

## 機械手臂 順向運動學

# Manipulator Forward Kinematics

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# 引言 -1

□ 運動學(Kinematics)：討論運動狀態本身，未連結到產生運動的「力」

◆ 位置( $x$ )、速度( $v$ )、加速度( $a$ )、和時間( $t$ ) 之間的關係

◆ 移動/轉動

◦ 位置/姿態

◦ 速度/角速度

◦ 加速度/角加速度

$$\begin{aligned} v &= \frac{d}{dt}x & a &= \frac{d}{dt}v \\ a &= \frac{d^2}{dt^2}x & vdv &= ads \end{aligned}$$

□ 動力學(Dynamics)：討論力/力矩如何產生運動

◆ Newton's 2<sup>nd</sup> Law

$$\sum F = ma$$

◆ Work & energy

$$T_1 + V_1 + U_{1-2}' = T_2 + V_2$$

◆ Impulse & momentum

$$\int \sum F dt = G_2 - G_1$$

# 引言 -2

## □ 機械手臂

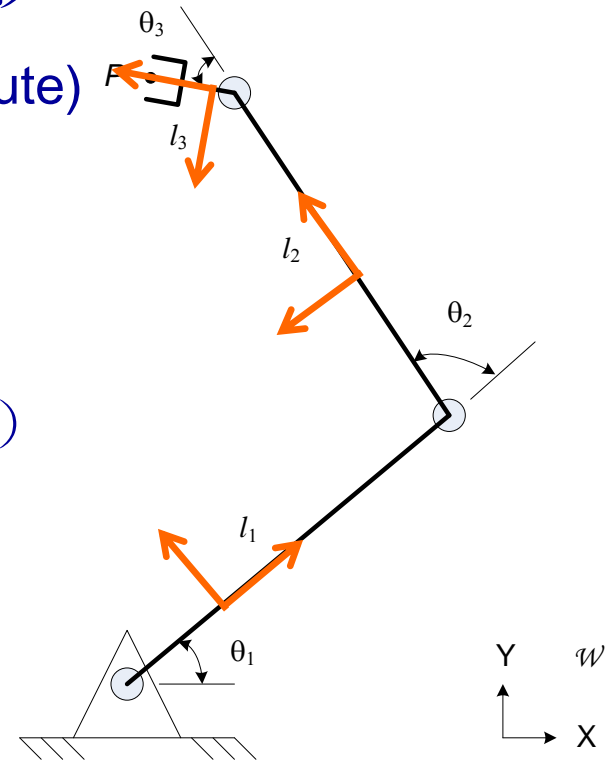
- ◆ 多個桿件(link)相串連，具有複雜的幾何外形
- ◆ 桿件間可相對移動(prismatic)或轉動(revolute)  
由致動器驅動來達成

## □ 對應關係

- ◆ 需求：手臂末端點狀態（位置 $^W P$ 、速度...
- ◆ 達成方式：驅動各致動器  
$$^W P = f(\theta_1, \theta_2, \dots, \theta_n)$$

## □ 描述手臂狀態方法

- ◆ 找出桿件間的相對幾何狀態
- ◆ 在各桿件上建立frame，以frame狀態來代表桿件狀態



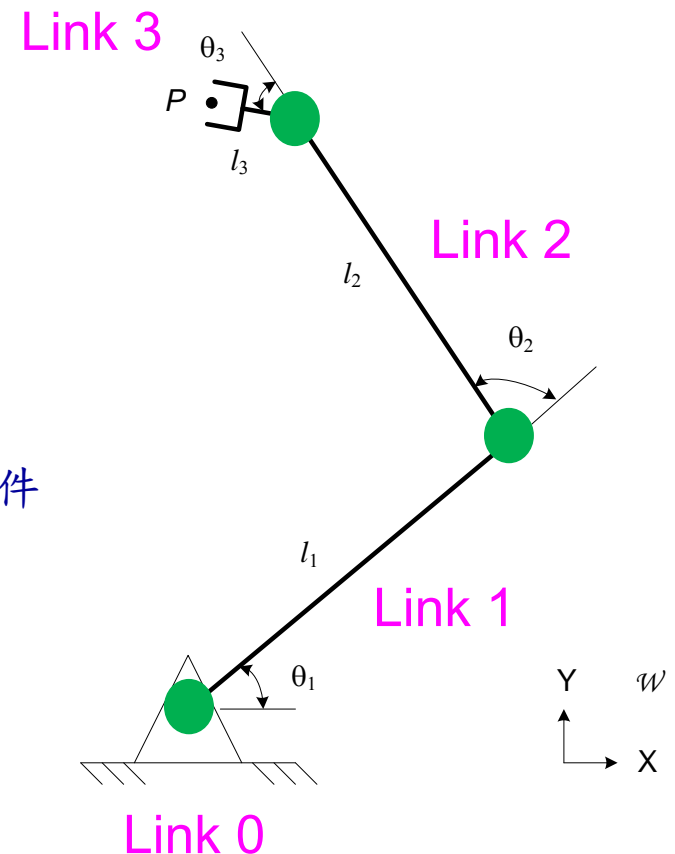
# 手臂幾何描述方式 -1

## □ Joint

- ◆ 每個revolute或prismatic的joint具有 1 DOF
- ◆ 每個joint對 某特定axis 進行rotation或translation

## □ Link

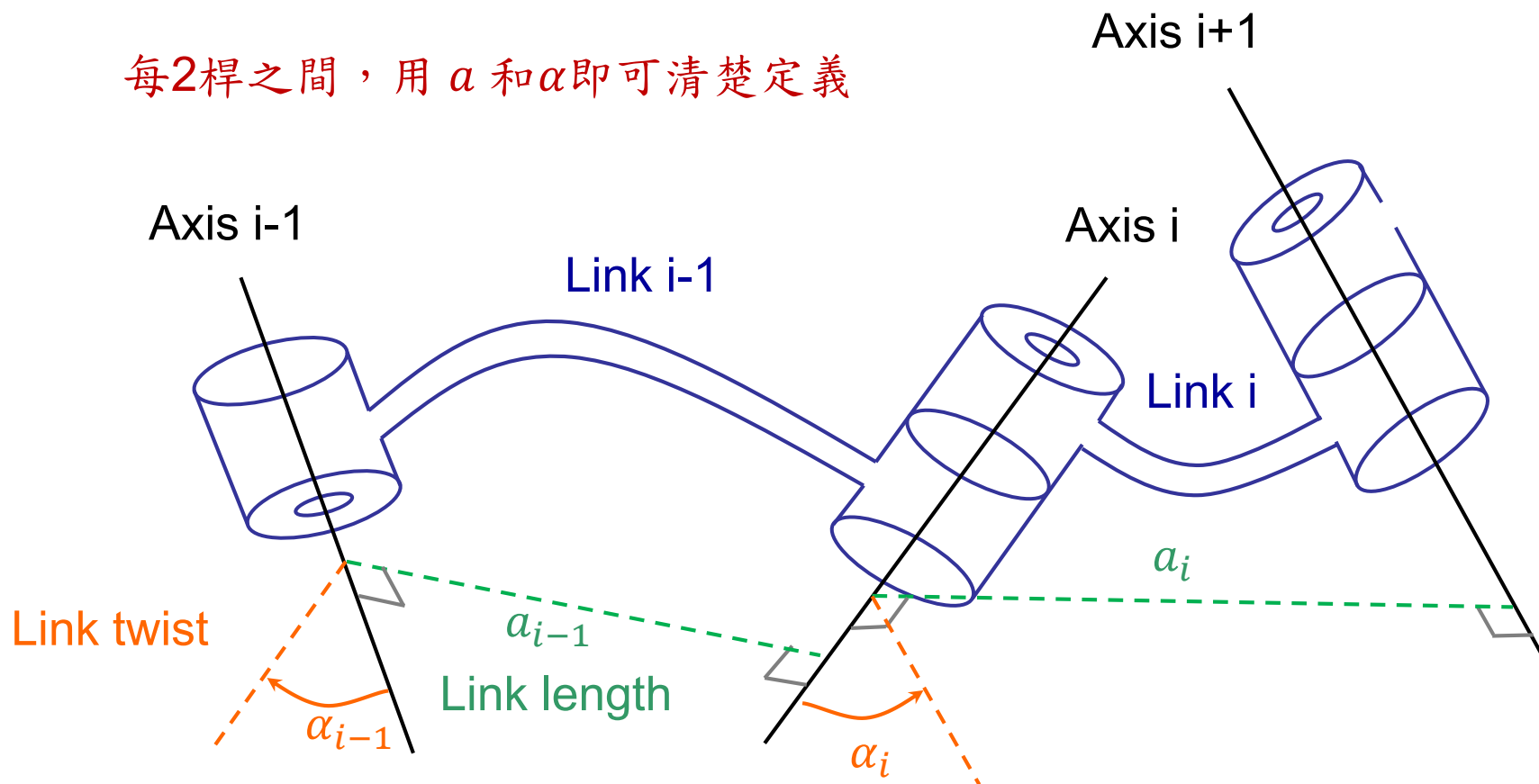
- ◆ 連接joints的桿件，為剛體(rigid body)
- ◆ 編號方式
  - Link 0: 地桿，不動的桿件
  - Link 1: 和Link 0相連，第一個可動的桿件
  - Link 2: 第二個可動的桿件
  - 依序下去...



## 手臂幾何描述方式 -2

- 對空間中2個任意方向的axes，兩axes之間具有一線段和此2個axes都相互垂直

每2桿之間，用  $a$  和  $\alpha$  即可清楚定義



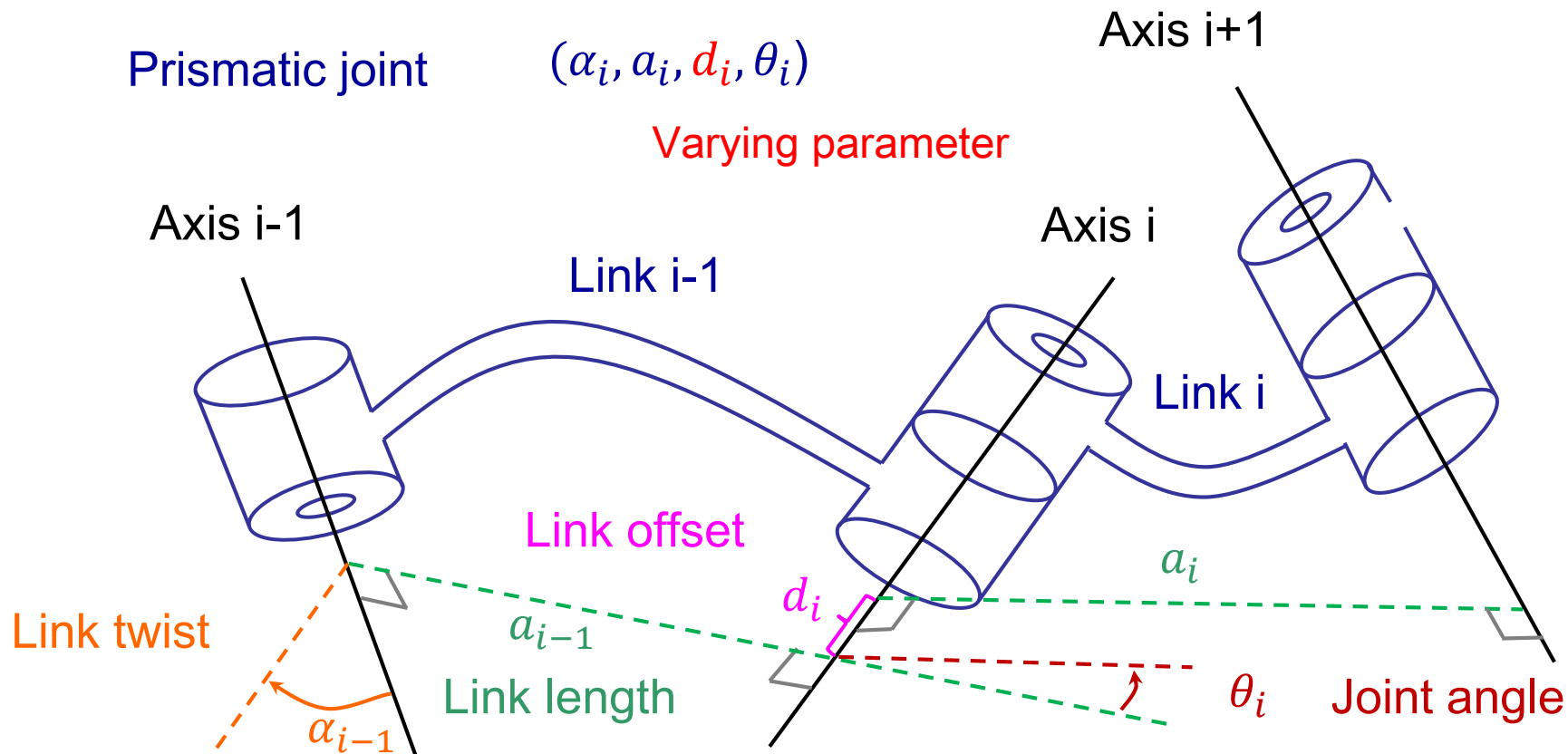
## 手臂幾何描述方式 -3

- 但若要多桿串連，則另需要兩個參數( $d_i, \theta_i$ )，來描述相鄰線段 $a_{i-1}$ 和 $a_i$ 間的相對幾何關係

Revolute joint      ( $\alpha_i, a_i, d_i, \theta_i$ )

Prismatic joint      ( $\alpha_i, a_i, d_i, \theta_i$ )

Varying parameter



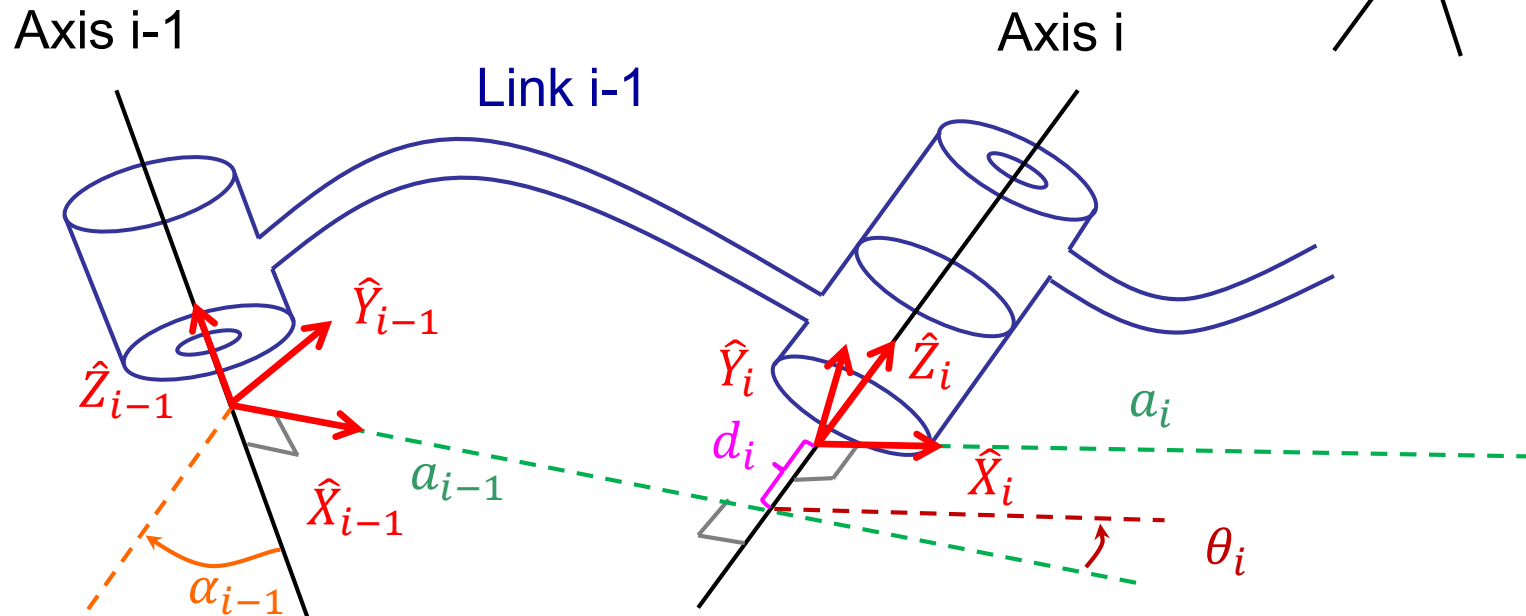
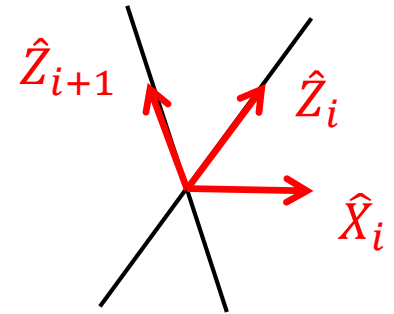
# 桿件上建立Frames -1

