Geometry Homework 12

B96201044 黃上恩, *B98901182* 時丕勳, *K0020100x* 劉士瑋

December 22, 2011

Problem 3 (Ex p294 3.). *If p is a point of a regular surface S, prove that*

$$K(p) = \lim_{r\to 0} \frac{12}{\pi} \frac{\pi r^2 - A}{r^4},$$

where $K(p)$ is the Gaussian curvature of S at p, r is the radius of a geodesic circle $S_r(p)$ centered in p, and A is the area of the region bounded by $S_r(p)$.

Proof.

$$\begin{aligned}
A_R &= \int_0^R \int_0^{2\pi} \sqrt{EG - F^2} d\theta dr \\
&= \int_0^R \int_0^{2\pi} \sqrt{G} d\theta dr \\
&\approx \int_0^R \int_0^{2\pi} r - \frac{K}{6} r^3 d\theta dr \\
&= \int_0^{2\pi} \frac{1}{2} R^2 - \frac{K}{24} R^4 d\theta \\
&= \pi R^2 - \frac{R^4}{24} \int_0^{2\pi} K d\theta \\
\to \frac{1}{2\pi} \int_0^{2\pi} K d\theta &= \frac{12}{r^4} (r^2 - \frac{1}{\pi} A_r) \\
\to K(p) &= \lim_{r\to 0} \frac{12}{r^4} (r^2 - \frac{1}{\pi} A_r) \\
&= \lim_{r\to 0} \frac{12}{\pi} \frac{\pi r^2 - A_r}{r^4}
\end{aligned}$$

□

Problem 4 (Ex p295 4.). *Show that in a system of normal coordinates centered in p, all the Christoffel symbols are zero at p.*

Proof. Let (u, v) be normal coordinate centered at p, (r, θ) be the geodesic polar coordinate centered at p.

Let $\hat{E}, \hat{F}, \hat{G}$ be the first fundamental form of the coordinate (r, θ), E, F, G be the first fundamental form of the coordinate (u, v),

$$\begin{aligned}
\hat{E} = 1, \hat{F} = 0 \\
\lim_{r\to 0} \hat{G} = 0, \lim_{r\to 0} \sqrt{\hat{G}}_r = 1 \\
\to \hat{G} = r^2 + o(r^3)
\end{aligned}$$

$$\begin{aligned}
r &= \sqrt{u^2+v^2} \\
\theta &= \tan^{-1}\frac{v}{u} \\
\mathbb{X}_u &= \frac{u}{r}\mathbb{X}_r - \frac{v}{r^2}\mathbb{X}_\theta \\
\mathbb{X}_v &= \frac{v}{r}\mathbb{X}_r + \frac{u}{r^2}\mathbb{X}_\theta \\
\to E &= \frac{u^2}{r^2} + \frac{v^2}{r^4}\hat{G} \\
F &= \frac{uv}{r^2} - \frac{uv}{r^4}\hat{G} \\
G &= \frac{v^2}{r^2} + \frac{u^2}{r^4}\hat{G}
\end{aligned}$$

When $r \to 0$:

$$\begin{aligned}
\hat{G} &\to r^2 \\
\hat{G}_u &\to 2u \\
\hat{G}_v &\to 2v
\end{aligned}$$

$$\begin{aligned}
E_u &= \frac{2uv^2}{r^4} - \frac{4uv^2}{r^6}\hat{G} + \frac{v^2}{r^4}\hat{G}_u \\
&= \frac{2uv^2}{r^4} - \frac{4uv^2}{r^4} + \frac{2uv^2}{r^4} = 0 \\
E_v &= -\frac{2u^2v}{r^4} + \frac{2v(u^2-v^2)}{r^6}\hat{G} + \frac{v^2}{r^4}\hat{G}_v \\
&= -\frac{2u^2v}{r^4} + \frac{2v(u^2-v^2)}{r^4} + \frac{2v^3}{r^4} = 0 \\
F_u &= \frac{v^3-vu^2}{r^4} - \frac{v^3-3u^2v}{r^6}\hat{G} - \frac{uv}{r^4}\hat{G}_u \\
&= \frac{v^3-vu^2}{r^4} - \frac{v^3-3u^2v}{r^4} - \frac{2u^2v}{r^4} = 0 \\
F_v &= \frac{u^3-uv^2}{r^4} - \frac{u^3-3v^2u}{r^6}\hat{G} - \frac{uv}{r^4}\hat{G}_v \\
&= \frac{u^3-uv^2}{r^4} - \frac{u^3-3v^2u}{r^4} - \frac{2uv^2}{r^4} = 0
\end{aligned}$$

