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Some mathematical properties of superoscillations

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Abstract

In this paper, we give a possible mathematical setting for superoscillations. We define the set of superoscillation in terms of the uniform convergence of functions on such a set and study the problem of the approximation of a function by superoscillating functions.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The recent introduction of superoscillatory functions [2, 4–6, 10, 14] has demonstrated that a superposition of small Fourier components, with a bounded Fourier spectrum $|k| < 1$, can nevertheless result in a shift by an arbitrarily large a , well outside the spectrum:

$$\sum_j c_j e^{ik_j x} \rightarrow e^{iax}. \quad (1)$$

Consider a particular example

$$f(x) = \left(\cos\left(\frac{2\pi x}{n}\right) + ia \sin\left(\frac{2\pi x}{n}\right) \right)^n = \left(\frac{1+a}{2} e^{i\frac{2\pi x}{n}} + \frac{1-a}{2} e^{-i\frac{2\pi x}{n}} \right)^n, \quad (2)$$

where $a > 1$, $a \in \mathbb{R}$. If we perform a binomial expansion of $f(x)$, we see that the smallest wavelength in the expansion is 1. However, around $|x| < \sqrt{n}$, $f(x)$ can be approximated as $f(x) \approx e^{i2\pi ax}$, that is, with a wavelength much shorter than 1, seemingly a violation of the Fourier theorem. This surprising phenomenon is very general and holds for a wide range of functions and coefficients.

The superoscillation literature has been growing rapidly. For example, it is known that regions of superoscillations are typical in random fields [13]. It is believed that part of the key to the superoscillatory phenomenon are the extremely rapid oscillations in the coefficients c_j . It is also known that superoscillations are always found in regions of relatively low intensity. Indeed, the regions of superoscillations are created at the expense of having the

function grow exponentially in other regions. It would therefore be natural to conclude that the superoscillations would be quickly ‘overtaken’ by tails coming from the exponential regions and would thus be short-lived. However, it has been shown that superoscillations are remarkably robust [7] and can last for a surprisingly long time. From the perspective of communication theory, it has been shown that this relationship is also related to a tradeoff between the signal-to-noise ratio and bandwidth [11], making it easier to engineer superoscillatory signals [12, 34]. These theoretical developments have led to the practical applications of superoscillations to situations which were previously probed with evanescent waves as occurs in the superresolution of very fine features. Superoscillations do not require a media substrate (in contrast to evanescent waves) and can therefore penetrate much deeper into the media than evanescent waves. This has been used to beat the diffraction limit [7].

Many physical and mathematical questions concerning superoscillations remain unanswered. For example, it is not known how to optimize superoscillations with Fourier integrals. In this paper we discuss a number of mathematical issues related to superoscillations. Throughout this paper, we study detailed properties of the superoscillations in the sequence of functions

$$F_n(x) := \left(\cos\left(\frac{x}{nL}\right) + ia \sin\left(\frac{x}{nL}\right) \right)^n. \quad (3)$$

Let \mathbb{R} be the set of real numbers and \mathbb{N} be the set of natural numbers including zero. Let L and $a \in \mathbb{R}$ be such that $L > 0$ and $a > 1$ and let $n \in \mathbb{N}$. While we consider this form, we argue that our results are general. We also discuss the case of a function $F_n(A)$ of a self-adjoint operator A in a Hilbert space \mathcal{H} .

The plan of the paper is as follows. Since superoscillations are, at first blush, very counter-intuitive, we briefly review in section 2 the physics that originally led to their discovery. This should prepare the reader for an analysis of a refinement of the conditions necessary for superoscillations. In section 3, we give a simple mathematical explanation of superoscillating phenomena in terms of the Fourier and the Taylor coefficients of a function. In section 4, we formulate the phenomenon of superoscillating functions in terms of uniform convergence and we study the behavior of F_n showing that the sequence F_n has a superoscillation property on every compact set in \mathbb{R} . In section 5, we replace x in equation (3) by a self-adjoint operator A using the von Neumann spectral theorem and study the convergence in a Hilbert space. In section 6, we prove that this approximation is possible for some families of functions in the space of rapidly decreasing functions.

2. Review of superoscillations and intuition leading to their discovery

The original insights which eventually led to superoscillations started with the observation by Aharonov, Bergmann and Lebowitz [1] that, as a result of the uncertainty principle, the initial boundary condition of a quantum mechanical system can be selected independently of the final boundary conditions. Subsequently, it was demonstrated by Aharonov, Albert and Vaidman (AAV) [2, 3, 29–33] that if non-disturbing measurements are performed on such pre- and post-selected systems, then strange outcomes will be obtained during the intermediate time. Such outcomes depend on both the pre- and post-selections, can lie outside the allowed eigenvalue spectrum, and are related to superoscillations. This was subsequently developed as the notion of superoscillation [4] and by Berry as the concept of super-Fourier [5].

Traditionally, it was believed that if a measurement interaction is weakened so that there is no disturbance on the system, then no information will be obtained. However, it has been shown that information can be obtained even though not a single particle (in an ensemble) was disturbed [16].

Consider a general theorem for any vector (state) in a Hilbert space.

Theorem 2.1. $\hat{A}|\psi\rangle = \langle\hat{A}\rangle|\psi\rangle + \Delta A|\psi_\perp\rangle$, where $\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$, $|\psi\rangle$ is any vector in a Hilbert space, $\Delta A^2 = \langle\psi|(\hat{A} - \langle\hat{A}\rangle)^2|\psi\rangle$ and $|\psi_\perp\rangle$ is a vector (state) in the perpendicular Hilbert space such that $\langle\psi|\psi_\perp\rangle = 0$.

