剛體運動狀態描述

林沛群
國立台灣大學
機械工程學系

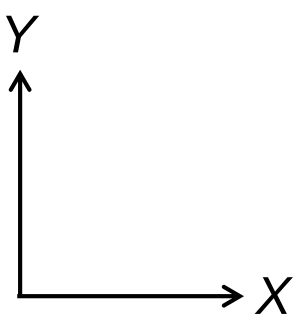
導讀 -1

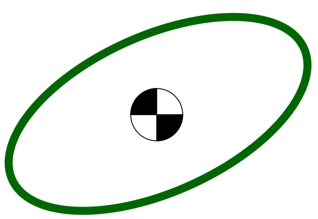
- 一個剛體(Rigid body)的狀態該如何描述？
  - 平面：

    移動 2 DOFs、轉動 1 DOF

    Degree of freedom

$\{W\}$ world frame





導讀 -2

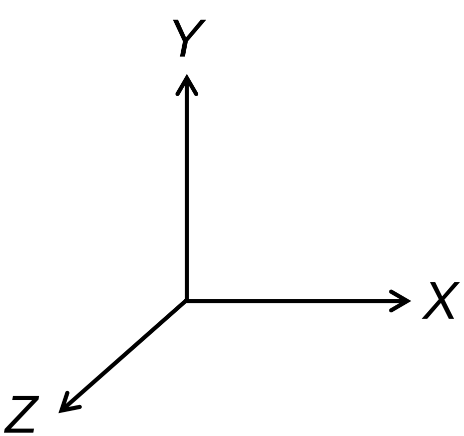
- 一個剛體(Rigid body)的狀態該如何描述？
  - 平面：

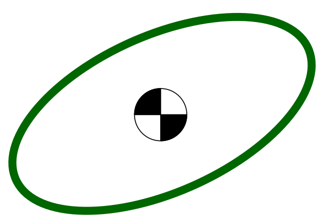
    移動 2 DOFs、轉動 1 DOF

    Degree of freedom
  - 空間：

    移動 3 DOFs、轉動 3 DOFs

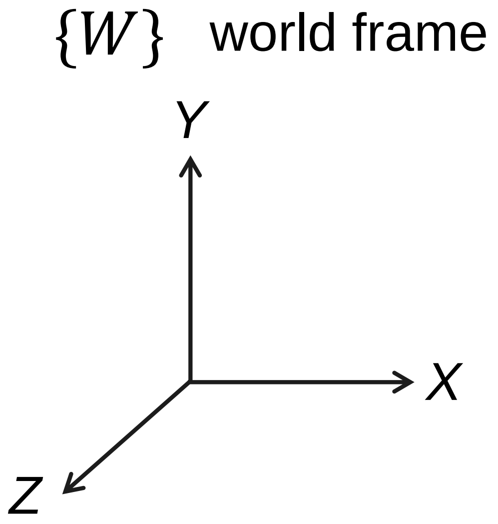
$\{W\}$ world frame

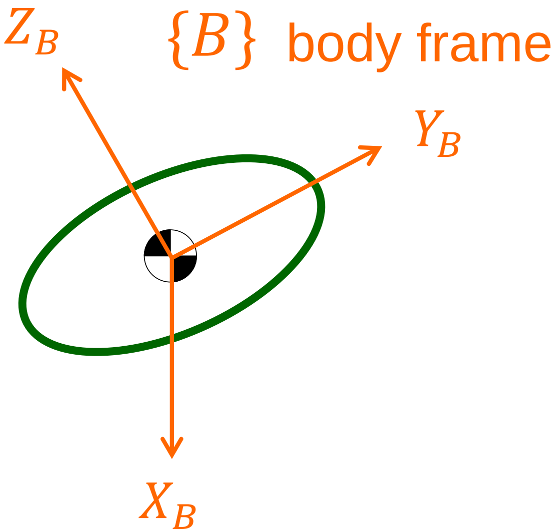




導讀 -3

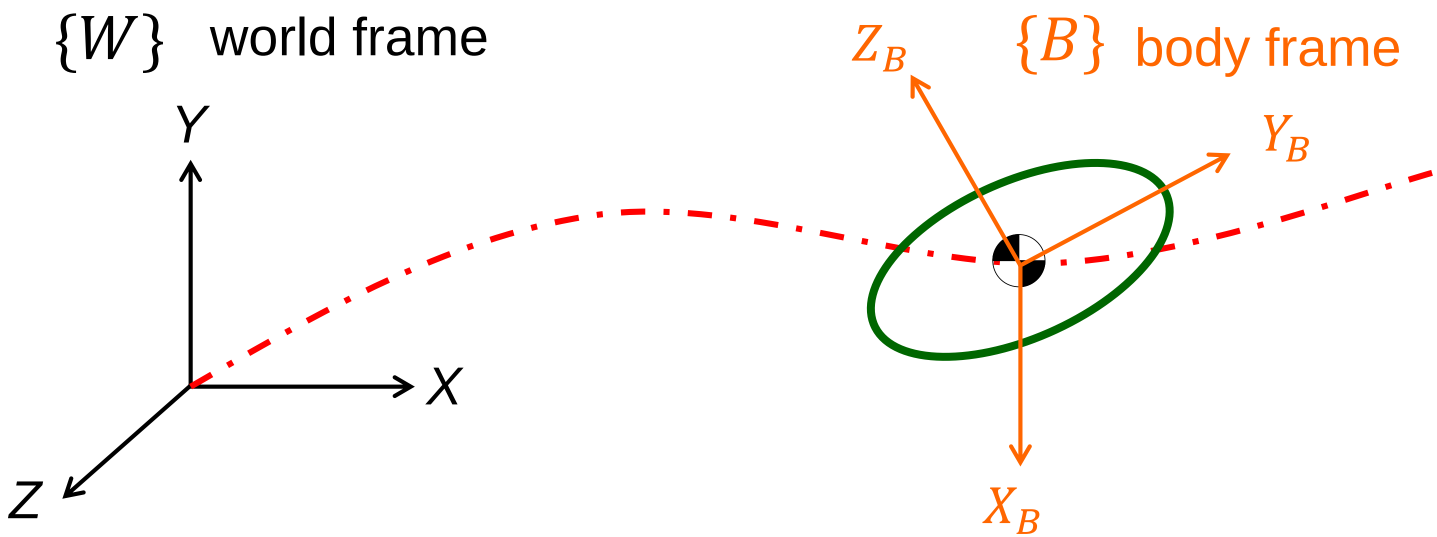
- 該如何整合表達剛體的狀態？
  - ⇨ 在剛體(Rigid body)上建立frame，常建立在質心上
    - 移動：由body frame 的原點位置判定
    - 轉動：由body frame的姿態判定





