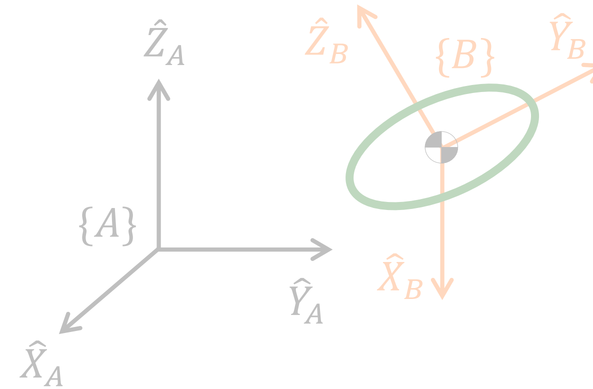
Rotation Matrix與轉角

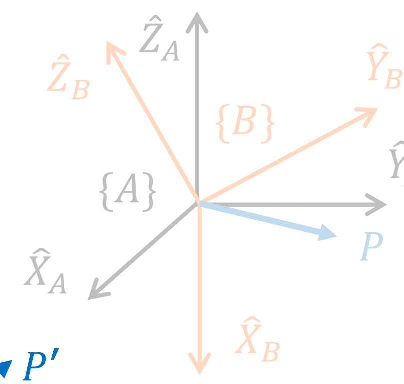
- Rotation matrix 的三種用法
  - 描述一個frame(相對於另一個frame)的姿態

$$
{}^{A}_{B}R = \begin{bmatrix} | & | & | \\ {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \\ | & | & | \end{bmatrix}
$$



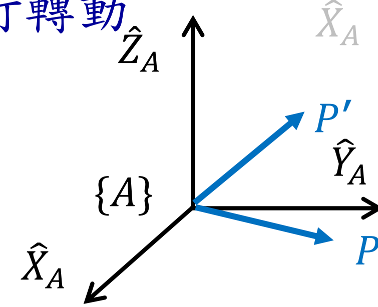
  - 將point由某一個frame的表達換到另一個和此frame僅有相對轉動的frame來表達

$$
{}^{A}P = {}^{A}_{B}R\, {}^{B}P
$$



  - 將point(vector)在同一個frame中進行轉動

$$
{}^{A}P' = R(\theta)\, {}^{A}P
$$

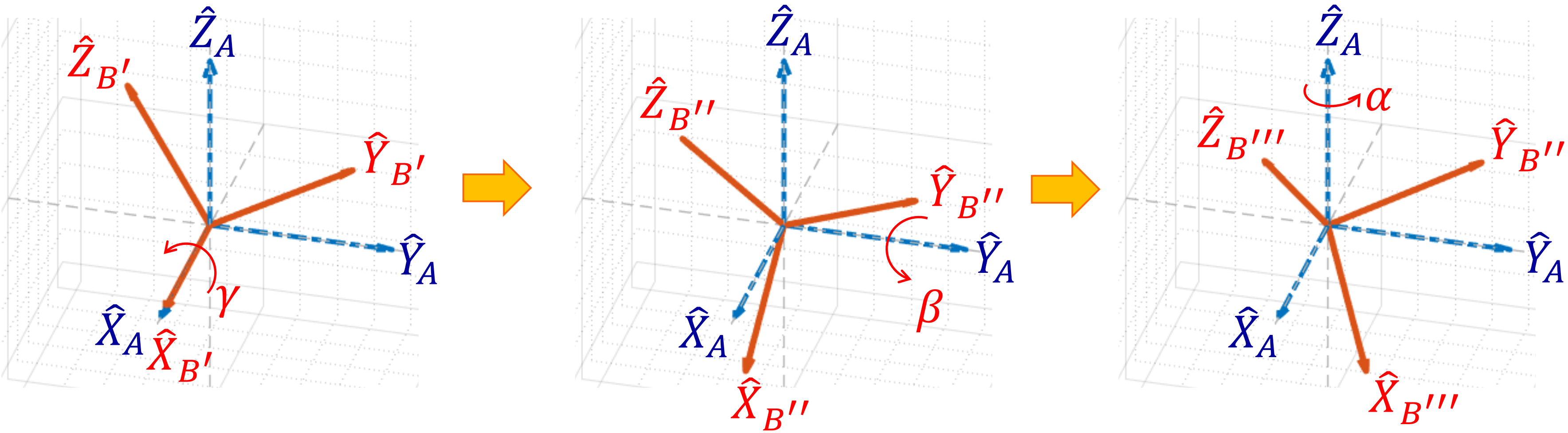


Rotation Matrix與轉角

- 空間中的Rotation是3 DOFs，那要如何把一般rotation matrix所表達的姿態，拆解成3次旋轉角度，以對應到3個DOFs？
- 拆解成「三次旋轉連乘」所需注意事項
  - Rotation不是commutable，所以多次旋轉的先後順序需要明確定義
  - 旋轉轉軸也需要明確定義。是對「固定不動」的轉軸旋轉？或是對「轉動的frame當下所在」的轉軸旋轉？
- 兩個拆解方式
  - 對方向「固定不動」的轉軸旋轉：Fixed angles
  - 對「轉動的frame當下所在」的轉軸方向旋轉：Euler angles

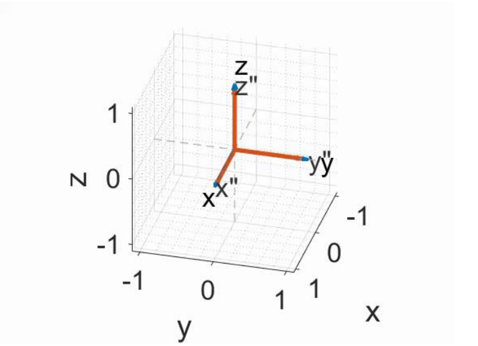
Fixed Angles -1

- X-Y-Z Fixed Angles – 由angles推算R



$${}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma) \quad v' = {}^{A}_{B}Rv = R_3R_2R_1v$$

先轉的放「後面」：以operator來想，對某一個向量，「以同一個座標為基準」，進行轉動或移動的操作

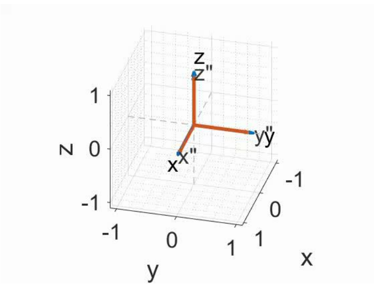


$$= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}$$

$$= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

Fixed Angles -2

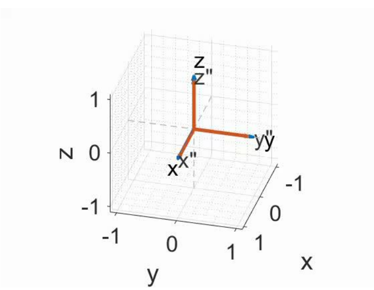
- Ex: 以Fixed Angles旋轉：「先對X軸旋轉60度，後對Y軸旋轉30度」和「先對Y軸旋轉30度，後對X軸旋轉60度」各自的 ${}^{A}_{B}R$ 分別是?



先對X轉60度，再對Y轉30度

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha)=R_Z(0)R_Y(30)R_X(60)=\begin{bmatrix}0.866 & 0.433 & 0.25\\ 0 & 0.5 & -0.866\\ -0.5 & 0.75 & 0.433\end{bmatrix} $$

$$\neq$$



先對Y轉30度，再對X轉60度

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha)=R_Z(0)R_X(60)R_Y(30)=\begin{bmatrix}0.866 & 0 & 0.5\\ 0.433 & 0.5 & -0.75\\ -0.25 & 0.866 & 0.433\end{bmatrix} $$

Fixed Angles -3

- X-Y-Z Fixed Angles – 由R推算angles

$$
{}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha)=\begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}
$$

