their curvature radii. According to Eq. (30) we have

$$\frac{w_A^2}{w_B^2} = \frac{R_A(R_B + D)}{R_B(R_A - D)} = (\cos\alpha + \varepsilon\sin\alpha)^2, \tag{41}$$

so that Eq. (20) becomes ($\cos\alpha + \varepsilon\sin\alpha$ has been shown to be positive [11])

$$V_B(\boldsymbol{\sigma}) = \frac{w_A}{w_B} e^{i\alpha} \mathscr{F}_\alpha[V_A](\boldsymbol{\sigma}) = \frac{w_A}{w_B} e^{i\alpha} e^{-\pi\sigma^2}. \tag{42}$$

Now we consider the sphere $\mathscr{C}$ at a distance D_1 from $\mathscr{A}$, whose curvature radius is defined by Eq. (32). The field transfer from $\mathscr{A}$ to $\mathscr{C}$ is described as a fractional Fourier transform whose order is α_1. The analysis of Section 4 shows that scaled variables on $\mathscr{A}$ are the same for $\mathscr{F}_\alpha$ and $\mathscr{F}_{\alpha_1}$ so that w_A can be used for the transfer from $\mathscr{A}$ to $\mathscr{B}$ as well as for the transfer from $\mathscr{A}$ to $\mathscr{C}$. We conclude that the field on $\mathscr{C}$ takes the form

$$V_C(\boldsymbol{\rho}_1) = \frac{w_A}{w_C} e^{i\alpha_1} e^{-\pi\rho_1^2}, \tag{43}$$

where w_C is such that

$$\frac{w_A^2}{w_C^2} = \frac{R_A(R_C + D_1)}{R_C(R_A - D_1)} = (\cos\alpha_1 + \varepsilon_1\sin\alpha_1)^2. \tag{44}$$

According to Eq. (43), the scaled field amplitude on $\mathscr{C}$ is Gaussian: we conclude that the field amplitude on $\mathscr{C}$ is also Gaussian and $\mathscr{C}$ is the wave surface of the Gaussian beam at a distance D_1 from $\mathscr{A}$.

We introduce α_2 which is the order of the transform associated to the field transfer from $\mathscr{C}$ to $\mathscr{B}$. According to the analysis of Section 4.2 we have $\alpha = \alpha_1 + \alpha_2$. Scaled variables for this transfer are also scaled variables for the transfer from $\mathscr{C}$ to $\mathscr{B}$. We conclude that w_B can be used for both transfers. Moreover, α_1, α_2 and α are such that Eq. (29) holds and we conclude that w_C as defined by Eq. (44) is such that

$$\frac{w_C^2}{w_B^2} = (\cos\alpha_2 + \varepsilon_2\sin\alpha_2)^2. \tag{45}$$

Then

$$V_B(\boldsymbol{\sigma}) = \frac{w_C}{w_B} e^{i\alpha_2} \mathscr{F}_{\alpha_2}[V_C](\boldsymbol{\sigma}) = \frac{w_C}{w_B} e^{i\alpha_2} \frac{w_A}{w_C} e^{i\alpha_1} e^{-\pi\sigma^2} = \frac{w_A}{w_B} e^{i\alpha} e^{-\pi\sigma^2}, \tag{46}$$

and we obtain Eq. (42) once more. We conclude that the field amplitude on $\mathscr{B}$ can be directly deduced from the field amplitude on $\mathscr{A}$ or can be seen as the composition of two transfers: one from $\mathscr{A}$ to $\mathscr{C}$ and the other from $\mathscr{C}$ to $\mathscr{B}$, in agreement with the Huygens–Fresnel principle. All transfers are expressed in similar ways with respect to distances of propagation and radii of curvature of emitters and receivers.

It seems difficult to deal with the former example by methods that need lenses to adapt the fractional Fourier transform to light propagation [13–15]: we should set a lens in place of our intermediate receiver $\mathscr{C}$ and, in so doing, we would not describe strict free space propagation any more.

6. Conclusion

The composition law of fractional order Fourier transforms is in accordance with the Huygens–Fresnel principle only for a given intermediate sphere between the emitter and the receiver. This fact represents a restriction when compared with the general integral representation of diffraction which preserves the Huygens–Fresnel principle whatever the intermediate sphere is (see Section 2.2). However, it also introduces a physical point of view related to Huygens wavelets. Given a wave surface, the wave surface obtained at a given distance after light wave propagation takes place is not arbitrary: it is unique among many surfaces. This leads us to believe that an adequate diffraction representation by fractional Fourier transforms, such as the one proposed in Section 4, introduces a physical insight that reflects light propagation from an emitter as Huygens wavelets. This holds true if the field amplitude on the emitter is Gaussian, as shown in Section 5.

