GEOMETRY HOMEWORK 2

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Problem 3 (P47: 5). *If a closed plane curve C is contained inside a disk of radius r, prove that there exists a point $p \in C$ such that the curvature κ of C at p satifies $|\kappa| \geq 1/r$.*

Proof. Let $X(s)$ denote the curve C, where $s \in [0, l]$ is an arc-length parameter, that is, $\|X'(s)\| \equiv 1$. Since C is contained inside a disk of radius r, let A be the centre of the disk. So we have

$$\|X(s) - A\| \leq r \tag{1}$$

Consider $f(s) = \langle X(s) - A, X(s) - A \rangle$. Since $[0, l]$ is compact, the maximum exists, denoting by $f(s') = \max_{s\in[0,l]} f(s)$. Therefore, we have $f'(s') = 0$ and $f''(s') \leq 0$. Now

$$f''(s) = 2\left(\|X'(s)\|^2 + \kappa(s)\langle X(s) - A, N(s)\rangle\right), \tag{2}$$

where $X''(s) = \kappa(s)N(s)$ and $N(s)$ is the normal vector. Take $s = s'$ in (2) we have $f''(s') \leq 0$ and hence

$$\kappa(s')\langle X(s) - A, N(s)\rangle \leq -1 \tag{3}$$

This implies

$$|\kappa(s')\langle X(s) - A, N(s)\rangle| \geq 1 \tag{4}$$

By (1), $|\langle X(s) - A, N(s)\rangle| \leq \|X(s) - A\| \cdot \|N(s)\| \leq r$. We have $|\kappa(s')| \geq 1/r$ as desired. □

Problem 4 (P23: 4, 僅討論平面情形). *Assume that all normals of a parametrized curve pass through a fixed point. Prove that the trace of the curve is contained in a circle.*

Proof. Let P be the fixed point, and let $X(s)$ be this curve. Then from description, $\langle X(s) - P, X'(s)\rangle \equiv 0$ for all s. Let $f(s) = \|X(s) - P\|^2$, then we have $f'(s) = 2\langle X(s) - P, X'(s)\rangle = 0$ for all s. This implies the trace of the curve is contained in a circle centered at point P with radius $\sqrt{f(s_0)}$ for some s_0. □

Problem 5. 以 $t = 0$ 開始將曲線 (t^2, t^3) 化成長度參數。並討論 $t = 0$ 時的曲率。

Proof. Consider $t > 0$, the length of the curve of t is

$$\int_0^t 3t\sqrt{(4/9)+t^2}\ dt = \int_0^t \frac{3}{2}\sqrt{(4/9)+t^2}\ dt^2 = \left(\frac{4}{9}+t^2\right)^{3/2} - \frac{8}{27} \tag{5}$$

Let $s > 0$ be the arc-length parameter, note that $s > 0$ equivalent to $t > 0$, so we have $s = (4/9+t^2)^{3/2} - 8/27$ and hence

$$t = \sqrt{\left(s+\frac{8}{27}\right)^{2/3} - \frac{4}{9}} \tag{6}$$

Therefore the curve with arc-length parameter $s > 0$ is

$$\left((s+8/27))^{2/3} - 4/9, \left((s+8/27)^{2/3} - 4/9\right)^{3/2}\right) \tag{7}$$

By symmetry, for the case $s < 0$ the corresponding curve is

$$\left((8/27-s))^{2/3} - 4/9, -\left((8/27-s)^{2/3} - 4/9\right)^{3/2}\right) \tag{8}$$

We can write them together to get the result,

$$\left((8/27+|s|))^{2/3} - 4/9, sign(s)\cdot\left((8/27+|s|)^{2/3} - 4/9\right)^{3/2}\right) \tag{9}$$

□

Problem 6.

(a) 以原點為中心，將 $y = f(x)$ 的圖形縮放 λ 倍，並說明新的圖形是 $y = \lambda f(\frac{x}{\lambda})$ 的函數圖形。

(b) 討論曲率的變化。

Proof. □

Problem 7. 如圖，有一橢圓，其焦點為 O_1 和 O_2，設 L 切橢圓於 P，且 L 與 $\overline{O_2P}$ 之夾角為 θ。以 θ 為參數，說明曲率 $\kappa \propto \sin^3\theta$

Proof. □

Problem 9. 如圖，有 *regular curve* $\gamma(t)$，$\gamma_0 = \gamma(0)$，$N_0 = N(0)$，$L_0 = \{\gamma_0 + vN_0\}$。現考慮直線 $L_t = \{\gamma(t) + uN(t)\}$，令 $P(t) = L_t \cap L_0$ 證明

$$\kappa(0) \neq 0 \Rightarrow \lim_{t\to 0} P(t) = \gamma_0 + \frac{1}{\kappa(0)}N_0$$

Proof. $\gamma_0 + vN_0 = \gamma(t) + uN(t) \ \Rightarrow\ v = \dfrac{\begin{vmatrix} \gamma(t)-\gamma_0 & N(t) \end{vmatrix}}{\begin{vmatrix} N_0 & N(t) \end{vmatrix}}$ □