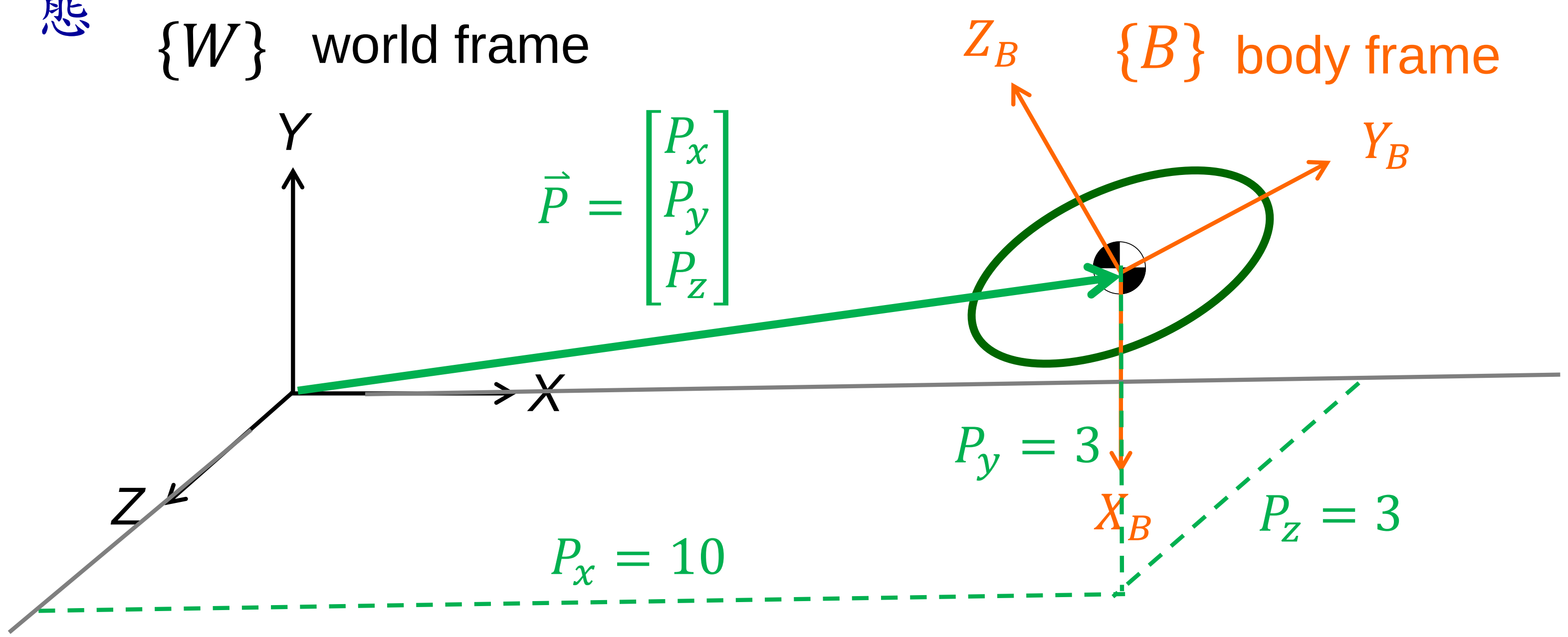
導讀 -4

- 一個剛體(Rigid body)的「運動」狀態該如何描述？
  - 利用各個DOF的微分，將位移和姿態（displacement / orientation）轉換到速度（velocity）和加速度（acceleration）等運動狀態



移動 -1

- 移動：以向量（vector）$\vec{P}$ 來描述$\{B\}$的原點相對於$\{A\}$的狀態



- Ex: $\vec{P} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$

移動 -2

- 向量可表達空間關係的兩個方式
  - A position in space (i.e., position vector)

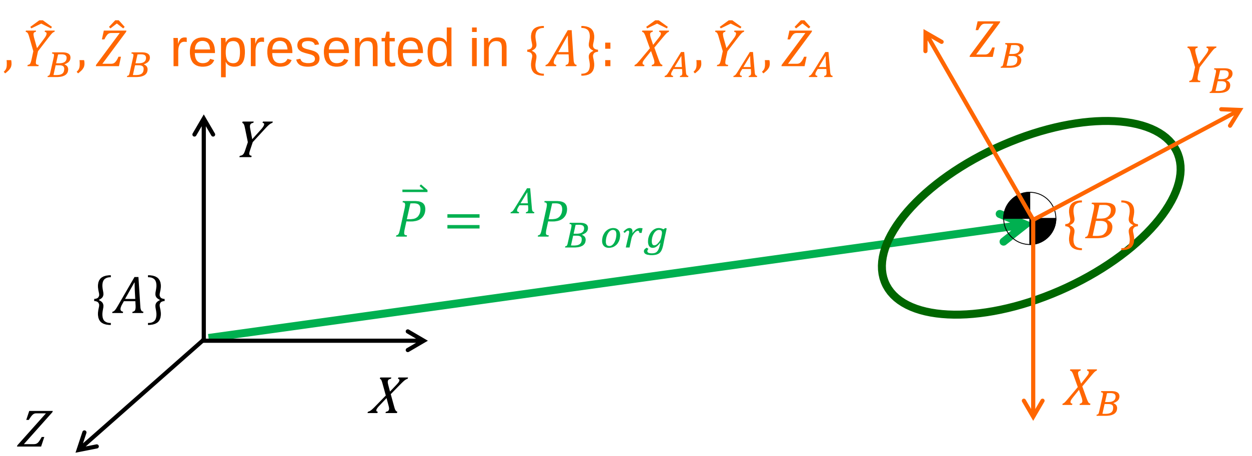
    如同前一頁內容，以此方式描述body frame原點

    $$\vec{P} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = {}^{A}P_{B\,org} = \text{origin of } \{B\} \text{ represented in } \{A\}$$

  - A vector (i.e., displacement, frame basis)

    以此方式表達body frame上principal axes的方向

    $\{B\}$: $\hat{X}_B, \hat{Y}_B, \hat{Z}_B$ represented in $\{A\}$: $\hat{X}_A, \hat{Y}_A, \hat{Z}_A$



轉動 -1

- 轉動：描述$\{B\}$相對於$\{A\}$之姿態---Rotation Matrix

$$
{}^{A}_{B}R = \begin{bmatrix} | & | & | \\ {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \\ | & | & | \end{bmatrix}
$$

