Geometry Homework 5

B96201044 黃上恩, *B98901182* 時丕勳, *K0020100x* 劉士瑋

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Problem 1 (參見 P67 Ex16). 考慮

$$\begin{array}{cccc} \mathbb{X}: & \mathbb{R}^2 & \to & S^2\setminus\{N\} \\ & (u,v) & \mapsto & \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right) \end{array}, N=(0,0,1)$$

(a) 檢查這的確是 $S^2\setminus\{N\}$ 的參數式

(b) 計算 E, F, G，$E=G$ 嗎？

(c) 計算 $\mathbb{X}_u, \mathbb{X}_v$

(d) 若 W_1, W_2 是 $\mathbb{R}^2$ 兩以 a 為起點的向量，說明 W_1, W_2 的夾角 $= d\mathbb{X}(W_1)$ 與 $d\mathbb{X}(W_2)$ 的夾角

Proof.

(a) 考慮 $S^2\setminus\{N\}$ 上面的點 (x,y,z)，它必須滿足 $x^2+y^2+z^2=1$ 且 $z\neq 1$。則令 $u=\frac{x}{1-z}, v=\frac{y}{1-z}$，於是有 $\frac{2u}{u^2+v^2+1} = 2\frac{x}{1-z}\cdot\frac{(1-z)^2}{x^2+y^2+(1-z)^2} = 2\frac{x}{1-z}\cdot\frac{(1-z)^2}{2-2z} = x$，類似地 $\frac{2v}{u^2+v^2+1} = 2\frac{y}{1-z}\cdot\frac{(1-z)^2}{x^2+y^2+(1-z)^2} = y$，$\frac{u^2+v^2-1}{u^2+v^2+1} = \frac{1-z^2-(1-z)^2}{1-z^2+(1-z)^2} = \frac{2z-2z^2}{2-2z} = z$。因此這的確是 $S^2\setminus\{N\}$ 的參數式。

(c) $\mathbb{X}_u = \left(\frac{2}{u^2+v^2+1} - \frac{4u^2}{(u^2+v^2+1)^2}, -\frac{4uv}{(u^2+v^2+1)^2}, \frac{2u}{u^2+v^2+1} - \frac{2u(u^2+v^2-1)}{(u^2+v^2+1)^2}\right)$

$\mathbb{X}_v = \left(-\frac{4uv}{u^2+v^2+1}, \frac{2}{u^2+v^2+1} - \frac{4v^2}{(u^2+v^2+1)^2}, \frac{2v}{u^2+v^2+1} - \frac{2v(u^2+v^2-1)}{(u^2+v^2+1)^2}\right)$

(b) $E = \langle\mathbb{X}_u,\mathbb{X}_u\rangle = \frac{4}{(u^2+v^2+1)^4}(((u^2+v^2+1)-2u^2)^2 + (2uv)^2 + (u(u^2+v^2+1)-u(u^2+v^2-1))^2) = \frac{4}{(u^2+v^2+1)^4}((v^2-u^2+1)^2+4u^2v^2+4u^2) = \frac{4}{(u^2+v^2+1)^2}$

$F = \langle\mathbb{X}_u,\mathbb{X}_v\rangle = \frac{4}{(u^2+v^2+1)^4}(-2uv(v^2-u^2+1) - 2uv(u^2-v^2+1) + 4uv) = \frac{4}{(u^2+v^2+1)^4}\cdot 0 = 0$

$G = \langle\mathbb{X}_v,\mathbb{X}_v\rangle = \frac{4}{(u^2+v^2+1)^4}((2uv)^2 + ((u^2+v^2+1)-2v^2)^2 + (v(u^2+v^2+1) - v(u^2+v^2-1))^2) = \frac{4}{(u^2+v^2+1)^4}((u^2-v^2+1)^2 + 4u^2v^2 + 4v^2) = \frac{4}{(u^2+v^2+1)^2} = E$

