Design and Analysis of Algorithms (VII)

The Network Flow Problem

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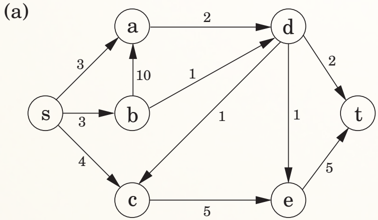
Shipping Oil

We have a network of pipelines along which oil can be sent.

The goal is to ship as much oil as possible from the source to the sink.

Each pipeline has a maximum capacity it can handle, and there are no opportunities for storing oil en route.

A Flow Example



Maximizing Flow

The networks consist of a directed graph $G = (V, E)$; two special nodes $s, t \in V$, a source and sink of G; and capacities $c_e > 0$ on the edges.

Aim to send as much oil as possible from s to t without exceeding the capacities of any of the edges.

Maximizing Flow

A flow consists of a variable f_e for each edge e of the network, satisfying the following two properties:

❶ It doesn't violate edge capacities: $0 \leq f_e \leq c_e$ for all $e \in E$.

❷ For all nodes u except s and t, the amount of flow entering u equals the amount leaving

$$\sum_{(w,v)\in E} f_{wu} = \sum_{(u,z)\in E} f_{uz}$$

In other words, flow is conserved.

Maximizing Flow

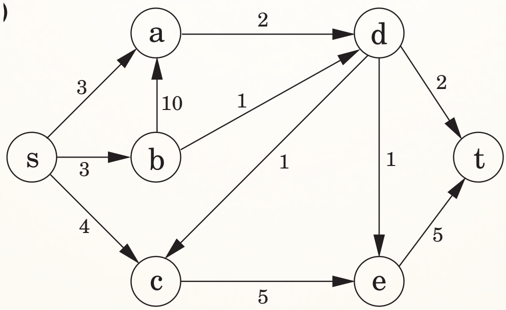
The size of a flow is the total quantity sent from s to t and, by the conservation principle, is equal to the quantity leaving s:

$$\texttt{size}(f) = \sum_{(s,u)\in E} f_{su}$$

Our goal is to assign values to $\{f_e | e \in E\}$ that will satisfy a set of linear constraints and maximize a linear objective function.

This is a linear program. The maximum-flow problem reduces to linear programming.

The Example



LP

11 variables, one per edge.

maximize $f_{sa} + f_{sb} + f_{sc}$

27 constraints:

- 11 for nonnegativity (such as $f_{sa} \geq 0$),
- 11 for capacity (such as $f_{sa} \leq 3$),
- 5 for flow conservation (one for each node of the graph other than s and t, such as $f_{sc} + f_{dc} = f_{ce}$).

Another Representation

First, introduce a fictitious edge of infinite capacity from t to s thus converting the flow to a circulation;

The objective is to maximize the flow on this edge, denoted by f_{ts}.

The advantage of making this modification is that we can now require flow conservation at s and t as well.

Another Representation

$$\max f_{ts}$$

$$f_{ij} \leq c_{ij} \qquad (i,j) \in E$$

$$\sum_{(w,i)\in E} f_{wi} - \sum_{(i,z)\in E} f_{iz} \leq 0 \quad i \in V$$

$$f_{ij} \geq 0 \qquad (i,j) \in E$$

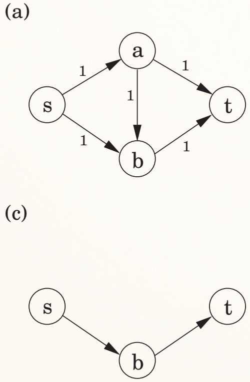
A Closer Look at the Algorithm

All we know so far of the simplex algorithm is the vague geometric intuition that it keeps making local moves on the surface of a convex feasible region, successively improving the objective function until it finally reaches the optimal solution.

The behavior of simplex has an elementary interpretation:

- Start with zero flow.
- Repeat: choose an appropriate path from s to t, and increase flow along the edges of this path as much as possible.

A Flow Example

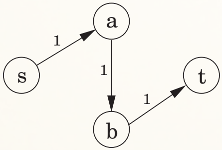


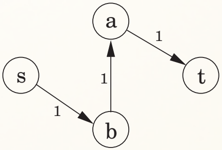
A Closer Look at the Algorithm

There is just one complication.

What if we choose a path that blocks all other paths?

Simplex gets around this problem by also allowing paths to cancel existing flow.