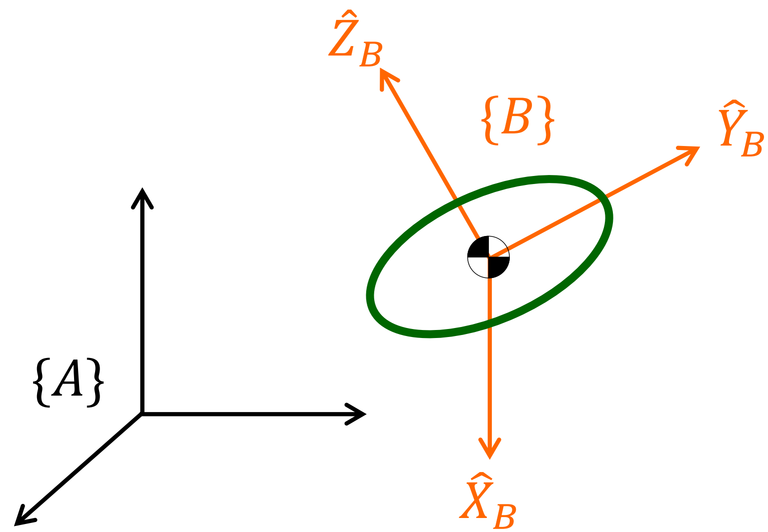
B relative to A

"column vector"

$$
= \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}
$$

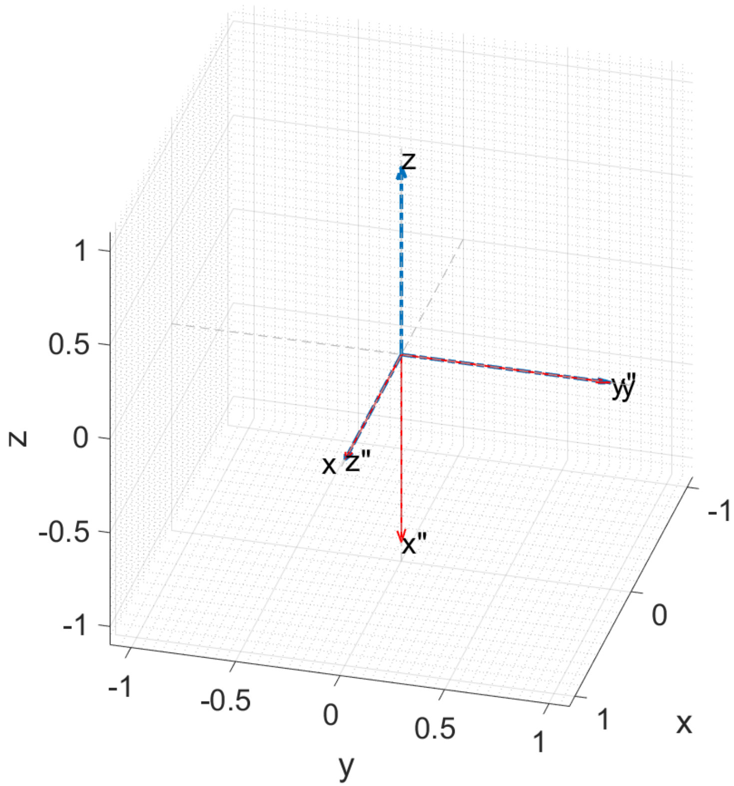
"direct cosines"

R的三個columns即為frame $\{B\}$的$basis$: $\hat{X}_B, \hat{Y}_B, \hat{Z}_B$（由$\{A\}$看）



轉動 -2

- Ex: $\{B\}$相對於$\{A\}$之姿態 ${}^{A}_{B}R = ?$



藍虛線: World Frame $\{A\}$

紅實線: Body Frame $\{B\}$

$\{B\}$的x″軸為$\{A\}$的z軸反向 $\Rightarrow\ {}^{A}\hat{X}_B = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

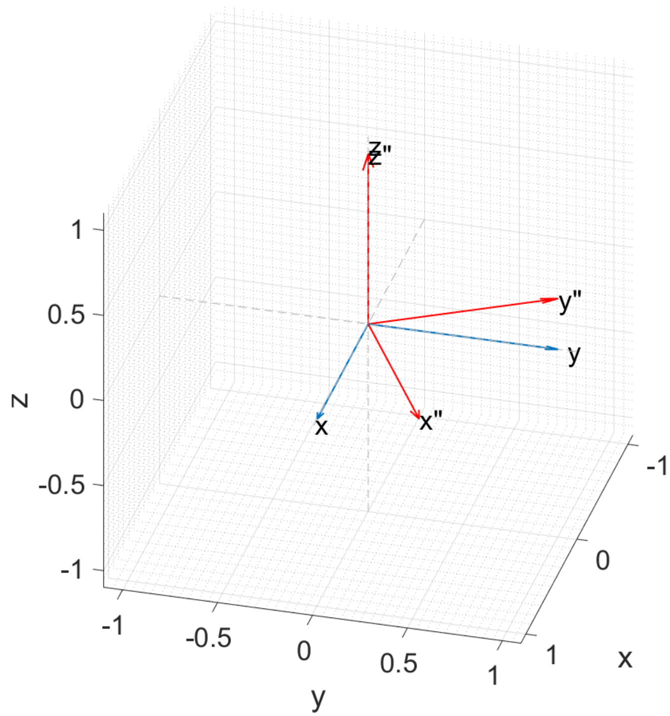
$\{B\}$的y″軸與$\{A\}$的y軸重疊 $\Rightarrow\ {}^{A}\hat{Y}_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\{B\}$的z″軸與$\{A\}$的x軸重疊 $\Rightarrow\ {}^{A}\hat{Z}_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

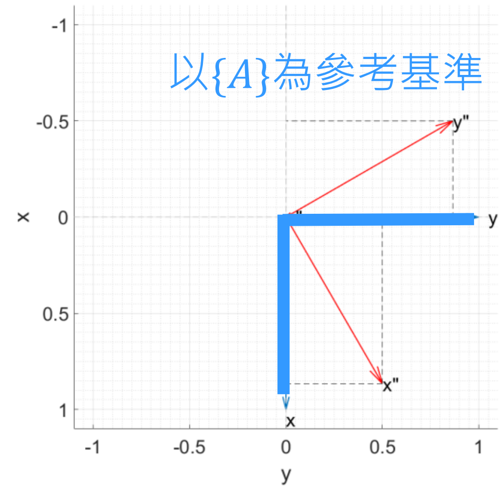
因此，$\{B\}$相對於$\{A\}$之姿態： ${}^{A}_{B}R = \begin{bmatrix} | & | & | \\ {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