□  $\hat{Z}_i$  轉動或移動axis的方向

□  $\hat{X}_i$  沿著 $a_i$ 方向 (if  $a_i \neq 0$ )

和 $\hat{Z}_i$ 和 $\hat{Z}_{i+1}$ 兩者垂直 (if  $a_i = 0$ )

□  $\hat{Y}_i$  與 $\hat{X}_i$ 和 $\hat{Z}_i$ 兩者垂直，依循右手定則



## 桿件上建立Frames -2

### □ 地桿 link (0)

Frame {0} coincides with frame {1}

$$a_0 = 0$$

$$\alpha_0 = 0$$

Revolute joint

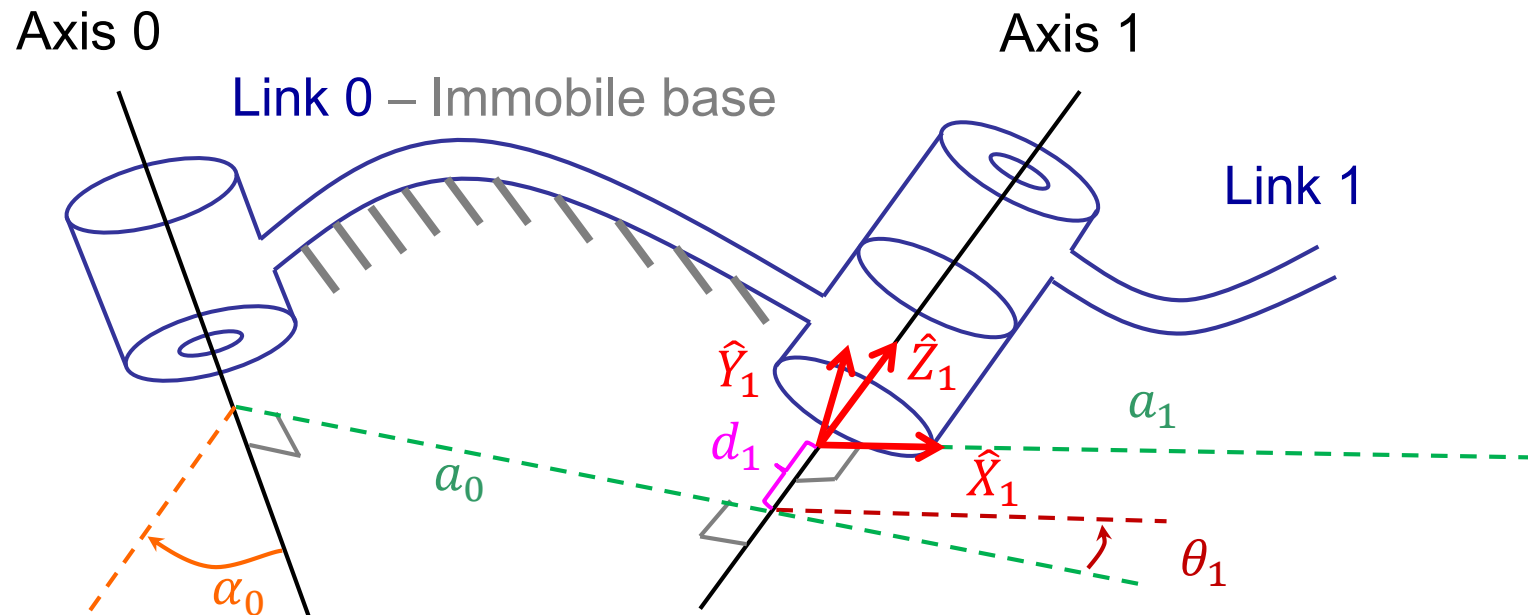
$\theta_1$  arbitrary

$$d_1 = 0$$

Prismatic joint

$d_1$  arbitrary

$$\theta_1 = 0$$





## 桿件上建立Frames -3

### □ Last link ( $n$ )

取和  $\hat{X}_{n-1}$  同方向

$$a_n = 0 \quad \alpha_n = 0$$

Revolute joint

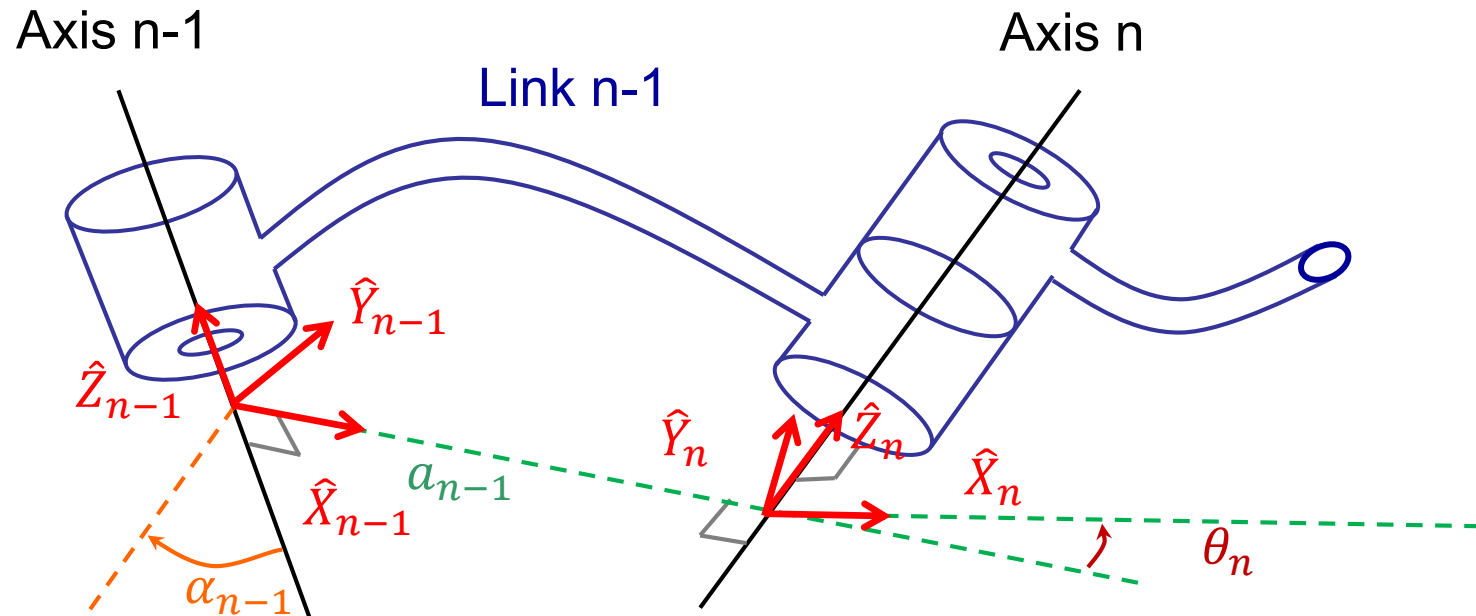
$\theta_n$  variable

$$d_n = 0$$

Prismatic joint

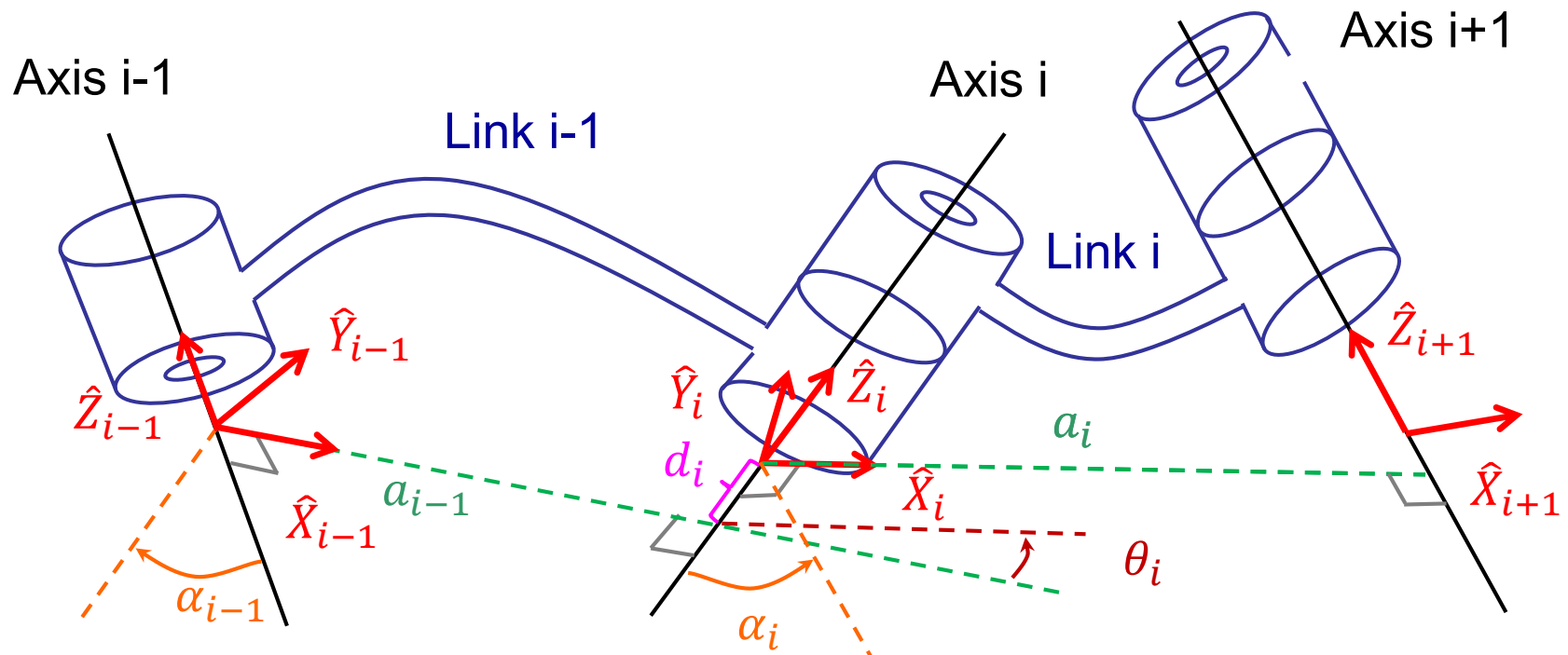
$d_n$  variable

$$\theta_n = 0$$



# Denavit-Hartenberg表達法 (Craig version)

- $\alpha_{i-1}$ : 以 $\hat{X}_{i-1}$ 方向看， $\hat{Z}_{i-1}$ 和 $\hat{Z}_i$ 間的夾角
- $a_{i-1}$ : 沿著 $\hat{X}_{i-1}$ 方向， $\hat{Z}_{i-1}$ 和 $\hat{Z}_i$ 間的距離 ( $a_i > 0$ )
- $\theta_i$ : 以 $\hat{Z}_i$ 方向看， $\hat{X}_{i-1}$ 和 $\hat{X}_i$ 間的夾角
- $d_i$ : 沿著 $\hat{Z}_i$ 方向， $\hat{X}_{i-1}$ 和 $\hat{X}_i$ 間的距離

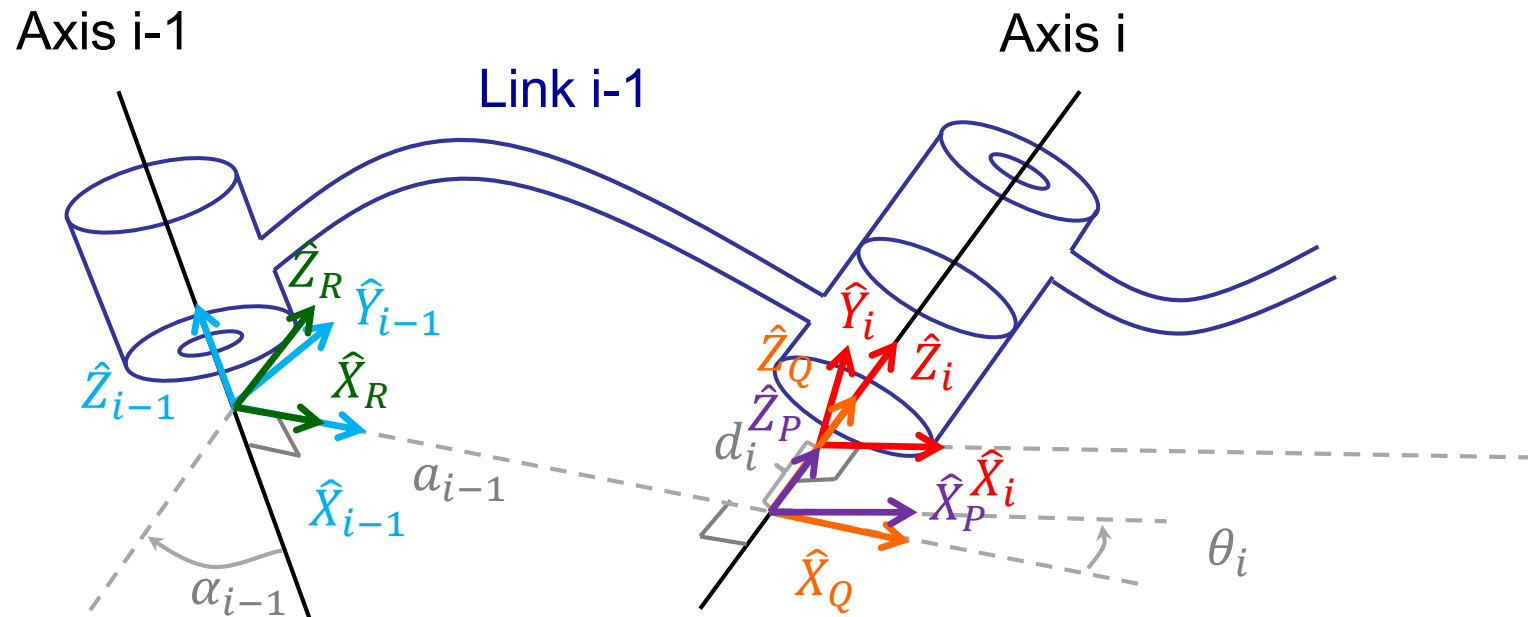


# Link Transformations -1

$${}^{i-1}P = {}^{i-1}_iT^iP$$

$${}^{i-1}P = {}^{i-1}_RT^R_Q T^Q_P T^P_i T^iP$$

$$\begin{aligned} {}^{i-1}_iT &= {}^{i-1}_RT^R_Q T^Q_P T^P_i T \\ &= T_{\hat{X}_{i-1}}(\alpha_{i-1}) T_{\hat{X}_R}(a_{i-1}) T_{\hat{Z}_Q}(\theta_i) T_{\hat{Z}_P}(d_i) \end{aligned}$$