If $\beta \neq 90°$

$$\beta = Atan2\left(-r_{31}, \sqrt{{r_{11}}^2 + {r_{21}}^2}\right)$$

$$\alpha = Atan2(r_{21}/c\beta, r_{11}/c\beta)$$

$$\gamma = Atan2(r_{32}/c\beta, r_{33}/c\beta)$$

$-90° \leq \beta \leq 90°$

Single solution

If $\beta = 90°$

$$\alpha = 0°$$

$$\gamma = Atan2(r_{12}, r_{22})$$

If $\beta = -90°$

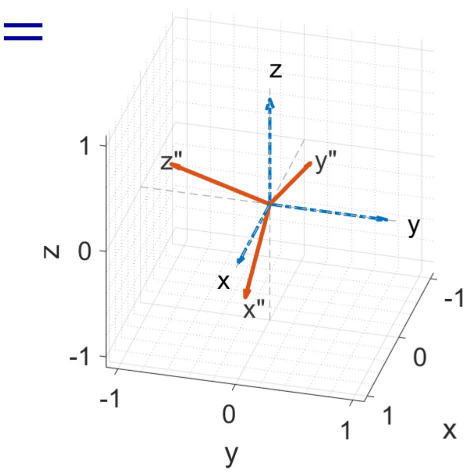
$$\alpha = 0°$$

$$\gamma = -Atan2(r_{12}, r_{22})$$

Fixed Angles -4

- Ex: 以X-Y-Z Fixed Angles方法，反算$R=$

$$\begin{bmatrix} 0.866 & 0.433 & 0.25 \\ 0 & 0.5 & -0.866 \\ -0.5 & 0.75 & 0.433 \end{bmatrix}$$ 的angles



$$\beta = Atan2\left(-r_{31}, \sqrt{{r_{11}}^2 + {r_{21}}^2}\right) = Atan2\left(-(-0.5), \sqrt{0.866^2 + 0^2}\right) = 30°$$

$$\alpha = Atan2\left(\frac{r_{21}}{c\beta}, \frac{r_{11}}{c\beta}\right) = Atan2\left(\frac{0}{\cos 30}, \frac{0.866}{\cos 30}\right) = 0°$$

$$\gamma = Atan2\left(\frac{r_{32}}{c\beta}, \frac{r_{33}}{c\beta}\right) = Atan2\left(\frac{0.75}{\cos 30}, \frac{0.433}{\cos 30}\right) = 60°$$

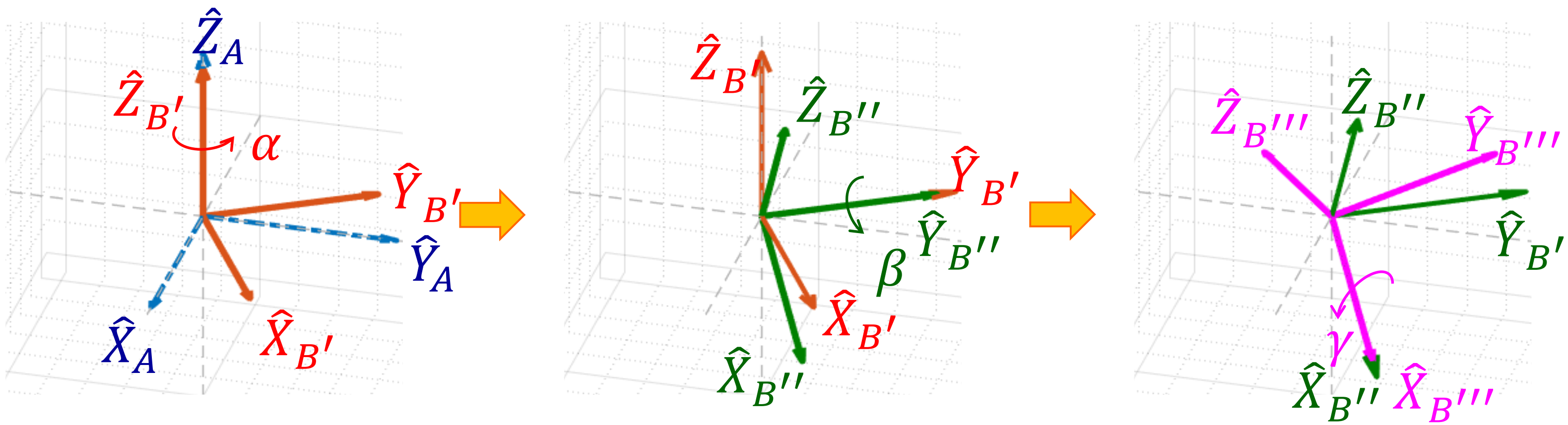
$\boldsymbol{R_Z(0)R_Y(30)R_X(60)}$

先對X轉60度，再對Y轉30度

和Fixed Angles -2的結果相同

Euler Angles -1

- Z-Y-X Euler Angles - 由angles推算R



$$ {}^A_BR_{Z'Y'X'}(\alpha,\beta,\gamma) = {}^A_{B'}R\,{}^{B'}_{B''}R\,{}^{B''}_{B}R = R_{Z'}(\alpha)R_{Y'}(\beta)R_{X'}(\gamma) $$

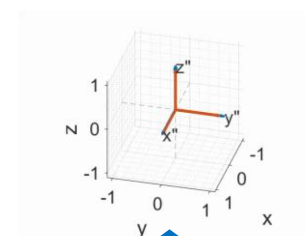
先轉的放「前面」：以mapping來想，對某一個向量，從最後一個frame「逐漸轉動或移動」來回到第一個frame

$$ {}^AP = {}^A_BR\,{}^BP = R_1R_2R_3\,{}^BP $$

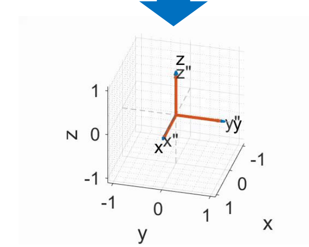
$$ = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

$$ = R_Z(\alpha)R_Y(\beta)R_X(\gamma) = {}^A_BR_{XYZ}(\gamma,\beta,\alpha) $$

和X-Y-Z Fixed angle得到一樣的R

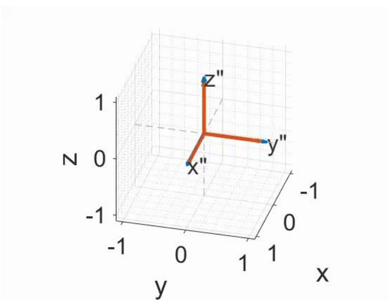


最後得出相同的R



Euler Angles -2

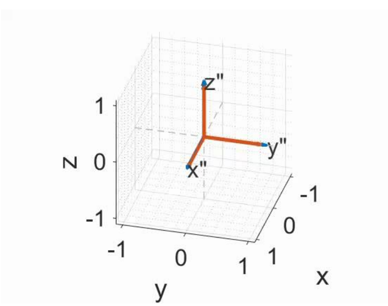
- Ex: 以Euler Angles旋轉：「先對X軸旋轉60度，後對Y軸旋轉30度」和「先對Y軸旋轉30度，後對X軸旋轉60度」各自的 ${}^{A}_{B}R$ 分別是?



先對X轉60度，再對Y轉30度

$${}^{A}_{B}R_{X'Y'Z'}(\gamma,\beta,\alpha)=R_{X'}(60)R_{Y'}(30)=\begin{bmatrix}0.866 & 0 & 0.5\\ 0.433 & 0.5 & -0.75\\ -0.25 & 0.866 & 0.433\end{bmatrix}$$

$$\neq$$

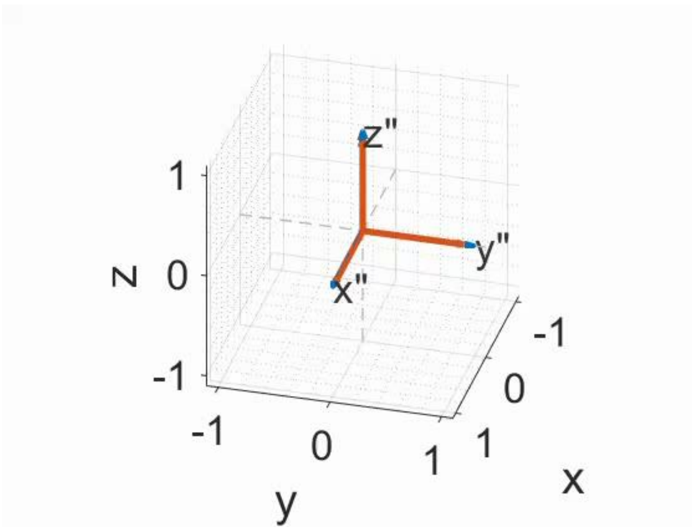


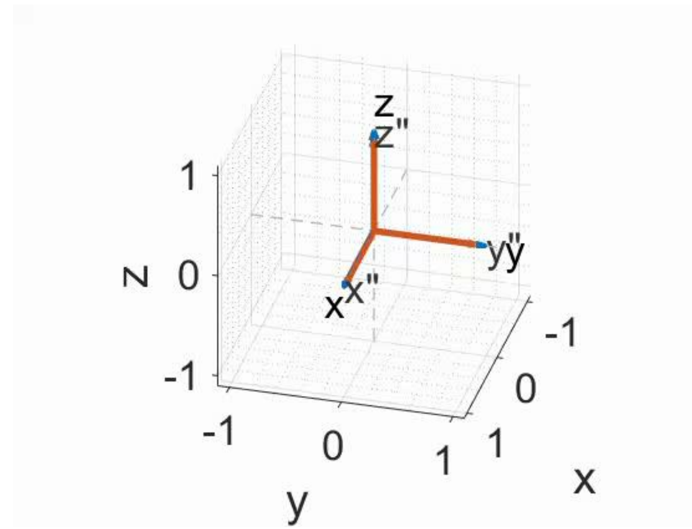
先對Y轉30度，再對X轉60度

$${}^{A}_{B}R_{X'Y'Z'}(\gamma,\beta,\alpha)=R_{Y'}(30)R_{X'}(60)=\begin{bmatrix}0.866 & 0.433 & 0.25\\ 0 & 0.5 & -0.866\\ -0.5 & 0.75 & 0.433\end{bmatrix}$$

