# Geometry Homework 5

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**Problem 1** (Ex P151 2). *Show that if a surface is tangent to a plane along a curve, then the points of this curve are either parabolic or planar.*

*Proof.* Assume that the curve is $\gamma(s)$, then along this curve, $N(\gamma(s))$ is perpendicular to the plane, so it is constant.

At point $\gamma(s)$, $[dN](\gamma'(s)) = \left(\frac{dN(\gamma(t))}{dt}\right)_{t=s} = 0$, so $\gamma'(s)$ is one of the principal direction of the surface at $\gamma(s)$, and it's associated principal curvature is $0$.

So the gaussian curvature of the surface at $\gamma(s)$ is $K = 0$, and this means that the point $\gamma(s)$ is either parabolic or planar. $\qquad\square$

**Problem 3** (Ex P151 3).

(a) Let $C \subset S$ be a regular curve on a surface $S$ with Gaussian curvature $K > 0$. Show that the curvature $\kappa$ of $C$ at $p$ satisfies

$$\kappa \geq \min(|\kappa_1|, |\kappa_2|),$$

where $\kappa_1, \kappa_2$ are the principal curvatures of $S$ at $p$.

(b) 為什麼上一小題需要 $\kappa > 0$ 的條件，$\kappa \geq 0$ 不可以嗎？

*Proof.* (a)

$$\begin{aligned}
\kappa &\geq |\kappa_n| \\
&= |\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta| \\
&= |\kappa_1| \cos^2\theta + |\kappa_2| \sin^2\theta (\because \kappa_1, \kappa_2 \text{ has equal sign.}) \\
&\geq \min(|\kappa_1|, |\kappa_2|)(\cos^2\theta + \sin^2\theta) \\
&= \min(|\kappa_1|, |\kappa_2|)
\end{aligned}$$

(b)

$\qquad\square$

**Problem 7.**

(a) $T_\lambda$ 是縮放 $\lambda$ 倍的映射，$\lambda > 0$。$\mathbb{X} : \Omega \to \mathbb{R}^3$ *regular surface*。討論 $T_\lambda \circ \mathbb{X} : \Omega \to \mathbb{R}^3$ 上對應點 $\kappa_n, H, K$ 的變化。

*(b)* $\mathbb{X}: \begin{array}{c} \Omega \\ (u,v) \end{array} \to \mathbb{R}^3$，若定義 $\overline{\mathbb{X}}(u,v) = \mathbb{X}(v,u)$ *(因此 $N$ 轉向)*。討論 $\overline{\mathbb{X}}(\Omega)$
上相對應點的 $K_n, H, K$ 變化。

*Proof.* $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

**Problem 9** (旋轉面). $\mathbb{X}(u,v) = (f(u)\cos v, f(u)\sin v, g(u)),\ f > 0$

*(a)* 計算其 $e, f, g, H, K$

*(b)* 討論其 *principal direction* 與 *principal curvature* $K_1, K_2$。

*Proof.* To avoid the notational ambiguity, let $\mathbb{X}(u,v) = (s(u)\cos v, s(u)\sin v, t(u))$, and that $s > 0$.