Proof. Multiplying the left by $\langle\psi|$ yields the first term; evaluating $|(A - \langle A\rangle)|\psi\rangle|^2 = \Delta A^2$ yields the second. \square

Now, the average of any operator $\langle\hat{A}\rangle \equiv \langle\Psi|\hat{A}|\Psi\rangle$ can be measured as the ‘eigenvalue’ of a single ‘collective operator,’ $\hat{A}^{(N)} \equiv \frac{1}{N} \sum_{i=1}^N \hat{A}_i$ without causing a disturbance (with \hat{A}_i being the same operator \hat{A} acting on the i th particle). To see this, we apply theorem 2.1 to the N -particle product state $|\Psi^{(N)}\rangle = |\psi\rangle_1 |\psi\rangle_2 \dots |\psi\rangle_N$, with all particles being in the same state $|\psi\rangle$. We see that

$$\hat{A}^{(N)}|\Psi^{(N)}\rangle = \frac{1}{N} \left[N\langle\hat{A}\rangle|\Psi^{(N)}\rangle + \Delta A \sum_i |\Psi_\perp^{(N)}(i)\rangle \right], \quad (4)$$

where $\langle\hat{A}\rangle$ is the average for any one particle and the N states $|\Psi_\perp^{(N)}(i)\rangle = |\psi\rangle_1 |\psi\rangle_2 \dots |\psi_\perp\rangle_i \dots |\psi\rangle_N$ are mutually orthogonal. With a normalized state, $|\Psi_\perp^{(N)}\rangle = \sum_i \frac{1}{\sqrt{N}} |\Psi_\perp^{(N)}(i)\rangle$, the last term of equation (4) is $\frac{\Delta A}{\sqrt{N}} |\Psi_\perp^{(N)}\rangle$ and $|\frac{\Delta A}{\sqrt{N}} |\Psi_\perp^{(N)}\rangle|^2 \propto \frac{1}{N}$. The probability that measuring \hat{A}_i/N changes the state of the i th system is proportional to $1/N^2$, and therefore the probability that it changes the state of any system is proportional to $1/N$. Thus, as $N \rightarrow \infty$, $|\Psi^{(N)}\rangle$ becomes an eigenstate of $\hat{A}^{(N)}$ with a value $\langle\hat{A}\rangle$ and not even a single particle has been disturbed.

To actually make a measurement of an observable such as $\hat{A}^{(N)}$, we switch on an interaction $H_{\text{int}} = \lambda g(t) \hat{Q}_{\text{md}} \hat{A}^{(N)}$, where \hat{Q}_{md} is an observable of the measuring device (i.e. position), λ is a coupling constant which determines the strength of the measurement and $g(t)$ is a normalized time profile $\int g(t) dt = 1$ which determines the duration of the measurement (setting $\hbar = 1$). We fix $\Delta P_{\text{md}} = 1$ for the distribution in the momentum \hat{P}_{md} (i.e. the pointer) which is conjugate to \hat{Q}_{md} . We can then take $\lambda \gg 1$, in order to distinguish the shift $\lambda\langle\hat{A}\rangle$ from the width. In addition, fixing $\lambda \ll \sqrt{N}$ along with $|\hat{A}_i| < 1$ ensures that the measurement does not shift any particle into an orthogonal state. While the coupling to any individual member of the ensemble is reduced by $\frac{1}{N}$ and therefore the probability that a measurement will disturb any member of the ensemble approaches zero as $\frac{1}{N}$, nevertheless, information about the average is obtained.

By adding a post-selection to these ordinary—yet weakened—von Neumann measurements, the measuring device will register a weak value [2]:

$$\hat{A}_w = \frac{\langle\Psi_{\text{fin}}|\hat{A}|\Psi_{\text{in}}\rangle}{\langle\Psi_{\text{fin}}|\Psi_{\text{in}}\rangle}, \quad (5)$$

with $|\Psi_{\text{in}}\rangle$ and $|\Psi_{\text{fin}}\rangle$ being the initial and final (post-selected) states. The weak value, A_w , is an unusual quantity and is not in general an eigenvalue of \hat{A} . We have used such limited disturbance measurements to explore many paradoxes (see e.g. [19, 25, 18, 22, 23]). Equation (5) can also be motivated by inserting a complete set of states $\{|\Psi_{\text{fin}}\rangle_j\}$ into $\langle\hat{A}\rangle$:

$$\langle\hat{A}\rangle = \langle\Psi_{\text{in}}|\hat{A}|\Psi_{\text{in}}\rangle = \sum_j |\langle\Psi_{\text{fin}}|\Psi_{\text{in}}\rangle_j|^2 \underbrace{\frac{\langle\Psi_{\text{fin}}|\hat{A}|\Psi_{\text{in}}\rangle_j}{\langle\Psi_{\text{fin}}|\Psi_{\text{in}}\rangle_j}}_{A_w^j \equiv \text{weak value}}, \quad (6)$$

with $|\Psi_{\text{fin}}\rangle_j$ being the states corresponding to the outcome of a final ideal measurement on the system (i.e. the post-selection). The average $\langle\hat{A}\rangle$ over all post-selections j is thus constructed

out of pre- and post-selected sub-ensembles in which the weak value (A_w^i) is multiplied by a probability to obtain the particular post-selection $|\Psi_{\text{fin}}\rangle_j$. While this suggests that weak values become less likely as they go further outside the eigenvalue spectrum, it has also recently been shown that such eccentric weak values (and thus superoscillations) can be quite common (if one averages over all pre- and post-selections) [8, 9, 21].

While AAV called such measurements ‘weak measurements’ (after their non-disturbing nature) [15], these measurements can be quite precise. Let us consider a physical implementation of equation (3). Consider a large number N of spin-1/2 particles with pre- and post-selections $|\hat{S}_z = \frac{N}{2}\rangle = \prod_{j=1}^N |\uparrow_z\rangle_j$ and $|\hat{S}_x = \frac{N}{2}\rangle = \prod_{j=1}^N |\uparrow_x\rangle_j$, respectively. One may measure the magnetic field with an error of \sqrt{N} , while not disturbing more than \sqrt{N} of the spins. By way of example, one may consider measuring the spin in a direction $\xi = 45^\circ$ relative to the x - z plane during $t \in [t_{\text{in}}, t_{\text{fin}}]$ using such weak measurements. This can be modeled by a collective observable $\hat{S}_\xi^{(N)} \equiv \frac{1}{N} \sum_{i=1}^N \left\{ \frac{\hat{S}_x + \hat{S}_z}{\sqrt{2}} \right\}$. In this regime, $\hat{S}_z^{(N)}$ and $\hat{S}_x^{(N)}$ can both be measured without ‘disturbing’ the pre- and post-selections (since they effectively commute). AAV therefore predicted that the weak measurement of $\hat{S}_{45}^{(N)}$ will yield the weak value:

$$\hat{S}_{45}^{(N)}{}_w = \frac{\prod_{k=1}^N \langle \uparrow_z | \{ \hat{S}_z^{(N)} + \hat{S}_x^{(N)} \} | \prod_{j=1}^N |\uparrow_x\rangle_j}{\sqrt{2}(\langle \uparrow_z | \uparrow_x \rangle)^N} = \frac{\frac{N}{2} + \frac{N}{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}N \pm O(\sqrt{N}), \quad (7)$$

i.e. a value completely outside the spectrum of the spin operator. The possible values for $\hat{S}_{45}^{(N)}$ extend only from $-\frac{N}{2}$ to $\frac{N}{2}$, while the weak measurement registers a result $\sqrt{2}$ times bigger than the maximum allowed value.

