# Assignment1

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*2/3/2019*

## Problem 1

## Question 1

$$
\begin{aligned}
f(y,\lambda) &= \lambda e^{-\lambda y} \\
&= \exp(-\lambda y + \log\lambda) \\
&= \exp[-(\lambda y - \log\lambda)]
\end{aligned}
$$

Let $\theta = \lambda$, then

$$f(y,\theta,\phi) = \exp[-(\theta y - \log\theta)]$$

So scale parameter $\phi = -1$, $b(\theta) = log\theta$.

$E(Y) = b'(\theta) = \frac{1}{\theta}$, $Var(Y) = \phi b''(\theta) = -(\frac{1}{\theta})' = -\frac{1}{\theta^2}$.

As $b'(\theta) = \frac{1}{\theta}$, $g = b'^{-1}(\mu) = \frac{1}{\mu}$.

### Question 2

$$
\begin{aligned}
f(y,\pi) &= \binom{n}{y}\pi^y(1-\pi)^{(n-y)} \\
&= exp[\log\binom{n}{y} + y\log\pi + (n-y)\log(1-\pi)] \\
&= exp[y\log(\frac{\pi}{1-\pi}) + n\log(1-\pi) + \log\binom{n}{y}]
\end{aligned}
$$

Let $\theta = \log(\frac{\pi}{1-\pi})$, then $e^\theta = \frac{\pi}{1-\pi}$, $\pi = \frac{e^\theta}{e^\theta+1}$,

$$
\begin{aligned}
f(y,\theta,\phi) &= exp[y\theta + n\log(1-\frac{e^\theta}{e^\theta+1}) + \log\binom{n}{y}] \\
&= exp[y\theta - (-n\log\frac{1}{e^\theta+1}) + \log\binom{n}{y}]
\end{aligned}
$$

So scale parameter $\phi = 1$, $b(\theta) = -n\log\frac{1}{e^\theta+1}$.

$E(Y) = b'(\theta) = n(e^\theta+1)[-\frac{e^\theta}{(e^\theta+1)^2}] = \frac{ne^\theta}{e^\theta+1}$, $Var(Y) = \phi b''(\theta) = n(1-\frac{1}{e^\theta+1})' = \frac{ne^\theta}{(e^\theta+1)^2}$.

As $b'(\theta) = \frac{ne^\theta}{e^\theta+1}$, $g = b'^{-1}(\mu) = \log\frac{\mu}{n-\mu}$

## Question 3

$$
\begin{aligned}
f(y,\lambda) &= \frac{1}{y!}\lambda^y e^{-\lambda} \\
&= \exp[-\lambda + y\log\lambda + \log(\frac{1}{y!})]
\end{aligned}
$$

Let $\theta = \log \lambda$, then $\lambda = e^{\theta}$,

$$f(y, \theta, \phi) = \exp[\theta y - e^{\theta} + \log(\frac{1}{y!})]$$

So scale parameter $\phi = 1$, $b(\theta) = e^{\theta}$.

$E(Y) = b'(\theta) = e^{\theta}$, $Var(Y) = \phi b''(\theta) = e^{\theta}$.

As $b'(\theta) = e^{\theta}$, $g = b'^{-1}(\mu) = \log \mu$.

## Question 4

$$\begin{aligned}
f(y, k) &= \frac{1}{\Gamma(\frac{k}{2})2^{\frac{k}{2}}} y^{\frac{k}{2}-1} e^{-\frac{y}{2}} \\
&= \exp\{-\frac{y}{2} + (\frac{k}{2} - 1)\log y - \log[\Gamma(\frac{k}{2})2^{\frac{k}{2}}]\} \\
&= \exp[\frac{k}{2}\log y - \log\Gamma(\frac{k}{2}) - \frac{k}{2}\log 2 - \log y - \frac{y}{2}]
\end{aligned}$$

Let $\theta = \frac{k}{2}$, then

$$f(y, \theta, \phi) = \exp\{\theta \log y - [\log\Gamma(\theta) + \theta \log 2] - \log y - \frac{y}{2}\}$$

So scale parameter $\phi = 1$, $b(\theta) = \log\Gamma(\theta) + \theta \log 2$.