(d) Let $a = (u, v)$. Then $\|d\mathbb{X}(W_1)\|^2 = W_1^T \begin{bmatrix} E & F \\ F & G \end{bmatrix} W_1 = \frac{4}{(u^2+v^2+1)^2}\|W_1\|^2$. Similarly $\|d\mathbb{X}(W_2)\| = \frac{4}{(u^2+v^2+1)^2}\|W_2\|^2$. Now the cosine of angle between them is

$$\begin{aligned}
\cos\theta &= \langle d\mathbb{X}(W_1), d\mathbb{X}(W_2)\rangle / (\frac{4}{(u^2+v^2+1)^2} \cdot \|W_1\| \cdot \|W_2\|) \\
&= \left(W_1^T \begin{bmatrix} E & F \\ F & G \end{bmatrix} W_2\right) / \left(\frac{4}{(u^2+v^2+1)^2} \cdot \|W_1\| \cdot \|W_2\|\right) \\
&= \langle W_1, W_2\rangle / (\|W_1\| \cdot \|W_2\|)
\end{aligned}$$

which is cosine of angle between W_1 and W_2.

□

Problem 2 (旋轉面)**.** $\mathbb{X}(\theta, s) = (a(s)\cos\theta, a(s)\sin\theta, b(s))$，其中 $(a(s), b(s))$ 為長度參數之平面曲線。計算 E, F, G 並討論其 *regular* 的條件。

Proof.

$$\begin{aligned}
\mathbb{X}_\theta &= (-a(s)\sin\theta, a(s)\cos\theta, 0) \\
\mathbb{X}_s &= (a'(s)\cos\theta, a'(s)\sin\theta, b'(s)) \\
\Rightarrow E &= a(s)^2\sin^2\theta + a(s)^2\cos^2\theta \\
&= a(s)^2 \\
F &= -a(s)a'(s)\sin\theta\cos\theta + a(s)a'(s)\cos\theta\sin\theta \\
&= 0 \\
G &= a'(s)^2\sin^2\theta + a'(s)^2\cos^2\theta \\
&= a'(s)^2 \\
&= 1 \\
|\mathbb{X}_\theta \times \mathbb{X}_s| &= \sqrt{EG - F^2} \\
&= \sqrt{a(s)^2} \\
&= |a(s)|
\end{aligned}$$

So $\mathbb{X}$ is regular iff $a(s) \neq 0$. □

Problem 3 (管面)**.** 設空間曲線 $\gamma(s)$，s 長度參數，$\vec{t}, \vec{n}, \vec{b}$ 為 *Frenet frame*。令 $\mathbb{X}_l(s, \theta) = \gamma(s) + l\cos\theta\vec{n}(s) + l\sin\theta\vec{b}(s), l > 0$，計算 E, F, G 並討論其 *regular* 條件。

Proof.

$$\begin{aligned}
\mathbb{X}_{ls} &= \gamma'(s) + l\cos\theta\vec{n}'(s) + l\sin\theta\vec{b}'(s) \\
&= \vec{t}(s) + l\cos\theta\left(-\kappa(s)\vec{t}(s) - \tau(s)\vec{b}(s)\right) + l\tau(s)\sin\theta\vec{n}(s) \\
&= (1 - l\kappa(s)\cos\theta)\vec{t}(s) + l\tau(s)\sin\theta\vec{n}(s) - l\tau(s)\cos\theta\vec{b}(s) \\
\mathbb{X}_{l\theta} &= -l\sin\theta\vec{n}(s) + l\cos\theta\vec{b}(s) \\
\Rightarrow E &= (1 - l\kappa(s)\cos\theta)^2 + (l\tau(s)\sin\theta)^2 + (l\tau(s)\cos\theta)^2 \\
&= (1 - l\kappa(s)\cos\theta)^2 + l^2\tau(s)^2 \\
F &= -l^2\tau(s)\sin^2\theta - l^2\tau(s)\cos^2\theta \\
&= -l^2\tau(s) \\
G &= l^2\sin^2\theta + l^2\cos^2\theta \\
&= l^2 \\
|\mathbb{X}_{ls} \times \mathbb{X}_{l\theta}| &= \sqrt{EG - F^2} \\
&= \sqrt{l^2\left((1 - l\kappa(s)\cos\theta)^2 + l^2\tau(s)^2\right) - l^4\tau(s)^2} \\
&= \sqrt{l^2(1 - l\kappa(s)\cos\theta)^2} \\
&= l\,|1 - l\kappa(s)\cos\theta|
\end{aligned}$$

So $\mathbb{X}$ is regualr iff $l\kappa(s)\cos\theta \neq 1 \forall s, \theta$
$\Leftrightarrow \frac{1}{l\kappa(s)} \neq \cos\theta$
$\Leftrightarrow \left|\frac{1}{l\kappa(s)}\right| > 1$
$\Leftrightarrow |\kappa(s)| < \frac{1}{l}$

