Assignment 2

Deadline: Apr. 16

1. Prove the complementary slackness property of linear programs.

2. In the single-source shortest-paths problem, we want to find the shortest-path weights from a source vertex s to all vertices $v \in V$. Given a graph G, write a linear program for which the solution has the property that d_v is the shortest-path weight from s to v for each vertex $v \in V$.

3. Prove the König theorem: Let G be bipartite, then cardinality of maximum matching = cardinality of maximum vertex cover.

4. Show how to find a minimum cut of a graph (not only the cost of minimum cut, but also the set of edges in the cut).

5. Consider the following optimization problem

$$\begin{array}{rl} \text{minimize} & f_0(\mathbf{x}), \\ \text{subject to} & f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \end{array}$$

with variable $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$. Note that the program may not be linear. The *Lagrangian* $L : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ associated with the program is defined as

$$L(\mathbf{x}, \lambda) = f_0(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i f_i(\mathbf{x})$$

where $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbf{R}^m$.

Define the *Lagrange dual function* $g : \mathbf{R}^m \to \mathbf{R}$ as the minimum value of the Lagrangian over x: for $\lambda \in \mathbf{R}^m$,

$$g(\lambda) = \inf_{\mathbf{x}} L(\mathbf{x}, \lambda)$$

We write $\lambda \geq 0$ if $\lambda_i \geq 0$ for all $1 \leq i \leq m$ and let p^* be the optimal value of original program. Show that for every $\lambda \geq 0$,

$$g(\lambda) \leq p^*$$

6. Consider the following optimization problem

$$\begin{array}{rl} \text{maximize} & g(\lambda) \\ \text{subject to} & \lambda \geq 0 \end{array}$$

Show that if the program in previous exercise is a linear program, then this program is its dual program.

7. Show that if the gap between the values of an integer program and its dual (the ”duality gap”) is zero, then the linear programming relaxations of the integer program and the dual of the relaxation, both admit integral solutions.

How about the converse? Do you think it holds?

8. Show that the Primal-Dual algorithm for shortest s-t path is equivalent to Dijkstra algorithm.