Design and Analysis of Algorithms (V)

An Introduction to Linear Programming

Guoqiang Li

School of Software, Shanghai Jiao Tong University

Linear Programming

Linear Programming

A linear programming problem gives a set of variables, and assigns real values to them so as to

1. satisfy a set of linear equations and/or linear inequalities involving these variables, and
2. maximize or minimize a given linear objective function.

Example: Profit Maximization

A boutique chocolatier has two products:

- triangular chocolates, called Pyramide,
- and the more decadent and deluxe Pyramide Nuit.

Q: How much of each should it produce to maximize profits?

- Every box of Pyramide has a a profit of $1.
- Every box of Nuit has a profit of $6.
- The daily demand is limited to at most 200 boxes of Pyramide and 300 boxes of Nuit.
- The current workforce can produce a total of at most 400 boxes of chocolate per day.

LP Formulation

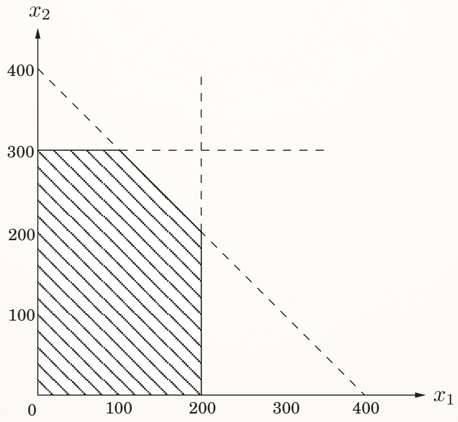
$$
\begin{array}{ll}
\texttt{Objective function} & \max x_1 + 6x_2 \\
\texttt{Constraints} & x_1 \leq 200 \\
 & x_2 \leq 300 \\
 & x_1 + x_2 \leq 400 \\
 & x_1, x_2 \geq 0
\end{array}
$$

A linear equation in x_1 and x_2 defines a line in the two-dimensional (2D) plane, and a linear inequality designates a half-space, the region on one side of the line.

The set of all feasible solutions of this linear program is the intersection of five half-spaces.

It is a convex polygon.

The Convex Polygon



The Optimal Solution

We want to find the point in this polygon at which the objective function is maximized.

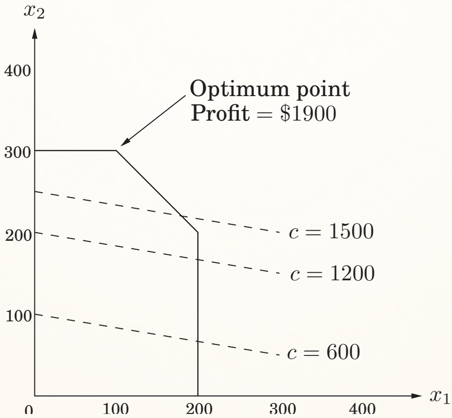
The points with a profit of c dollars lie on the line $x_1 + 6x_2 = c$, which has a slope of $-1/6$.

As c increases, this “profit line” moves parallel to itself, up and to the right.

Since the goal is to maximize c, we must move the line as far up as possible, while still touching the feasible region.

The optimum solution will be the very last feasible point that the profit line sees and must therefore be a vertex of the polygon.

The Convex Polygon



The Optimal Solution

It is a general rule of linear programs that the optimum is achieved at a vertex of the feasible region.

The only exceptions are cases in which there is no optimum; this can happen in two ways:

1. The linear program is infeasible; that is, the constraints are so tight that it is impossible to satisfy all of them.
   - For instance, $x \leq 1$, $x \geq 2$.
2. The constraints are so loose that the feasible region is unbounded, and it is possible to achieve arbitrarily high objective values.
   - For instance, $\max x_1 + x_2$
   - $x_1, x_2 \geq 0$

Solving Linear Programs

Linear programs (LPs) can be solved by the simplex method, devised by George Dantzig in 1947.