Euler Angles -3

- Ex: Euler(Y30, X60) v.s. Fixed(X60, Y30)





Euler Angles:

先對Y轉30度，再對X轉60度

$$
{}^{A}_{B}R_{X'Y'Z'}(\gamma,\beta,\alpha) \\
= R_{Y'}(30)R_{X'}(60) \\
= \begin{bmatrix} 0.866 & 0.433 & 0.25 \\ 0 & 0.5 & -0.866 \\ -0.5 & 0.75 & 0.433 \end{bmatrix}
$$

=

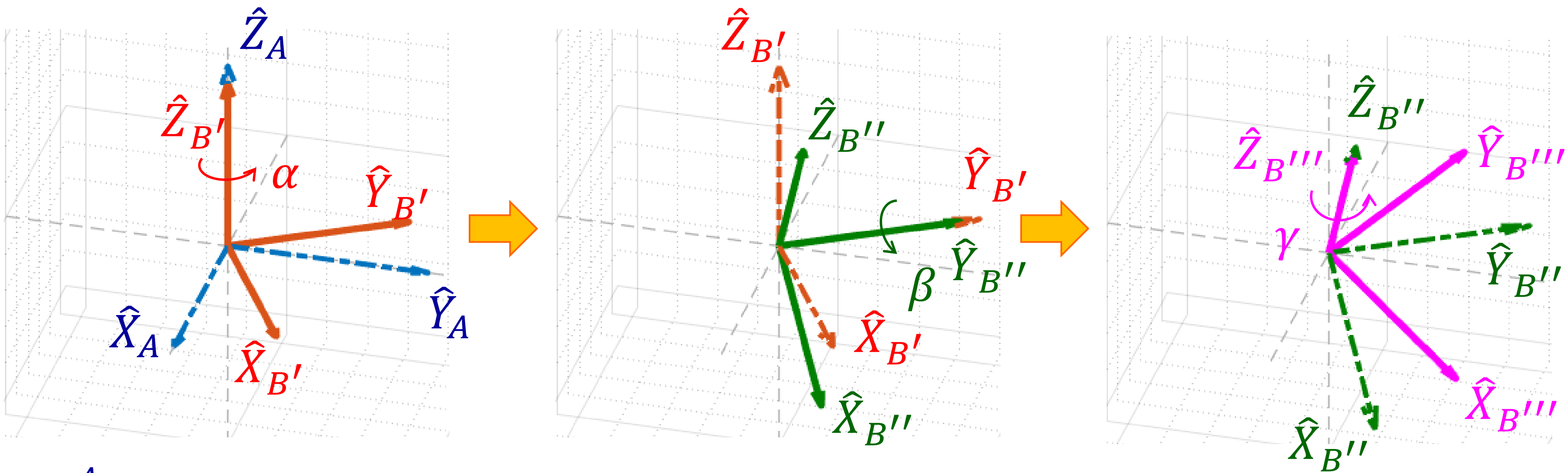
Fixed Angles:

先對X轉60度，再對Y轉30度

$$
{}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) \\
= R_{Y}(30)R_{X}(60) \\
= \begin{bmatrix} 0.866 & 0.433 & 0.25 \\ 0 & 0.5 & -0.866 \\ -0.5 & 0.75 & 0.433 \end{bmatrix}
$$

Euler Angles -4

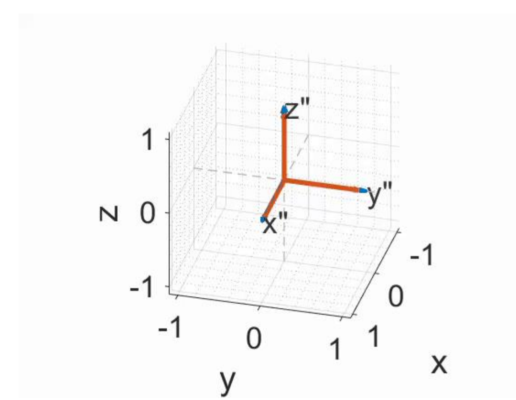
- Z-Y-Z Euler Angles - 由angles推算R



$${}^{A}_{B}R_{Z'Y'Z'}(\alpha,\beta,\gamma) = R_{Z'}(\alpha)R_{Y'}(\beta)R_{Z'}(\gamma)$$

先轉的放「前面」

$$= \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$



Euler Angles -5

- Z-Y-Z Euler Angles - 由R推算angles

$$ {}^{A}_{B}R_{Z'Y'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

If $\beta \neq 0°$

$$\beta = Atan2(\sqrt{{r_{31}}^2 + {r_{32}}^2}, r_{33})$$

$$\alpha = Atan2(r_{23}/s\beta, r_{13}/s\beta)$$

$$\gamma = Atan2(r_{32}/s\beta, -r_{31}/s\beta)$$

If $\beta = 0°$

$$\alpha = 0°$$

$$\gamma = Atan2(-r_{12}, r_{11})$$

If $\beta = 180°$

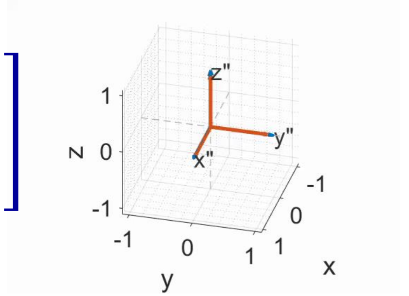
$$\alpha = 0°$$

$$\gamma = Atan2(r_{12}, -r_{11})$$

Euler Angles -6

- Ex: Revisit Euler Angles-2的範例

$$ {}^A_BR_{X'Y'Z'}(60{,}30{,}0) = R_{X'}(60)R_{Y'}(30) = \begin{bmatrix} 0.866 & 0 & 0.5 \\ 0.433 & 0.5 & -0.75 \\ -0.25 & 0.866 & 0.433 \end{bmatrix} $$



$R_{X'}(60)R_{Y'}(30)$

- 若以ZYZ的公式反算，Euler Angles 為何？

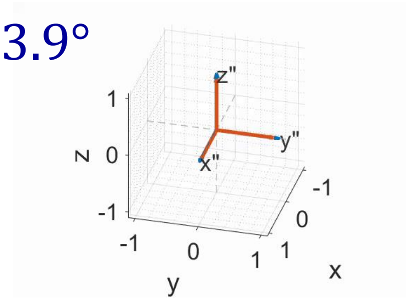
$$ \beta = Atan2\left(\sqrt{r_{31}{}^2 + r_{32}{}^2}, r_{33}\right) = Atan2\left(\sqrt{(-0.25)^2 + 0.866^2}, 0.433\right) = 64.3° $$

$$ \alpha = Atan2\left(\frac{r_{23}}{s\beta}, \frac{r_{13}}{s\beta}\right) = Atan2\left(\frac{-0.75}{s\beta}, \frac{0.5}{s\beta}\right) = -56.3° $$

$$ \gamma = Atan2(r_{32}/s\beta, -r_{31}/s\beta) = Atan2(0.866/s\beta, 0.25/s\beta) = 73.9° $$

$\Rightarrow$ $\boldsymbol{R_{Z'}(-56.3)R_{Y'}(64.3)R_{Z'}(73.9)}$

先對Z轉 -56.3°,對Y轉64.3°,最後對Z轉73.9°



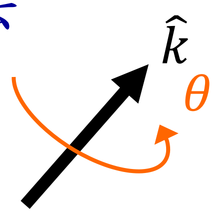
$R_{Z'}(-56.3)R_{Y'}(64.3)R_{Z'}(73.9)$

Rotation Matrix與轉角 小結

- Euler/Fixed angles
  - 12種 Euler angles 和 12種 fixed angles
  - 存在Duality – 共12種對principal axes連三次轉動的拆解方法
- Angle-axis表達法

對$\hat{k}$旋轉θ

unit vector



Unit vector裡2個參數，轉角1個參數，
也為3 DOFs

- Quaternion表達法

$$q = \epsilon_4 + \epsilon_1 \hat{\imath} + \epsilon_2 \hat{\jmath} + \epsilon_3 \hat{k}$$

$$= cos\frac{\theta}{2} + sin\frac{\theta}{2}(k_x \hat{\imath} + k_y \hat{\jmath} + k_z \hat{k})$$