A Closer Look at the Algorithm

To summarize, in each iteration simplex looks for an $s - t$ path whose edges (u, v) can be of two types:

1. (u, v) is in the original network, and is not yet at full capacity.
2. The reverse edge (v, u) is in the original network, and there is some flow along it.

If the current flow is f, then in the first case, edge (u, v) can handle up to $c_{uv} - f_{uv}$ additional units of flow;

in the second case, up to f_{vu} additional units (canceling all or part of the existing flow on (v, u)).

A Closer Look at the Algorithm

These flow-increasing opportunities can be captured in a residual network $G^f = (V, E^f)$, which has exactly the two types of edges listed, with residual capacities c^f:

$$\begin{cases} c_{uv} - f_{uv} & \text{if } (u,v) \in E \text{ and } f_{uv} < c_{uv} \\ f_{vu} & \text{if } (v,u) \in E \text{ and } f_{vu} > 0 \end{cases}$$

Thus we can equivalently think of simplex as choosing an $s - t$ path in the residual network.

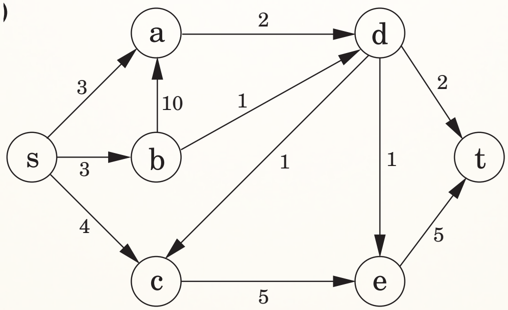
By simulating the behavior of simplex, we get a direct algorithm for solving max-flow.

It proceeds in iterations, each time explicitly constructing G^f, finding a suitable $s - t$ path in G^f by using, say, a linear-time BFS.

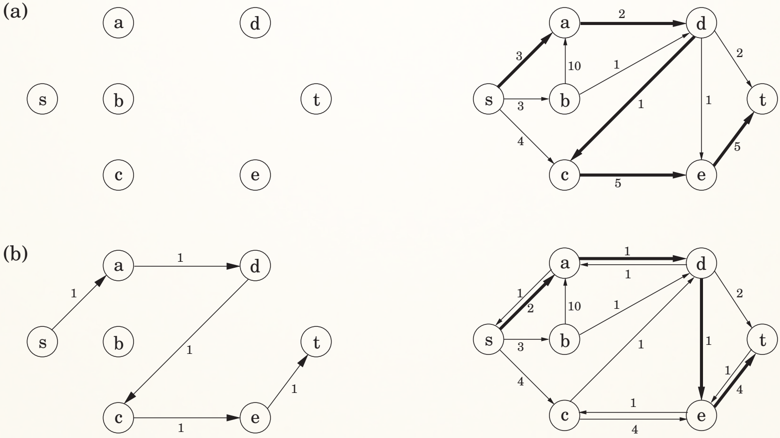
Remark

Ford-Fulkersons algorithm can be regarded as a special algorithm of linear programs.

The Example

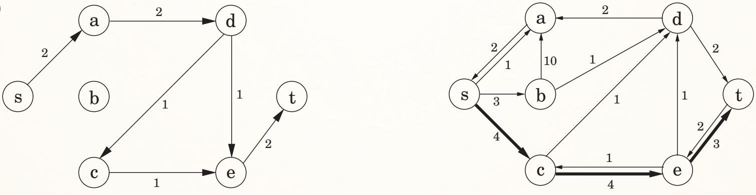


A Flow Example

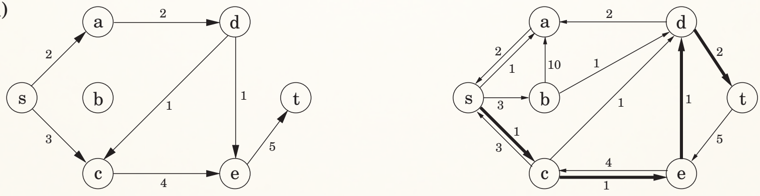


A Flow Example

(c)



(d)



Cuts

An (s,t)-cut partitions the vertices into two disjoint groups L and R such that $s \in L$ and $t \in R$. Its capacity is the total capacity of the edges from L to R, and as argued previously, is an upper bound on any flow:

Pick any flow f and any (s,t)-cut (L,R),

$$\texttt{size}(f) \leq \texttt{capacity}(L,R)$$

Max-Flow Min-Cut

Theorem (Max-flow min-cut)
The size of the maximum flow in a network equals the capacity of the smallest (s, t)-cut.