# Link Transformations -2

□ Thus

$${}^{i-1}_iT = T_{\hat{x}_{i-1}}(\alpha_{i-1})T_{\hat{x}_R}(a_{i-1})T_{\hat{z}_Q}(\theta_i)T_{\hat{z}_P}(d_i)$$

$$= \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

□ 連續link transformations

$${}^0_nT = {}^0_1T {}^1_2T {}^2_3T \dots {}^{n-2}_{n-1}T {}^{n-1}_nT$$

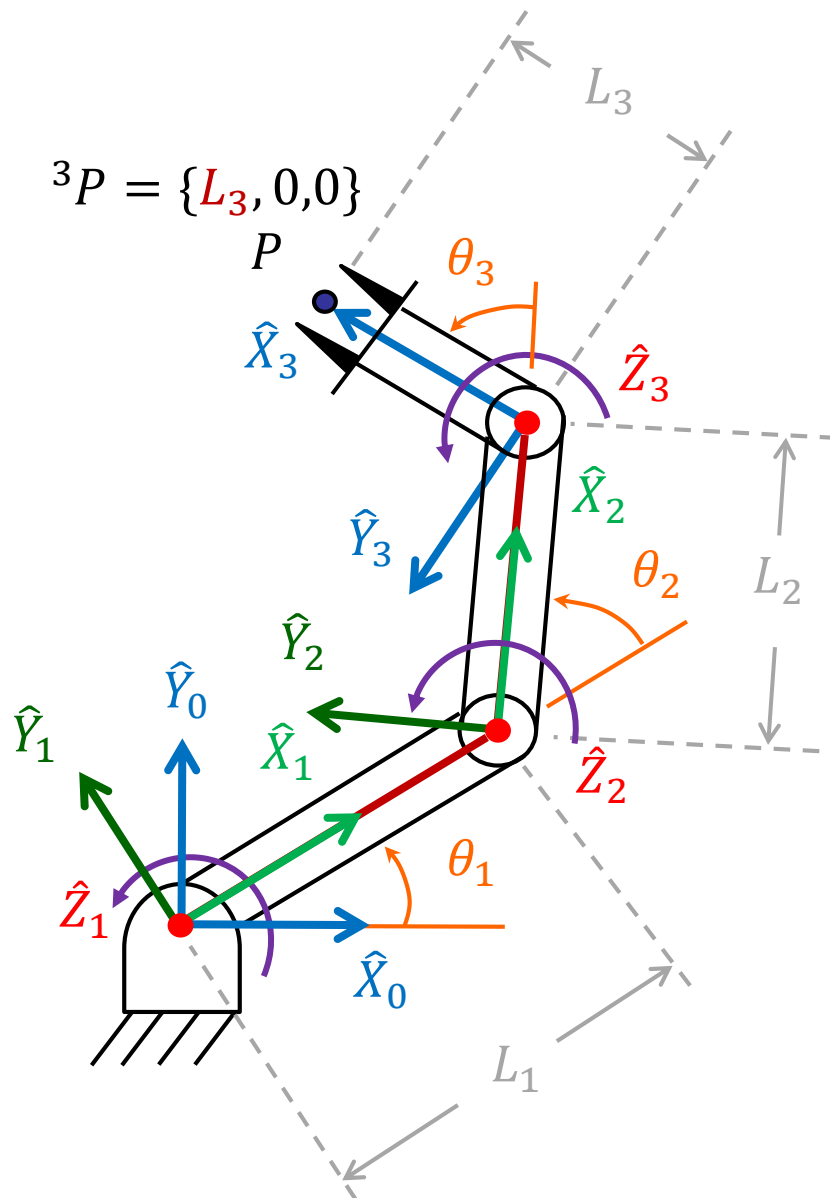
Frame {n} 相對於 Frame {0} 的空間幾何關係具清楚且量化之定義

在Frame {n} 下表達的向量可轉回 Frame {0} 下來表達

# Example: A RRR Manipulator

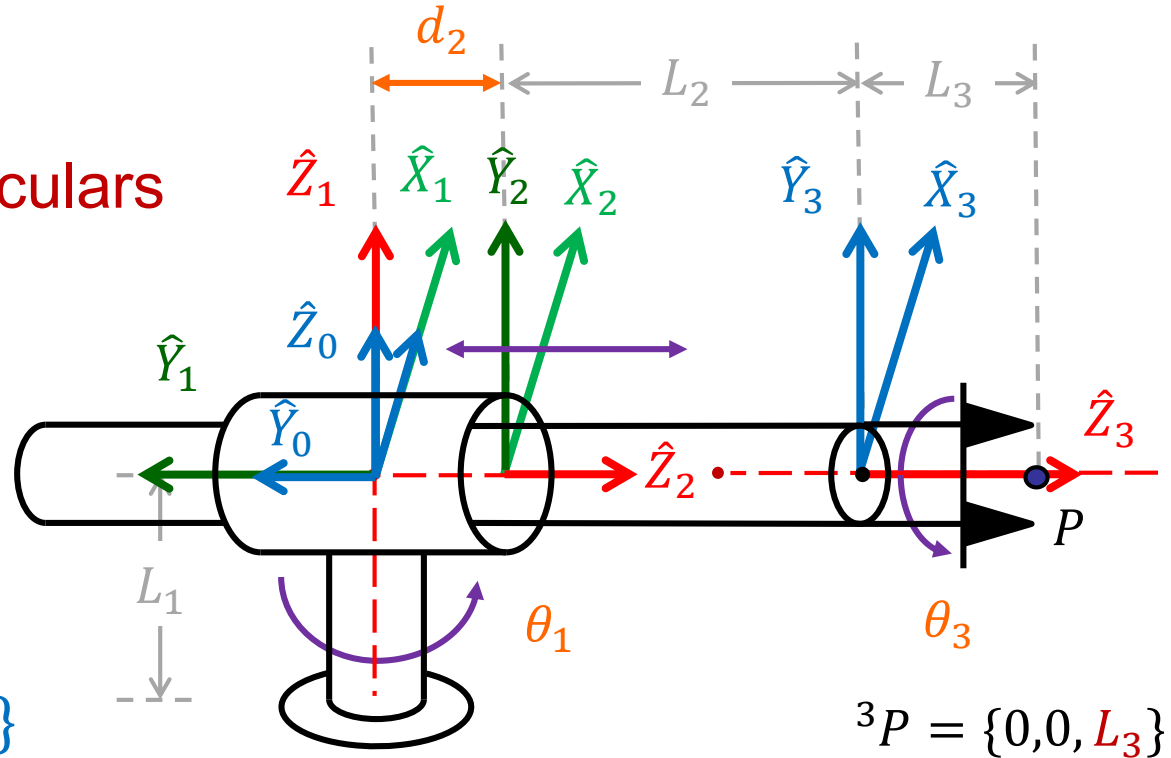
- Joint axes
- Common perpendiculars
- $\hat{Z}_i$
- $\hat{X}_i$
- $\hat{Y}_i$
- Frames  $\{0\}$  and  $\{n\}$

i	$\alpha_{i-1}$	$a_{i-1}$	$d_i$	$\theta_i$
1	0	0	0	$\theta_1$
2	0	$L_1$	0	$\theta_2$
3	0	$L_2$	0	$\theta_3$



# Example: A RPR Manipulator -2

- Joint axes
- Common perpendiculars
- $\hat{Z}_i$
- $\hat{X}_i$
- $\hat{Y}_i$
- Frames  $\{0\}$  and  $\{n\}$



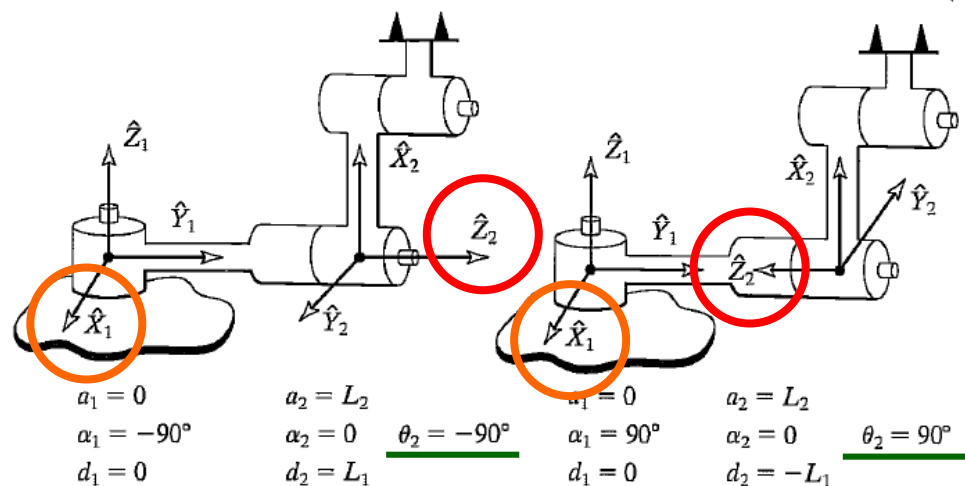
$i$	$\alpha_{i-1}$	$a_{i-1}$	$d_i$	$\theta_i$
1	0	0	0	$\theta_1$
2	$90^\circ$	0	$d_2$	0
3	0	0	$L_2$	$\theta_3$