$$
\begin{aligned}
G_u &= -\frac{2v^2u}{r^4} + \frac{2u(v^2-u^2)}{r^6}\hat{G} + \frac{u^2}{r^4}\hat{G}_u \\
&= -\frac{2v^2u}{r^4} + \frac{2u(v^2-u^2)}{r^4} + \frac{2u^3}{r^4} = 0 \\
G_v &= \frac{2vu^2}{r^4} - \frac{4vu^2}{r^6}\hat{G} + \frac{u^2}{r^4}\hat{G}_v \\
&= \frac{2vu^2}{r^4} - \frac{4vu^2}{r^4} + \frac{2vu^2}{r^4} = 0
\end{aligned}
$$

So $[i,j,k]=0$ and $\Gamma_{ij}^k = 0$. □

Problem 5 (Ex p295 5.). *For which of the pair of surfaces given below does there exist a local isometry?*

(a) Torus of revolution and cone.

(b) Cone and sphere.

(c) Cone and cylinder.

Problem 8.

(a) 在半徑 R 的球面上，計算 *geodesic circle* 的長度，並驗證 *P292* 課文中間 $K(p)$ 的公式。

(b) 用一樣的精神，檢驗 *P294 3.* 的公式。

Proof. (a) WLOG, let $p=(0,0,R)$, If $q \in T_p$ with $q=(l,\theta)$, then

$$\exp(q) = \left(R\sin\frac{l}{R}\cos\theta, R\sin\frac{l}{R}\sin\theta, R\cos\frac{l}{R}\right),$$

and thus the length of the image of the circle $\{q \in T_p : d(q,p) = l\}$ is

$$2\pi\left\langle \frac{\partial \exp(q)}{\partial\theta}, \frac{\partial \exp(q)}{\partial\theta}\right\rangle^{1/2} = 2\pi\left|R\sin\frac{l}{R}\right|$$

. When $l \to 0$, it is

$$2\pi R\sin\frac{l}{R}.$$

, which is the length of the geodesic circle. By the formula,

$$
\begin{aligned}
K(p) &= \lim_{r\to 0}\frac{3}{\pi}\frac{2\pi r - L}{r^3} = \lim_{r\to 0}\frac{3}{\pi}\frac{2\pi r - 2\pi R\sin\frac{l}{R}}{r^3} \\
&\approx \lim_{r\to 0}\frac{3}{\pi}\frac{2\pi r - 2\pi R\left(r/R - \left(\frac{r}{R}\right)^3/6\right)}{r^3} \\
&= \lim_{r\to 0}\frac{3}{\pi}\frac{2\pi r\left(\frac{r}{R}\right)^3/6}{r^3} = \frac{1}{R^2}.
\end{aligned}
$$

(b) The area bounded by the geodesic circle is

$$2\pi R \int_0^l \left|\sin \frac{r}{R}\right| dr.$$

When $l \to 0$, it is

$$2\pi R^2 - 2\pi R^2 \cos \frac{l}{R}.$$

By the formula,

$$\begin{aligned} K(p) &= \lim_{r\to 0} \frac{12}{\pi} \frac{\pi r^2 - A}{r^4} = \lim_{r\to 0} \frac{12}{\pi} \frac{\pi r^2 + 2\pi R^2 \cos \frac{r}{R} - 2\pi R^2}{r^4} \\ &\approx \lim_{r\to 0} \frac{12}{\pi} \frac{\pi r^2 + 2\pi R^2 \left(1 - \left(\frac{r}{R}\right)^2 /2 + \left(\frac{r}{R}\right)^4 /24\right) - 2\pi R^2}{r^4} \\ &= \lim_{r\to 0} \frac{12}{\pi} \frac{\pi R^2 \left(\frac{r}{R}\right)^4 /12}{r^4} = \frac{1}{R^2} \end{aligned}$$

□