轉動 -3

- Ex: $\{B\}$相對於$\{A\}$之姿態 ${}^{A}_{B}R = ?$



上視圖



藍虛線: World Frame $\{A\}$

紅實線: Body Frame $\{B\}$

$$
{}^{A}\hat{X}_B = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A \end{bmatrix} = \begin{bmatrix} 0.866 \\ 0.5 \\ 0 \end{bmatrix}
$$

$$
{}^{A}\hat{Y}_B = \begin{bmatrix} \hat{Y}_B \cdot \hat{X}_A \\ \hat{Y}_B \cdot \hat{Y}_A \\ \hat{Y}_B \cdot \hat{Z}_A \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.866 \\ 0 \end{bmatrix}
$$

$$
{}^{A}\hat{Z}_B = \begin{bmatrix} \hat{Z}_B \cdot \hat{X}_A \\ \hat{Z}_B \cdot \hat{Y}_A \\ \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

$\{B\}$相對於$\{A\}$之姿態：

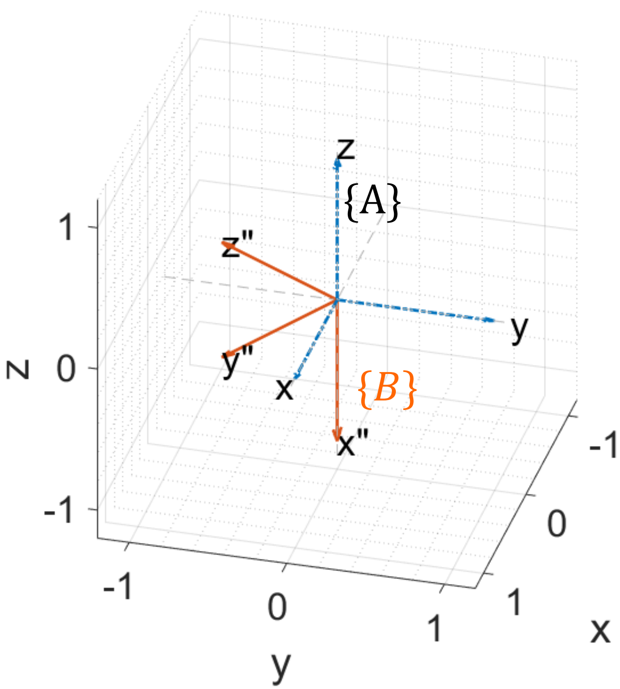
$$
{}^{A}_{B}R = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

轉動 -4

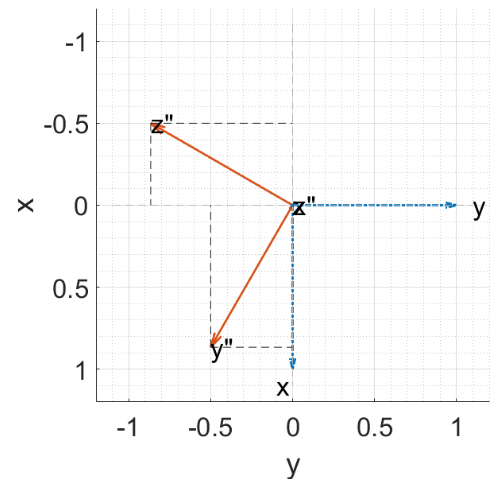
- In-video Quiz: $\{B\}$相對於$\{A\}$之姿態 ${}^{A}_{B}R = ?$

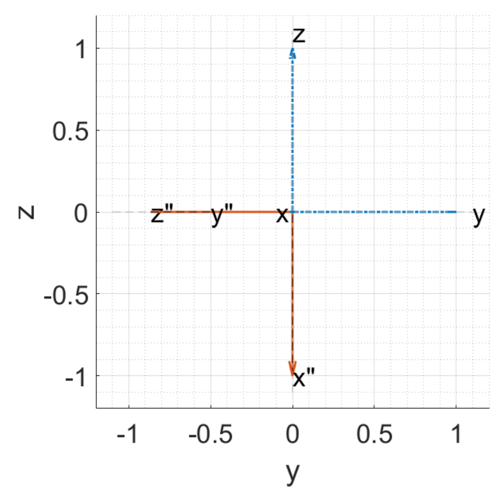
藍虛線: World Frame $\{A\}$

紅實線: Body Frame $\{B\}$



投影到
XY/YZ平面





A. $\begin{bmatrix} 0 & -0.866 & 0.5 \\ 0 & 0.5 & 0.866 \\ -1 & 0 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 0.866 & -0.5 \\ 0 & -0.5 & -0.866 \\ -1 & 0 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 0 & -1 \\ 0.866 & -0.5 & 0 \\ -0.5 & -0.866 & 0 \end{bmatrix}$

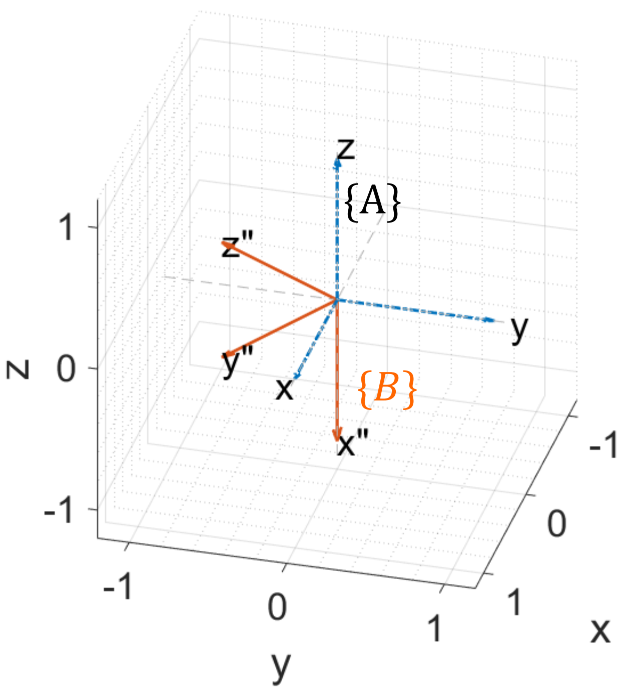
D. $\begin{bmatrix} 0 & 0 & -1 \\ -0.866 & 0.5 & 0 \\ 0.5 & 0.866 & 0 \end{bmatrix}$