Fresnel and fractional Fourier transforms are special cases of the linear canonical transform [7,16]. It is an open question to decide whether our method can be stated in such a framework.

Finally we note that there is some interest in computing fractional Fourier transforms for simulation purposes. Sampling the functions may also introduce a limitation in the choice of the orders of the fractional Fourier transforms to be composed [18], as for preserving the Huygens–Fresnel principle, but in a different way.

Appendix A

A.1. Proof of Eq. (9)

We introduce the following functions

$$u_A(\boldsymbol{r}) = \exp\left[\frac{i\pi r^2}{\lambda R_A}\right] U_A(\boldsymbol{r}), \tag{47}$$

$$u_B(\boldsymbol{s}) = \exp\left[\frac{i\pi s^2}{\lambda R_B}\right] U_B(\boldsymbol{s}), \tag{48}$$

$$u_C(\boldsymbol{r}') = \exp\left[\frac{i\pi r'^2}{\lambda R_C}\right] U_C(\boldsymbol{r}'), \tag{49}$$

and

$$g(\boldsymbol{r}) = \frac{\mathrm{i}}{\lambda D} \exp\left[-\frac{\mathrm{i}\pi r^2}{\lambda D}\right], \tag{50}$$

$$g_1(\boldsymbol{r}) = \frac{\mathrm{i}}{\lambda D_1} \exp\left[-\frac{\mathrm{i}\pi r^2}{\lambda D_1}\right], \tag{51}$$

$$g_2(\boldsymbol{r}) = \frac{\mathrm{i}}{\lambda D_2} \exp\left[-\frac{\mathrm{i}\pi r^2}{\lambda D_2}\right], \tag{52}$$

so that Eq. (9) is equivalent to

$$g * u_{\mathrm{A}} = u_{\mathrm{B}} = g_2 * u_{\mathrm{C}} = g_2 * (g_1 * u_{\mathrm{A}}), \tag{53}$$

that is, by using a Fourier transform,

$$\hat{g}\hat{u}_{\mathrm{A}} = \hat{u}_{\mathrm{B}} = \hat{g}_2\hat{u}_{\mathrm{C}} = \hat{g}_2\hat{g}_1\hat{u}_{\mathrm{A}}. \tag{54}$$

The Huygens–Fresnel principle is preserved if $\hat{g} = \hat{g}_2\hat{g}_1$. If a is a real number, we have the Fourier pair

$$\frac{1}{\mathrm{i}a} \exp\left[\frac{\mathrm{i}\pi r^2}{a}\right] \rightleftharpoons \exp[-\mathrm{i}\pi a F^2], \tag{55}$$

and we obtain

$$\begin{aligned}\hat{g}_2(\boldsymbol{F})\hat{g}_1(\boldsymbol{F}) &= \exp[\mathrm{i}\pi\lambda D_2 F^2]\exp[\mathrm{i}\pi\lambda D_1 F^2] \\ &= \exp[\mathrm{i}\pi\lambda D F^2] = \hat{g}(\boldsymbol{F}),\end{aligned} \tag{56}$$

and the proof is complete.

A.2. Proof of Eq. (30)

We define $\mu = D/R_{\mathrm{A}}$, and from Eqs. (14) and (15) we obtain

$$\cot^2\alpha = \varepsilon^2 \frac{(R_{\mathrm{A}} - D)^2}{D^2} = \varepsilon^2 \frac{(1-\mu)^2}{\mu^2}, \tag{57}$$

and then

$$\sin^2\alpha = \frac{\mu^2}{\mu^2 + \varepsilon^2(1-\mu)^2}. \tag{58}$$

Since $\varepsilon R_{\mathrm{A}} > 0$ and since $\cot\alpha$ has the sign of $R_{\mathrm{A}} D(R_{\mathrm{A}} - D)$, we deduce from Eq. (57)

$$\cot\alpha = \varepsilon \frac{R_{\mathrm{A}} - D}{D} = \varepsilon \frac{1-\mu}{\mu}. \tag{59}$$

Then

$$\frac{\cos\alpha + \varepsilon\sin\alpha}{\varepsilon\sin\alpha} = \frac{1}{\mu}. \tag{60}$$

