1. As $f(Y=t \mid X=x) = 0.01x \cdot e^{-0.01xt}$

$f(X=x) = \theta e^{-\theta x}$

the joint pdf of x, y is

$$f(X=x, Y=t) = f(Y=t \mid X=x) \cdot f(X=x)$$

$$= 0.01x \cdot e^{-0.01xt} \cdot \theta \cdot e^{-\theta x}$$

$$= 0.01\theta x e^{-(0.01t+\theta)x}$$

So the marginal pdf of Y is

$$f(Y=t) = \int_0^{+\infty} 0.01x \cdot \theta \cdot e^{-(0.01t+\theta)x} dx$$

$$= 0.01\theta \int_0^{+\infty} \frac{1}{(0.01t+\theta)} (0.01t+\theta) x e^{-(0.01t+\theta)x} dx$$

$$= \frac{0.01\theta}{0.01t+\theta} E[X \mid X \sim \exp(0.01t+\theta)]$$

$$= \frac{0.01\theta}{0.01t+\theta} \cdot \frac{1}{(0.01t+\theta)} = \frac{1}{(0.01t+\theta)^2}$$

So likelihood function of Y_i is

$$L = \prod_{i=1}^{n} \frac{0.01\theta}{(0.01t+\theta)^2}.$$

2. We need to find out $\hat{\theta}$ that ~~minize~~ maximize $L(Y_i)$.

Use Golden Search method.

the loglikelihood function is.

$l = \sum_{i=1}^{n} \log(0.01\theta) - 2\log(0.01t+\theta)$

and $\hat{\theta}$ also maximize l.

0 we assume θ belongs to (a,b)

1 calculate $l(a)$, $l(b)$.

2 choose $x_1 = a+(a-b)\times(1-w)$ where $w=0.618$

3 ~~ct~~ calculate $l(x_1)$

4 choose $x_2 = x_1+(b-a)\times(1-w)\times w$

5 calculate $l(x_2)$

6 if $f(x_2) < f(x_1)$ then $b=x_1$ ~~and repeats~~

else $a=x_1$, $x_1=x_2$

7 repeats. 1 - 6 n times.

3. As $f(Y=t \mid X=x) = 0.01\, x\, e^{-0.01xt}$

$f(X=x) = \theta e^{-\theta x}$

the joint pdf of x, y:

$f(x,y) = 0.01\theta x \cdot e^{-(0.01y+\theta)x}$

So complete log likelihood function of X, Y:

$l(x,y,\theta) = \sum_{i=1}^{n} [\log(0.01\theta x) - (0.01y+\theta)x]$

E-step:

$$E[X \mid Y,\theta] = \int_0^{+\infty} x f(x \mid Y,\theta)\, dx$$
$$= \int_0^{+\infty} x f(x \mid \theta)\, dx$$
$$= \frac{1}{\theta} = \delta$$

M-step (replace x by δ)

$l(y,\theta) = \sum_{i=1}^{n} [\log(0.01\theta\delta) - (0.01y+\theta)\delta]$

$\frac{\partial l}{\partial \theta} = \sum_{i=1}^{n} [\frac{0.01\delta}{0.01\theta\delta} - \delta] = \sum_{i=1}^{n} [\frac{1}{\theta} - \delta]$

let $\frac{\partial l}{\partial \theta} = 0 \Rightarrow \hat{\theta} = n / \sum_{i=1}^{n} \delta_i$