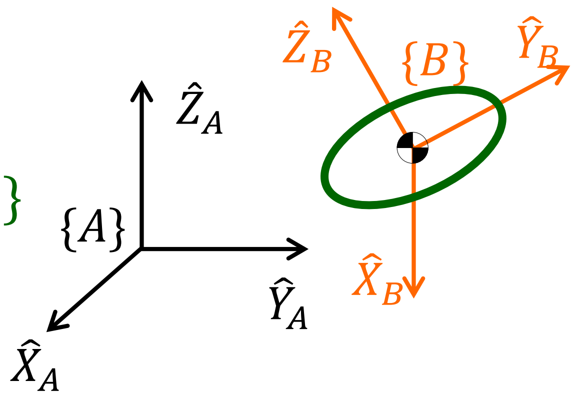
note $\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 = 1$

4個參數+1個限制條件，也為3 DOFs

剛體狀態的表達 -1

- 「導讀-3」的問題：該如何整合表達剛體的狀態？
- ⇨在剛體(Rigid body)上建立frame，常建立在質心上
  - 移動：由body frame 的原點位置判定

$$ {}^{A}P_{B\,org} = \begin{bmatrix} P_{B\,x} \\ P_{B\,y} \\ P_{B\,z} \end{bmatrix} = \text{origin of } \{B\} \text{ represented in } \{A\} $$



  - 轉動：由body frame的姿態判定

$$ {}^{A}_{B}R = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} = \begin{bmatrix} | & | & | \\ {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \\ | & | & | \end{bmatrix} $$

  - 彙整後：

$$ \{B\} = \{ {}^{A}_{B}R,\ {}^{A}P_{B\,org} \} $$

但無法進行量化計算

剛體狀態的表達 -2

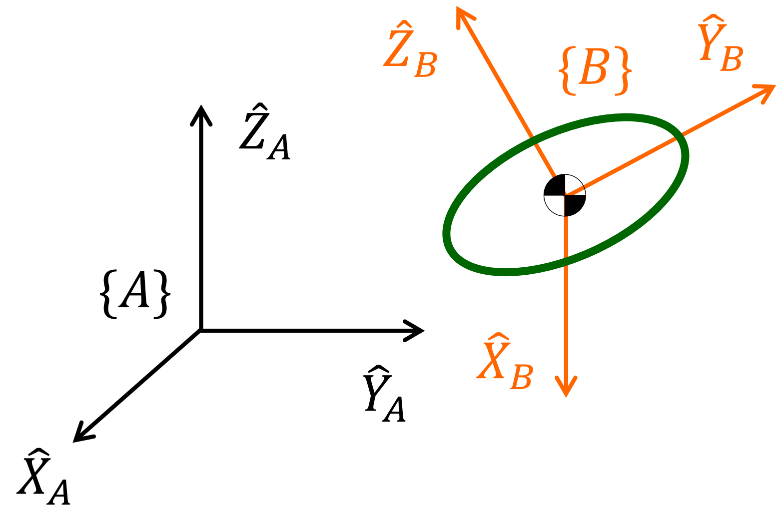
- 如何將移動和轉動整合在一起描述？

  ⇨ Homogeneous transformation matrix

$$\left[\begin{array}{ccc|c} & {}^{A}_{B}R_{3\times 3} & & {}^{A}P_{B\,org\,3\times 1} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]_{4\times 4}$$

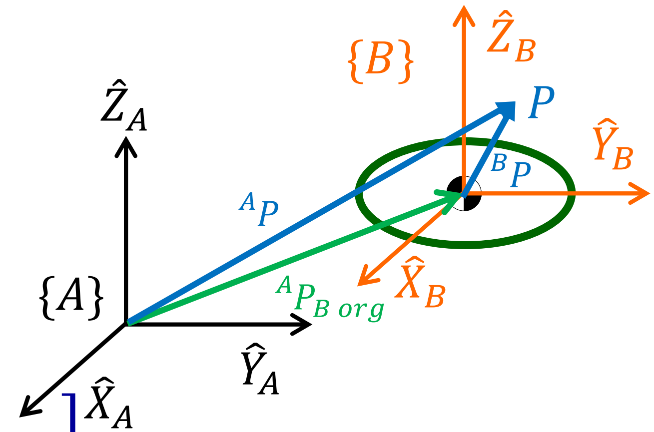
$$= \left[\begin{array}{ccc|c} | & | & | & | \\ {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B & {}^{A}P_{B\,org} \\ | & | & | & | \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$= {}^{A}_{B}T$$



Mapping -1

- 以Mapping，轉換向量（或點）之座標系的方式來確認 ${}^{A}_{B}T$ 運算之正確性



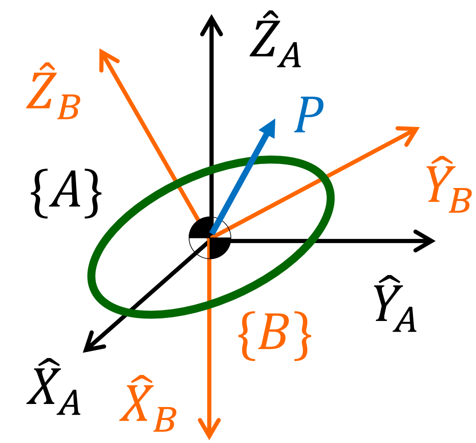
  - 僅有 移動

$$\underset{3\times1}{{}^{A}P} = \underset{3\times1}{{}^{B}P} + \underset{3\times1}{{}^{A}P_{B\,org}}$$

$$\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \left[\begin{array}{ccc:c} & I_{3\times3} & & {}^{A}P_{B\,org} \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{B}P + {}^{A}P_{B\,org} \\ 1 \end{bmatrix}$$

  - 僅有 轉動

$${}^{A}P_{3\times1} = \underset{3\times3}{{}^{A}_{B}R}\;{}^{B}P_{3\times1}$$



$$\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \left[\begin{array}{ccc:c} & & & 0 \\ & {}^{A}_{B}R & & 0 \\ & & & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R\,{}^{B}P \\ 1 \end{bmatrix}$$

Mapping -2

- 移動和轉動複合

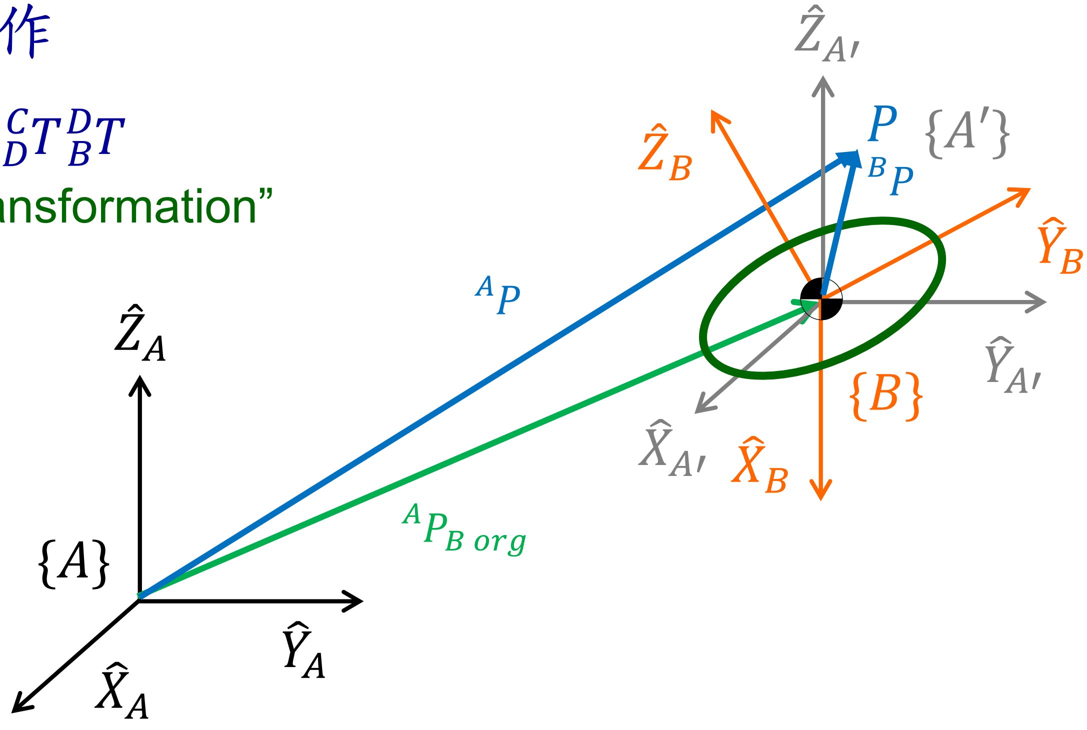
$$^{A}P_{3\times 1} = {}^{A}_{B}R\,{}^{B}P_{3\times 1} + {}^{A}P_{B\,org\ 3\times 1}$$

$$\begin{bmatrix} ^{A}P \\ 1 \end{bmatrix} = \begin{bmatrix} & {}^{A}_{B}R & & {}^{A}P_{B\,org} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ^{B}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{B\,org} \\ 1 \end{bmatrix}$$