We can see the phenomenon of superoscillation in this example if we focus on the measuring device rather than the system. How can a superposition of shifts in the pointer of the measuring device by amounts within the eigenvalue spectrum $[-\frac{N}{2}, \frac{N}{2}]$ results in a shift of the pointer that is arbitrarily far outside this spectrum (e.g. $\sqrt{2}N/2$)? The answer is that the pointer states of the measuring device interfere constructively around the ‘impossible’ value and destructively for all other values. This is a superoscillation in the *Fourier transform* of the pointer basis of the measuring device. To be more precise, the final state of the measuring device is

$$|\Phi_{\text{fin}}^{\text{MD}}\rangle = \prod_{j=1}^N \langle \uparrow_z | \exp \left\{ \frac{\lambda}{N} \hat{Q}_{\text{md}} \sum_{k=1}^N \hat{S}_\xi^k \right\} | \prod_{i=1}^N |\uparrow_x\rangle_i | \Phi_{\text{in}}^{\text{MD}} \rangle \quad (8)$$

$$= [\langle \uparrow_z | \uparrow_x \rangle]^N \left\{ \cos \frac{\lambda \hat{Q}_{\text{md}}}{N} - i \alpha_w \sin \frac{\lambda \hat{Q}_{\text{md}}}{N} \right\}^N |\Phi_{\text{in}}^{\text{MD}}\rangle \quad (9)$$

$$= \underbrace{\left\{ 1 - \frac{\lambda^2 (\hat{Q}_{\text{md}})^2}{N^2} - \frac{i \lambda \alpha_w \hat{Q}_{\text{md}}}{N} \right\}^N}_{\equiv \psi(Q_{\text{md}})} |\Phi_{\text{in}}^{\text{MD}}\rangle \approx e^{i \lambda \alpha_w \hat{Q}_{\text{md}}} |\Phi_{\text{in}}^{\text{MD}}\rangle, \quad (10)$$

where we have substituted the weak value $\alpha_w \equiv (\hat{S}_\xi)_w = \frac{\langle \uparrow_z | \hat{S}_\xi | \uparrow_x \rangle}{\langle \uparrow_z | \uparrow_x \rangle}$. When projected onto the pointer P_{md} , we see it shifted by the weak value $\hat{S}_{45}^{(N)} = \frac{\sqrt{2}}{2}N \pm O(\sqrt{N})$. One can see from equation (9) that we have derived the same general form introduced in section 1 (equation (2)) which we use repeatedly throughout this paper.

Alternatively, we can also view the expression in the brackets of equation (10) (i.e. $\psi(Q_{\text{md}})$) in a very different way, by performing a binomial expansion:

$$\psi(Q_{\text{md}}) = \sum_{n=0}^N c_n \exp \left\{ \frac{i \lambda \hat{Q}_{\text{md}} (2n - N)}{N} \right\}. \quad (11)$$

Since the exponentials in $\psi(Q_{\text{md}})$ act as translation operators on the wavefunction of the measuring device, we see that this wavefunction is a superposition of waves with small wavenumbers $|k| \leq 1$ ($k = \frac{(2n-N)}{N}$). For a small region (which can include several wavelengths, $2\pi/\alpha_w$, depending on how large one chooses N), $\psi(Q_{\text{md}})$ appears to have a very large momentum, since α_w (from equation (8)) can be arbitrarily large, i.e. the superoscillation phenomenon.

3. An example of a superoscillating function

As the brief review in the previous section indicated, the interference phenomena responsible for superoscillations were originally discovered in the context of quantum waves, though many aspects can also be mimicked by classical waves. In this section we study equation (3), an important example of a superoscillating function. In the following, we will denote by $\binom{n}{j} := \frac{n!}{j!(n-j)!}$ Newton's binomial coefficients.

We collect in the following remark some elementary properties of the sequence in (3).

Remark 3.1. Consider the sequence in (3). Then,

- (1) For every $x \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} F_n(x) = e^{ia\frac{x}{L}}.$$

- (2) Functions $F_n(x)$ can be written in terms of their Fourier coefficients $C_j(n, a)$ as

$$F_n(x) = \sum_{j=0}^n C_j(n, a) e^{i(1-2j/n)\frac{x}{L}},$$

where

$$C_j(n, a) := \frac{(-1)^j}{2^n} \binom{n}{j} (a+1)^{n-j} (a-1)^j.$$

- (3) For every $p \in \mathbb{N}$, we have the following relation:

$$F_n^{(p)}(0) = \sum_{j=0}^n C_j(n, a) [i(1-2j/n)/L]^p$$

between the Taylor and the Fourier coefficients of (3).

The previous remark explains the mathematical behavior of superoscillations in terms of the Taylor and Fourier coefficients. In fact,

- For every $a > 1$ and for every $n \in \mathbb{N}$, a direct computation of F'_n written in the form of (3) gives $F'_n(0) = \frac{ia}{L}$.
- By point (3) in the previous remark, we have $F'_n(0) = \sum_{j=0}^n C_j(n, a) (1 - \frac{2j}{n}) \frac{i}{L}$, where $C_j(n, a)$ are the Fourier coefficients that depend on a .
- As a consequence, we obtain $\frac{ia}{L} = \sum_{j=0}^n C_j(n, a) [i(1-2j/n)\frac{1}{L}]$.

We observe that $F_n(x) \approx e^{\frac{ia}{L}x}$, as $x \rightarrow 0$, even if $F_n(x) = \sum_{j=0}^n C_j(n, a) e^{ix(n-2j)\frac{1}{L}}$ has frequencies that do not depend on a and are smaller than a (which can be arbitrarily large). This is a consequence of the fact that the Taylor coefficients contain local information, while the Fourier coefficients contain global information on the function.