□

Problem 6 (Ex6, p100). *Show that*

$$\mathbf{x}(u, v) = (u\sin\alpha\cos v, u\sin\alpha\sin v, u\cos\alpha)$$

where $0 < u < \infty, 0 < v < 2\pi, \alpha = const.$, *is a parametrization of the cone with* 2α *as the angle of the vertex. In the corresponding coordinate neighborhood, prove that the curve*

$$\mathbf{x}(c\exp(v\sin\alpha\cot\beta), v), \quad c = const., \beta = const.,$$

intersects the generators of the cone ($v = const.$) *under the constant angle* β.

Proof.

$$d\mathbf{x} = \begin{bmatrix} \sin\alpha\cos v & \sin\alpha\sin v & \cos\alpha \\ -u\sin\alpha\sin v & u\sin\alpha\cos v & 0 \end{bmatrix}$$

We have $\mathbf{x}_u \times \mathbf{x}_v = (-, -, u) \neq 0$ whenever $u > 0$. To show that the angle is β, define

$$\begin{aligned}
E &:= \langle \mathrm{x}_u, \mathrm{x}_u \rangle = \sin^2\alpha\cos^2 v + \sin^2\alpha\sin^2 v + \cos^2\alpha) = 1 \\
F &:= \langle \mathrm{x}_u, \mathrm{x}_v \rangle = -u\sin^2\alpha\cos v\sin v + u\sin^2\alpha\sin v\cos v + 0 = 0 \\
G &:= \langle \mathrm{x}_v, \mathrm{x}_v \rangle = u^2\sin^2\alpha\sin^2 v + u^2\sin^2\alpha\cos^2 v + 0 = u^2\sin^2\alpha
\end{aligned}$$

$$\langle\langle (a,b),(c,d)\rangle\rangle := \begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} E & F \\ F & G \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix}$$

to be the first fundamental form of x. Then

$$\begin{aligned}
&A := \partial(u,v)/\partial u = (1,0) \\
&B := \partial(c\exp(v\sin\alpha\cot\beta), v)/\partial v = (c\sin\alpha\cot\beta\exp(v\sin\alpha\cot\beta), 1).
\end{aligned}$$

Fixing the intersection at $(c\exp(v\sin\alpha\cot\beta), v)$, we got

$$\begin{aligned}
\langle\langle A,B\rangle\rangle &= \begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} E & F \\ F & G \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & c^2\exp(2v\sin\alpha\cot\beta)\sin^2\alpha \end{bmatrix}\begin{bmatrix} c\sin\alpha\cot\beta\exp(v\sin\alpha\cot\beta) \\ 1 \end{bmatrix} \\
&= c\sin\alpha\cot\beta\exp(v\sin\alpha\cot\beta) \\
\langle\langle A,A\rangle\rangle &= \begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} E & F \\ F & G \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & c^2\exp(2v\sin\alpha\cot\beta)\sin^2\alpha \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= 1 \\
\langle\langle B,B\rangle\rangle &= \begin{bmatrix} c & d \end{bmatrix}\begin{bmatrix} E & F \\ F & G \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} \\
&= \begin{bmatrix} c\sin\alpha\cot\beta\exp(v\sin\alpha\cot\beta) & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & c^2\exp(2v\sin\alpha\cot\beta)\sin^2\alpha \end{bmatrix} \\
&\quad \begin{bmatrix} c\sin\alpha\cot\beta\exp(v\sin\alpha\cot\beta) \\ 1 \end{bmatrix} \\
&= c^2\sin^2\alpha\cot^2\beta\exp(2v\sin\alpha\cot\beta) + c^2\sin^2\alpha\exp(2v\sin\alpha\cot\beta) \\
&= c^2\sin^2\alpha\csc^2\beta\exp(2v\sin\alpha\cot\beta) \\
\cos\theta &= \frac{\langle\langle A,B\rangle\rangle}{\sqrt{\langle\langle A,A\rangle\rangle\langle\langle B,B\rangle\rangle}} = \frac{c\sin\alpha\cot\beta\exp(v\sin\alpha\cot\beta)}{|c\sin\alpha\csc\beta\exp(v\sin\alpha\cot\beta)|} = \pm\cos\beta
\end{aligned}$$

□