- 可連續操作

$${}^{A}_{B}T = {}^{A}_{C}T\,{}^{C}_{D}T\,{}^{D}_{B}T$$

"sequential transformation"

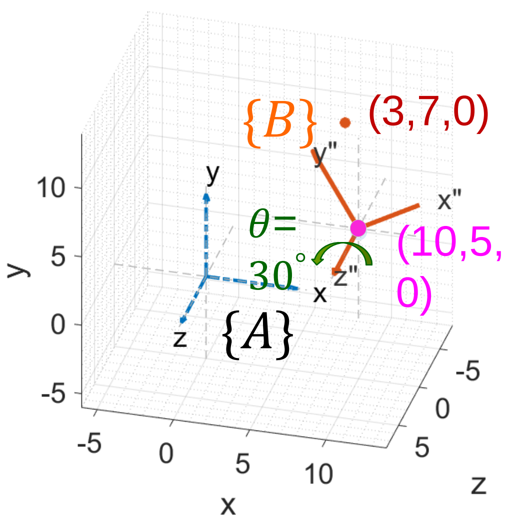


Mapping -3

- Ex:

$$ {}^{B}P = \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix} \quad {}^{A}P_{Borg} = \begin{bmatrix} 10 \\ 5 \\ 0 \end{bmatrix} \quad {}^{A}\hat{X}_B = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \quad {}^{A}\hat{Y}_B = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix} \quad {}^{A}\hat{Z}_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow {}^{A}P = ? $$

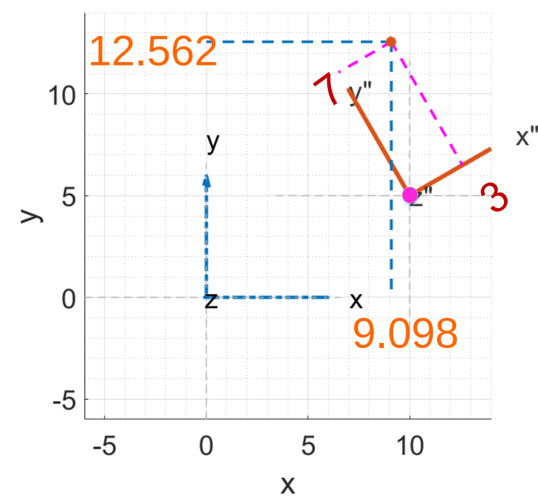
$$ \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{B\,org} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} = \underbrace{\left[\begin{array}{ccc|c} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 & 10 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 5 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]}_{\text{單純看 } {}^{A}_{B}T\text{: 表達}\{B\}\text{ 相對於}\{A\}\text{ 的方法}} \begin{bmatrix} 3 \\ 7 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9.098 \\ 12.562 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} | \\ {}^{A}P \\ | \\ 1 \end{bmatrix} $$



看整個操作：

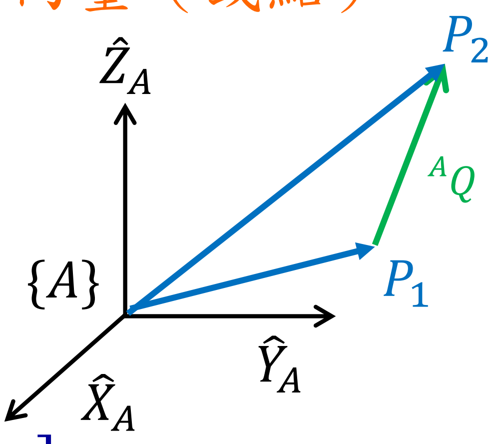
轉換point在不同frame下的表達

投影至XY平面驗證答案



Operators -1

- ${}^A_BT$除了Mapping之外，也可當Operator，對向量（或點）進行移動或轉動



  - 僅有 移動

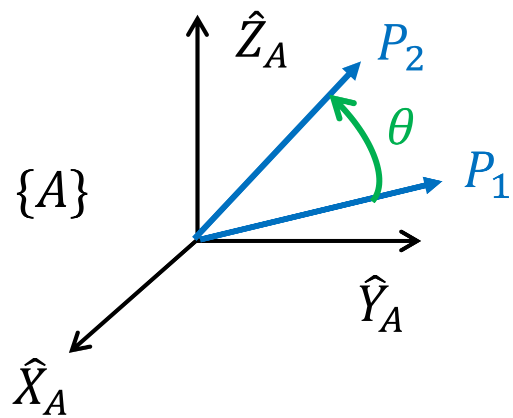
$${}^AP_{2\,3\times1} = {}^AP_{1\,3\times1} + {}^AQ_{\,3\times1}$$

$$\begin{bmatrix} {}^AP_2 \\ 1 \end{bmatrix} = D(Q)\begin{bmatrix} {}^AP_1 \\ 1 \end{bmatrix} = \begin{bmatrix} & I & & {}^AQ \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} {}^AP_1 \\ 1 \end{bmatrix} = \begin{bmatrix} {}^AP_1 + {}^AQ \\ 1 \end{bmatrix}$$

  - 僅有 轉動

$${}^AP_{2\,3\times1} = R_{\hat{K}}(\theta)_{3\times3}\,{}^AP_{1\,3\times1}$$

$$\begin{bmatrix} {}^AP_2 \\ 1 \end{bmatrix} = \begin{bmatrix} & & & 0 \\ & R_{\hat{K}}(\theta) & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} {}^AP_1 \\ 1 \end{bmatrix} = \begin{bmatrix} R_{\hat{K}}(\theta)\,{}^AP_1 \\ 1 \end{bmatrix}$$

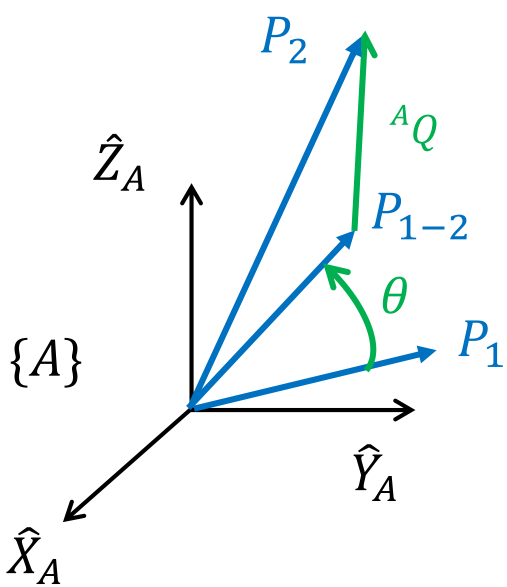


Operators -2

- 移動和轉動複合

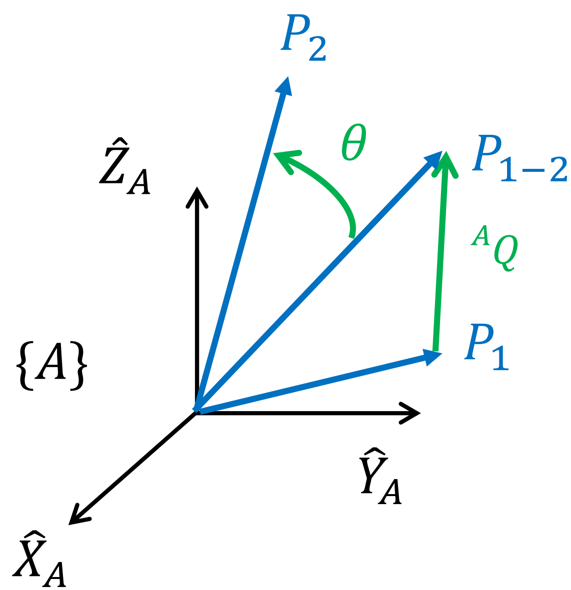
$$ {}^{A}P_{2\,{3\times1}} = \underset{3\times3}{R_{\hat{K}}(\theta)}\,{}^{A}P_{1\,{3\times1}} + {}^{A}Q_{\,3\times1} $$ 先轉動再移動

$$ \begin{bmatrix} {}^{A}P_2 \\ 1 \end{bmatrix} = \begin{bmatrix} R_{\hat{K}}(\theta) & & & {}^{A}Q \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{A}P_1 \\ 1 \end{bmatrix} = \begin{bmatrix} R_{\hat{K}}(\theta)\,{}^{A}P_1 + {}^{A}Q \\ 1 \end{bmatrix} = T \begin{bmatrix} {}^{A}P_1 \\ 1 \end{bmatrix} $$