$$\begin{aligned}
E(Y) = \int y f(y) dy &= \int_0^{\infty} y \frac{1}{\Gamma(\frac{k}{2})2^{\frac{k}{2}}} y^{\frac{k}{2}-1} e^{-\frac{y}{2}} dy \\
&= \frac{\Gamma(\frac{k}{2}+1)2^{\frac{k}{2}+1}}{\Gamma(\frac{k}{2})2^{\frac{k}{2}}} \int_0^{\infty} \frac{1}{\Gamma(\frac{k}{2}+1)2^{\frac{k}{2}+1}} y^{\frac{k}{2}+1-1} e^{-\frac{y}{2}} dy \\
&= \frac{k}{2} \times 2 = k
\end{aligned}$$

$$\begin{aligned}
E(Y^2) = \int y^2 f(y) dy &= \int_0^{\infty} y \frac{1}{\Gamma(\frac{k}{2})2^{\frac{k}{2}}} y^{\frac{k}{2}-1} e^{-\frac{y}{2}} dy \\
&= \frac{\Gamma(\frac{k}{2}+2)2^{\frac{k}{2}+2}}{\Gamma(\frac{k}{2})2^{\frac{k}{2}}} \int_0^{\infty} \frac{1}{\Gamma(\frac{k}{2}+2)2^{\frac{k}{2}+2}} y^{\frac{k}{2}+2-1} e^{-\frac{y}{2}} dy \\
&= k^2 + 2k
\end{aligned}$$

$$Var(Y) = E(Y^2) - (EY)^2 = 2k$$

So $b'(\theta) = E(Y) = k = 2\theta$, $g = b'^{-1}(\mu) = \frac{\mu}{2}$.

## Question 5

$$\begin{aligned}
f(y, \beta) &= \binom{y+m-1}{m-1} \beta^m (1-\beta)^y \\
&= \exp[y \log(1-\beta) + m \log \beta + log\binom{y+m-1}{m-1}]
\end{aligned}$$

Let $\theta = \log(1-\beta)$, then $\beta = 1 - e^\theta$,

$$f(y,\theta,\phi) = \exp[\theta y - (-m\log(1-e^\theta)) + log\binom{y+m-1}{m-1}]$$

So scale parameter $\phi = 1$, $b(\theta) = -m\log(1-e^\theta)$.

$E(Y) = b'(\theta) = m\frac{1}{1-e^\theta}(-e^\theta) = \frac{me^\theta}{1-e^\theta}$, $Var(Y) = \phi b''(\theta) = m(1+\frac{1}{e^\theta-1}) = \frac{me^\theta}{(e^\theta-1)^2}$.

As $b'(\theta) = \frac{me^\theta}{1-e^\theta}$, $g = b'^{-1}(\mu) = \log\frac{\mu}{\mu+m}$.

## Question 6

$$\begin{aligned}
f(y,\lambda) &= \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} \\
&= \exp[-\beta y + (\alpha-1)\log y + \alpha\log\beta - \log\Gamma(\alpha)] \\
&= \exp[-(y\beta - \alpha\log\beta) + (\alpha-1)\log y - \log\Gamma(\alpha)]
\end{aligned}$$

Let $\theta = \beta$, then

$$f(y,\theta,\phi) = \exp[-(y\theta - \alpha\log\theta) + (\alpha-1)\log y - \log\Gamma(\alpha)]$$

So scale parameter $\phi = -1$, $b(\theta) = \alpha\log\theta$.

$E(Y) = b'(\theta) = \frac{\alpha}{\theta}$, $Var(Y) = \phi b''(\theta) = \frac{\alpha}{\theta^2}$.

As $b'(\theta) = \frac{\alpha}{\theta}$, $g = b'^{-1}(\mu) = \frac{\alpha}{\mu}$.