This algorithm starts at a vertex, and repeatedly looks for an adjacent vertex of better objective value.

It does hill-climbing on the vertices of the polygon, walking from neighbor to neighbor so as to steadily increase profit along the way.

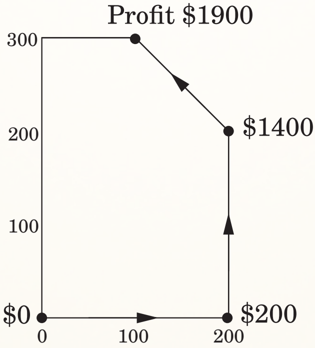
Upon reaching a vertex that has no better neighbor, simplex declares it to be optimal and halts.

Solving Linear Programs

Q: Why does this local test imply global optimality?

By simple geometry. Since all the vertex's neighbors lie below the line, the rest of the feasible polygon must also lie below this line.

The Example



More Products

The chocolatier introduces a third and even more exclusive chocolates, called Pyramide Luxe. One box of these will bring in a profit of \$13.

Let x_1, x_2, x_3 denote the number of boxes of each chocolate produced daily, with x_3 referring to Luxe.

The old constraints on x_1 and x_2 persist. The labor restriction now extends to x_3 as well: the sum of all three variables is at most 400.

Nuit and Luxe require the same packaging machinery. Luxe uses it three times as much, which imposes another constraint $x_2 + 3x_3 \leq 600$.

LP

$$\max x_1 + 6x_2 + 13x_3$$
$$x_1 \leq 200$$
$$x_2 \leq 300$$
$$x_1 + x_2 + x_3 \leq 400$$
$$x_2 + 3x_3 \leq 600$$
$$x_1, x_2, x_3 \geq 0$$

LP

The space of solutions is now three-dimensional.

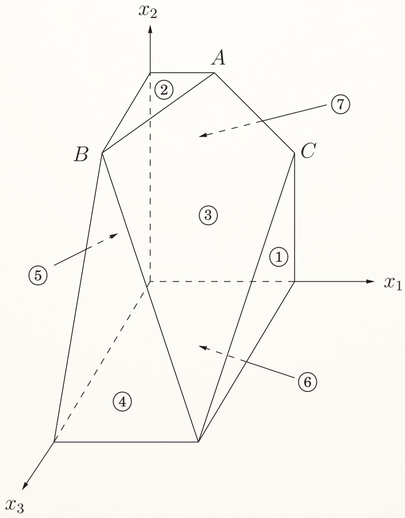
Each linear equation defines a 3D plane, and each inequality a half-space on one side of the plane.

The feasible region is an intersection of seven half-spaces, a polyhedron.

A profit of c corresponds to the plane $x_1 + 6x_2 + 13x_3 = c$.

As c increases, this profit-plane moves parallel to itself, further into the positive orthant until it no longer touches the feasible region.

The Example



LP

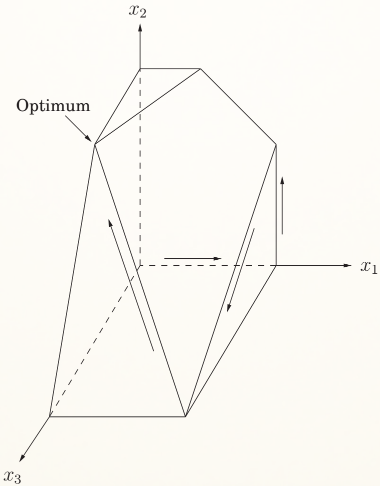
The point of final contact is the optimal vertex: $(0, 300, 100)$, with total profit $\$3100$.

Q: How would the simplex algorithm behave on this modified problem?