先轉動再移動

$\neq$



先移動再轉動（ ${}^{A}Q$也會被轉動到 ）

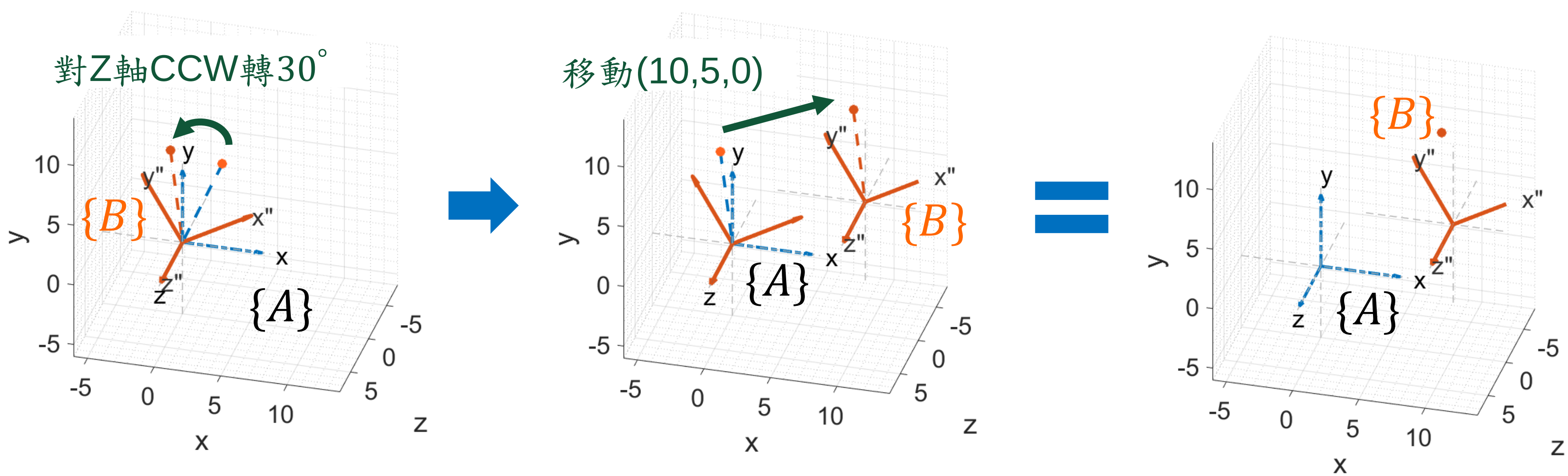
$$ {}^{A}P_2 = R_{\hat{K}}(\theta)\left({}^{A}P_1 + {}^{A}Q\right) = R_{\hat{K}}(\theta)\,{}^{A}P_1 + R_{\hat{K}}(\theta)\,{}^{A}Q $$

Operators -3

- Ex: Point $P_1 = \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix}$，先對Z軸CCW轉30°, 然後移動$\begin{bmatrix} 10 \\ 5 \\ 0 \end{bmatrix}$到$P_2$ ⇨ P_2 =?

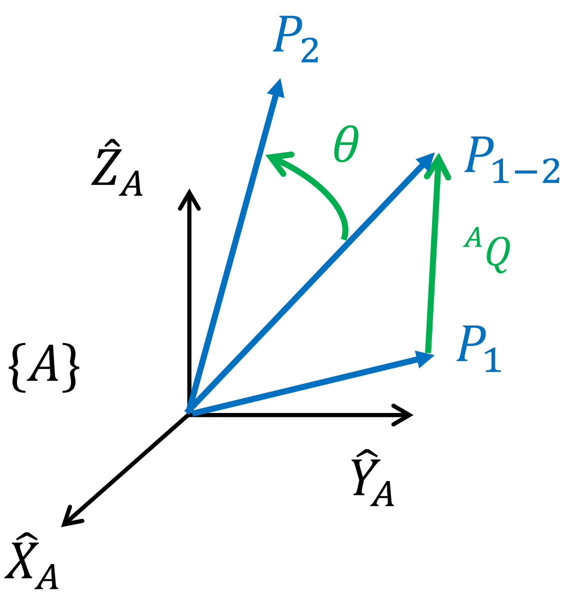
$$\begin{bmatrix} {}^AP_2 \\ 1 \end{bmatrix} = \begin{bmatrix} R_{\hat{K}}(\theta) & & & {}^AQ \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^AP_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 10 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 9.098 \\ 12.562 \\ 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} | \\ {}^AP_2 \\ | \\ 1 \end{bmatrix}$$

和「Mapping -3」的答案相同，Why?



Operators -4

- In-video Quiz: 如果要如下圖所示的先移動再轉動，那T應該如何表達？



A. $\begin{bmatrix} R_{\hat{K}}(\theta) & & & {}^{A}Q \\ 0 & 0 & 0 & 1 \end{bmatrix}$

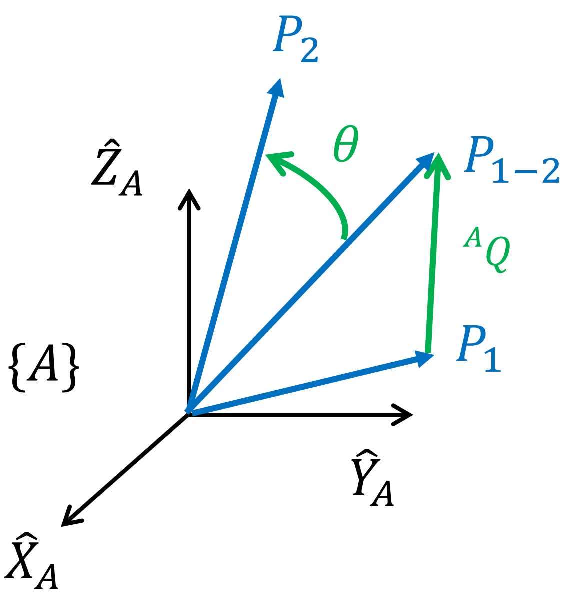
B. $\begin{bmatrix} & I & & {}^{A}Q \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} & & & 0 \\ & R_{\hat{K}}(\theta) & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} & & & 0 \\ & R_{\hat{K}}(\theta) & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} & I & & {}^{A}Q \\ 0 & 0 & 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} R_{\hat{K}}(\theta) & & & {}^{A}Q \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} & & & 0 \\ & R_{\hat{K}}(\theta) & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Operators -4

- In-video Quiz: 如果要如下圖所示的先移動再轉動，那T應該如何表達？



A. $\begin{bmatrix} R_{\hat{K}}(\theta) & & & {}^{A}Q \\ 0 & 0 & 0 & 1 \end{bmatrix}$

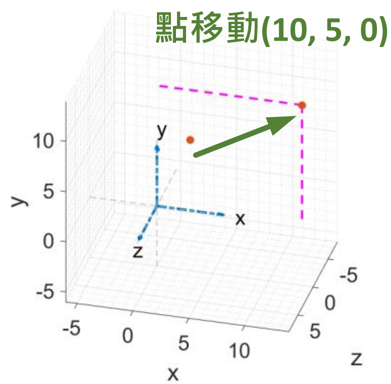
B. $\begin{bmatrix} & I & & {}^{A}Q \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} & & & 0 \\ & R_{\hat{K}}(\theta) & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

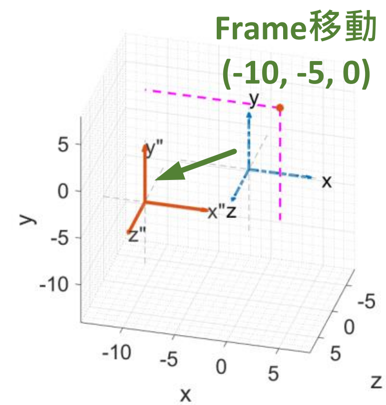
C. $\begin{bmatrix} & & & 0 \\ & R_{\hat{K}}(\theta) & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} & I & & {}^{A}Q \\ 0 & 0 & 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} R_{\hat{K}}(\theta) & & & {}^{A}Q \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} & & & 0 \\ & R_{\hat{K}}(\theta) & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

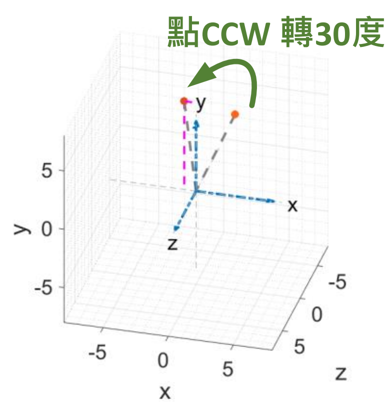
Operators -5

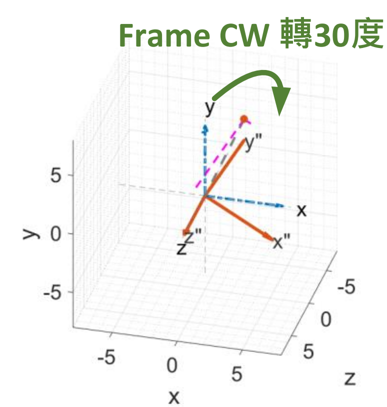
- 因為運動是相對的，${}^{A}_{B}T$當Operator時對向量（或點）進行移動或轉動的操作，也可以想成是對frame進行「反向」的移動或轉動的操作
  - Point往前移 = frame往後移