Remark 3.2. Observe that taking the derivative of $F_n(x)$ yields

$$F'_n(x) = \frac{1}{L} \left(\cos\left(\frac{x}{nL}\right) + ia \sin\left(\frac{x}{nL}\right) \right)^{n-1} \left(-\sin\left(\frac{x}{nL}\right) + ia \cos\left(\frac{x}{nL}\right) \right),$$

so that

$$F'_n(x) = \frac{1}{L} F_n(x) \frac{-\sin\left(\frac{x}{nL}\right) + ia \cos\left(\frac{x}{nL}\right)}{\cos\left(\frac{x}{nL}\right) + ia \sin\left(\frac{x}{nL}\right)}.$$

Denoting by $F(x) := \lim_{n \rightarrow \infty} F_n(x)$, which exists and is finite, we obtain from the above relation

$$F'(x) = \frac{ia}{L} F(x).$$

We now integrate and obtain

$$F(x) = C e^{iax/L}$$

and since $F_n(0) = 1$ for every $n \in \mathbb{N}$, we obtain

$$F(x) = e^{iax/L}.$$

4. The superoscillation set

We point out that the material in this section is based on a precise definition of superoscillation phenomenon in terms of the uniform convergence of sequences of functions. Here, we follow a different approach with respect to the one in section 2 of [7] where superoscillations are not studied in terms of the uniform convergence of functions. Moreover, our setting is suitable for superoscillating functions $F_n(x)$ that have a limit for $n \rightarrow \infty$, for every $x \in \mathbb{R}$. In [7], the authors treat a different case by describing superoscillations with wavenumbers different from a , in the region away from the origin when n is large but finite. Consequently, they consider the superoscillating function

$$G_n(x) := (\cos x + ia \sin x)^n$$

that converges only at $x = 0$, so that our strategy does not apply.

The precise definition of superoscillation set can be formulated in terms of the uniform convergence of the functions F_n to a limit function in a suitable interval $I_{so} \subseteq \mathbb{R}$.

Definition 4.1. Let $I \subseteq \mathbb{R}$ be an open interval. Let $g_n : I \rightarrow \mathbb{C}$ be a sequence of functions and let $M > 0$. Let $g_\alpha(x) = e^{i\alpha x}$ for some $\alpha \in \mathbb{R}$. We say that $I_{so} := \{x \in \mathbb{R} : |x| \leq M\}$ is the superoscillation set for g_n if $\forall \varepsilon > 0$ and $\forall x \in I_{so}$, there exists N_ε in \mathbb{N} such that $|g_n(x) - g_\alpha(x)| < \varepsilon$ for all $n > N_\varepsilon$.

We now show that the sequence $F_n(x)$ does not converge uniformly on \mathbb{R} , but it converges uniformly on every compact set in \mathbb{R} .

Proposition 4.2. The sequence $F_n(x)$ converges to $e^{ia\frac{x}{L}}$ for all $x \in \mathbb{R}$, but it does not converge uniformly.

Proof. For the uniform convergence we have to show that

$$\sup_{x \in \mathbb{R}} |F_n(x) - e^{ia\frac{x}{L}}| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Observe that if $x = 0$ we have $F_n(0) - e^0 = 0$, but if we consider the points $x_n = j\pi nL$ for $j \in \mathbb{Z} \setminus \{0\} = \{\pm 1, \pm 2, \dots\}$ we have

$$|F_n(x_n) - e^{ia\frac{x_n}{L}}| = |(\pm 1)^n - e^{ia\frac{x_n}{L}}| \not\rightarrow 0 \quad \text{as } n \rightarrow \infty$$

if $a \in \mathbb{R} \setminus (\mathbb{Z} \setminus \{0\})$, so the convergence cannot be uniform. \square

Theorem 4.3. *Let $M > 0$ be a fixed real number. Then, for every x such that $|x| \leq M$ the sequence $F_n(x)$ converges uniformly to $e^{ia\frac{x}{L}}$.*

Proof. We show that for every x such that $|x| \leq M$, we have

$$\sup_{|x| \leq M} |F_n(x) - e^{ia\frac{x}{L}}| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Let us estimate precisely the modulus of the function $F_n(x) - e^{ia\frac{x}{L}}$. Define the quantities

$$w := F_n(x) \quad \text{and} \quad z = e^{ia\frac{x}{L}},$$

and we observe that the modulus and the angles of w and z are, respectively,

$$\rho_w = \left(\cos^2 \left(\frac{x}{nL} \right) + a^2 \sin^2 \left(\frac{x}{nL} \right) \right)^{n/2}, \quad \theta_w = n \arctan \left(a \tan \left(\frac{x}{nL} \right) \right)$$

and

$$\rho_z = 1, \quad \theta_z = a\frac{x}{L}.$$

By the Carnot theorem for triangles, we have that

$$|w - z|^2 = 1 + \rho_w^2 - 2\rho_w \cos(\theta_w - \theta_z),$$

so we obtain

$$\begin{aligned} |F_n(x) - e^{ia\frac{x}{L}}|^2 &= 1 + \left(\cos^2 \left(\frac{x}{nL} \right) + a^2 \sin^2 \left(\frac{x}{nL} \right) \right)^n \\ &\quad - 2 \left(\cos^2 \left(\frac{x}{nL} \right) + a^2 \sin^2 \left(\frac{x}{nL} \right) \right)^{n/2} \cos \left[n \arctan \left(a \tan \left(\frac{x}{nL} \right) \right) - a\frac{x}{L} \right]. \end{aligned}$$

Now consider the function

$$\begin{aligned} \mathcal{E}_n^2(x, a, L) &:= 1 + \left(\cos^2 \left(\frac{x}{nL} \right) + a^2 \sin^2 \left(\frac{x}{nL} \right) \right)^n \\ &\quad - 2 \left(\cos^2 \left(\frac{x}{nL} \right) + a^2 \sin^2 \left(\frac{x}{nL} \right) \right)^{n/2} \cos \left[n \arctan \left(a \tan \left(\frac{x}{nL} \right) \right) - a\frac{x}{L} \right] \end{aligned} \quad (12)$$

and observe that for any x such that $|x| \leq M$ we have

$$\left(\cos^2 \left(\frac{x}{nL} \right) + a^2 \sin^2 \left(\frac{x}{nL} \right) \right)^n \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

and

$$\cos \left[n \arctan \left(a \tan \left(\frac{x}{nL} \right) \right) - a\frac{x}{L} \right] \rightarrow 1 \quad \text{as } n \rightarrow \infty,$$

so $\mathcal{E}_n^2(x, a, L) \rightarrow 0$. We observe that $\mathcal{E}_n^2(x, a, L)$, as a function of x , is continuous on the compact set $[-M, M]$ for any

$$n > \frac{2M}{\pi L}$$

so, by the Weierstrass theorem, $\mathcal{E}_n^2(x, a, L)$ has the absolute maximum. Set

$$\varepsilon(n, a, L) = \max_{x \in [-M, M]} \mathcal{E}_n(x, a, L).$$

It is now easy to see that $\varepsilon(n, a, L) \rightarrow 0$ as $n \rightarrow \infty$. Since

$$\sup_{|x| \leq M} |F_n(x) - e^{ia\frac{x}{L}}| = \varepsilon(n, a, L),$$

we have uniform convergence on every compact set in \mathbb{R} . \square

Remark 4.4. For any fixed $M \in \mathbb{R}$, the set $\{x \in \mathbb{R} : |x| \leq M\}$ is a superoscillation set for F_n .