A Certificate of Optimality

Proof

- Suppose f is the final flow when the algorithm terminates.
- We know that node t is no longer reachable from s in the residual network G^f.
- Let L be the nodes that are reachable from s in G^f, and let $R = V \backslash L$ be the rest of the nodes.
- We claim that $\texttt{size}(f) = \texttt{capacity}(L, R)$.
- To see this, observe that by the way L is defined, any edge going from L to R must be at full capacity (in the current flow f), and any edge from R to L must have zero flow.
- Therefore the net flow across (L, R) is exactly the capacity of the cut.

Efficiency

- Each iteration is efficient, requiring $O(|E|)$ time if a DFS or BFS is used to find an $s - t$ path.

But how many iterations are there?

- Suppose all edges in the original network have integer capacities $\leq C$. Then on each iteration of the algorithm, the flow is always an integer and increases by an integer amount.
- Therefore, since the maximum flow is at most $C|E|$, the number of iterations is at most this much.
- If paths are chosen in a sensible manner - in particular, by using a BFS, which finds the path with the fewest edges - then the number of iterations is at most $O(|V| \cdot |E|)$, no matter what the capacities are. *Edmonds-Karp algorithm*
- This latter bound gives an overall running time of $O(|V| \cdot |E|^2)$ for maximum flow.

Efficiency

Lemma:

If the *Edmonds-Karp algorithm* is run on a flow network $G = (V, E)$ with source s and sink t, then for all vertices $v \in V - \{s, t\}$, the shortest path distance $\delta_f(s, v)$ in the residual network G_f increases monotonically with each flow augmentation.

proof:

- Prove by contradiction. Let f be the flow just before the first augmentation that decreases some shortest path distance, and let f' be the flow just afterward.
- Let v be the shortest vertex with the minimum $\delta_{f'}(s, v)$ whose distance was decreased by the augmentation, so that $\delta_{f'}(s, v) < \delta_f(s, v)$. Let $s \rightsquigarrow u \rightarrow v$ be a shortest path from s to v in $G_{f'}$, so that $(u, v) \in E_{f'}$ and

$$\delta_{f'}(s, u) = \delta_{f'}(s, v) - 1$$

Efficiency

proof:

- Because of how we chose v, we know that the distance label of vertex u did not decrease, i.e.,

$$\delta_{f'}(s,u) \geq \delta_f(s,u)$$

- $(u,v) \notin E_f$.
  - otherwise $\delta_f(s,v) \leq \delta_f(s,u) + 1 \leq \delta_{f'}(s,u) + 1 = \delta_{f'}(s,v)$
- $(u,v) \notin E_f$ but $(u,v) \in E_{f'}$? The augmentation must have increased the flow from v to u.
- The Edmonds-Karp algorithm always augments flow along shortest paths, and therefore the shortest path from s to u in G_f has (v,u) as its last edge.
- $\delta_f(s,v) = \delta_f(s,u) - 1 \leq \delta_{f'}(s,u) - 1 = \delta_{f'}(s,v) - 2$.
- Contradict to $\delta_{f'}(s,v) < \delta_f(s,v)$.

Efficiency

Theorem:

If the *Edmonds-Karp algorithm* is run on a flow network $G = (V, E)$ with source s and sink t, then the total number of flow augmentations performed by the algorithm is $O(V \cdot E)$.

Efficiency

proof:

An edge (u, v) in a residual network G_f is critical p if the residual capacity of p is the residual capacity of (u, v), $c_f(p) = c_f(u, v)$. Each of the $|E|$ edges can become critical at most $|V|/2 - 1$ times. Let $(u, v) \in E$,

- Since augmenting paths are shortest paths, when (u, v) is critical for the first time, we have

$$\delta_f(s, v) = \delta_f(s, u) + 1$$

- It reappears after the flow from u to v is decreased, which occurs only if (v, u) appears on an augmenting path. We have

$$\delta_{f'}(s, u) = \delta_{f'}(s, v) + 1$$

Efficiency

proof:

- $\delta_{f'}(s, u) \geq \delta_f(s, u) + 2$.
- The distance of u from the source is initially at least 0. Until u becomes unreachable from the source, if ever, its distance is at most $|V| - 2$. Thus, (u, v) can become critical at most $(|V| - 2)/2 = |V|/2 - 1$ times.
- Since there are $O(E)$ pairs of vertices that can have an edge between them in a residual graph, the total number of critical edges during the entire execution of the Edmonds-Karp algorithm is $O(V \cdot E)$.