  - Point逆時針轉= frame順時針轉



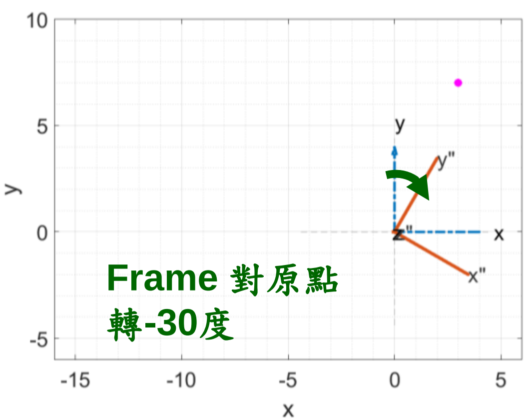


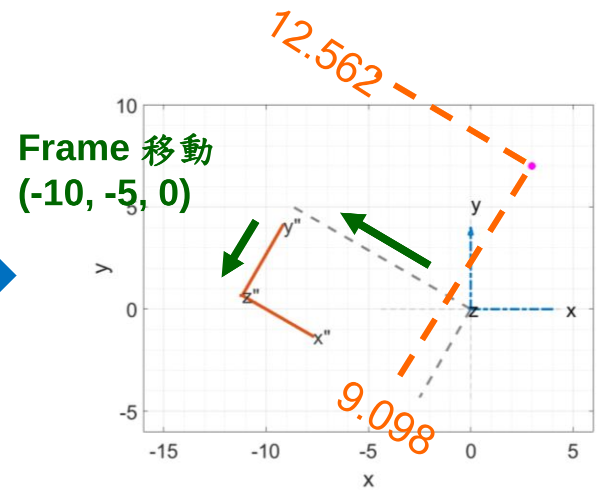
Operators -6

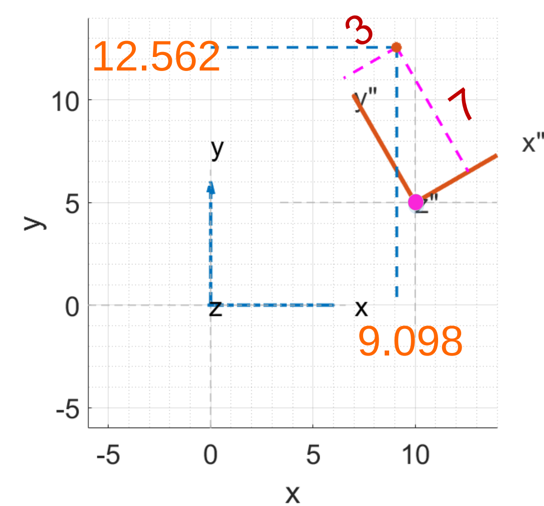
- Ex: Revisit Operator-3的範例，改以frame轉動的角度來想

$$\text{Point } P_1 = \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix}\text{，先對Z軸CCW轉}30^\circ\text{, 然後移動}\begin{bmatrix} 10 \\ 5 \\ 0 \end{bmatrix}\text{到}P_2 \quad \Rightarrow P_2 = ?$$

投影至XY平面來看





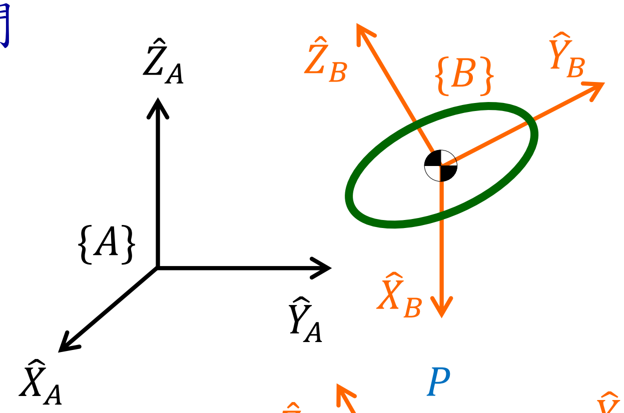


與Operator -3的答案相同

Transformation Matrix小結

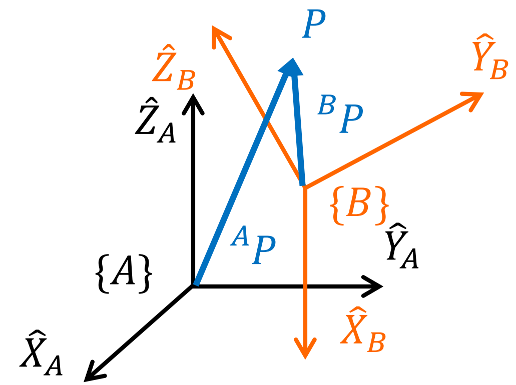
- Homogeneous transformation matrix 的三種用法
  - 描述一個frame(相對於另一個frame)的空間狀態

$$ {}^{A}_{B}T = \begin{bmatrix} | & | & | & | \\ {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} & {}^{A}P_{B\,org} \\ | & | & | & | \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



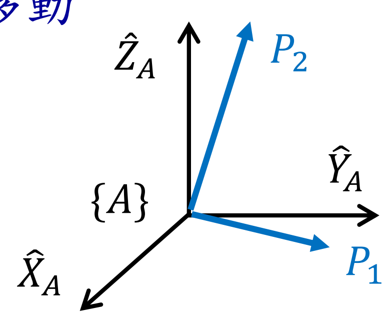
  - 將point由某一個frame的表達換到另一個frame來表達

$$ \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = {}^{A}_{B}T \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} $$



  - 將point(vector)在同一個frame中進行移動和轉動

$$ \begin{bmatrix} {}^{A}P_2 \\ 1 \end{bmatrix} = T \begin{bmatrix} {}^{A}P_1 \\ 1 \end{bmatrix} $$



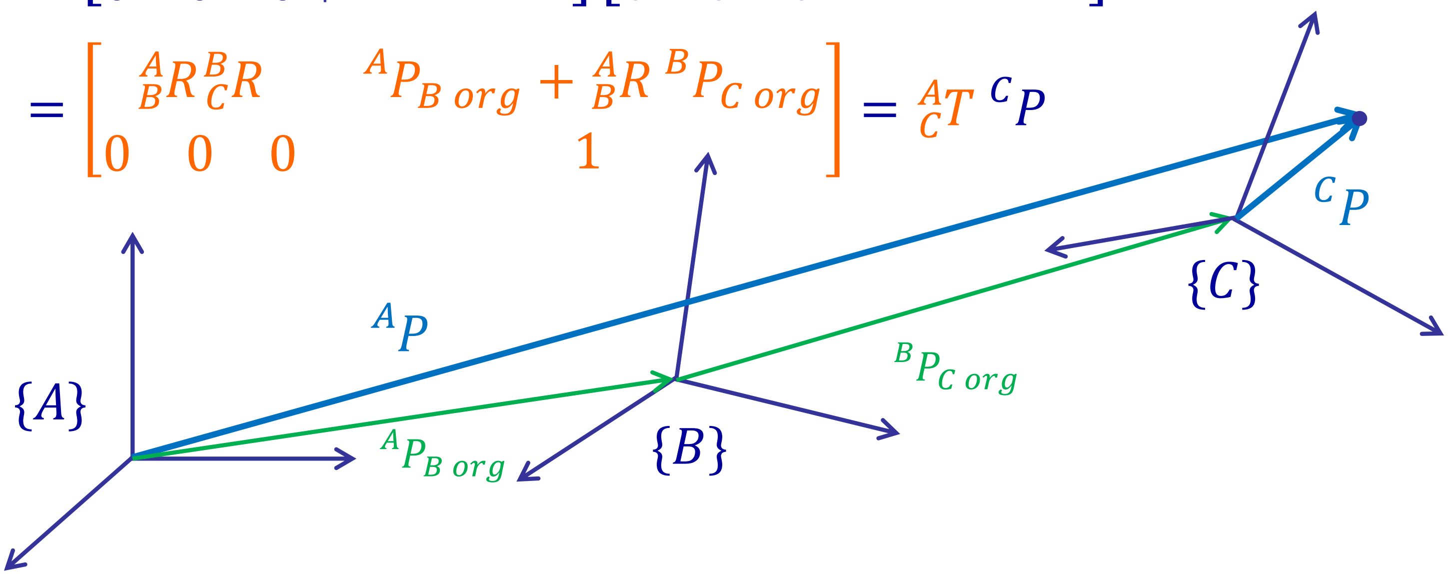
Transformation Matrix運算 -1

- 連續運算

$$ {}^{A}P = {}^{A}_{B}T\,{}^{B}P = {}^{A}_{B}T({}^{B}_{C}T\,{}^{C}P) = {}^{A}_{B}T\,{}^{B}_{C}T\,{}^{C}P $$

$$ = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{B\,org} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]\left[\begin{array}{ccc|c} & {}^{B}_{C}R & & {}^{B}P_{C\,org} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]{}^{C}P $$

