Geometry Homework 4

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Problem 3.

(a) 假設 $\kappa(s) \neq 0, \tau(s) \neq 0$，由四點決定一球，討論空間曲線 $\gamma(s)$ 的密切球，並決定球心與半徑。

(b) 討論螺線 $(a\cos t, a\sin t, bt)$ 的密切球，$a > 0$。

Proof. (a) Assume that the sphere is $|X - C| = R$

$$
\begin{aligned}
\to \langle X-C, X-C\rangle &= R^2 \\
\to \langle X-C, X-C\rangle' &= 0 \\
&= 2\langle X-C, T\rangle \\
\to \langle X-C, T\rangle' &= 0 \\
&= \langle T, T\rangle + \langle X-C, T'\rangle \\
&= 1 + \kappa\langle X-C, N\rangle \\
\to (\kappa\langle X-C, N\rangle)' &= 0 \\
&= \kappa'\langle X-C, N\rangle + \kappa\langle T, N\rangle + \kappa\langle X-C, N'\rangle \\
&= \kappa'\langle X-C, N\rangle + \kappa\langle X-C, -\kappa T - \tau B\rangle \\
&= \kappa'\langle X-C, N\rangle - \kappa\tau\langle X-C, B\rangle \\
\to \langle X-C, T\rangle &= 0 \\
\langle X-C, N\rangle &= -\frac{1}{\kappa} \\
\langle X-C, B\rangle &= -\frac{\kappa'}{\kappa^2\tau} \\
\to X - C &= -\frac{1}{\kappa}N - \frac{\kappa'}{\kappa^2\tau}B \\
\to C &= X + \frac{1}{\kappa}N + \frac{\kappa'}{\kappa^2\tau}B \\
R &= |X - C| \\
&= \sqrt{\frac{1}{\kappa^2} + \frac{\kappa'^2}{\kappa^4\tau^2}}
\end{aligned}
$$

□

Problem 4. $\kappa \neq 0, \tau \neq 0$ 為兩常數，請決定 $\kappa(s) = \kappa, \tau(s) = \tau$ 的曲線方程式。*(長度參數 s)*

Proof. Upto translations and rotations, all space curves $\alpha(s)$ satisfying the condition are of the following form

$$\begin{aligned}
\alpha(s) &= (\frac{\kappa}{\kappa^2+\tau^2}\sin\sqrt{\kappa^2+\tau^2}s, \frac{\kappa}{\kappa^2+\tau^2}\cos\sqrt{\kappa^2+\tau^2}s, \frac{\tau}{\sqrt{\kappa^2+\tau^2}}s) \\
T(s) &= (\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\cos\sqrt{\kappa^2+\tau^2}s, -\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\sin\sqrt{\kappa^2+\tau^2}s, \frac{\tau}{\sqrt{\kappa^2+\tau^2}})
\end{aligned}$$

$$\|T(s)\| = 1 \quad \text{(arc-length)}$$

$$\begin{aligned}
T'(s) &= (-\kappa\sin\sqrt{\kappa^2+\tau^2}s), -\kappa\cos\sqrt{\kappa^2+\tau^2}s, 0) \\
\kappa(s) &= \|T(s)\| = \kappa \\
N(s) &= (-\sin\sqrt{\kappa^2+\tau^2}s), -\cos\sqrt{\kappa^2+\tau^2}s, 0) \\
B(s) &= T(s)\times N(s) = (\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\cos\sqrt{\kappa^2+\tau^2}s, -\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\sin\sqrt{\kappa^2+\tau^2}s, -\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}) \\
B'(s) &= (-\tau\sin\sqrt{\kappa^2+\tau^2}s, -\tau\cos\sqrt{\kappa^2+\tau^2}s, 0) \\
\tau(s) &= B'(s)/N(s) = \tau
\end{aligned}$$

□

Problem 5 (Darboux vector). $\gamma(s)$ *arc length*

(a) 說明 $\exists$ *vector* $\omega(s)$ *(called* Darboux vector*) such that*

$$\begin{cases} T' &= \omega\times T \\ N' &= \omega\times N \\ B' &= \omega\times B \end{cases}$$

(b) $V(s)$ *is a vector along* $\gamma(s)$ 且 *w.r.t* (T,N,B), $V(s) = (v_1(s), v_2(s), v_3(s))$ $\Rightarrow V' = (v_1', v_2', v_3') + \omega\times V$

