Geometry Homework 2

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Problem 3 (P47: 5)**.** *If a closed plane curve C is contained inside a disk of radius r, prove that there exists a point $p \in C$ such that the curvature κ of C at p satifies $|\kappa| \geq 1/r$.*

Proof. Let $X(s)$ denote the curve C, where $s \in [0, l]$ is an arc-length parameter, that is, $\|X'(s)\| \equiv 1$. Since C is contained inside a disk of radius r, let A be the centre of the disk. So we have

$$\|X(s) - A\| \leq r \tag{1}$$

Consider $f(s) = \langle X(s) - A, X(s) - A\rangle$. Since $[0, l]$ is compact, the maximum exists, denoting by $f(s') = \max_{s\in[0,l]} f(s)$. Therefore, we have $f'(s') = 0$ and $f''(s') \leq 0$. Now

$$f''(s) = 2\left(\|X'(s)\|^2 + \kappa(s)\langle X(s) - A, N(s)\rangle\right), \tag{2}$$

where $X''(s) = \kappa(s)N(s)$ and $N(s)$ is the normal vector. Take $s = s'$ in (2) we have $f''(s') \leq 0$ and hence

$$\kappa(s')\langle X(s) - A, N(s)\rangle \leq -1 \tag{3}$$

This implies

$$|\kappa(s')\langle X(s) - A, N(s)\rangle| \geq 1 \tag{4}$$

□

Problem 4 (P23: 4, 僅討論平面情形)**.** *Assume that all parametrized curve α has the property that all its tangent lines pass through a fixed point.*

(a) Prove that the trace of α is a (segment of a) straight line.

(b) Does the conclusion in part (a) still hold if α is not regular?

Problem 5. 以 $t = 0$ 開始將曲線 (t^2, t^3) 化成長度參數。並討論 $t = 0$ 時的曲率。

Problem 6.

(a) 以原點為中心，將 $y = f(x)$ 的圖形縮放 λ 倍，並說明新的圖形是 $y = \lambda f(\frac{x}{\lambda})$ 的函數圖形。

(b) 討論曲率的變化。

Problem 7. 如圖，有一橢圓，其焦點為 O_1 和 O_2，設 L 切橢圓於 P，且 L 與 $\overline{O_2P}$ 之夾角為 θ。以 θ 為參數，說明曲率 $\kappa \propto \sin^3\theta$

Problem 9. 如圖，有 *regular curve* $\gamma(t)$，$\gamma_0 = \gamma(0)$，$N_0 = N(0)$，$L_0 = \{\gamma_0 + vN_0\}$。現考慮直線 $L_t = \{\gamma(t) + uN(t)\}$，令 $P(t) = L_t \cap L_0$ 證明

$$\kappa(0) \neq 0 \Rightarrow \lim_{t\to 0} P(t) = \gamma_0 + \frac{1}{\kappa(0)} N_0$$