We obtain

$$(\cos\alpha + \varepsilon\sin\alpha)^2 = \frac{\varepsilon^2\sin^2\alpha}{\mu^2} = \frac{\varepsilon^2}{\mu^2 + \varepsilon^2(1-\mu)^2} = \frac{R_{\mathrm{A}}(R_{\mathrm{B}} + D)}{R_{\mathrm{B}}(R_{\mathrm{A}} - D)}. \tag{61}$$

A.3. Proof of Eq. (33)

According to Eq. (15) we have

$$\varepsilon_1^2 = \frac{D_1(R_{\mathrm{C}} + D_1)}{(R_{\mathrm{A}} - D_1)(D_1 - R_{\mathrm{A}} + R_{\mathrm{C}})}. \tag{62}$$

We introduce

$$\Delta = D_1(R_{\mathrm{A}} - D) + D_2(R_{\mathrm{B}} + D). \tag{63}$$

Then

$$(D_1 + R_{\mathrm{C}})\Delta = D(D + R_{\mathrm{B}})(R_{\mathrm{A}} - D_1), \tag{64}$$

and

$$(D_1 - R_{\mathrm{A}} + R_{\mathrm{C}})\Delta = D_1(R_{\mathrm{A}} - D)(D - R_{\mathrm{A}} + R_{\mathrm{B}}), \tag{65}$$

so that

$$\begin{aligned}\varepsilon_1^2 &= \frac{D_1(D_1 + R_{\mathrm{C}})\Delta}{(R_{\mathrm{A}} - D_1)(D_1 - R_{\mathrm{A}} + R_{\mathrm{C}})\Delta} \\ &= \frac{D_1 D(D + R_{\mathrm{B}})(R_{\mathrm{A}} - D_1)}{D_1(R_{\mathrm{A}} - D_1)(R_{\mathrm{A}} - D)(D - R_{\mathrm{A}} + R_{\mathrm{B}})} \\ &= \frac{D(D + R_{\mathrm{B}})}{(R_{\mathrm{A}} - D)(D - R_{\mathrm{A}} + R_{\mathrm{B}})} = \varepsilon^2.\end{aligned} \tag{66}$$

A.4. Proof of Eq. (34)

According to Eq. (15), ε_2 is such that

$$\varepsilon_2^2 = \frac{D_2(R_{\mathrm{B}} + D_2)}{(R_{\mathrm{C}} - D_2)(D_2 - R_{\mathrm{C}} + R_{\mathrm{B}})}. \tag{67}$$

We calculate

$$(D - D_1 - R_{\mathrm{C}} + R_{\mathrm{B}})\Delta = (D + R_{\mathrm{B}})(D - D_1)(D - R_{\mathrm{A}} + R_{\mathrm{B}}), \tag{68}$$

and

$$(R_{\mathrm{C}} - D + D_1)\Delta = D(D - D_1 + R_{\mathrm{B}})(R_{\mathrm{A}} - D), \tag{69}$$

so that

$$\begin{aligned}\varepsilon_2^2 &= \frac{(D - D_1)(D - D_1 + R_{\mathrm{B}})\Delta^2}{D(R_{\mathrm{A}} - D)(D - D_1 + R_{\mathrm{B}})(D + R_{\mathrm{B}})(D - D_1)(D - R_{\mathrm{A}} + R_{\mathrm{B}})} \\ &= \frac{\Delta^2}{D(R_{\mathrm{A}} - D)(D + R_{\mathrm{B}})(D - R_{\mathrm{A}} + R_{\mathrm{B}})} \\ &= \frac{D(D + R_{\mathrm{B}})\Delta^2}{(R_{\mathrm{A}} - D)(D - R_{\mathrm{A}} + R_{\mathrm{B}})D^2(D + R_{\mathrm{B}})^2} \\ &= \varepsilon^2\left[\frac{D_1(R_{\mathrm{A}} - D) + D_2(R_{\mathrm{B}} + D)}{D(D + R_{\mathrm{B}})}\right]^2.\end{aligned} \tag{70}$$

A.5. Proof of the relation $\alpha = \alpha_1 + \alpha_2$

We proceed in three steps

(i) Since $\varepsilon_1 = \varepsilon$, we have

$$\cot\alpha_1 = \varepsilon_1 \frac{R_{\mathrm{A}} - D_1}{D_1} = \varepsilon \frac{R_{\mathrm{A}} - D_1}{D_1}. \tag{71}$$