(c) 說明 $\omega = \frac{1}{2}(T\times T' + N\times N' + B\times B')$

Proof. (a) Let $\omega(s) = -\tau T + \kappa B$, then:

$$\begin{aligned}
\omega\times T &= (-\tau T + \kappa B)\times T \\
&= \kappa N \\
&= T' \\
\omega\times N &= (-\tau T + \kappa B)\times N \\
&= -\kappa T - \tau B \\
&= N' \\
\omega\times B &= (-\tau T + \kappa B)\times B \\
&= \tau N \\
&= B'
\end{aligned}$$

So $\omega(s)$ satisfy the conditions.

(b)

$$\begin{aligned}
V &= v_1T + v_2N + v_3B \\
\to V' &= v_1'T + v_2'N + v_3'B + v_1T' + v_2N' + v_3B' \\
&= v_1'T + v_2'N + v_3'B + v_1\omega \times T + v_2\omega \times N + v_3\omega \times B \\
&= v_1'T + v_2'N + v_3'B + \omega \times (v_1T) + \omega \times (v_2N) + \omega \times (v_3B) \\
&= v_1'T + v_2'N + v_3'B + \omega \times (v_1T + v_2N + v_3B) \\
&= (v_1', v_2', v_3') + \omega \times V
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{1}{2}(T \times T' + N \times N' + B \times B') &= \frac{1}{2}(T \times \kappa N + N \times (-\kappa T - \tau B) + B \times \tau N) \\
&= \frac{1}{2}(\kappa B + \kappa B - \tau T - \tau T) \\
&= -\tau T + \kappa B \\
&= \omega
\end{aligned}$$

□

Problem 8.

(a) 令函數 $x_i : \begin{array}{c} \mathbb{R}^n \to \mathbb{R}. \\ (x_1, \cdots, x_n) \mapsto x_i \end{array}$ 。計算 $[dx_i]$，在不同的 $a \in \mathbb{R}^n$，dx_i 如何隨 a 變化。

(b) 由上題將微分式 $df = \frac{\partial f}{\partial x_1}dx_1 + \cdots + \frac{\partial f}{\partial x_n}dx_n$ 與映射 df 結合起來。

(c) $f : \mathbb{R}^n \to \mathbb{R}^m$，怎麼利用上題幫你計算 df

Proof.

(a) $[dx_i](v) = \left(\frac{d(x_i(\gamma(t)))}{dt}\right)_{t=0}$, where $\gamma(0) = a, \gamma'(0) = v$
令 $\gamma(t) = (\gamma_1(t), \cdots, \gamma_n(t))$, 則 $x_i(\gamma(t)) = \gamma_i(t)$
$\to \left(\frac{d(x_i(\gamma(t)))}{dt}\right)_{t=0} = \left(\frac{d(\gamma_i(t))}{dt}\right)_{t=0} = v_i = x_i(v)$
$\to [dx_i] = e_i^T$. 故 dx_i 不隨 a 變化.

(b) 因為 $[df](e_i) = \frac{d}{dt}(f \circ \gamma(t))\Big|_{t=0}$，其中 $\gamma(0) = a, \gamma'(0) = e_i$，因此左式會變成 $\lim_{t\to 0}(f(a + te_i) - f(a))/t = \frac{\partial f}{\partial x_i}(a)$。於是對於一般的 $v = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$，$df : T_p\mathbb{R}^n \to T_p\mathbb{R}$，而且 $[df](v) = [df](\sum_{k=1}^n v_k e_k) = \sum_{k=1}^n [df](e_k) \cdot v_k = \sum_{k=1}^n \frac{\partial f}{\partial x_k}(a) \cdot [dx_k](v)$。於是可以得到我們要的結果 $df = \frac{\partial f}{\partial x_1}dx_1 + \cdots + \frac{\partial f}{\partial x_n}dx_n$。

(c)

$$
\begin{aligned}
df &= \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n \\
&= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_m}{\partial x_1} \end{bmatrix} e_1^T + \cdots + \begin{bmatrix} \frac{\partial f_1}{\partial x_n} \\ \vdots \\ \frac{\partial f_m}{\partial x_n} \end{bmatrix} e_n^T \\
&= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}
\end{aligned}
$$

□