A possible trajectory

$$\frac{(0,0,0)}{\$0} \rightarrow \frac{(200,0,0)}{\$200} \rightarrow \frac{(200,200,0)}{\$1400} \rightarrow \frac{(200,0,200)}{\$2800} \rightarrow \frac{(0,300,100)}{\$3100}$$

The Example



ILP and Rounding

Example: Production Planning

The company makes handwoven carpets, a product for which the demand is extremely seasonal.

Our analyst has just obtained demand estimates for all months of the next calendar year: $d_1, d_2, \ldots, d_{12}$, ranging from 440 to 920.

Currently with 30 employees, each of whom makes 20 carpets per month and gets a monthly salary of $2000.

With no initial surplus of carpets.

Example: Production Planning

Q: How can we handle the fluctuations in demand? There are three ways:

1. Overtime. Overtime pay is 80% more than regular pay. Workers can put in at most 30% overtime.
2. Hiring and firing, costing \$320 and \$400, respectively, per worker.
3. Storing surplus production, costing \$8 per carpet per month. Currently without stored carpets on hand, and without any carpets stored at the end of year.

LP Formulations

$$
\begin{aligned}
w_i &= \text{number of workers during } i\text{-th month; } w_0 = 30. \\
x_i &= \text{number of carpets made during } i\text{-th month.} \\
o_i &= \text{number of carpets made by overtime in month } i. \\
h_i, f_i &= \text{number of workers hired and fired, respectively,} \\
& \text{ at beginning of month } i. \\
s_i &= \text{number of carpets stored at end of month } i\text{; } s_0 = 0.
\end{aligned}
$$

LP Formulation

All variables must be nonnegative:

$$w_i, x_i, o_i, h_i, f_i, s_i \geq 0, i = 1, \ldots, 12$$

The total number of carpets made per month consists of regular production plus overtime:

$$x_i = 20w_i + o_i$$

$i = 1, \ldots, 12$.

The number of workers can potentially change at the start of each month:

$$w_i = w_{i-1} + h_i - f_i$$

LP Formulation

The number of carpets stored at the end of each month is what we started with, plus the number we made, minus the demand for the month:

$$s_i = s_{i-1} + x_i - d_i$$

And overtime is limited:

$$o_i \leq 6w_i$$

LP Formulation

The objective function is to minimize the total cost:

$$\min 2000 \sum_i w_i + 320 \sum_i h_i + 400 \sum_i f_i + 8 \sum_i s_i + 180 \sum_i o_i$$

Integer Linear Programming

The optimum solution might turn out to be fractional; for instance, it might involve hiring 10.6 workers in the month of March.

This number would have to be rounded to either 10 or 11 in order to make sense, and the overall cost would then increase correspondingly.

In the example, most of the variables take on fairly large values, and thus rounding is unlikely to affect things too much.

Integer Linear Programming

There are other LPs, in which rounding decisions have to be made very carefully to end up with an integer solution of reasonable quality.

There is a tension in linear programming between the ease of obtaining fractional solutions and the desirability of integer ones.

In NP problems, finding the optimum integer solution of an LP is an important but very hard problem, called integer linear programming.

Quiz

Shortest path problem gives a weighted, directed graph $G = (V, E)$, with weight function $w : E \to \mathbb{R}$ mapping edges to real-valued weights, a source vertex s, and destination vertex t. We wish to compute the weight of a shortest path from s to t.

Shortest Path in LP

$$\max d_t$$

$$\begin{aligned} d_v \leq d_u + w(u,v) &\quad (u,v) \in E \\ d_s = 0 & \\ d_i \geq 0 &\quad i \in V \end{aligned}$$

Q: Another formalization?

Shortest Path in LP

Let $\mathcal{S} = \{S \subseteq V : s \in S, t \notin S\}$; that is, $\mathcal{S}$ is the set of all s-t cuts in the graph. Then we can model the shortest s-t path problem with the following integer program,

$$\min \sum_{e \in E} w_e x_e$$

$$\begin{aligned} \sum_{e \in \delta(S)} x_e \geq 1 \quad & S \in \mathcal{S} \\ x_e \in \{0, 1\} \quad & e \in E \end{aligned}$$

where $\delta(S)$ is the set of all edges that have one endpoint in S and the other endpoint not in S.