Since the expression of $\mathcal{E}_n^2(x, a, L)$ in (12) is quite complicated, to give a simpler expression for the error it is useful to replace it by its principal part.

Remark 4.5. Observe that in the proof of theorem 4.3 the term $|F_n(x) - e^{ia\frac{x}{L}}|^2$ can be estimated by formula (12). If we fix the error ε , then the precise dependence on $n \in \mathbb{N}$, $x \in [-M, M]$, $L > 0$, $a > 1$ is given by $\mathcal{E}_n^2(x, a, L) = \varepsilon$. So we can give a first approximation of $\mathcal{E}_n^2(x, a, L)$ in (12) by considering the principal part of the infinitesimal $\mathcal{E}_n^2(x, a, L) \rightarrow 0$ as $n \rightarrow \infty$ for $|x| \leq M$. We look for two constants j and $\beta \in \mathbb{R}^+$ such that

$$\mathcal{E}_n^2(x, a, L) = \beta \left(\frac{x}{nL} \right)^j + o\left(\frac{x}{nL} \right)^j, \quad \text{as } n \rightarrow \infty,$$

so as a first approximation we can choose

$$\mathcal{E}_n^2(x, a, L) \approx \beta \left(\frac{x}{nL} \right)^j \quad \text{as } n \rightarrow \infty.$$

With some computations we obtain

$$\beta \left(\frac{x}{nL} \right)^j = \frac{3}{2}(a^2 - 1) \left(\frac{x}{nL} \right)^2,$$

so we obtain

$$\varepsilon \approx \frac{x}{nL} \sqrt{\frac{3}{2}(a^2 - 1)}.$$

We conclude the section with a final observation about the sequence F_n .

Remark 4.6. Consider a superoscillation set $|x| \leq M$ for the functions

$$F_n(x) := \left(\cos\left(\frac{x}{nL}\right) + ia \sin\left(\frac{x}{nL}\right) \right)^n.$$

(This was derived from the quantum example in section 2, equation (9).) We consider a point x_0 inside the superoscillation set. Let δx be an increment inside the same superoscillation region and such that $x_0 + \delta x$ is outside the superoscillation region. Then we have

$$\begin{aligned}
F_n(x_0 + \delta x) &= \left(\cos\left(\frac{x_0 + \delta x}{nL}\right) + ia \sin\left(\frac{x_0 + \delta x}{nL}\right) \right)^n \\
&= \left(\cos\left(\frac{x_0}{nL}\right) \cos\left(\frac{\delta x}{nL}\right) - \sin\left(\frac{x_0}{nL}\right) \sin\left(\frac{\delta x}{nL}\right) \right. \\
&\quad \left. + ia \left[\sin\left(\frac{x_0}{nL}\right) \cos\left(\frac{\delta x}{nL}\right) + \cos\left(\frac{x_0}{nL}\right) \sin\left(\frac{\delta x}{nL}\right) \right] \right)^n \\
&= \left\{ \cos\left(\frac{\delta x}{nL}\right) \left[\cos\left(\frac{x_0}{nL}\right) + ia \sin\left(\frac{x_0}{nL}\right) \right] \right. \\
&\quad \left. + ia \sin\left(\frac{\delta x}{nL}\right) \left[\frac{i}{a} \sin\left(\frac{x_0}{nL}\right) + \cos\left(\frac{x_0}{nL}\right) \right] \right\}^n \\
&= \left\{ \cos\left(\frac{\delta x}{nL}\right) + ia \sin\left(\frac{\delta x}{nL}\right) \left(\frac{\cos\left(\frac{x_0}{nL}\right) + \frac{i}{a} \sin\left(\frac{x_0}{nL}\right)}{\cos\left(\frac{x_0}{nL}\right) + ia \sin\left(\frac{x_0}{nL}\right)} \right) \right\}^n \\
&\quad \times \left[\cos\left(\frac{x_0}{nL}\right) + ia \sin\left(\frac{x_0}{nL}\right) \right]^n \\
&= \left\{ \cos\left(\frac{\delta x}{nL}\right) + i\tilde{a} \sin\left(\frac{\delta x}{nL}\right) \right\}^n \left[\cos\left(\frac{x_0}{nL}\right) + ia \sin\left(\frac{x_0}{nL}\right) \right]^n,
\end{aligned}$$

where

$$\tilde{a} = a \frac{[\cos(\frac{x_0}{nL}) + \frac{i}{a} \sin(\frac{x_0}{nL})][\cos(\frac{x_0}{nL}) - ia \sin(\frac{x_0}{nL})]}{\cos^2(\frac{x_0}{nL}) + a^2 \sin^2(\frac{x_0}{nL})}$$

or

$$\tilde{a} = a \frac{1 - \frac{i}{2} \frac{a^2 - 1}{a} \sin(2\frac{x_0}{nL})}{\cos^2(\frac{x_0}{nL}) + a^2 \sin^2(\frac{x_0}{nL})}.$$

This indicates that the behavior of $F_n(x)$ remains superoscillatory also outside the original set of superoscillation. However, the limit function is now $e^{i\tilde{a}x}$ where, since \tilde{a} is a complex number, the modulus of the limit function now grows as a grows. The amplitude of the superoscillations decreases when a increases. We see that $a = 1$ is a fixed point. When a is large, then we obtain large variations.

5. The sequence $F_n(A)$ of a self-adjoint operator A

Let A be a self-adjoint operator on a Hilbert space \mathcal{H} and let $\mathcal{D}(A)$ be the domain of A . By the spectral theorem every self-adjoint operator A in \mathcal{H} has a unique spectral decomposition E^A , see [24], which satisfies

$$(A\phi, \psi) = \int_{\sigma(A)} \lambda \, dE_{\phi, \psi}^A(\lambda), \quad \text{for all } \phi \in \mathcal{D}(A), \quad \psi \in \mathcal{H}, \quad (13)$$

where $\sigma(A)$ denotes the spectrum of the operator A . In a shorter way, the spectral theorem (13) may be expressed more simply as

$$A = \int_{\sigma(A)} \lambda \, dE^A(\lambda). \quad (14)$$

Consider the function in (3) and write it and function $F(x)$ as

$$F_n(x) = \left(\frac{a+1}{2} e^{\frac{i}{nL}x} + \frac{1-a}{2} e^{-\frac{i}{nL}x} \right)^n, \quad (15)$$

$$F(x) = e^{i\frac{a}{L}x}. \quad (16)$$

Definition 5.1. Let A be a self-adjoint operator in a Hilbert space \mathcal{H} and let F_n and F be the functions in (15) and (16), respectively. By the spectral theorem, we define the operators

$$F_n(A) = \int_{I_{os} \cap \sigma(A)} F_n(\lambda) dE^A(\lambda), \quad F(A) = \int_{I_{os} \cap \sigma(A)} F(\lambda) dE^A(\lambda), \quad (17)$$

where I_{os} is a superoscillation set for F_n .