# Example: A RRR Manipulator -3

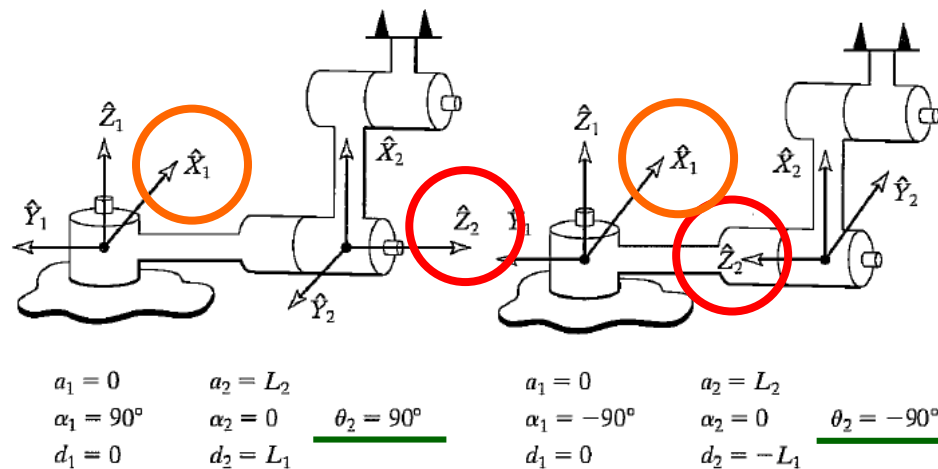
□ 當  $a_1 = 0$

$\hat{Z}_1$  和  $\hat{Z}_2$  相交

◆  $\hat{Z}_2$  兩個選擇

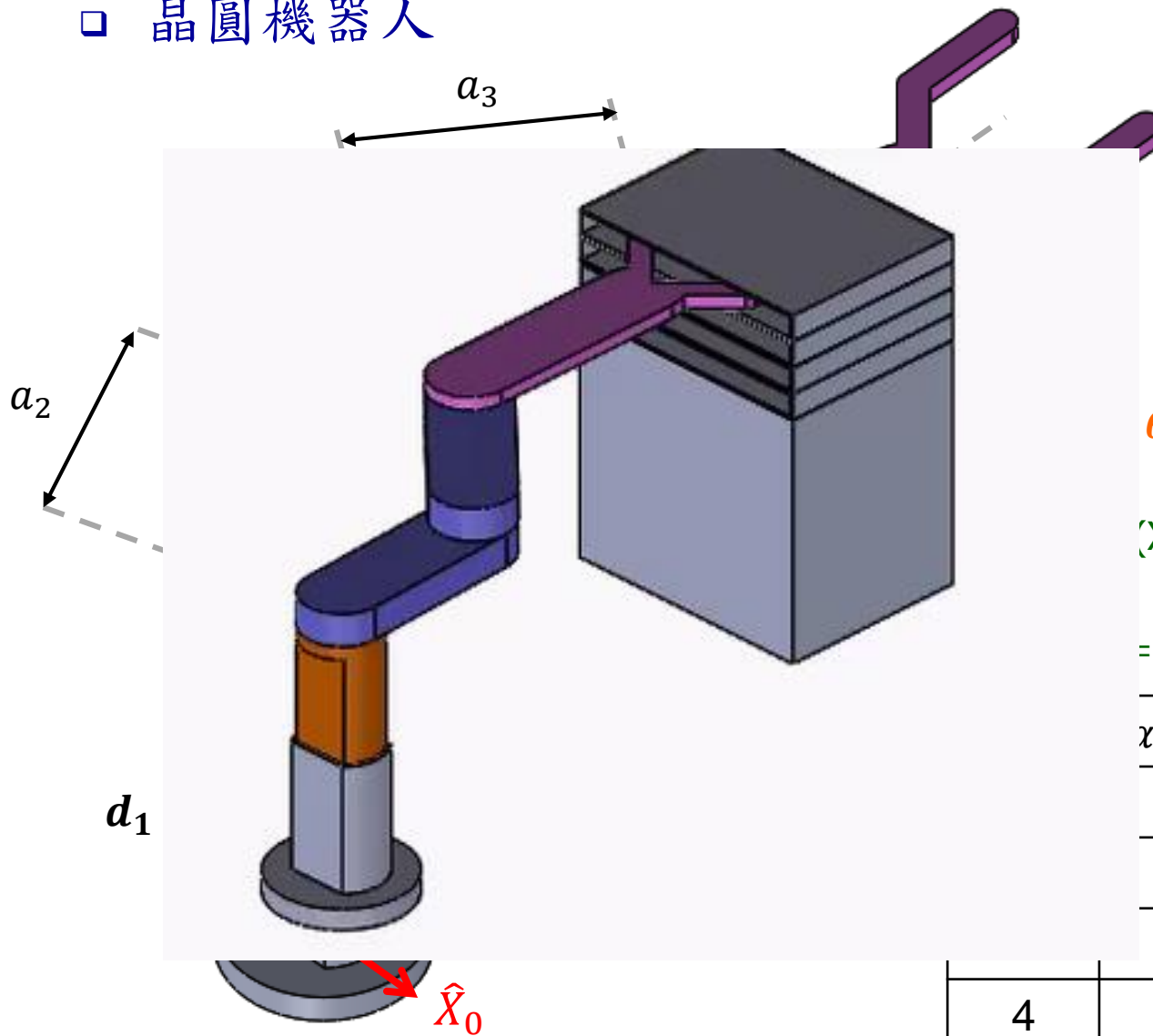


◆  $\hat{X}_1$  兩個選擇



# Example: A PRRR Manipulator

## 晶圓機器人



$\theta_3, \theta_4$  為驅動關節

$(X_0 // X_1)$

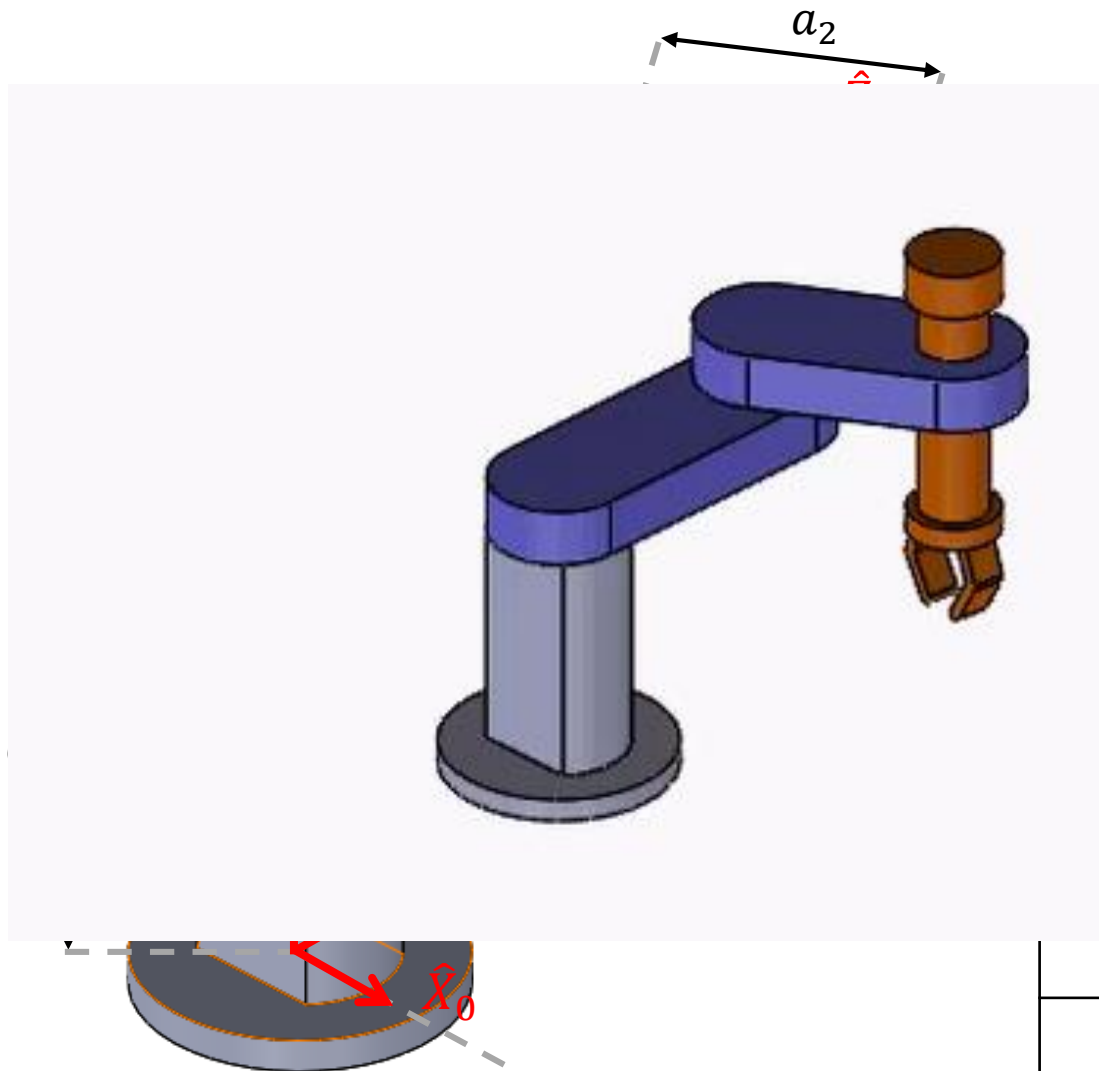
$\alpha_2, \alpha_3 = 0$  (所有Z軸相互平行)  
 $= 0$

$\chi_{i-1}$	$a_{i-1}$	$d_i$	$\theta_i$
0	0	$d_1$	0
0	0	$d_2$	$\theta_2$
0	$a_2$	$d_3$	$\theta_3$
4	0	$d_4$	$\theta_4$



# Example: A RRRP Manipulator

## SCARA 機器人



$\theta_1, \theta_2, \theta_3, d_4$  為驅動關節

$\theta_4 = 0$  ( $X_0 \parallel X_1$ )

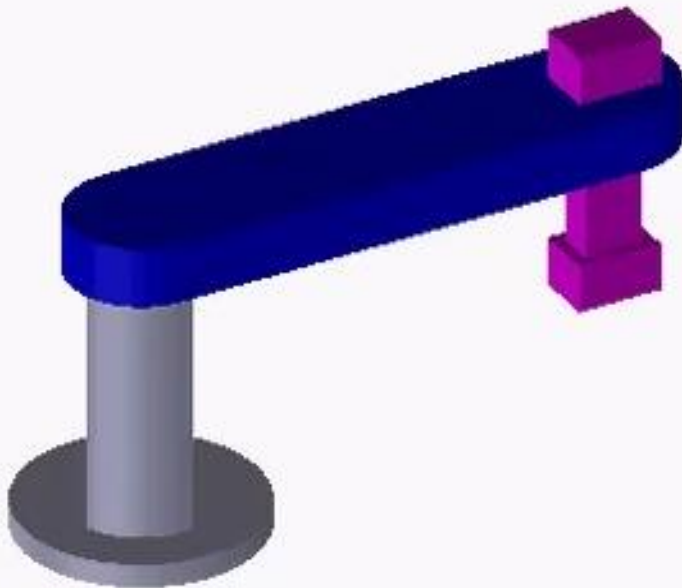
$\alpha_0, \alpha_1, \alpha_2, \alpha_3 = 0$  (所有Z軸相互平行)

$a_0, a_3 = 0$

i	$\alpha_{i-1}$	$a_{i-1}$	$d_i$	$\theta_i$
1	0	0	$d_1$	$\theta_1$
2	0	$a_1$	$d_2$	$\theta_2$
3	0	$a_2$	0	$\theta_3$
4	0	0	$d_4$	0

## Example: A RP manipulator

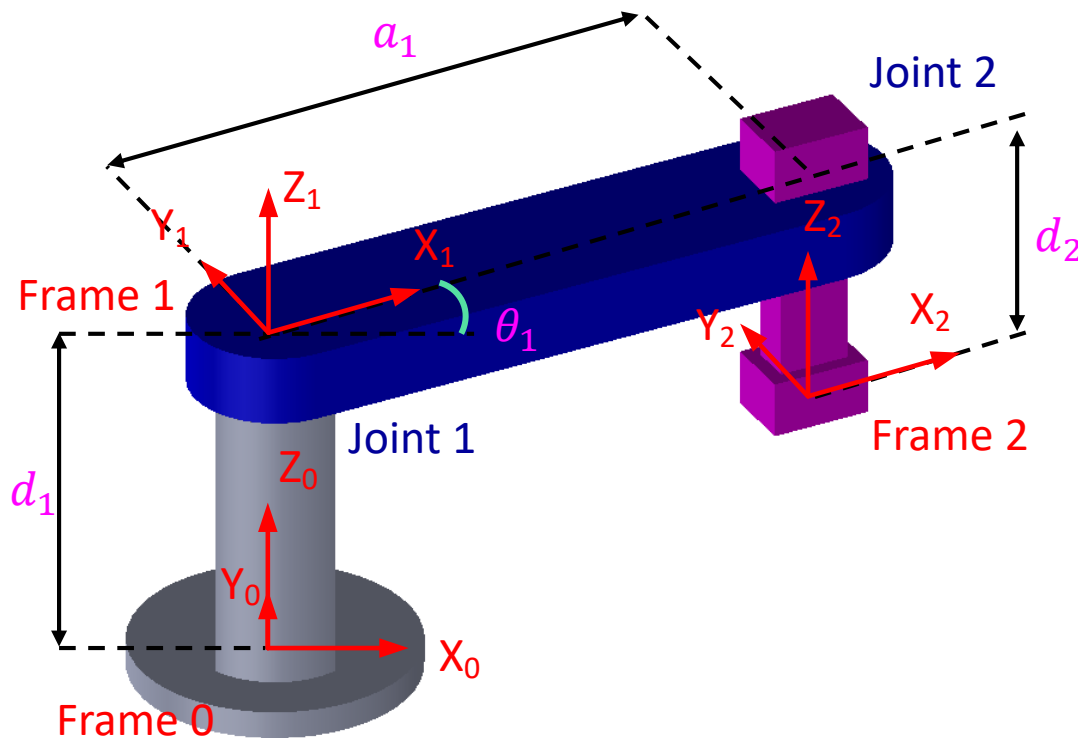
- In-video Quiz：下方手臂由一個revolute joint和一個prismatic joint組成，在所有的DH參數  $(a_{i-1} \alpha_{i-1} d_i \theta_i)$  中，哪兩個參數為驅動關節？



- (A)  $\alpha_1, a_2$
- (B)  $\alpha_1, d_2$
- (C)  $\theta_1, a_2$
- (D)  $\theta_1, d_2$

# Example: A RP manipulator (Solution)

- In-video Quiz：下方手臂由一個revolute joint和一個 prismatic joint組成，在所有的DH參數  $(a_{i-1} \alpha_{i-1} d_i \theta_i)$  中，哪兩個參數為驅動關節？



For Joint 1:

$a_0 = 0$  ( $Z_0$   $Z_1$  are concurrent)

$\alpha_0 = 0$  ( $Z_0 \parallel Z_1$ )

For Joint 2:

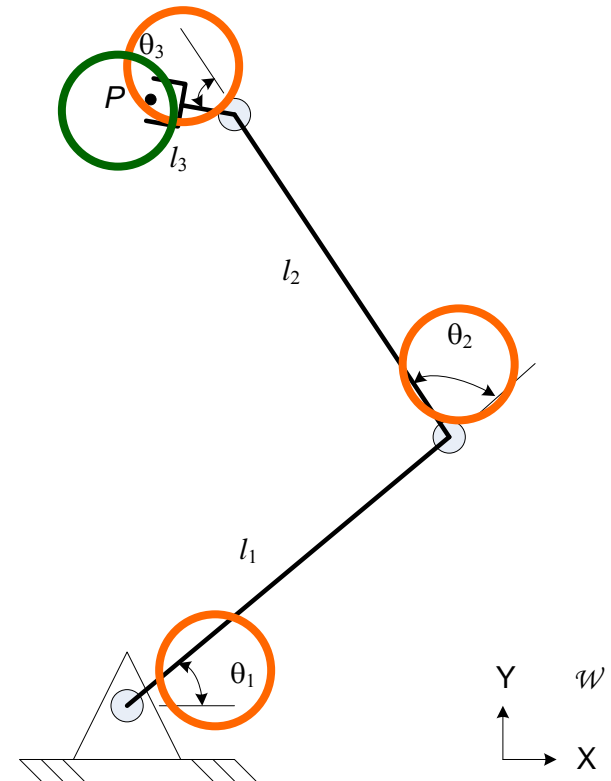
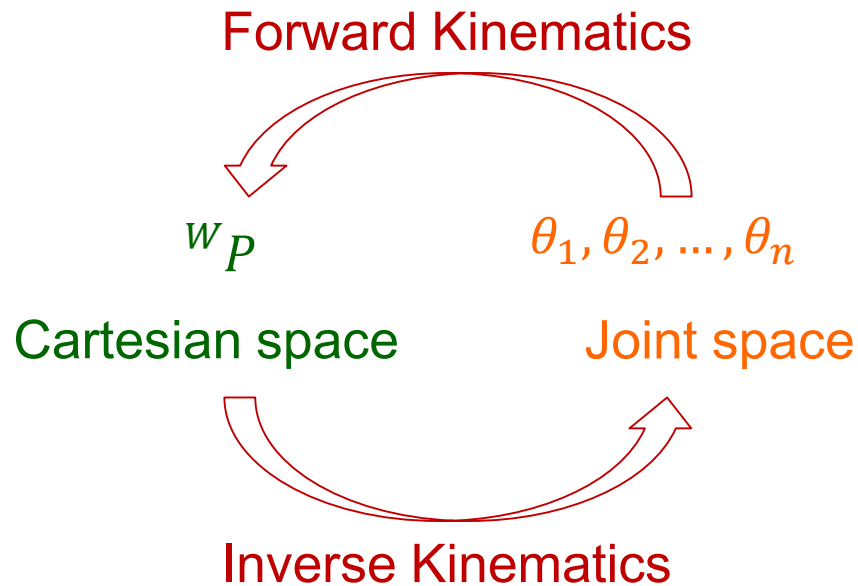
$\alpha_1 = 0$  ( $Z_1 \parallel Z_2$ )

$\theta_2 = 0$  ( $X_1 \parallel X_2$ )

$\theta_1, d_2$  are varying parameters

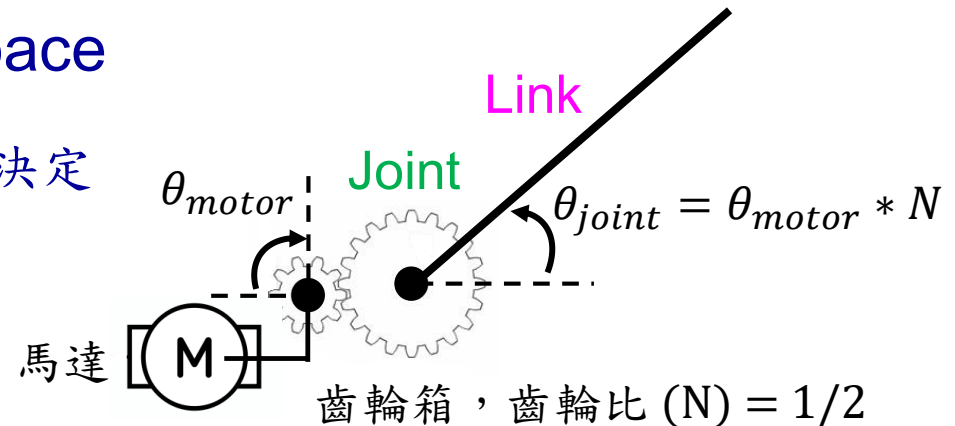
# Actuator, Joint, and Cartesian Spaces -1