$$ = \begin{bmatrix} {}^{A}_{B}R\,{}^{B}_{C}R & {}^{A}P_{B\,org} + {}^{A}_{B}R\,{}^{B}P_{C\,org} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} = {}^{A}_{C}T\,{}^{C}P $$



$$ {}^{A}P = {}^{A}_{B}T\,{}^{B}_{C}T\,{}^{C}_{D}T\,{}^{D}P $$

$$ = \begin{bmatrix} {}^{A}_{B}R\,{}^{B}_{C}R\,{}^{C}_{D}R & {}^{A}P_{B\,org} + {}^{A}_{B}R\,{}^{B}P_{C\,org} + {}^{A}_{B}R\,{}^{B}_{C}R\,{}^{C}P_{D\,org} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} = {}^{A}_{D}T\,{}^{D}P $$

Transformation Matrix運算 -2

- 反矩陣 $^{A}_{B}T = \begin{bmatrix} & ^{A}_{B}R & & ^{A}P_{B\,org} \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\qquad ^{B}_{A}T = {^{A}_{B}T}^{-1} = ?$

$$^{A}_{B}T\,^{B}_{A}T = {^{A}_{B}T}\,{^{A}_{B}T}^{-1} = I_{4\times4}$$

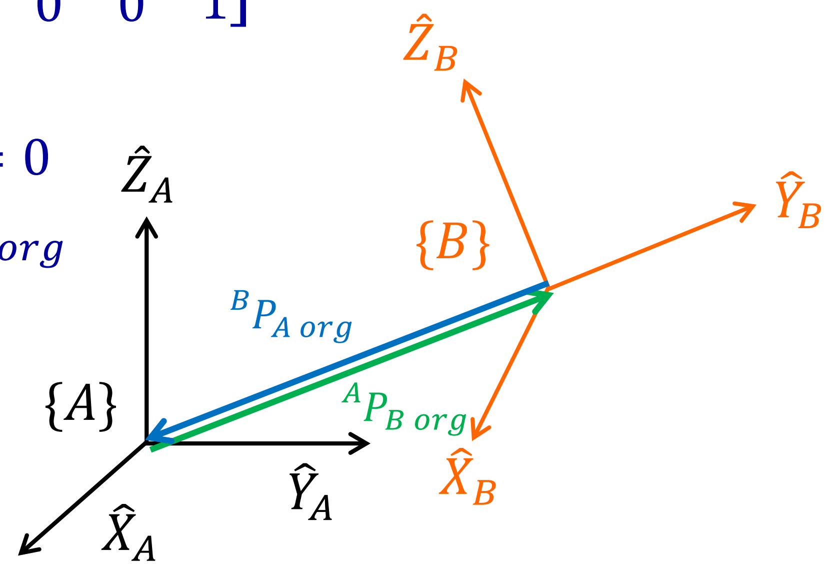
$$\begin{bmatrix} & ^{A}_{B}R & & ^{A}P_{B\,org} \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} & ^{B}_{A}R & & ^{B}P_{A\,org} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} ^{A}_{B}R\,^{B}_{A}R & & & ^{A}P_{B\,org} + {^{A}_{B}R}\,^{B}P_{A\,org} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & & & 0 \\ & I_{3\times3} & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{A}_{B}R\,^{B}_{A}R = I_{3\times3} \qquad ^{A}P_{B\,org} + {^{A}_{B}R}\,^{B}P_{A\,org} = 0$$

$$\Rightarrow\ ^{B}_{A}R = {^{A}_{B}R}^{T} \qquad \Rightarrow\ ^{B}P_{Aorg} = -{^{A}_{B}R}^{T}\,^{A}P_{Borg}$$

$$\Rightarrow\ {^{A}_{B}T}^{-1} = \begin{bmatrix} & {^{A}_{B}R}^{T} & & -{^{A}_{B}R}^{T}\,^{A}P_{B\,org} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Transformation Matrix運算 -3

- 連續運算，求未知之相對關係

$${}^{U}_{D}T = {}^{U}_{A}T\,{}^{A}_{D}T = {}^{U}_{B}T\,{}^{B}_{C}T\,{}^{C}_{D}T$$

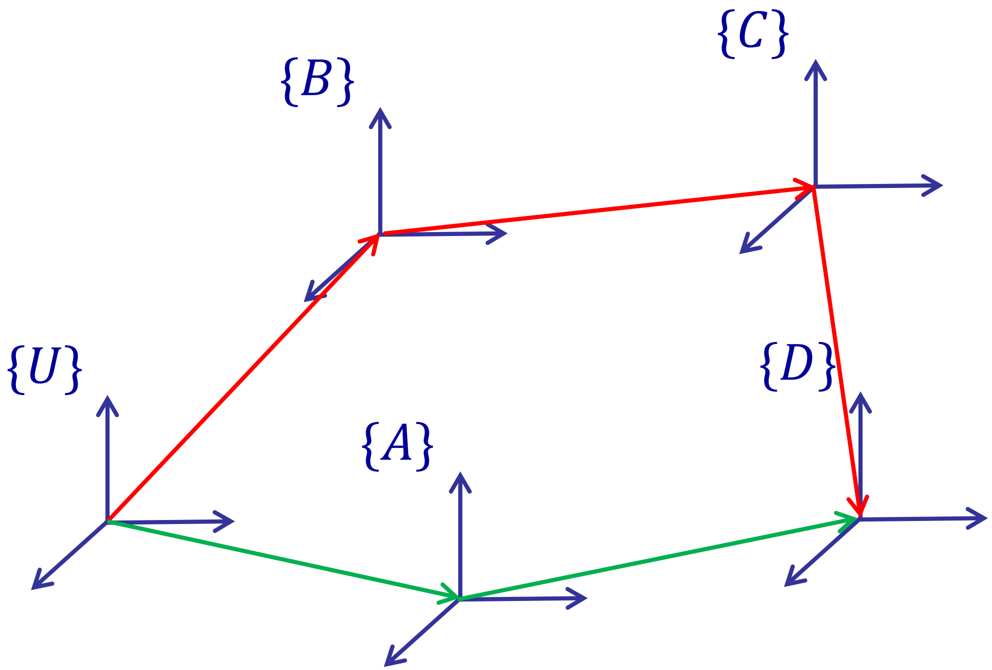
if ${}^{C}_{D}T$ unknown

$$= ({}^{U}_{B}T\,{}^{B}_{C}T)^{-1}\,{}^{U}_{A}T\,{}^{A}_{D}T$$

$$= {}^{B}_{C}T^{-1}\,{}^{U}_{B}T^{-1}\,{}^{U}_{A}T\,{}^{A}_{D}T$$

if ${}^{B}_{C}T$ unknown

$$= {}^{U}_{B}T^{-1}\,{}^{U}_{A}T\,{}^{A}_{D}T\,{}^{C}_{D}T^{-1}$$



Transformation Matrix運算 -4

- 連續運算 法則
  - Initial condition: $\{A\}$ and $\{B\}$ coincide $\quad {}^{A}_{B}T = I_{4\times 4}$
  - $\{B\}$ 對 $\{A\}$ 的轉軸旋轉：用 “premultiply”
    - 以operator來想，對某一個向量，「以同一個座標為基準」，進行轉動或移動的操作
    - Ex: $\{B\}$依序經過T_1、T_2、T_3三次transformations

      $${}^{A}_{B}T = T_3T_2T_1I \qquad v' = {}^{A}_{B}Tv = T_3T_2T_1v$$
  - $\{B\}$ 對 $\{B\}$ 自身的轉軸旋轉：用“postmultiply”
    - 以mapping來想，對某一個向量，從最後一個frame「逐漸轉動或移動」來回到第一個frame
    - Ex: $\{B\}$依序經過T_1、T_2、T_3三次transformation

      $${}^{A}_{B}T = IT_1T_2T_3 \qquad {}^{A}P = {}^{A}_{B}T\,{}^{B}P = IT_1T_2T_3\,{}^{B}P$$

Transformation Matrix運算 -5

- 連續運算 小結
  - 以固定的$\{A\}$或移動的$\{B\}$為基準進行移動換轉動操作，transformation matrix應用不同的連乘方式
  - 思考邏輯和考量Fixed angles vs. Euler angles的連續旋轉順序相似