The following theorem shows in what sense the sequence of operators $F_n(A)$ converges to $F(A)$.

Theorem 5.2. Let A be a self-adjoint operator on a Hilbert space \mathcal{H} and let $F_n(A)$ and $F(A)$ be the operators defined in (17). Then we have

$$\lim_{n \rightarrow \infty} (F_n(A)\phi, \psi) = (F(A)\phi, \psi), \quad \text{for all } \phi \in \mathcal{D}(A), \quad \psi \in \mathcal{H}.$$

Proof. We take the limit

$$\begin{aligned} \lim_{n \rightarrow \infty} F_n(A) &= \lim_{n \rightarrow \infty} \int_{\sigma(A)} F_n(\lambda) dE(\lambda) \\ &= \int_{\sigma(A)} \lim_{n \rightarrow \infty} F_n(\lambda) dE(\lambda) = \int_{\sigma(A)} F(\lambda) dE(\lambda) = F(A), \end{aligned}$$

where we have used the Lebesgue-dominated convergence theorem to pass to the limit under the integral, and where $F(x) = \lim_{n \rightarrow \infty} F_n(x)$. Since $F(x) = e^{i\frac{a}{L}x}$, the limit operator is $F(A) = e^{i\frac{a}{L}A}$. The operator $F(A)$ is unique because the spectral decomposition guarantees that there is a unique spectral measure E^A . \square

Remark 5.3. In particular, if A is the momentum operator $A = -iD_x = \hat{P}$, we have

$$F_n(\hat{P}) = \left(\frac{a+1}{2} e^{\frac{i}{n}L\hat{P}} + \frac{1-a}{2} e^{-\frac{i}{n}L\hat{P}} \right)^n \quad \text{and} \quad F(\hat{P}) = e^{iaL\hat{P}}, \quad (18)$$

and the convergence is as follows:

$$\lim_{n \rightarrow \infty} (F_n(\hat{P})\phi, \psi) = (F(\hat{P})\phi, \psi), \quad \text{for all } \phi \in \mathcal{D}(\hat{P}), \quad \psi \in \mathcal{H}.$$

We now consider the sequence of operators in (18) and write it in the equivalent form

$$F_n(\hat{P}) = \sum_{j=0}^n C_j(n, a) e^{i(1-2j/n)L\hat{P}}.$$

The action of $F_n(\hat{P})$ on functions ψ belonging to suitable function spaces gives a linear combination of the function ψ computed at the points $x + (1 - 2j/n)L$:

$$\phi_n(x) := F_n(\hat{P})\psi(x) = \sum_{j=0}^n C_j(n, a)\psi(x + (1 - 2j/n)L).$$

Moreover, the limit function is given by

$$F(\hat{P})\psi(x) = \psi(x + aL), \quad (19)$$

since $F(\hat{P}) = e^{iaL\hat{P}}$ simply translates the wavefunction $\psi(x)$ by aL . A natural question is to ask in which cases it is possible to approximate the value of a function ψ computed on the point $x + aL$, where a can be arbitrarily large, by suitably combining of the values of ψ computed in points of the form $x + (1 - 2j/n)L$. Consequently, we study the following problem.

Problem 5.4. Fix $a \in \mathbb{R}$ and $n \in \mathbb{N}$ and $\varepsilon > 0$. Determine those functions ψ , belonging to a suitable function space, such that we have

$$|\phi_n(x) - \psi(x + aL)| < \varepsilon, \quad \text{for every } x \in \mathbb{R}.$$

In the next section, we solve this problem for the set of rapidly decreasing functions $\mathcal{S}(\mathbb{R})$.

6. Approximations of functions in $\mathcal{S}(\mathbb{R})$ by superoscillations

In the function space $\mathcal{S}(\mathbb{R})$, the approximation functions ϕ_n are well defined and if $\psi \in \mathcal{S}(\mathbb{R})$ then also its Fourier transform $\hat{\psi}$ belongs to $\mathcal{S}(\mathbb{R})$, see [17] for more details.

Definition 6.1 (Approximating sequence). Let $\psi \in \mathcal{S}(\mathbb{R})$ and $n \in \mathbb{N}$. We call

$$\phi_{\psi,n}(x) := \sum_{j=0}^n C_j(n, a) \psi(x + (1 - 2j/n)L)$$

approximating sequence of ψ .

Proposition 6.2 (Integral representation of the approximating sequence). Suppose that $\psi \in \mathcal{S}(\mathbb{R})$. Then we have

$$\phi_{\psi,n}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} F_n(\lambda) \hat{\psi}(\lambda) e^{i\lambda x} d\lambda.$$

Proof. Taking the Fourier transform of

$$\phi_{\psi,n}(x) = \sum_{j=0}^n C_j(n, a) \psi(x + (1 - 2j/n)L),$$

we obtain

$$\hat{\phi}_{\psi,n}(\lambda) = \sum_{j=0}^n C_j(n, a) e^{i\lambda(1-2j/n)L} \hat{\psi}(\lambda).$$

Now observe that

$$\hat{\phi}_{\psi,n}(\lambda) = F_n(\lambda) \hat{\psi}(\lambda),$$

and taking the anti-Fourier transform we obtain the required result. \square

Remark 6.3. Observe that passing to the limit we obviously have

$$\lim_{n \rightarrow \infty} \phi_{\psi,n}(x) = \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{\mathbb{R}} F_n(\lambda) \hat{\psi}(\lambda) e^{i\lambda x} d\lambda = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ia\lambda L} \hat{\psi}(\lambda) e^{i\lambda x} d\lambda = \psi(x + aL).$$

Definition 6.4. The family of spaces \mathcal{X}_γ depending on a real parameter $\gamma > 0$ is defined by

$$\mathcal{X}_\gamma(\mathbb{R}) := \{u \in \mathcal{S}(\mathbb{R}) : \int_{\mathbb{R}} |\hat{u}(\lambda)| d\lambda \leq \gamma\}.$$

The following result is a direct consequence of proposition 6.2.