## □ Joint space $\Leftrightarrow$ Cartesian space



## □ Actuator space $\Leftrightarrow$ joint space

- ◆ 由連結致動器和joint的機構決定



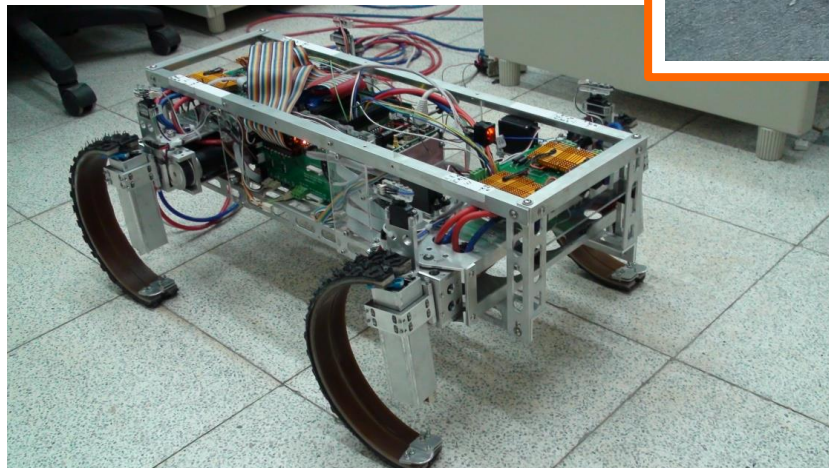
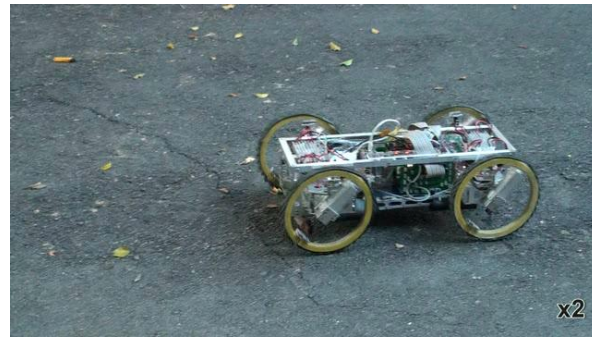
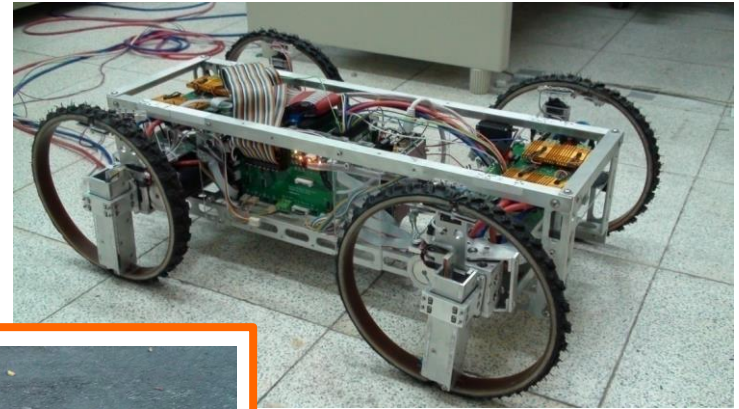
# Actuator, Joint, and Cartesian Spaces -2

- Example: A leg-wheel transformable robot

輪模式

平地上

快速、平穩、省能



腳模式

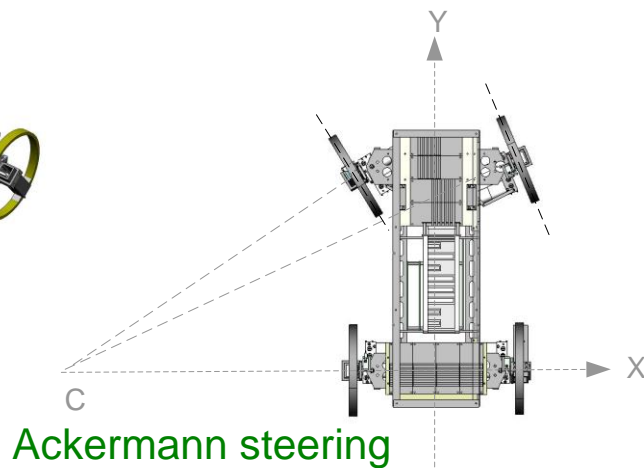
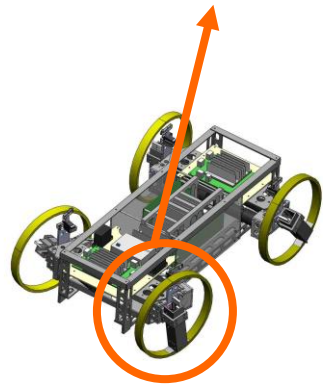
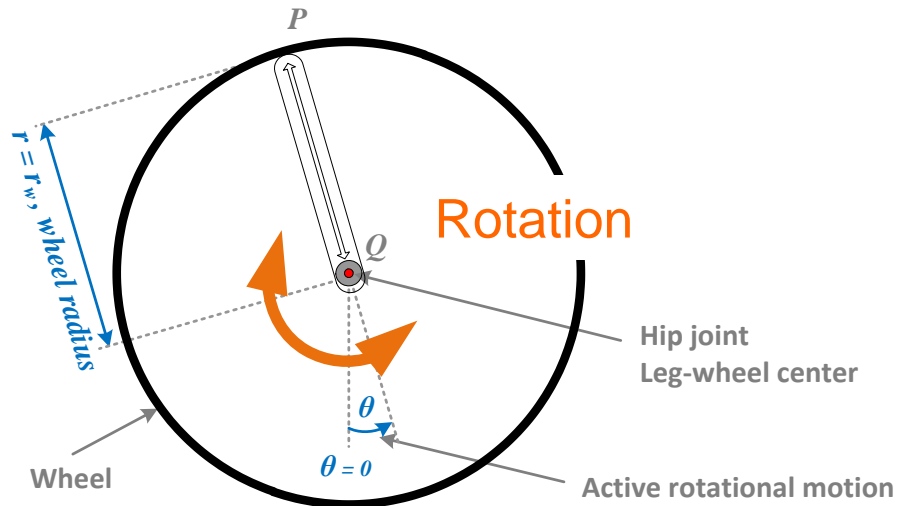
崎嶇地

越障、動態

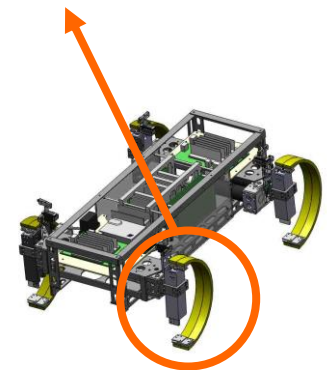
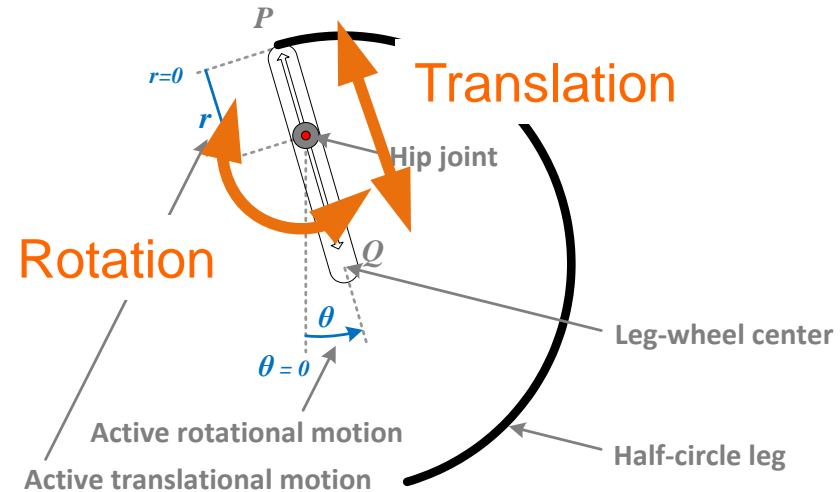
# Actuator, Joint, and Cartesian Spaces -3



輪模式



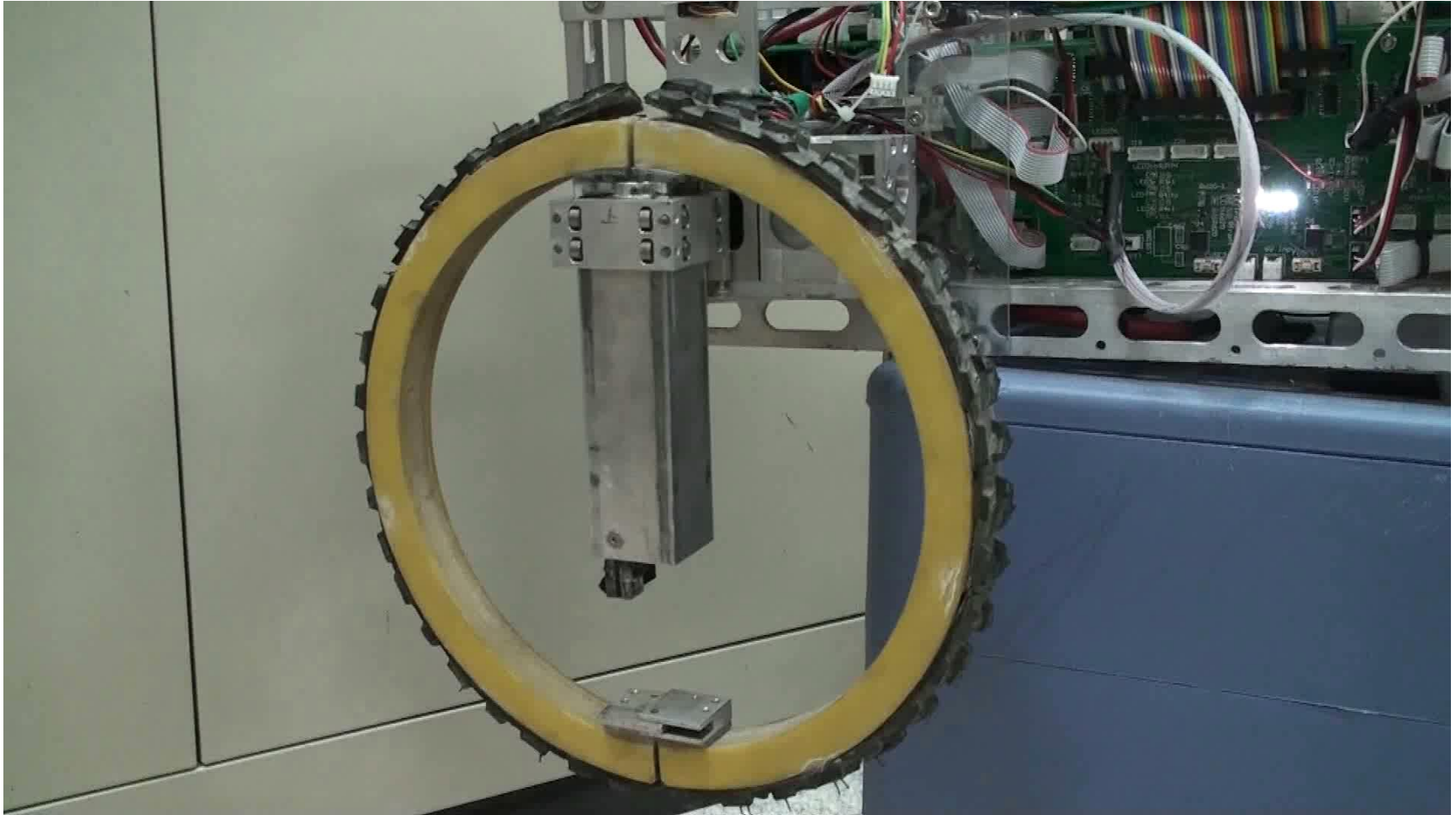
腳模式





# Actuator, Joint, and Cartesian Spaces -4

- Leg-wheel motion



# Actuator, Joint, and Cartesian Spaces -5

## □ Kinematic mapping

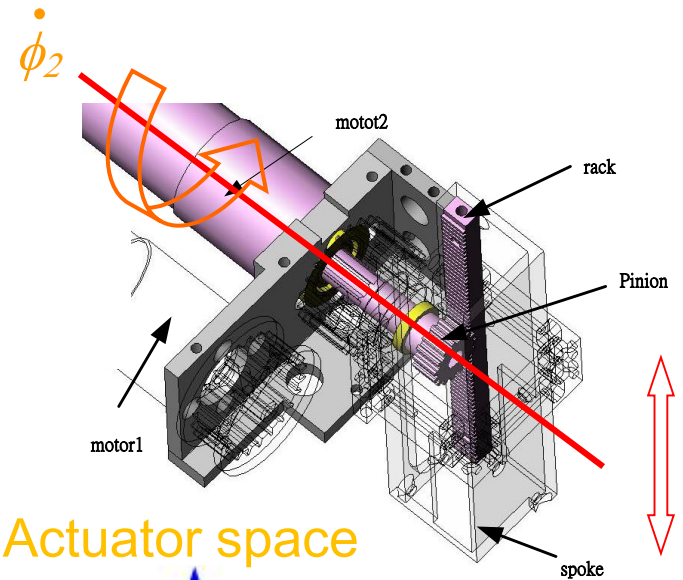
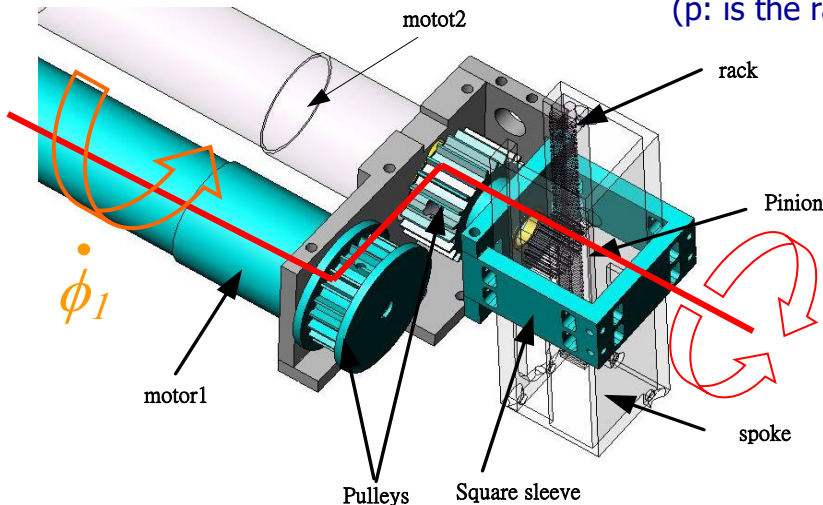
- ◆ Input: Motor speeds  $\dot{\phi}_1$   $\dot{\phi}_2$
- ◆ Output: Leg-wheel motion  $\dot{\theta}$   $\dot{r}$

in polar coordinate

Joint space  $\dot{\xi} = \begin{bmatrix} \dot{\theta} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -p & p \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}$

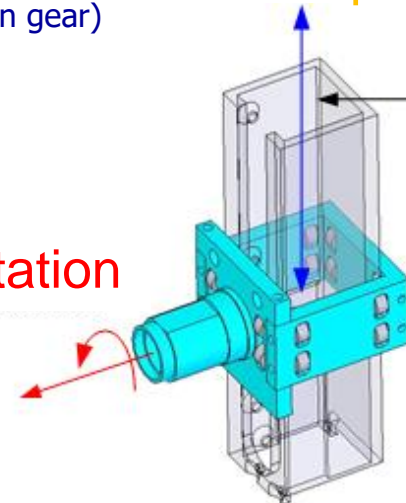
(p: is the radius of pinion gear)