## Problem 2

$$\begin{aligned}
l(y|\beta) &= \sum_{i=1}^n \log[\binom{m}{y_i}\pi_i^{y_i}(1-\pi_i)^{m-y_i}] \\
&= \sum_{i=1}^n [\log\binom{m}{y_i} + y_i\log\pi_i + (m-y_i)\log(1-\pi_i)] \\
&= \sum_{i=1}^n [y_i\log(\frac{\pi_i}{1-\pi_i}) + m\log(1-\pi_i) + \log\binom{m}{y_i}]
\end{aligned}$$

Since $Y_i \sim Bin(m,\pi_i)$, $\mu = m\pi_i$. Deviance :

$$l(y,\mu) = \sum_{i=1}^n [y_i\log(\frac{\mu_i}{m-\mu_i}) + m\log(\frac{m-\mu_i}{m}) + \log\binom{m}{y_i}] l(y,y) = \sum_{i=1}^n [y_i\log(\frac{y_i}{m-y_i}) + m\log(\frac{m-y_i}{m}) + \log\binom{m}{y_i}]$$

$$\begin{aligned}
D(y,\hat{\mu}) &= 2[l(y,y) - l(y,\mu)] \\
&= 2\sum_{i=1}^n [y_i\log(\frac{y_i}{m-y_i}\frac{m-\mu_i}{\mu_i}) + m\log(\frac{m-y_i}{m-\mu_i})] \\
&= 2\sum_{i=1}^n \{y_i\log(\frac{y_i}{e^{x_i\hat{\beta}}(m-y_i)}) + m\log[\frac{(e^{x_i\hat{\beta}}+1)(m-y_i)}{m}]\}
\end{aligned}$$

Pearson Residual :

$$rp_i = \frac{y_i - \hat{\mu}_i}{\sqrt{V(\hat{\mu_i})}} = (y_i - \frac{me^{x_i\hat{\beta}}}{e^{x_i\hat{\beta}}+1})/\sqrt{\frac{me^{x_i\hat{\beta}}}{(e^{x_i\hat{\beta}}+1)^2}}$$

Deviance residual :

$$\begin{aligned} rD_i &= sign(y_i - \hat{\mu_i})\sqrt{d_i} \\ &= sign(y_i - \frac{me^{x_i\hat{\beta}}}{e^{x_i\hat{\beta}}+1})\sqrt{2\{y_i \log(\frac{y_i}{e^{x_i\hat{\beta}}(m-y_i)}) + m\log[\frac{(e^{x_i\hat{\beta}}+1)(m-y_i)}{m}]\}} \end{aligned}$$

Pearson's $\chi^2$ statistic :

$$G = \sum_{i=1}^{n} \frac{(y_i - \hat{\mu_i})^2}{V(\hat{\mu_i})} = \sum_{i=1}^{n} \frac{(y_i - \frac{me^{x_i\hat{\beta}}}{e^{x_i\hat{\beta}}+1})^2(e^{x_i\hat{\beta}}+1)^2}{me^{x_i\hat{\beta}}}$$

## Problem 3

### Question 1

$$\begin{aligned} l(y,\pi) &= \sum_{i=1}^{n}[y_i \log\pi + (1-y_i)\log(1-\pi)] \\ &= \sum_{i=1}^{n}[y_i \log(\frac{\pi}{1-\pi}) + \log(1-\pi)] \end{aligned}$$

$$\begin{aligned} s(\pi) = \frac{\partial l(y,\pi)}{\partial \pi} &= \sum_{i=1}^{n}[yi\frac{1-\pi}{\pi}\frac{1}{(1-\pi)^2} + \frac{1}{1-\pi}(-1)] \\ &= \sum_{i=1}^{n}[y_i\frac{1}{\pi(1-\pi)} - \frac{1}{1-\pi}] \\ &= n\bar{y}\frac{1}{\pi(1-\pi)} - \frac{n}{1-\pi} \end{aligned}$$

$$\begin{aligned} I(\pi) = E(-\frac{\partial^2 l(y,\pi)}{\partial \pi^2}) &= E(-\sum_{i=1}^{n}[yi\frac{2\pi-1}{\pi^2(1-\pi)^2} - \frac{1}{(1-\pi)^2}]) \\ &= \frac{2\pi-1}{\pi^2(1-\pi)^2}E(-\sum_{i=1}^{n} yi) + \frac{n}{(1-\pi)^2}] \\ &= -\frac{(2\pi-1)n}{\pi(1-\pi)^2} + \frac{n}{(1-\pi)^2}] \\ &= -\frac{n(2\pi-1) - n\pi}{\pi(1-\pi)^2} \\ &= -\frac{n\pi - n}{\pi(\pi-1)^2} \\ &= \frac{n}{\pi(1-\pi)} \end{aligned}$$