(a) We have

$$\mathbb{X}_u = (s'(u)\cos v, s'(u)\sin v, t'(u));$$
$$\mathbb{X}_v = (-s(u)\sin v, s(u)\cos v, 0);$$
$$E = \langle \mathbb{X}_u, \mathbb{X}_u \rangle = s'(u)^2 + t'(u)^2$$
$$F = \langle \mathbb{X}_u, \mathbb{X}_v \rangle = 0$$
$$G = \langle \mathbb{X}_v, \mathbb{X}_v \rangle = s(u)^2$$
$$\mathbb{X}_{uu} = (s''(u)\cos v, s''(u)\sin v, t''(u));$$
$$\mathbb{X}_{uv} = (-s'(u)\sin v, s'(u)\cos v, 0);$$
$$\mathbb{X}_{vv} = (-s(u)\cos v, -s(u)\sin v, 0);$$
$$N = \frac{\mathbb{X}_u \times \mathbb{X}_v}{|\mathbb{X}_u \times \mathbb{X}_v|} = \frac{(-t'(u)s(u)\cos v, -t'(u)s(u)\sin v, s'(u)s(u))}{\sqrt{t'(u)^2 s(u)^2 + s'(u)^2 s(u)^2}}$$
$$= \frac{(-t'(u)\cos v, -t'(u)\sin v, s'(u))}{\sqrt{t'(u)^2 + s'(u)^2}};$$
$$e = \langle N, \mathbb{X}_{uu} \rangle = \frac{-s''(u)t'(u) + t''(u)s'(u)}{\sqrt{t'(u)^2 + s'(u)^2}}$$
$$f = \langle N, \mathbb{X}_{uv} \rangle = 0$$
$$g = \langle N, \mathbb{X}_{vv} \rangle = \frac{s(u)t'(u)}{\sqrt{t'(u)^2 + s'(u)^2}}$$
$$-dN^T = \begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} = \frac{1}{EG - F^2} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} = \begin{bmatrix} e/E & 0 \\ 0 & g/G \end{bmatrix}$$
$$K = \det(-dN) = \frac{eg}{EG}$$
$$H = \frac{1}{2}\text{tr}(-dN) = \frac{eG + gE}{2EG}$$

(b) Since $-dN$ is already a diagonal matrix, clearly,

$$
\begin{aligned}
K_1 &= e/E; \\
K_2 &= g/G; \\
V_1 &= \mathbb{X}_u; \\
V_2 &= \mathbb{X}_v;
\end{aligned}
$$

$\square$

**Problem 10 (管面).** $\mathbb{X}(s, \theta) = \gamma(s) + \cos\theta\, \vec{n}(s) + \sin\theta\, \vec{b}(s)$, $0 < \kappa < 1$

*(a) 計算其 $e, f, g, H, K$*

*(b) 討論曲面上 $K$ 的分佈。*

*Proof.*　(a) Let $\vec{t}(s), \vec{n}(s), \vec{b}(s)$ be the basis, and $\gamma(0)$ be the origin.

$$
\begin{aligned}
\mathbb{X}_s &= (1 - \kappa\cos\theta, \tau\sin\theta, -\tau\cos\theta); \\
\mathbb{X}_\theta &= (0, -\sin\theta, \cos\theta); \\
N &= \frac{\mathbb{X}_s \times \mathbb{X}_\theta}{|\mathbb{X}_s \times \mathbb{X}_\theta|} = \frac{(0, (\kappa\cos\theta - 1)\cos\theta, (\kappa\cos\theta - 1)\sin\theta)}{|\kappa\cos\theta - 1|} \\
&= (0, -\cos\theta, -\sin\theta); \quad \text{(since } \kappa < 1) \\
\mathbb{X}_{ss} &= (-\sin\theta\kappa\tau, \kappa + \cos\theta(-\kappa^2 - \tau^2), -\sin\theta\tau^2); \\
\mathbb{X}_{s\theta} &= (1 + \kappa\sin\theta, \tau\cos\theta, \tau\sin\theta); \\
\mathbb{X}_{\theta\theta} &= (0, -\cos\theta, -\sin\theta); \\
e &= -\kappa\cos\theta + \cos^2\theta(\kappa^2 + \tau^2) + \tau^2\sin^2\theta = -\kappa\cos\theta + \kappa^2\cos^2\theta + \tau^2; \\
f &= -\tau; \\
g &= 1; \\
dN(\mathbb{X}_s) &= N_s = (0, 0, 0); \\
dN(\mathbb{X}_\theta) &= N_\theta = (0, \sin\theta, -\cos\theta) = -\mathbb{X}_\theta; \\
\kappa_1 &= 0; \\
\kappa_2 &= 1; \\
K &= 0; \\
H &= 1/2.
\end{aligned}
$$

(b)

$$
K = 0 \quad \text{everywhere}
$$

$\square$