We also have

$$\cot\alpha_2 = \varepsilon_2 \frac{R_{\mathrm{C}} - D_2}{D_2}, \tag{72}$$

and then

$$\cot(\alpha_1+\alpha_2)=\frac{\cot\alpha_1\cot\alpha_2-1}{\cot\alpha_1+\cot\alpha_2}=\frac{\varepsilon_1\varepsilon_2(R_A-D_1)(R_C-D_2)-D_1D_2}{\varepsilon_1D_2(R_A-D_1)+\varepsilon_2D_1(R_C-D_2)}=\frac{\dfrac{\varepsilon^2\Delta(R_A-D_1)(R_C-D_2)}{D(D+R_B)}-D_1D_2}{\varepsilon D_2(R_A-D_1)+\dfrac{\varepsilon\Delta D_1(R_C-D_2)}{D(D+R_B)}}. \tag{73}$$

We write Eq. (73) in the form

$$\varepsilon\cot(\alpha_1+\alpha_2)=\frac{\dfrac{\Delta(R_A-D_1)(R_C-D_2)}{(R_A-D)(D-R_A+R_B)}-D_1D_2}{D_2(R_A-D_1)+\dfrac{\Delta D_1(R_C-D_2)}{D(D+R_B)}}=\frac{\varepsilon^2N}{D_2(R_A-D_1)D(D+R_B)+\Delta D_1(R_C-D_2)}, \tag{74}$$

where

$$N=\Delta(R_A-D_1)(R_C-D_2)-D_1D_2(R_A-D)(D-R_A+R_B). \tag{75}$$

We use Eq. (69) and write

$$R_C-D_2=R_C-D+D_1=\frac{D}{\Delta}(D-D_1+R_B)(R_A-D), \tag{76}$$

and Eq. (75) gives

$$\cot(\alpha_1+\alpha_2)=\varepsilon\frac{R_A-D}{D}\frac{N'}{N''}, \tag{77}$$

where

$$N'=D(R_A-D_1)(D+R_B)-D_1D(R_A-D_1)-D_1D_2(D-R_A+R_B), \tag{78}$$

and

$$N''=D(R_A-D_1)(D+R_B)-D_1(R_A-D_1)(D+R_B)+D_1(D-D_1+R_B)(R_A-D). \tag{79}$$

We note that $N'=N''$ and we come to

$$\cot(\alpha_1+\alpha_2)=\varepsilon\frac{R_A-D}{D}=\cot\alpha. \tag{80}$$

(ii) We write Eq. (60) as

$$\sin\alpha=\frac{D}{\varepsilon R_A}(\cos\alpha+\varepsilon\sin\alpha), \tag{81}$$

and we deduce from Eq. (59)

$$\cos\alpha=\frac{R_A-D}{R_A}(\cos\alpha+\varepsilon\sin\alpha). \tag{82}$$

Relations equivalent to Eqs. (81) and (82) can be written for α_1 by changing D into D_1 and ε into ε_1, and for α_2 by changing D into D_2, R_A into R_C and ε into ε_2: they provide $\sin\alpha_1$, $\cos\alpha_1$, $\sin\alpha_2$ and $\cos\alpha_2$. Since sphere $\mathscr{C}$ has been chosen so that Eq. (29) holds true, we can write

$$\sin(\alpha_1+\alpha_2)=\sin\alpha_1\cos\alpha_2+\sin\alpha_2\cos\alpha_1=\left(\frac{D_1}{\varepsilon_1R_A}\cdot\frac{R_C-D_2}{R_C}+\frac{D_2}{\varepsilon_2R_C}\cdot\frac{R_A-D_1}{R_A}\right)\times(\cos\alpha+\varepsilon\sin\alpha). \tag{83}$$

Since $\varepsilon_1=\varepsilon$, Eq. (35) leads us to

$$\frac{D_1}{\varepsilon_1R_A}\cdot\frac{R_C-D_2}{R_C}+\frac{D_2}{\varepsilon_2R_C}\cdot\frac{R_A-D_1}{R_A}=\frac{1}{\varepsilon R_AR_C}[D_1(R_C-D_2)+D_2(R_C+D_1)]=\frac{D}{\varepsilon R_A}, \tag{84}$$

so that Eq. (83) becomes

$$\sin(\alpha_1+\alpha_2)=\frac{D}{\varepsilon R_A}(\cos\alpha+\varepsilon\sin\alpha)=\sin\alpha. \tag{85}$$

(iii) Since α, α_1 and α_2 belong to $]-\pi,\pi[$, we conclude from Eqs. (80) and (85) that $\alpha=\alpha_1+\alpha_2$, and the proof is complete.

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