- Can we relax the restriction $x_e \in \{0, 1\}$ to $0 \leq x_e \leq 1$?
- How about $x_e \geq 0$?

Duality

Product Planning Revisit

Recall:

$$\max x_1 + 6x_2$$
$$x_1 \le 200$$
$$x_2 \le 300$$
$$x_1 + x_2 \le 400$$
$$x_1, x_2, x_3 \ge 0$$

Simplex declares the optimum solution to be $(x_1, x_2) = (100, 300)$, with objective value 1900.

Can this answer be checked somehow?

We take the first inequality and add it to six times the second inequality:

$$x_1 + 6x_2 \le 2000$$

Product Planning Revisit

Multiplying the three inequalities by 0, 5, and 1, respectively, and adding them up yields

$$x_1 + 6x_2 \leq 1900$$

Multipliers

Let's investigate the issue by describing what we expect of these three multipliers, call them y_1, y_2, y_3.

Multiplier	Inequality
y_1	$x_1 \leq 200$
y_2	$x_2 \leq 300$
y_3	$x_1 + x_2 \leq 400$

These y_i's must be nonnegative, otherwise they are unqualified to multiply inequalities.

After the multiplication and addition steps, we get the bound:

$$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$$

We want the left-hand side to look like the objective function $x_1 + 6x_2$ so that the right-hand side is an upper bound on the optimum solution.

Multipliers

$$x_1 + 6x_2 \leq 200y_1 + 300y_2 + 400y_3$$

if

$$\begin{gathered} y_1, y_2, y_3 \geq 0 \\ y_1 + y_3 \geq 1 \\ y_2 + y_3 \geq 6 \end{gathered}$$

The Dual Program

We can easily find y's that satisfy the inequalities on the right by simply making them large enough, for example $(y_1, y_2, y_3) = (5, 3, 6)$.

These particular multipliers tell us that the optimum solution of the LP is at most

$$200 \cdot 5 + 300 \cdot 3 + 400 \cdot 6 = 4300$$

What we want is a bound as tight as possible, so we minimize

$$200y_1 + 300y_2 + 400y_3$$

subject to the preceding inequalities. This is a new linear program!

The Dual Program

$$\begin{gathered}\min 200y_1 + 300y_2 + 400y_3 \\ y_1 + y_3 \geq 1 \\ y_2 + y_3 \geq 6 \\ y_1, y_2, y_3 \geq 0\end{gathered}$$

Any feasible value of this dual LP is an upper bound on the original primal LP.

If we find a pair of primal and dual feasible values that are equal, then they must both be optimal.

Here is just such a pair:

- Primal: $(x_1, x_2) = (100, 300)$;
- Dual: $(y_1, y_2, y_3) = (0, 5, 1)$.

They both have value 1900 and certify each other's optimality.

Matrix-Vector Form and Its Dual

Primal LP	Dual LP
$\max c^T \mathbf{x}$	$\min \mathbf{y}^T b$
$A\mathbf{x} \leq b$	$\mathbf{y}^T A \geq c^T$
$\mathbf{x} \geq 0$	$\mathbf{y} \geq 0$

Primal LP:

$$\begin{aligned} \max \ c_1x_1 + \cdots + c_nx_n \\ a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i \quad \text{for } i \in I \\ a_{i1}x_1 + \cdots + a_{in}x_n = b_i \quad \text{for } i \in E \\ x_j \geq 0 \quad \text{for } j \in N \end{aligned}$$

Dual LP:

$$\begin{aligned} \min \ b_1y_1 + \cdots + b_my_m \\ a_{1j}y_1 + \cdots + a_{mj}y_m \geq c_j \quad \text{for } j \in N \\ a_{1j}y_1 + \cdots + a_{mj}y_m = c_j \quad \text{for } j \notin N \\ y_i \geq 0 \quad \text{for } i \in I \end{aligned}$$