$$\hat{\pi}_{MLE} = \bar{y}$$

Wald : $TS_W = (\hat{\pi}_{MLE} - \pi_0) * I(\hat{\pi}_{MLE}) * (\hat{\pi}_{MLE} - \pi_0) = (\bar{y} - \pi_0)^2 \frac{n}{y_0(1-y_0)}$

Score : $TS_s = s(\pi_0) * I^{-1}(\pi_0) * s(\pi_0) = [\frac{n\bar{y}}{\pi_0(1-\pi_0)} - \frac{n}{1-\pi_0}]^2 \frac{\pi_0(1-\pi_0)}{n} = \frac{n(\bar{y}-\pi_0)^2}{\pi_0(1-\pi_0)}$

LR : $TS_{LR} = 2[l(y, \hat{\pi}_{MLE}) - l(y, \pi_0)] = 2\sum_{i=1}^{n}[y_i log(\frac{\bar{y}}{1-\bar{y}} \frac{1-\pi_0}{\pi_0}) + log(\frac{1-\bar{y}}{1-\pi_0})] = 2n[\bar{y}\log(\frac{\bar{y}(1-\pi_0)}{(1-\bar{y})\pi_0}) + log(\frac{1-\bar{y}}{1-\pi_0})]$

**Question 2**

When $\pi_0 = 0.1$,

Wald : $TS_W = (\bar{y} - \pi_0)^2 \frac{n}{y_0(1-y_0)} = 1.904$, corresponding $p_{wald} = 0.832$, fail to reject the null hypothesis.

Score : $TS_s = \frac{n(\bar{y}-\pi_0)^2}{\pi_0(1-\pi_0)} = 4.44$, corresponding $p_{score} = 0.965$, fail to reject the null hypothesis.

LR : $TS_{LR} = 2n[\bar{y}\log(\frac{\bar{y}(1-\pi_0)}{(1-\bar{y})\pi_0}) + log(\frac{1-\bar{y}}{1-\pi_0})] = 3.073$, corresponding $p_{lr} = 0.920$, fail to reject the null hypothesis.

When $\pi_0 = 0.3$,

Wald : $TS_W = (\bar{y} - \pi_0)^2 \frac{n}{y_0(1-y_0)} = 0$, corresponding $p_{wald} = 0$, reject the null hypothesis.

Score : $TS_s = \frac{n(\bar{y}-\pi_0)^2}{\pi_0(1-\pi_0)} = 0$, corresponding $p_{score} = 0$, reject the null hypothesis.

LR : $TS_{LR} = 2n[\bar{y}\log(\frac{\bar{y}(1-\pi_0)}{(1-\bar{y})\pi_0}) + log(\frac{1-\bar{y}}{1-\pi_0})] = 0$, corresponding $p_{lr} = 0$, reject the null hypothesis.

When $\pi_0 = 0.5$,

Wald : $TS_W = (\bar{y} - \pi_0)^2 \frac{n}{y_0(1-y_0)} = 1.904$, corresponding $p_{wald} = 0.832$, fail to reject the null hypothesis.

Score : $TS_s = \frac{n(\bar{y}-\pi_0)^2}{\pi_0(1-\pi_0)} = 1.6$, corresponding $p_{score} = 0.794$, fail to reject the null hypothesis.

LR : $TS_{LR} = 2n[\bar{y}\log(\frac{\bar{y}(1-\pi_0)}{(1-\bar{y})\pi_0}) + log(\frac{1-\bar{y}}{1-\pi_0})] = 1.646$, corresponding $p_{lr} = 0.800$, fail to reject the null hypothesis.

**Question 3**

In problem 3, the results of three statistic are equal when $\pi_0 = \bar{y} = 0.3$. When $\pi_0$ increases, the results of three statistic become closer. In all situations, three test statistics lead to same conclusions.