Proposition 6.5. *Let us fix $\gamma > 0, a > 1$ and $n \in \mathbb{N}$. Suppose that $\psi \in \mathcal{X}_\gamma(\mathbb{R})$. Then we have*

$$|\phi_{\psi,n}(x) - \psi(x + aL)| \leq (1 + a^n)\gamma.$$

Even though the above inequality is not helpful when $n \rightarrow \infty$ since $a > 1$, it is still interesting because it shows a uniform distance between $\phi_{\psi,n}(x)$ and $\psi(x + aL)$ for all $x \in \mathbb{R}$. It is however helpful for some families of functions as the next example shows.

Example 6.6. It is well known that the family of functions

$$u_\alpha(x) = xe^{-\alpha x^2}$$

belongs to $\mathcal{S}(\mathbb{R})$ for every $\alpha > 0$, see for example [17]. Its Fourier transform is given by

$$\hat{u}_\alpha(\lambda) = -\frac{\lambda i}{2\alpha} \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\lambda^2}{4\alpha}}.$$

Now observe that

$$\gamma = \int_{\mathbb{R}} |\hat{u}_\alpha(\lambda)| d\lambda = 2\sqrt{\frac{\pi}{\alpha}},$$

so we have

$$|\phi_{u_\alpha,n}(x) - u_\alpha(x + aL)| \leq \frac{1}{\pi} (1 + a^n) \sqrt{\frac{\pi}{\alpha}},$$

where

$$\phi_{u_\alpha,n}(x) := \sum_{j=0}^n C_j(n, a) u_\alpha(x + (1 - 2j/n)L).$$

Let us now return to the problem 5.4. Fix $a \in \mathbb{R}, n \in \mathbb{N}$ and $\varepsilon > 0$. The elements of the family $u_\alpha(x) = xe^{-\alpha x^2}$ such that

$$|\phi_{u_\alpha,n}(x) - u_\alpha(x + aL)| \leq \varepsilon \quad \text{for every } x \in \mathbb{R}$$

are those for which $\alpha > \alpha_0$, where

$$(1 + a^n) \sqrt{\frac{1}{\pi \alpha_0}} = \varepsilon. \quad (20)$$

Example 6.7. Let us consider the specific case $a = 1.2, n = 10$ and $\varepsilon = 10^{-3}, L = 1$. An easy computation shows that α_0 in (20) is $\alpha_0 \sim 16.5 \times 10^3$. We will consider two values of α : $\alpha_1 = 2 \times 10^6 > \alpha_0$ and $\alpha_2 = 2 \times 10^3 < \alpha_0$ and the corresponding functions $\phi_{u_\alpha,n}, u_\alpha$ (see figures 1 and 2) and their difference. The following figures illustrate how the approximation is good in the first case; indeed, the difference is less than 10^{-3} (see figure 3, left), while in the second case it is not accurate; indeed, it is not less than 10^{-3} (see figure 3, right).

Remark 6.8. Note that it is not appropriate to consider the Gaussian $v_\alpha(x) = e^{-\alpha x^2}$ instead of $u_\alpha(x)$; in fact, the constant γ appearing in proposition 6.5 cannot become arbitrarily small.

Remark 6.9. Observe that, as example 6.6 shows, to have a good uniform estimate on \mathbb{R} of a function ψ at the point $x + aL$ in terms of ψ computed at the points $x + (1 - 2j/n)L$, we have to require that the integral over \mathbb{R} of the modulus of its Fourier transform is small. If we limit ourselves to points in the interval $(-\frac{nL\pi}{2}, \frac{nL\pi}{2})$ then we have

$$|\phi_{\psi,n}(x) - \psi(x + aL)| \leq \mathcal{E}_n(\lambda, a, L) \int_{\mathbb{R}} |\hat{\psi}(\lambda)| d\lambda,$$

and in this case, if we fix n to be sufficiently large, we can approximate by ε even if λ is large since $\mathcal{E}_n(\lambda, a, L) \rightarrow 0$ as n becomes large.

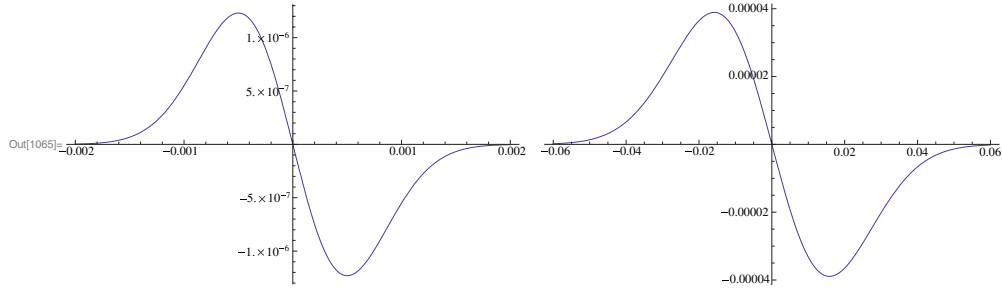


Figure 1. $\phi_{u_\alpha, n}(x)$, with $n = 10$, $a = 1.2$, α_1 (left), α_2 (right).

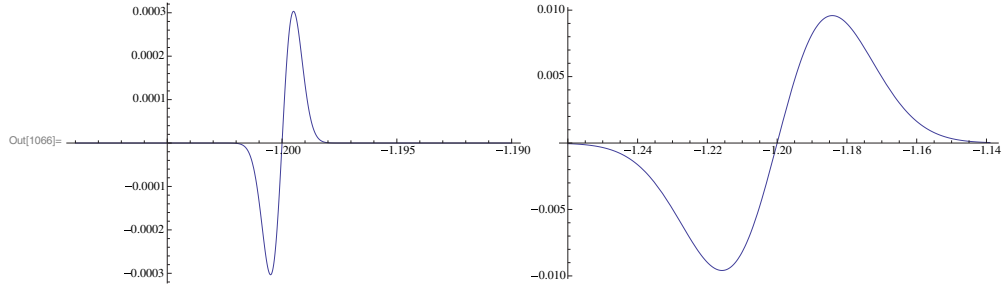


Figure 2. $u_\alpha(x + aL)$ with $a = 1.2$, α_1 (left), α_2 (right).