轉動 -4

- In-video Quiz: $\{B\}$相對於$\{A\}$之姿態 ${}^{A}_{B}R=?$

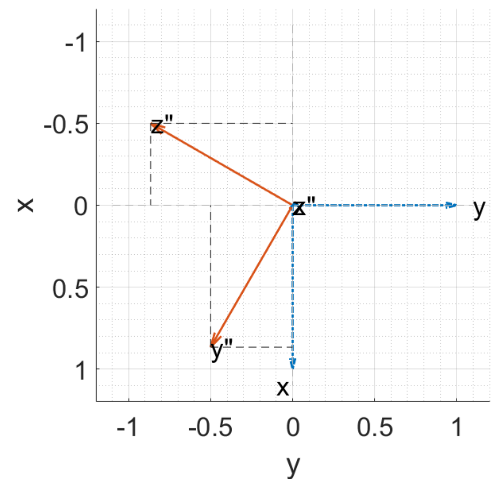
藍虛線: World Frame $\{A\}$

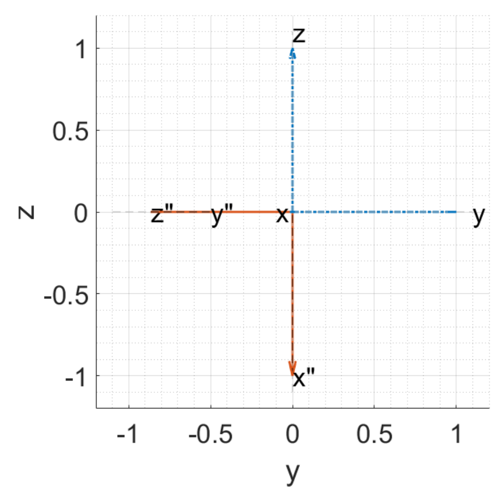
紅實線: Body Frame $\{B\}$



投影到

XY/YZ平面





A. $\begin{bmatrix} 0 & -0.866 & 0.5 \\ 0 & 0.5 & 0.866 \\ -1 & 0 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 0.866 & -0.5 \\ 0 & -0.5 & -0.866 \\ -1 & 0 & 0 \end{bmatrix}$

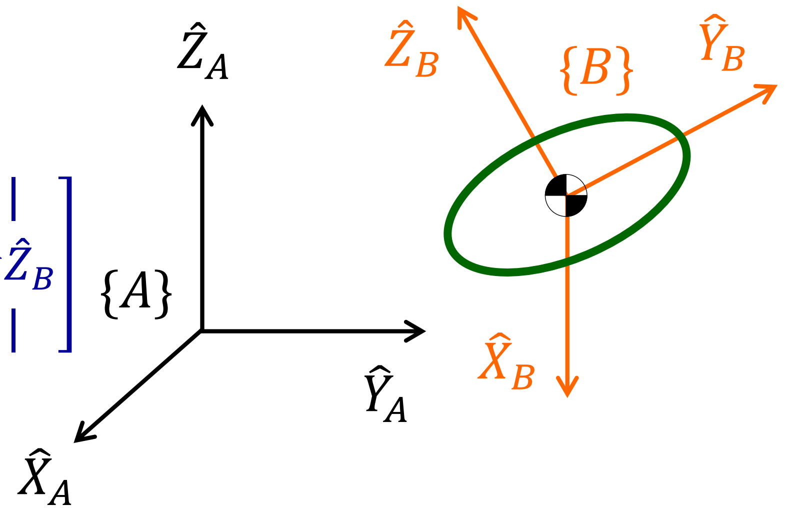
C. $\begin{bmatrix} 0 & 0 & -1 \\ 0.866 & -0.5 & 0 \\ -0.5 & -0.866 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 0 & -1 \\ -0.866 & 0.5 & 0 \\ 0.5 & 0.866 & 0 \end{bmatrix}$

Rotation Matrix -1

- 特性

$$ {}^{A}_{B}R = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} = \begin{bmatrix} | & | & | \\ {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \\ | & | & | \end{bmatrix} $$



前後向量互換

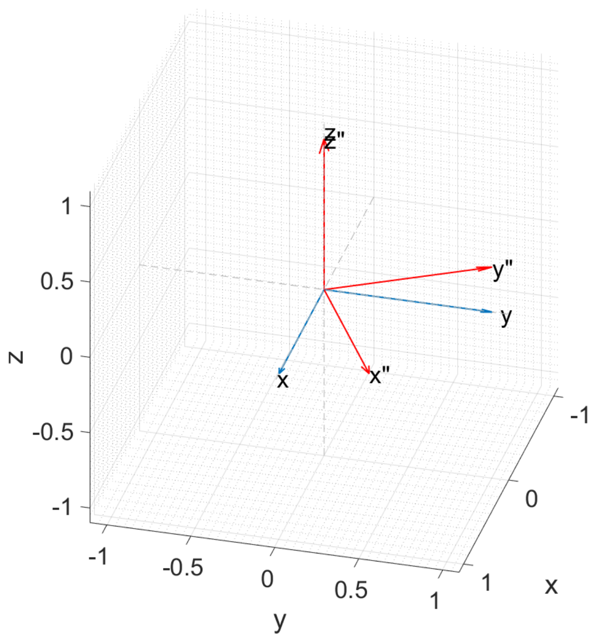
$$ = \begin{bmatrix} \hat{X}_A \cdot \hat{X}_B & \hat{X}_A \cdot \hat{Y}_B & \hat{X}_A \cdot \hat{Z}_B \\ \hat{Y}_A \cdot \hat{X}_B & \hat{Y}_A \cdot \hat{Y}_B & \hat{Y}_A \cdot \hat{Z}_B \\ \hat{Z}_A \cdot \hat{X}_B & \hat{Z}_A \cdot \hat{Y}_B & \hat{Z}_A \cdot \hat{Z}_B \end{bmatrix} = \begin{bmatrix} - & {}^{B}\hat{X}_A{}^{T} & - \\ - & {}^{B}\hat{Y}_A{}^{T} & - \\ - & {}^{B}\hat{Z}_A{}^{T} & - \end{bmatrix} $$

$$ = \begin{bmatrix} | & | & | \\ {}^{B}\hat{X}_A & {}^{B}\hat{Y}_A & {}^{B}\hat{Z}_A \\ | & | & | \end{bmatrix}^{T} = {}^{B}_{A}R^{T} $$