Matrix-Vector Form and Its Dual

$$\begin{aligned} \max x_1 + 6x_2 \\ x_1 \leq 200 \\ x_2 \leq 300 \\ x_1 + x_2 \leq 400 \\ x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \min 200y_1 + 300y_2 + 400y_3 \\ y_1 + y_3 \geq 1 \\ y_2 + y_3 \geq 6 \\ y_1, y_2, y_3 \geq 0 \end{aligned}$$

Matrix-Vector Form and Its Dual

Theorem (Duality)

If a linear program has a bounded optimum, then so does its dual, and the two optimum values coincide.

Complementary Slackness

The number of variables in the dual is equal to that of constraints in the primal and the number of constraints in the dual is equal to that of variables in the primal.

An inequality constraint has slack if the slack variable is positive.

The complementary slackness refers to a relationship between the slackness in a primal constraint and the associated dual variable.

LP and Its Dual

$$
\begin{aligned}
\max x_1 + 6x_2 \\
x_1 \leq 200 \\
x_2 \leq 300 \\
x_1 + x_2 \leq 400 \\
x_1, x_2 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\min 200y_1 + 300y_2 + 400y_3 \\
y_1 + y_3 \geq 1 \\
y_2 + y_3 \geq 6 \\
y_1, y_2, y_3 \geq 0
\end{aligned}
$$

$$x_1 = 100, x_2 = 300$$

$$y_1 = 0, y_2 = 5, y_3 = 1$$

Complementary Slackness

Theorem

Assume LP problem (P) has a solution x^* and its dual problem (D) has a solution y^*.

1. *If $x_j^* > 0$, then the j-th constraint in (D) is binding.*
2. *If the j-th constraint in (D) is not binding, then $x_j^* = 0$.*
3. *If $y_i^* > 0$, then the i-th constraint in (P) is binding.*
4. *If the i-th constraint in (P) is not binding, then $y_i^* = 0$.*

Proof.

Assignment ! □

Shortest Path in LP

In the shortest path problem, we are given a weighted, directed graph $G = (V, E)$, with weight function $w : E \to R$ mapping edges to real-valued weights, a source vertex s, and destination vertex t. We wish to compute the weight of a shortest path from s to t.

Shortest Path in LP

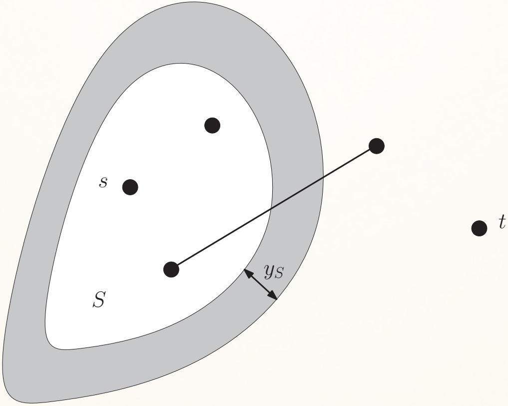
$$\min \sum_{e \in E} w_e x_e$$

$$\begin{aligned} \sum_{e \in \delta(S)} x_e &\geq 1 \quad & S \in \mathcal{S} \\ x_e &\geq 0 \quad & e \in E \end{aligned}$$

$$\max \sum_{S \in \mathcal{S}} y_S$$

$$\begin{aligned} \sum_{S \in \mathcal{S}, e \in \delta(S)} y_S &\leq w_e \quad & e \in E \\ y_S &\geq 0 \quad & S \in \mathcal{S} \end{aligned}$$

The Moat



Referred Materials

Content of this lecture comes from Section 7.1 and 7.4 in [DPV07], Section 29.2 in [CLRS09], and Section 7.3 in [WS11].