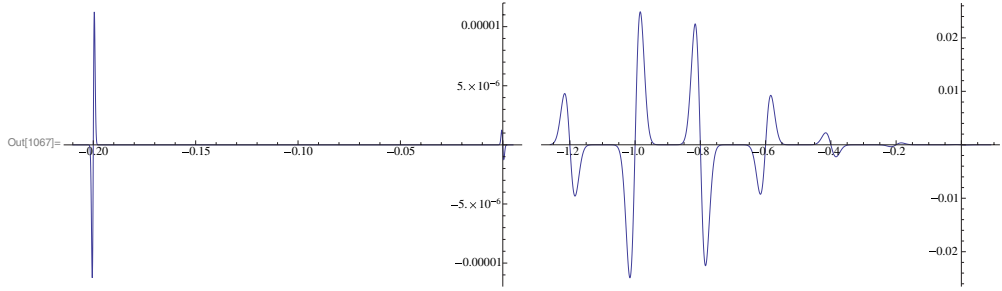


Figure 3. $\phi_{u_\alpha, n}(x) - u_\alpha(x + aL)$, with $n = 10$, $a = 1.2$, α_1 (left), α_2 (right).

We have therefore introduced a new kind of expansion which yields the global behavior. The condition is that the Fourier transform of the function needs to decrease at a sufficient rate (depending on the magnitude of the superoscillation).

7. Discussion

We have presented several new results in this paper including a new generalization of the conditions needed to create superoscillations (section 6) and the introduction of a new kind of expansion. The latter offers a new kind of global picture of a function. Instead of being based on derivatives at a point, it is based on a dense set of points along with sums of translation operators (say from 0 to L), producing the value of the function at a large distance $a \gg L$.

While we proved our results for particular classes of functions, we believe that the results are indeed very general as indicated by arguments presented in this section.

Under what conditions does the expansion yield the value of the function well outside the region? In essence, it worked when the product of the two functions in equation (19) was concentrated in the valid region of superoscillation. This resulted in an actual superoscillation in the localized region. This contrasts when the entire function is considered, in which case no superoscillatory component would be found.

Of course, if we are considering an analytic function, and if we know the function in a dense set of points, then it is known everywhere. But here, we have gone beyond this statement by considering a given n and a given (large) distance away from the region.

We briefly discuss the intuition behind these results. (For convenience, we use the same spin- N example introduced in section 2.) Although the typical error for such imprecise measurements is of the order \sqrt{N} , we have previously shown that in order to achieve non-disturbance (and superoscillations), larger errors *must* be possible; in fact, there must be an exponentially decreasing tail of large errors. To see this, consider that the disturbance to the wavefunction of the system being measured can be bounded if we prepare the measuring device with Q_{md} bounded, i.e. $\Phi_{\text{in}}^{\text{MD}}(Q_{\text{md}})$ has compact support. But this implies that the Fourier transform of $\Phi_{\text{in}}^{\text{MD}}(Q)$, i.e. $\Phi_{\text{in}}^{\text{MD}}(P_{\text{md}})$ is analytic. Therefore, there is a non-zero probability that the pointer P_{md} produces the ‘erroneous’ values (corresponding to the weak value) even from the initial state $\Phi_{\text{in}}^{\text{MD}}(P_{\text{md}})$. That is, it must be possible to constructively produce interference (i.e. a superoscillation) in the tails of $\Phi_{\text{in}}^{\text{MD}}(P_{\text{md}})$ in order to reconstruct the initial wavefunction of the measuring device in the ‘forbidden’ region, i.e. $\Phi_{\text{in}}^{\text{MD}}(P_{\text{md}} - \langle S_{45}^{(N)} \rangle_w)$ centered around $S_{45}^{(N)}_w$. This corresponds to our statement that there must be an exponentially decreasing tail of large errors.

As an example of the exponential relationship between the superoscillatory and other regions, consider the following: if from a superposition of small k ($|k| \leq k_o$), we obtain a region of large k' (superoscillating), then we have proven that the contribution from the second function (e.g. $\psi(x)$ in equation (19)), must be exponentially small.

We can consider two classes of functions to illustrate the generality of our arguments.

- (i) Suppose $\psi(x)$ has compact support ($\psi(x) \neq 0$ for $|x| \leq L$). For such a function, the probability for large k' cannot fall faster than $e^{-k'L}$. Since we know that the product of functions has the large k' when we use the bounded $\psi(x)$ and since the product of those 2 functions in the superoscillating region is proportional to the superoscillating function $F(\hat{P})$, the size of $F(\hat{P})$ in that region must be exponentially small when compared to other regions of $F(\hat{P})$.
- (ii) Consider the opposite situation. Suppose we bound the Fourier transform of $\psi(x)$, so that it has compact support in k which excludes the superoscillating k' ($|k| \leq k' - k_o \equiv K$). So, now we know that the product of the two functions (e.g. in equation (19)) does not contain the superoscillating k' . One can prove that since this function is bounded in K , then the fastest rate which the function in the position basis, $\psi(x)$ can decrease to 0 is e^{-Kx} . This can also be illustrated with our large spin- N example. Consider the size of equation (8):

$$|\Phi_{\text{md}}^{\text{fin}}| = \underbrace{\left\{ 1 + (\alpha_w^2 - 1) \sin^2 \frac{\lambda \hat{Q}_{\text{md}}}{N} \right\}^{\frac{N}{2}}}_A \underbrace{\exp \left\{ -\frac{(Q_{\text{md}})^2}{2(\Delta Q_{\text{md}})^2} \right\}}_B. \quad (21)$$

As long as there are no regions (outside the superoscillating region) in which the size of the system portion of equation (8) (i.e. the A term in equation (21)) is less than the exponential of the measuring device (i.e. the B term in equation (21)), then it will be more likely that the function is peaked outside $\hat{Q}_{\text{md}} = 0$ so that the product of the functions

will not contain k' . Had $\psi(x)$ not declined quickly enough, then we would have an inconsistency: we would have the large k' , but the k' is not contained in the product of the functions. In general, any function which tries to convert the product into being localized around the superoscillating region must compete with $k' - k_o$. We therefore deduce the size relationship.

The legitimacy of the approximation discussed in sections 5 and 6 can now be understood as a competition between the (scalar product) A term and the (probability) B term. We can also see how this can be applied for the weak value approximation. If the quantity equation (21) goes to 0 for large Q_{md} , then the weak value approximation is valid. On the other hand, if the increase in A was not counter-balanced by the decline in B , then we could not restrict the weak value approximation around $Q_{\text{md}} = 0$ because equation (21) would be much more likely to be located around large Q_{md} . Thus, the meaning of the new regime for the approximation presented here is that there is no other region in which the size of equation (8), i.e. equation (21), is significant, except around $Q_{\text{md}} = 0$.

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