⇨ ${}^{A}_{B}R = {}^{B}_{A}R^{T}$

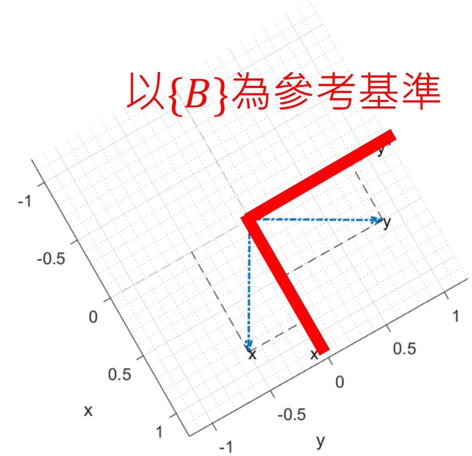
Rotation Matrix -2

- Ex: $\{A\}$相對於$\{B\}$之姿態 ${}^{B}_{A}R = ?$



上視圖

以$\{B\}$為參考基準



藍虛線: World Frame $\{A\}$

紅實線: Body Frame $\{B\}$

$${}^{B}\hat{X}_A = \begin{bmatrix} \hat{X}_A \cdot \hat{X}_B \\ \hat{Y}_A \cdot \hat{X}_B \\ \hat{Z}_A \cdot \hat{X}_B \end{bmatrix} = \begin{bmatrix} 0.866 \\ -0.5 \\ 0 \end{bmatrix}$$

$${}^{B}\hat{Y}_A = \begin{bmatrix} \hat{X}_A \cdot \hat{Y}_B \\ \hat{Y}_A \cdot \hat{Y}_B \\ \hat{Z}_A \cdot \hat{Y}_B \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.866 \\ 0 \end{bmatrix}$$

$${}^{B}\hat{Z}_A = \begin{bmatrix} \hat{X}_A \cdot \hat{Z}_B \\ \hat{Y}_A \cdot \hat{Z}_B \\ \hat{Z}_A \cdot \hat{Z}_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\{A\}$相對於$\{B\}$之姿態：

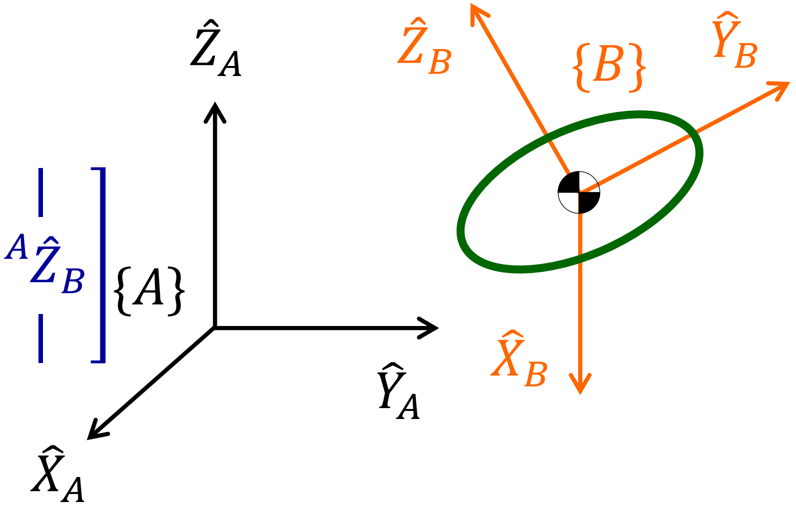
$${}^{B}_{A}R = \begin{bmatrix} 0.866 & 0.5 & 0 \\ -0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} = {}^{A}_{B}R^{T}$$

${}^{A}_{B}R$，「轉動 -3」頁面結果

Rotation Matrix -3

- 特性

$$
{}^A_BR^T\,{}^A_BR = \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix}^T \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix}
$$



$$
= \begin{bmatrix} - & {}^A\hat{X}_B{}^T & - \\ - & {}^A\hat{Y}_B{}^T & - \\ - & {}^A\hat{Z}_B{}^T & - \end{bmatrix} \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix}
$$

$$
= I_3
$$

3x3 identity matrix

$$
= {}^A_BR^{-1}\,{}^A_BR
$$

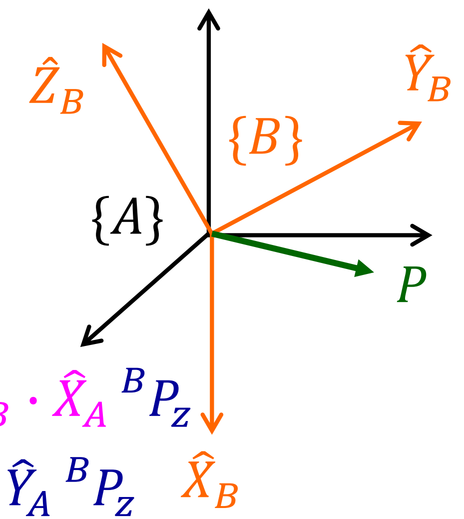
⇨ ${}^A_BR^T = {}^A_BR^{-1} = {}^B_AR$

Rotation Matrix -4

- A 3x3 orthogonal matrix Q $\quad QQ^T = Q^TQ = I$
  - Always invertible $\quad Q^{-1} = Q^T$
  - Columns: orthonormal basis
    - Length = 1
    - Mutually perpendicular
  - Rotation matrix (R)有9個數字，但上列兩個條件置入了6個 constraints，所以R只有3個DOFs，與空間中轉動具有3 DOFs相符
  - Determinant =1 (rotation); =-1 (reflection)

Rotation Matrix -5

- Rotation matrix 除描述$\{B\}$相對於$\{A\}$之姿態，也可用於轉換向量之座標



original coordinate $^{B}P = {}^{B}P_x \hat{X}_B + {}^{B}P_y \hat{Y}_B + {}^{B}P_z \hat{Z}_B$

new coordinate $^{A}P = {}^{A}P_x \hat{X}_A + {}^{A}P_y \hat{Y}_A + {}^{A}P_z \hat{Z}_A$

where $^{A}P_x = {}^{B}P \cdot \hat{X}_A = \hat{X}_B \cdot \hat{X}_A \, {}^{B}P_x + \hat{Y}_B \cdot \hat{X}_A \, {}^{B}P_y + \hat{Z}_B \cdot \hat{X}_A \, {}^{B}P_z$

$^{A}P_y = {}^{B}P \cdot \hat{Y}_A = \hat{X}_B \cdot \hat{Y}_A \, {}^{B}P_x + \hat{Y}_B \cdot \hat{Y}_A \, {}^{B}P_y + \hat{Z}_B \cdot \hat{Y}_A \, {}^{B}P_z$

$^{A}P_z = {}^{B}P \cdot \hat{Z}_A = \hat{X}_B \cdot \hat{Z}_A \, {}^{B}P_x + \hat{Y}_B \cdot \hat{Z}_A \, {}^{B}P_y + \hat{Z}_B \cdot \hat{Z}_A \, {}^{B}P_z$