Actuator space



$\dot{\theta}$  Rotation

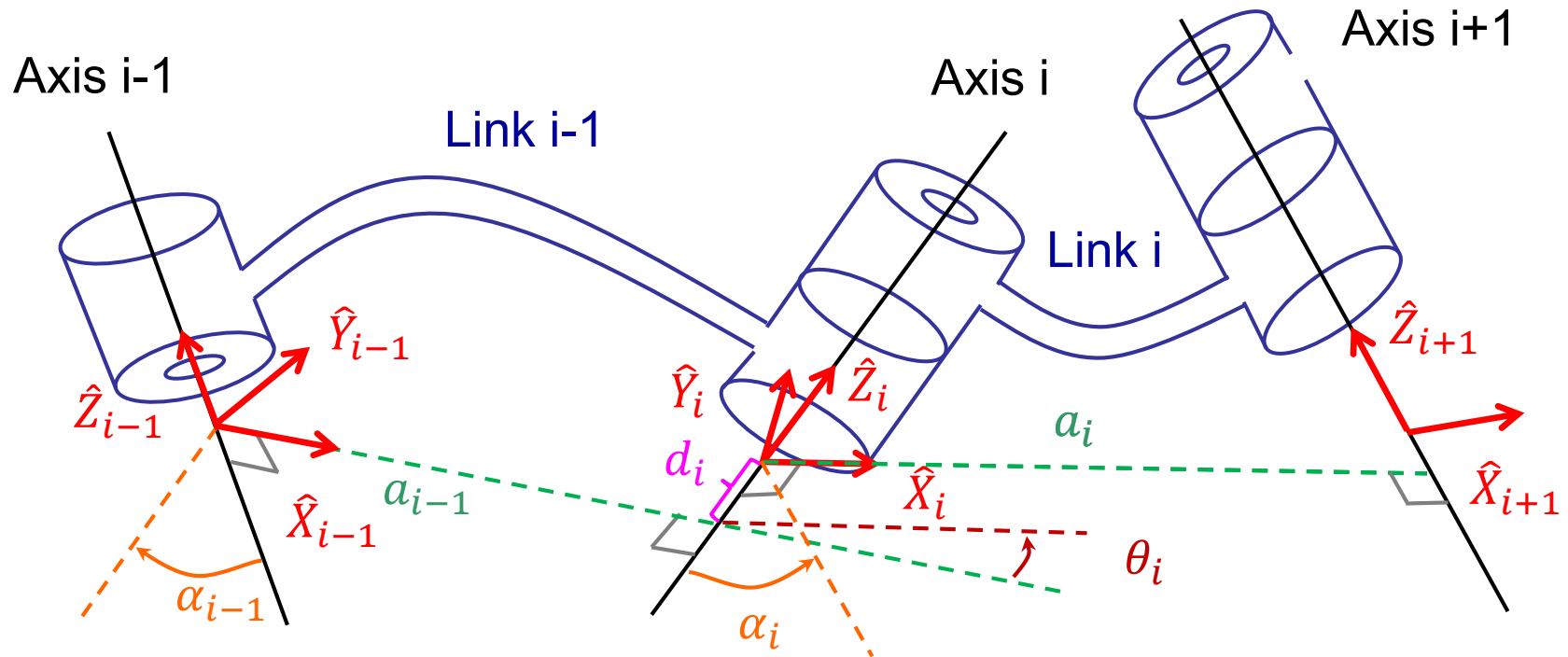
Translation  $\dot{r}$





# Denavit-Hartenberg 表達法小結 (Craig version) -1

- $\alpha_{i-1}$ : 以  $\hat{X}_{i-1}$  方向看,  $\hat{Z}_{i-1}$  和  $\hat{Z}_i$  間的夾角
- $a_{i-1}$ : 沿著  $\hat{X}_{i-1}$  方向,  $\hat{Z}_{i-1}$  和  $\hat{Z}_i$  間的距離 ( $a_i > 0$ )
- $\theta_i$ : 以  $\hat{Z}_i$  方向看,  $\hat{X}_{i-1}$  和  $\hat{X}_i$  間的夾角
- $d_i$ : 沿著  $\hat{Z}_i$  方向,  $\hat{X}_{i-1}$  和  $\hat{X}_i$  間的距離

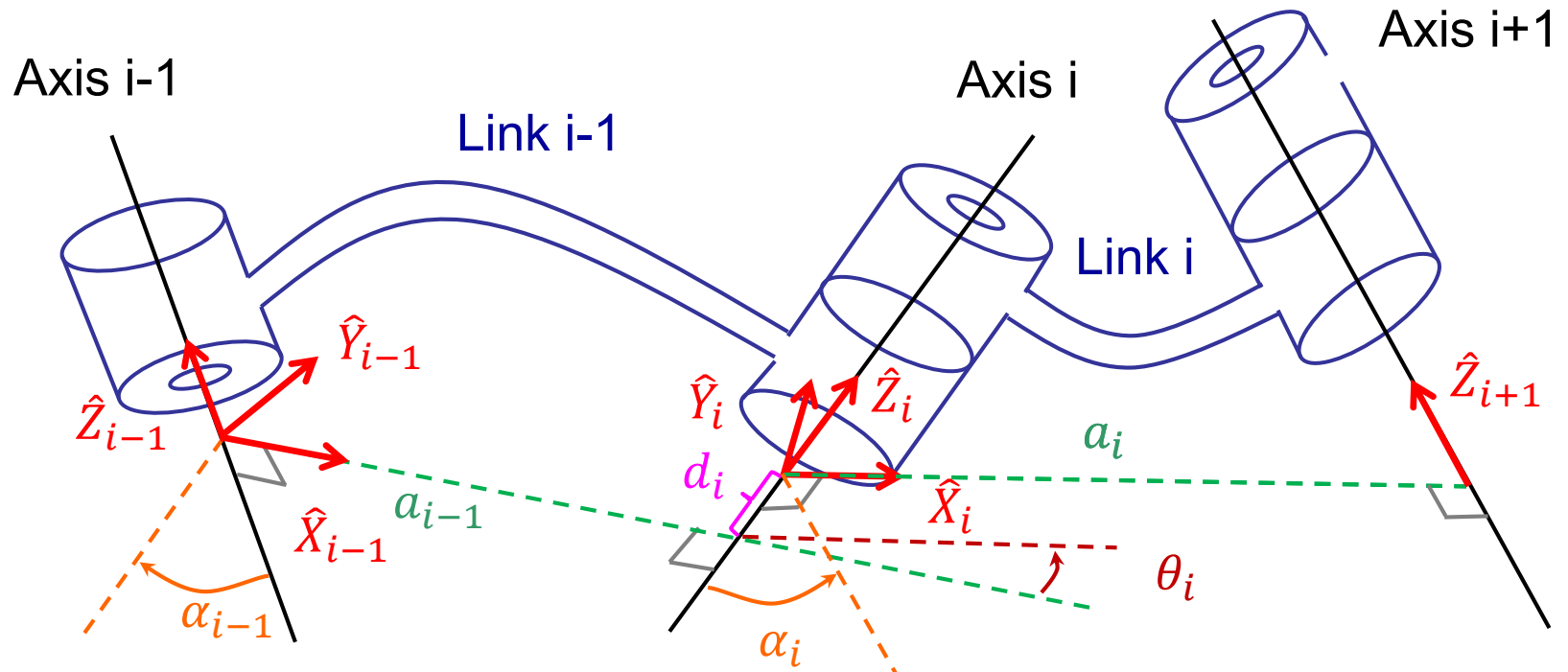


# Denavit-Hartenberg 表達法小結 (Craig version) -2

$${}^{i-1}_iT = {}^{i-1}_R T^R_Q T^Q_P T^P_i T$$

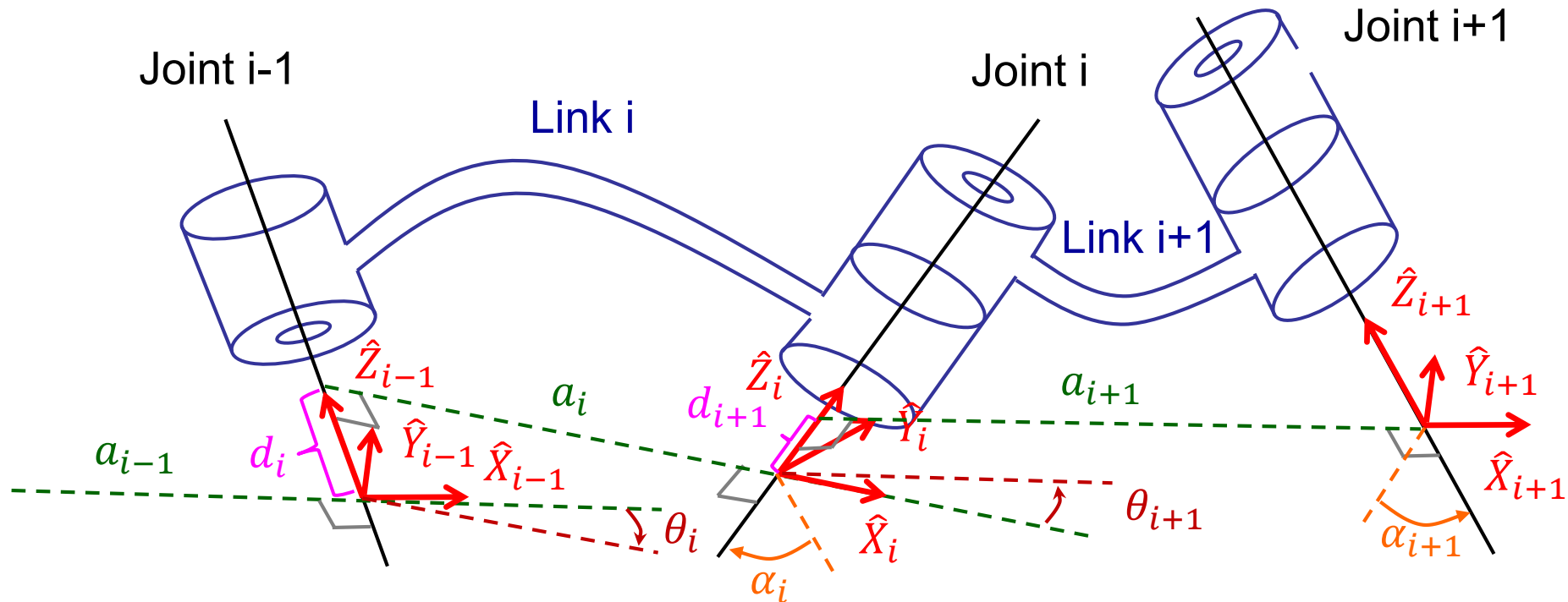
$$= T_{\hat{X}_{i-1}}(\alpha_{i-1}) T_{\hat{X}_R}(a_{i-1}) T_{\hat{Z}_Q}(\theta_i) T_{\hat{Z}_P}(d_i)$$

$$= \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



# Denavit-Hartenberg表達法 (Standard) -1

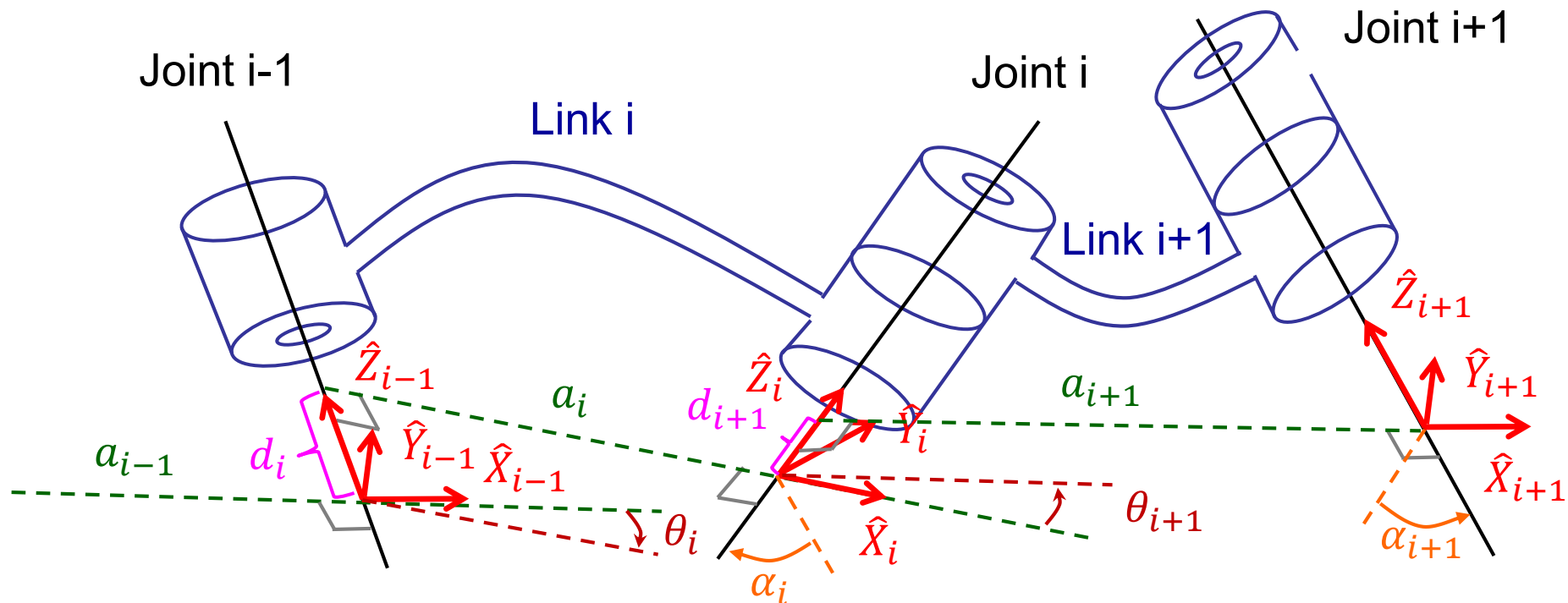
- $\theta_i$ : 以 $\hat{Z}_{i-1}$ 方向看， $\hat{X}_{i-1}$ 和 $\hat{X}_i$ 間的夾角
- $d_i$ : 沿著 $\hat{Z}_{i-1}$ 方向， $\hat{X}_{i-1}$ 和 $\hat{X}_i$ 間的距離
- $a_i$ : 沿著 $\hat{X}_i$ 方向， $\hat{Z}_{i-1}$ 和 $\hat{Z}_i$ 間的距離 ( $a_i > 0$ )
- $\alpha_i$ : 以 $\hat{X}_i$ 方向看， $\hat{Z}_{i-1}$ 和 $\hat{Z}_i$ 間的夾角