Min-Max Relations and Duality

LP for Max Flow

$$\max f_{ts}$$

$$f_{ij} \leq c_{ij} \qquad (i,j) \in E$$

$$\sum_{(w,i)\in E} f_{wi} - \sum_{(i,z)\in E} f_{iz} \leq 0 \quad i \in V$$

$$f_{ij} \geq 0 \qquad (i,j) \in E$$

LP-Duality

$$\max f_{ts}$$

$$f_{ij} \leq c_{ij} \qquad (i,j) \in E$$

$$\sum_{(w,i)\in E} f_{wi} - \sum_{(i,z)\in E} f_{iz} \leq 0 \qquad i \in V$$

$$f_{ij} \geq 0 \qquad (i,j) \in E$$

$$\min \sum_{(i,j)\in E} c_{ij} d_{ij}$$

$$\begin{aligned} d_{ij} - p_i + p_j &\geq 0 && (i,j) \in E \\ p_s - p_t &\geq 1 && \\ d_{ij} &\geq 0 && (i,j) \in E \\ p_i &\geq 0 && i \in V \end{aligned}$$

Explanation of the Dual

$$\min \sum_{(i,j)\in E} c_{ij} d_{ij}$$

$$\begin{aligned}
d_{ij} - p_i + p_j &\geq 0 \quad && (i,j) \in E \\
p_s - p_t &\geq 1 && \\
d_{ij} &\in \{0,1\} && (i,j) \in E \\
p_i &\in \{0,1\} && i \in V
\end{aligned}$$

To obtain the dual program we introduce variables d_{ij} and p_i corresponding to the two types of inequalities in the primal.

- d_{ij}: distance labels on edges;
- p_i: potentials on nodes.

Integer Program

$$\min \sum_{(i,j)\in E} c_{ij} d_{ij}$$

$$\begin{array}{ll} d_{ij} - p_i + p_j \geq 0 & (i,j) \in E \\ p_s - p_t \geq 1 & \\ d_{ij} \in \{0,1\} & (i,j) \in E \\ p_i \in \{0,1\} & i \in V \end{array}$$

Let $(\mathbf{d}^*, \mathbf{p}^*)$ be an optimal solution to this integer program.

The only way to satisfy the inequality $p_s^* - p_t^* \geq 1$ with a $0/1$ substitution is to set $p_s^* = 1$ and $p_t^* = 0$.

This solution defines an $s-t$ cut $(X, \overline{X})$, where X is the set of potential 1 nodes, and $\overline{X}$ the set of potential 0 nodes.

Integer Program

$$\min \sum_{(i,j)\in E} c_{ij} d_{ij}$$

$$\begin{aligned} d_{ij} - p_i + p_j \geq 0 \quad & (i,j) \in E \\ p_s - p_t \geq 1 \quad & \\ d_{ij} \in \{0,1\} \quad & (i,j) \in E \\ p_i \in \{0,1\} \quad & i \in V \end{aligned}$$

Consider an edge (i,j) with $i \in X$ and $j \in \overline{X}$, Since $p_i^* = 1$ and $p_j^* = 0$, and thus $d_{ij}^* = 1$.

The distance label for each of the remaining edges can be set to either 0 or 1 without violating the first constraints.

The objective function value is precisely the capacity of the cut $(X, \overline{X})$, and hence $(X, \overline{X})$ must be a minimum $s - t$ cut.

Relaxation of the Integer Program

The integer program is a formulation of the minimum $s - t$ cut problem.

The dual program can be viewed as a relaxation of the integer program where the integrality constraint on the variables is dropped.

This leads to the constraints $1 \geq d_{ij} \geq 0$ for $(i, j) \in E$ and $1 \geq p_i \geq 0$ for $i \in V$.

The upper bound constraints on the variables are redundant; their omission cannot give a better solution.

We will say that this program is the LP relaxation of the integer program.

Relaxation of the Integer Program

The best fractional $s - t$ cut could have lower capacity than the best integral cut. This does not happen here.

Now, it can be proven that each vertex solution is integral, with each coordinate being 0 or 1.

The constraint matrix of this program is totally unimodular, Thus, the dual program always has an integral optimal solution.

Referred Materials

Content of this lecture comes from Section 7.2 in [DPV07], Section 26.1 and 26.2 in [CLRS09] and Section 12.2 in [Vaz04].