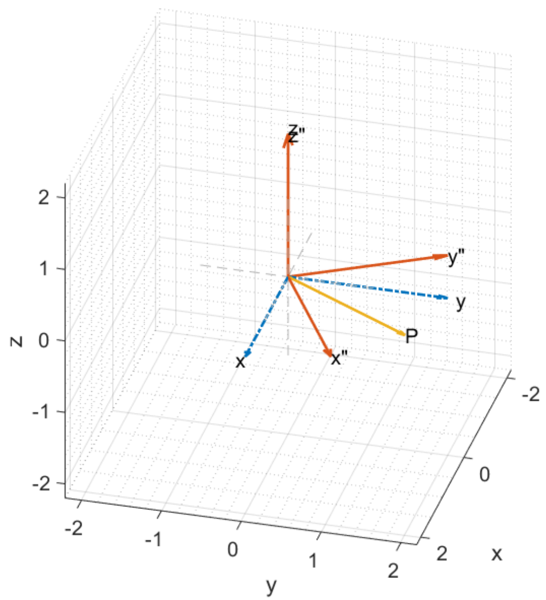
$$\Rightarrow \quad {}^{A}P = {}^{A}\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \boxed{\begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}} {}^{B}\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = {}^{A}_{B}R \, {}^{B}P$$

和「轉動-1」頁matrix相同，為rotation matrix

Rotation Matrix -6

- Ex: 若$\{B\}$和$\{A\}$ 的相對狀態同「轉動 -3」頁面所示，假設

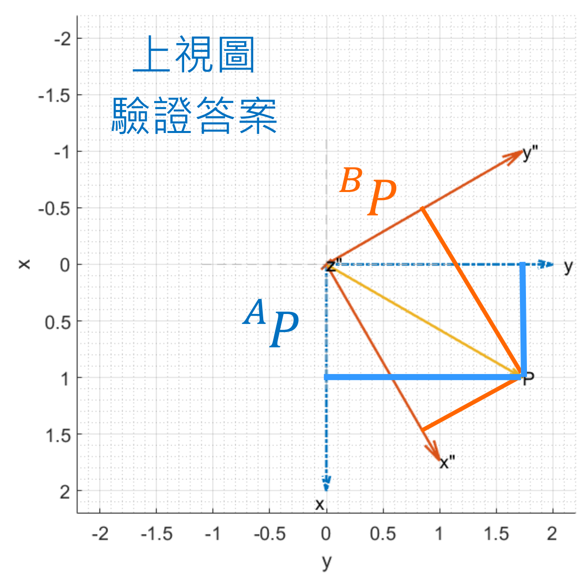
$$^{B}P = [1.732 \quad 1 \quad 0]^{T}，\ ^{A}P = ?$$



$$^{A}P = {}^{A}_{B}R\, ^{B}P$$

$$^{A}P = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.732 \\ 1 \\ 0 \end{bmatrix}$$

$$^{A}P = \begin{bmatrix} 1 \\ 1.732 \\ 0 \end{bmatrix}$$

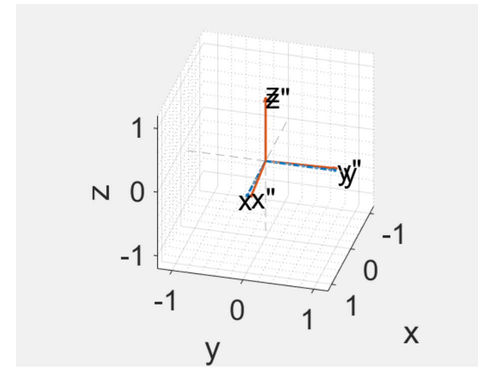


藍虛線: World Frame $\{A\}$

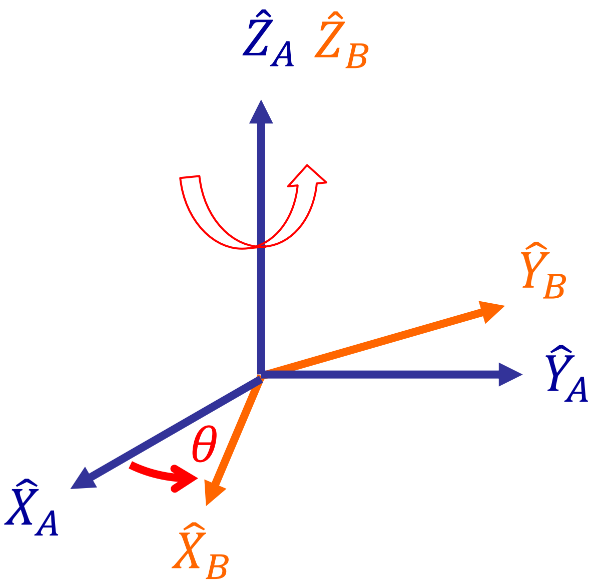
紅實線: Body Frame $\{B\}$

Rotation Matrix -7

- Rotation matrix的第三個功能，可進一步來描述物體「轉動」的狀態



- 以對三個principal axes旋轉的matrix為基礎
- About $\hat{Z}_A$ with θ



旋轉角度

$$R_{\hat{Z}_A}(\theta) = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

旋轉軸

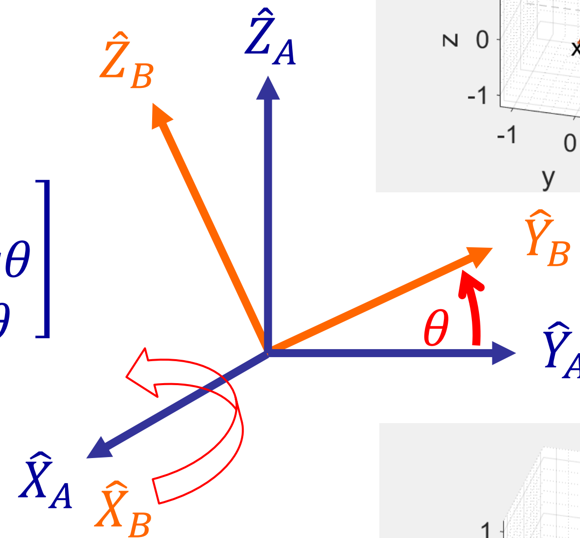
$$= \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

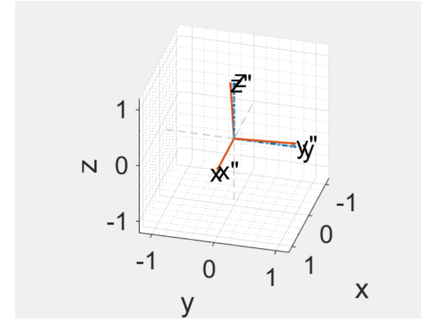
Note: $^{A}_{B}R = \begin{bmatrix} | & | & | \\ {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \\ | & | & | \end{bmatrix}$