# Summary of DH Notation (Standard) -2

$$\begin{aligned}
 {}^{i-1}_iT &= {}^{i-1}_R T_Q^R T_P^Q T_i^P T \\
 &= T_{\hat{Z}_{i-1}}(\theta_i) T_{\hat{Z}_R}(d_i) T_{\hat{X}_Q}(a_i) T_{\hat{X}_P}(\alpha_i)
 \end{aligned}$$

$$= \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



# Revisit Example: A RRR Manipulator -1

Craig DH

□ Joint axes

□ Common perpendiculars

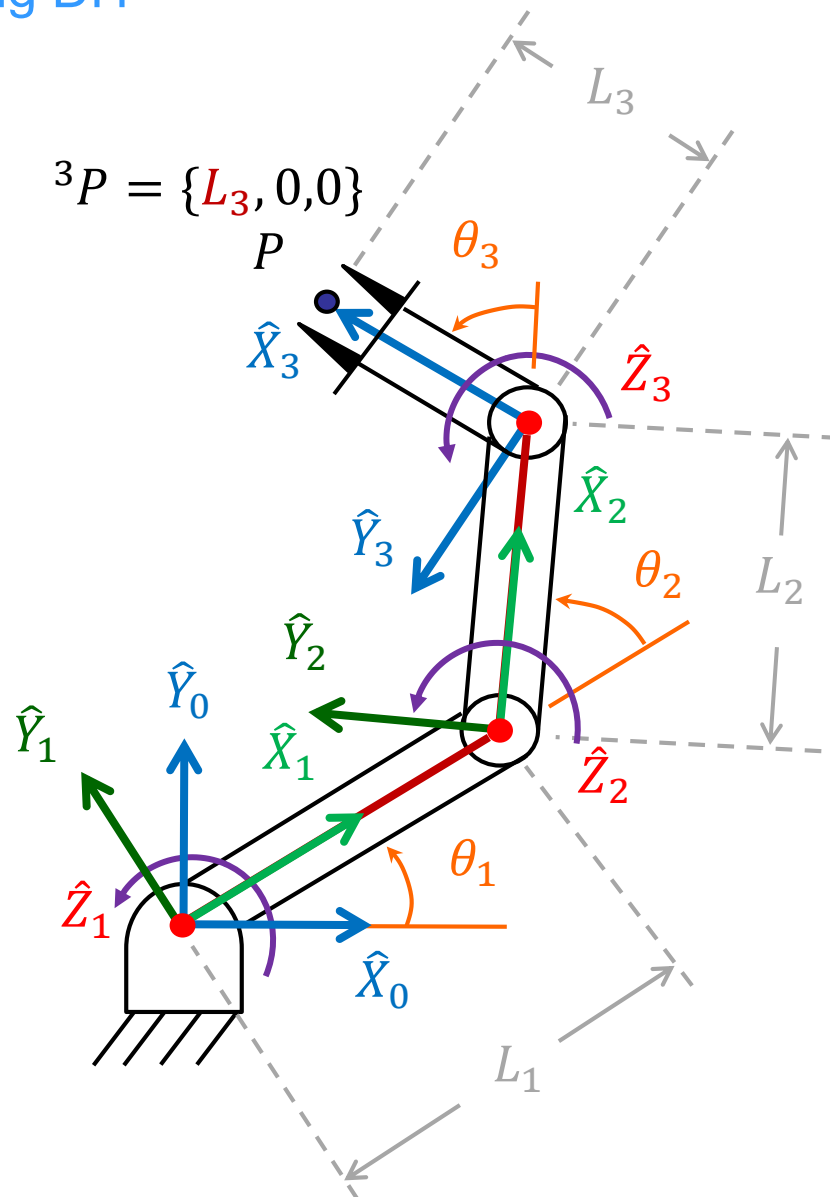
□  $\hat{Z}_i$

□  $\hat{X}_i$

□  $\hat{Y}_i$

□ Frames  $\{0\}$  and  $\{n\}$

i	$\alpha_{i-1}$	$a_{i-1}$	$d_i$	$\theta_i$
1	0	0	0	$\theta_1$
2	0	$L_1$	0	$\theta_2$
3	0	$L_2$	0	$\theta_3$



# Revisit Example: A RRR Manipulator -2

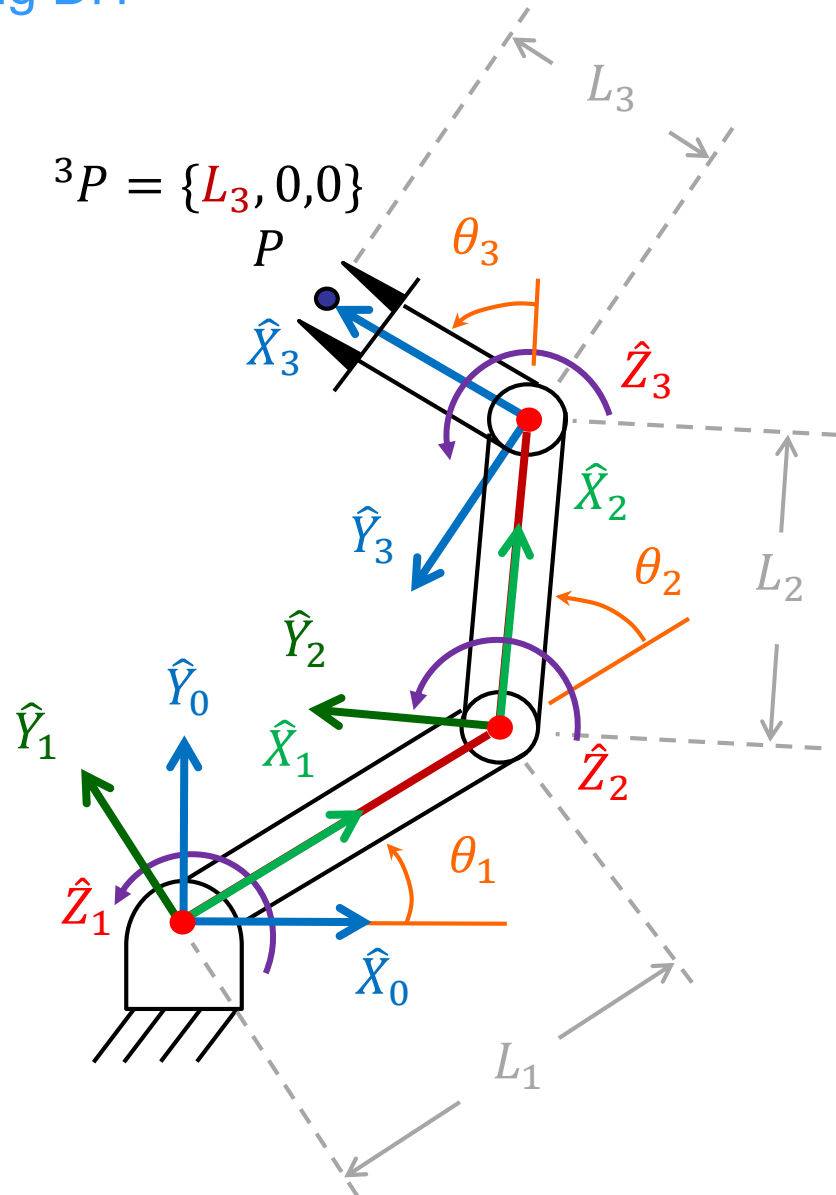
Craig DH

## Transformation matrices

$${}^0_1T = \begin{pmatrix} \cos[t1] & -\sin[t1] & 0 & 0 \\ \sin[t1] & \cos[t1] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^1_2T = \begin{pmatrix} \cos[t2] & -\sin[t2] & 0 & L1 \\ \sin[t2] & \cos[t2] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^2_3T = \begin{pmatrix} \cos[t3] & -\sin[t3] & 0 & L2 \\ \sin[t3] & \cos[t3] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

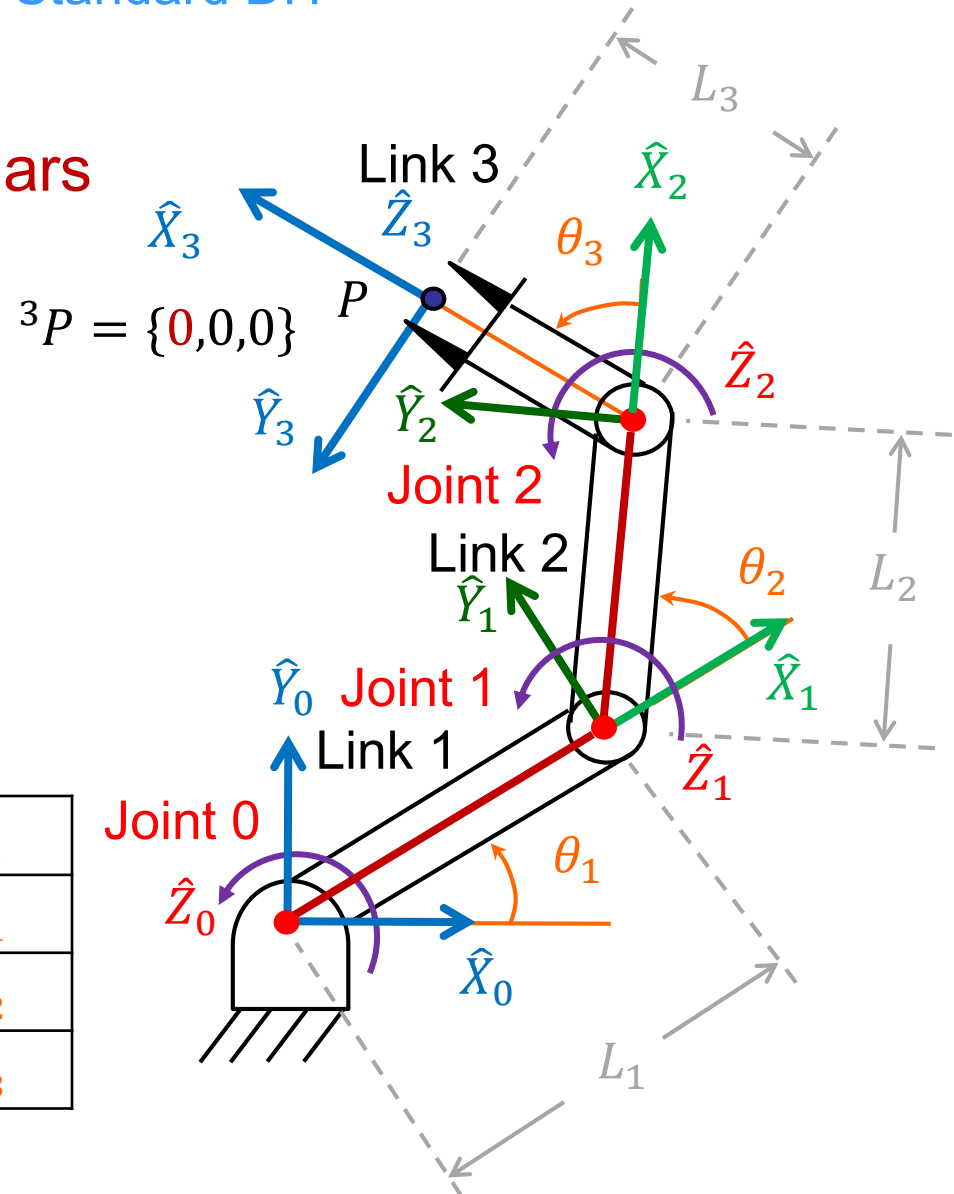


# Revisit Example: A RRR Manipulator -3

## Standard DH

- Joint axes
- Common perpendiculars
- $\hat{Z}_i$
- $\hat{X}_i$
- $\hat{Y}_i$
- Frames  $\{0\}$  and  $\{n\}$

i	$\alpha_i$	$a_i$	$d_i$	$\theta_i$
1	0	$L_1$	0	$\theta_1$
2	0	$L_2$	0	$\theta_2$
3	0	$L_3$	0	$\theta_3$



# Revisit Example: A RRR Manipulator -4

Standard DH

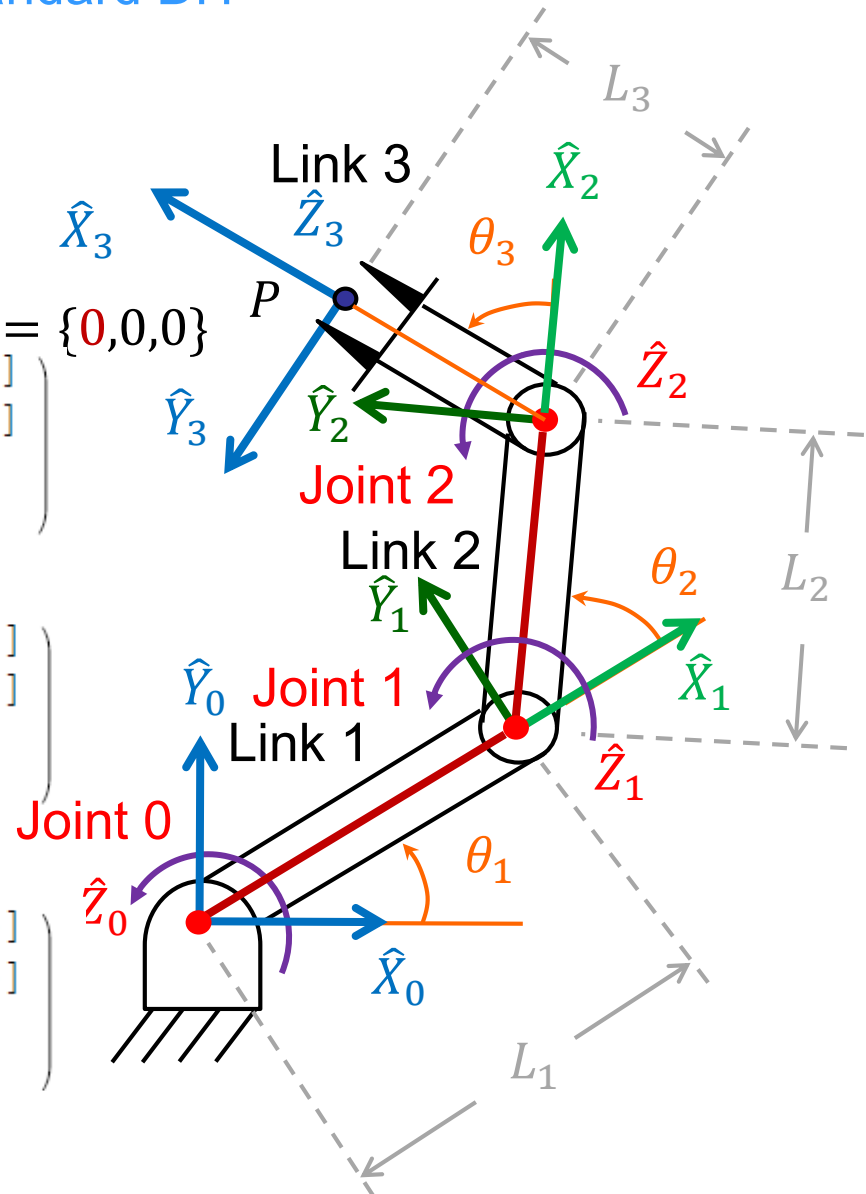
## Transformation matrices

$${}^0_1T = \begin{pmatrix} \cos[t1] & -\sin[t1] & 0 & L1 \cos[t1] \\ \sin[t1] & \cos[t1] & 0 & L1 \sin[t1] \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^1_2T = \begin{pmatrix} \cos[t2] & -\sin[t2] & 0 & L2 \cos[t2] \\ \sin[t2] & \cos[t2] & 0 & L2 \sin[t2] \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^2_3T = \begin{pmatrix} \cos[t3] & -\sin[t3] & 0 & L3 \cos[t3] \\ \sin[t3] & \cos[t3] & 0 & L3 \sin[t3] \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^3P = \{0,0,0\}$$