Rotation Matrix -8

- About $\hat{X}_A$ with θ

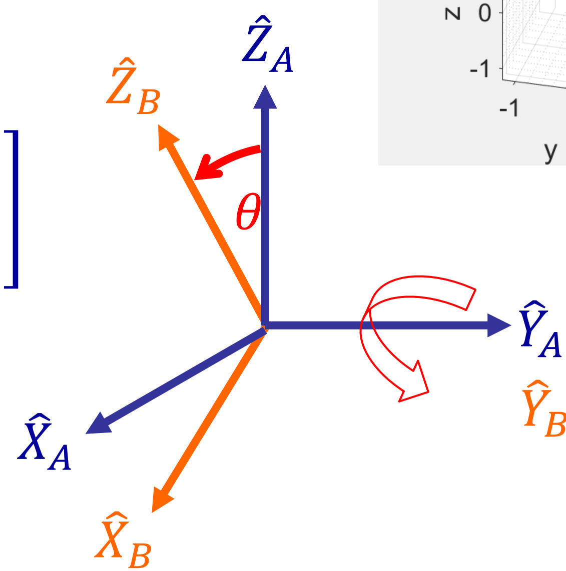
$$R_{\hat{X}_A}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}$$

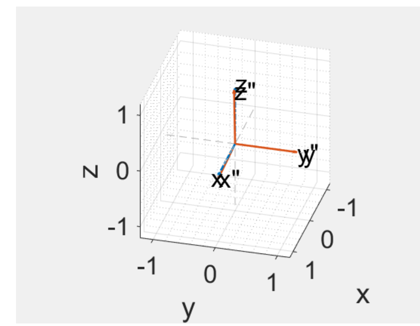




- About $\hat{Y}_A$ with θ

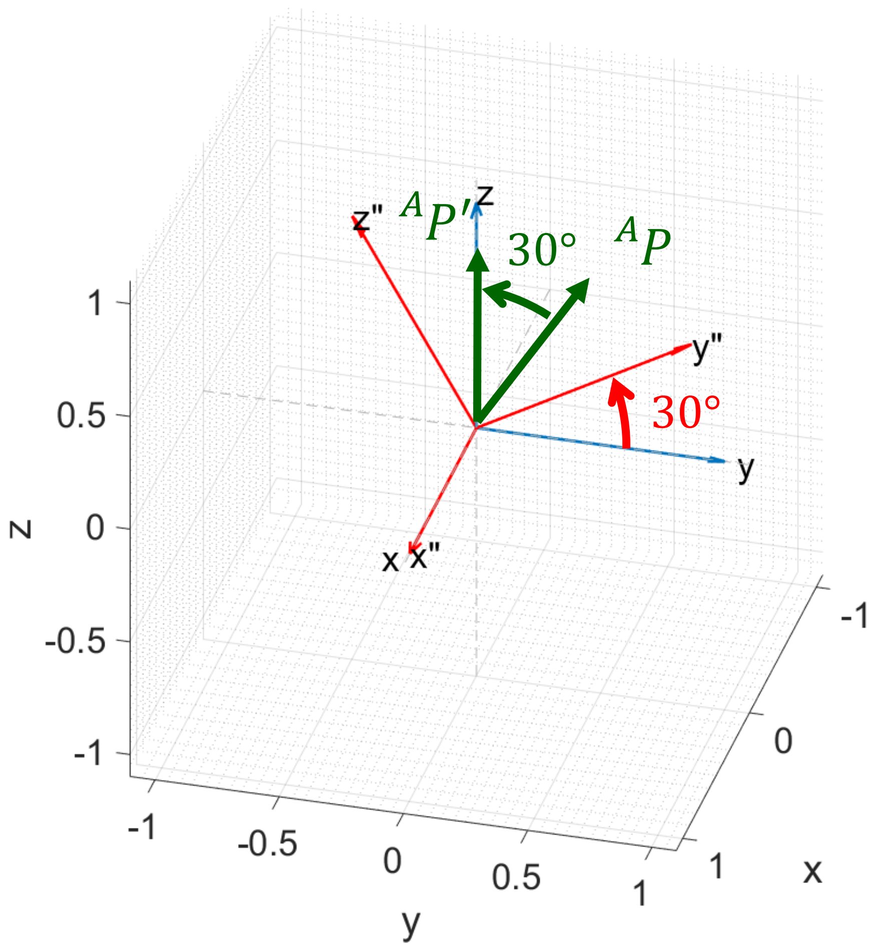
$$R_{\hat{Y}_A}(\theta) = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix} = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}$$





Rotation Matrix -9

- Ex: ${}^{A}P = [0 \quad 1 \quad 1.732]^T$ 對 $\hat{X}_A$ 軸旋轉 $30°$， ${}^{A}P' = ?$



$$R_{\hat{X}_A}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30° & -\sin 30° \\ 0 & \sin 30° & \cos 30° \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.866 & -0.5 \\ 0 & 0.5 & 0.866 \end{bmatrix}$$

$${}^{A}P' = R_{\hat{X}_A}(\theta)\, {}^{A}P$$

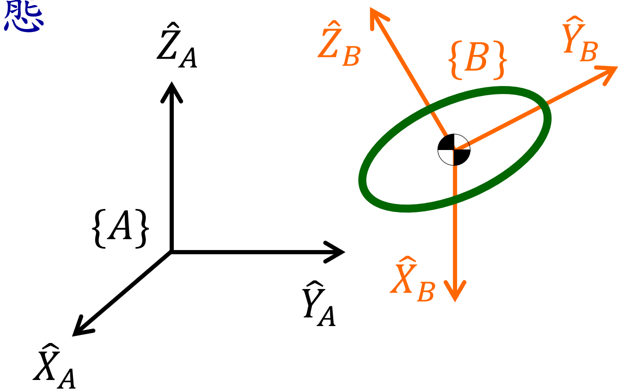
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.866 & -0.5 \\ 0 & 0.5 & 0.866 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1.732 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

Rotation Matrix -10

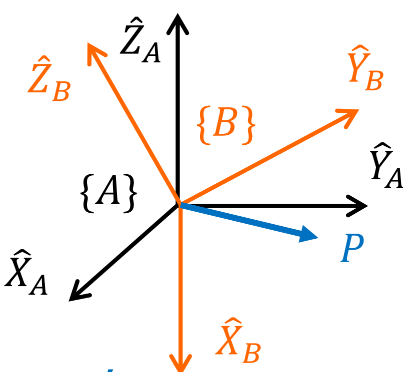
- Rotation matrix 的三種用法
  - 描述一個frame(相對於另一個frame)的姿態

$$ {}^{A}_{B}R = \begin{bmatrix} | & | & | \\ {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \\ | & | & | \end{bmatrix} $$



  - 將point由某一個frame的表達換到另一個和此frame僅有相對轉動的frame來表達

$$ {}^{A}P = {}^{A}_{B}R \, {}^{B}P $$



  - 將point(vector)在同一個frame中進行轉動

$$ {}^{A}P' = R(\theta) \, {}^{A}P $$