# Revisit Example: A RRR Manipulator -5

## ◆ Craig

$${}^0_3T \begin{pmatrix} \begin{matrix} \cos[t_1 + t_2 + t_3] & -\sin[t_1 + t_2 + t_3] & 0 \\ \sin[t_1 + t_2 + t_3] & \cos[t_1 + t_2 + t_3] & 0 \\ 0 & 0 & 1 \end{matrix} & \begin{matrix} L_1 \cos[t_1] + L_2 \cos[t_1 + t_2] \\ L_1 \sin[t_1] + L_2 \sin[t_1 + t_2] \\ 0 \end{matrix} \end{pmatrix}$$

$${}^0_3T \cdot T_{\hat{x}_3}([L_3, 0, 0])$$

$$\begin{pmatrix} \begin{matrix} \cos[t_1 + t_2 + t_3] & -\sin[t_1 + t_2 + t_3] & 0 \\ \sin[t_1 + t_2 + t_3] & \cos[t_1 + t_2 + t_3] & 0 \\ 0 & 0 & 1 \end{matrix} & \begin{matrix} L_1 \cos[t_1] + L_2 \cos[t_1 + t_2] + L_3 \cos[t_1 + t_2 + t_3] \\ L_1 \sin[t_1] + L_2 \sin[t_1 + t_2] + L_3 \sin[t_1 + t_2 + t_3] \\ 0 \end{matrix} \end{pmatrix}$$

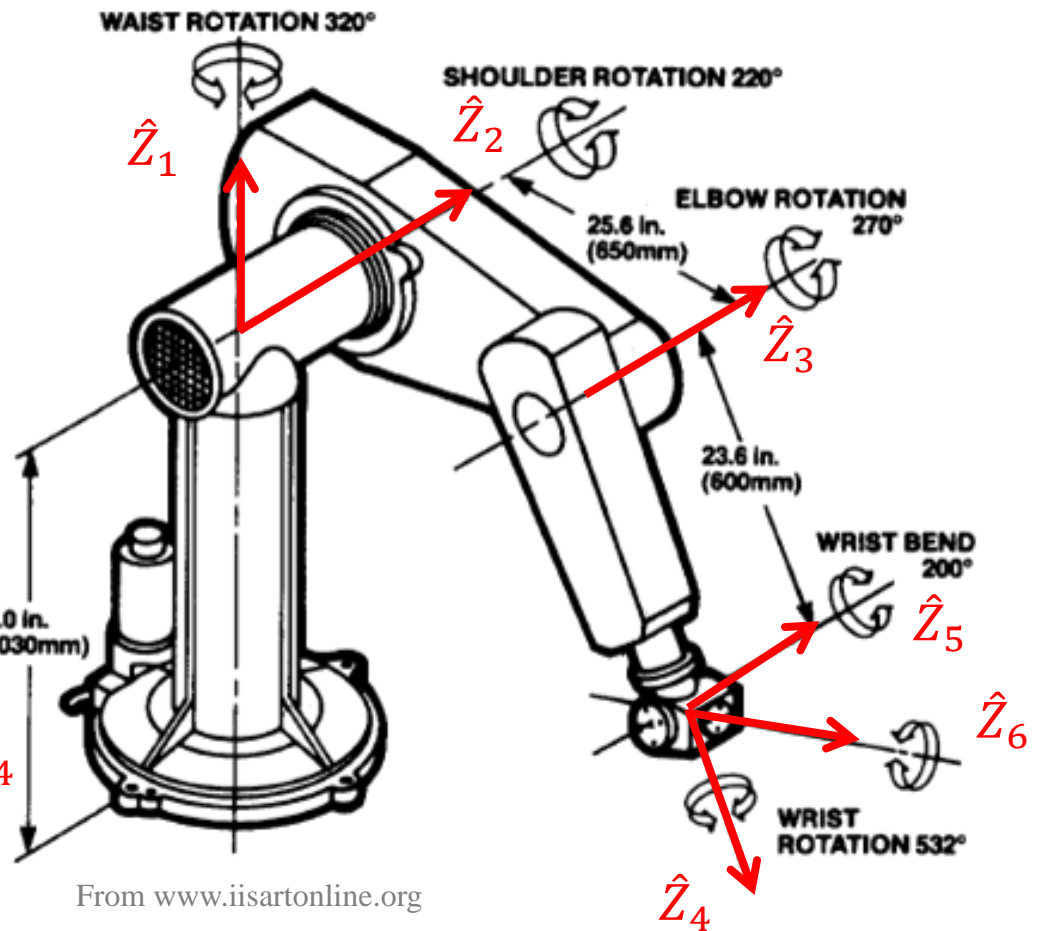
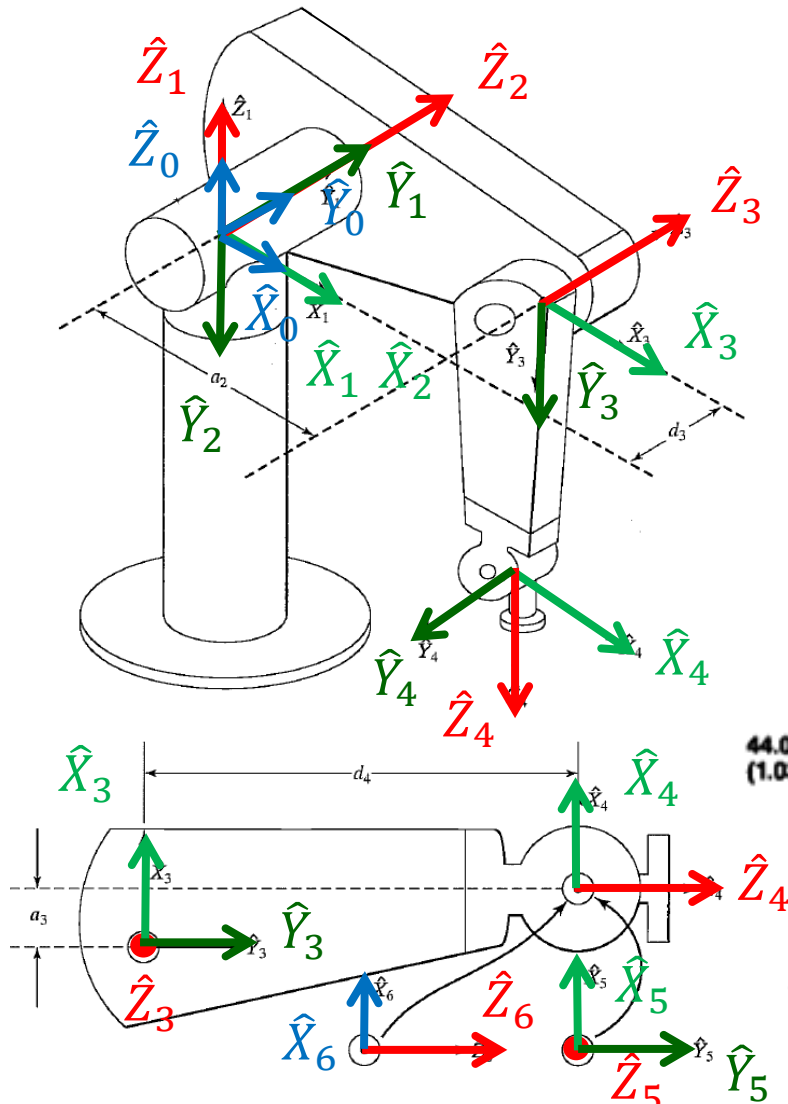
## ◆ Standard

${}^0_3T$

$$\begin{pmatrix} \begin{matrix} \cos[t_1 + t_2 + t_3] & -\sin[t_1 + t_2 + t_3] & 0 \\ \sin[t_1 + t_2 + t_3] & \cos[t_1 + t_2 + t_3] & 0 \\ 0 & 0 & 1 \end{matrix} & \begin{matrix} L_1 \cos[t_1] + L_2 \cos[t_1 + t_2] + L_3 \cos[t_1 + t_2 + t_3] \\ L_1 \sin[t_1] + L_2 \sin[t_1 + t_2] + L_3 \sin[t_1 + t_2 + t_3] \\ 0 \end{matrix} \end{pmatrix}$$

# Example: PUMA 560 -1

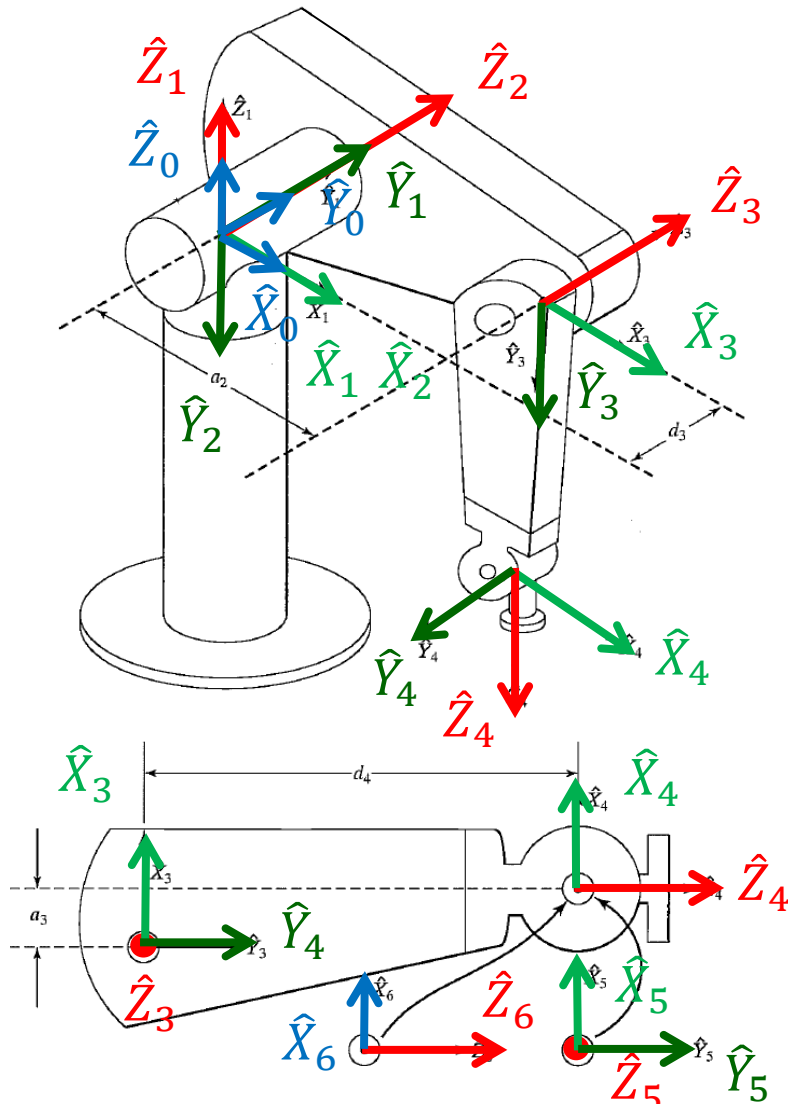
## □ Frames (Craig)



From [www.iisartonline.org](http://www.iisartonline.org)

# Example: PUMA 560 -2

## □ DH parameters (Craig)



$i$	$\alpha_{i-1}$	$a_{i-1}$	$d_i$	$\theta_i$
1	$0^\circ$	0	0	$\theta_1$
2	$-90^\circ$	0	0	$\theta_2$
3	$0^\circ$	$a_2$	$d_3$	$\theta_3$
4	$-90^\circ$	$a_3$	$d_4$	$\theta_4$
5	$90^\circ$	0	0	$\theta_5$
6	$-90^\circ$	0	0	$\theta_6$

## Example: PUMA 560 -3

### □ Transformation matrices

$${}^0_1T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1_2T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_2 & -c\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^2_3T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^3_4T = \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & a_3 \\ 0 & 0 & 1 & d_4 \\ -s\theta_4 & -c\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

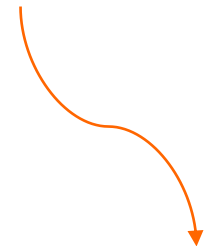
$${}^4_5T = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^5_6T = \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_6 & -c\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## Example: PUMA 560 -4

### □ Combining transformation matrices -1

$${}^4_6T = {}^4_5T {}^5_6T = \begin{bmatrix} c_5 c_6 & -c_5 s_6 & -s_5 & 0 \\ s_6 & c_6 & 0 & 0 \\ s_5 c_6 & -s_5 s_6 & c_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$


$${}^3_6T = {}^3_4T {}^4_6T = \begin{bmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & -c_4 s_5 & a_3 \\ s_5 c_6 & -s_5 s_6 & c_5 & d_4 \\ -s_4 c_5 c_6 - c_4 s_6 & s_4 c_5 s_6 - c_4 c_6 & s_4 s_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1_3T = {}^1_2T {}^2_3T = \begin{bmatrix} c_{23} & -s_{23} & 0 & a_2 c_2 \\ 0 & 0 & 1 & d_3 \\ -s_{23} & -c_{23} & 0 & -a_2 s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## Example: PUMA 560 -5

### □ Combining transformation matrices -2

$${}^1_6T = {}^1_3T {}^3_6T = \begin{bmatrix} {}^1r_{11} & {}^1r_{12} & {}^1r_{13} & {}^1p_x \\ {}^1r_{21} & {}^1r_{22} & {}^1r_{23} & {}^1p_y \\ {}^1r_{31} & {}^1r_{32} & {}^1r_{33} & {}^1p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1r_{11} = c_{23}[c_4c_5c_6 - s_4s_6] - s_{23}s_5s_6$$

$${}^1r_{21} = -s_4c_5c_6 - c_4s_6$$

$${}^1r_{31} = -s_{23}[c_4c_5c_6 - s_4s_6] - c_{23}s_5c_6$$

$${}^1r_{12} = -c_{23}[c_4c_5s_6 + s_4c_6] + s_{23}s_5s_6$$

$${}^1r_{22} = s_4c_5s_6 - c_4c_6$$

$${}^1r_{32} = s_{23}[c_4c_5s_6 + s_4c_6] + c_{23}s_5s_6$$

$${}^1r_{13} = -c_{23}c_4s_5 - s_{23}c_5$$

$${}^1r_{23} = s_4s_5$$

$${}^1r_{33} = s_{23}c_4s_5 - c_{23}c_5$$

$${}^1p_x = a_2c_2 + a_3c_{23} - d_4s_{23}$$

$${}^1p_y = d_3$$

$${}^1p_z = -a_3s_{23} - a_2s_2 - d_4c_{23}$$

## Example: PUMA 560 -6

### □ Combining transformation matrices -3

$${}^0_6T = {}^0_1T {}^1_6T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$r_{11} = c_1[c_{23}(c_4c_5c_6 - s_4s_5) - s_{23}s_5c_5] + s_1(s_4c_5c_6 + c_4s_6)$$

$$r_{21} = s_1[c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6] - c_1(s_4c_5c_6 + c_4s_6)$$

$$r_{31} = -s_{23}(c_4c_5c_6 - s_4s_6) - c_{23}s_5c_6$$

$$r_{12} = c_1[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] + s_1(c_4c_6 - s_4c_5s_6)$$

$$r_{22} = s_1[s_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] - c_1(c_4c_6 - s_4c_5s_6)$$

$$r_{32} = -s_{23}(-c_4c_5s_6 - s_4c_6) + c_{23}s_5s_6$$

$$r_{13} = -c_1(c_{23}c_4s_5 + s_{23}c_5) - s_1s_4s_5$$

$$r_{23} = -s_1(c_{23}c_4s_5 + s_{23}c_5) + c_1s_4s_5$$

$$r_{33} = s_{23}c_4s_5 - c_{23}c_5$$

$$p_x = c_1[a_2c_2 + a_3c_{23} - d_4s_{23}] - d_3s_1$$

$$p_y = s_1[a_2c_2 + a_3c_{23} - d_4s_{23}] + d_3c_1$$

$$p_z = -a_3s_{23} - a_2s_2 - d_4c_{23}$$