Geometry Homework 9

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Problem 2. 若 $E = 1, F = 0, G = 1, f = 0$，假設再加入函數 e, g 後是某 *surface* 的 *1st&2nd fundamental form*。

(a) 說明 e, g 中至少有一為 *0*

(b) 說明若 $e = g = 0$ 則此曲面為平面

(c) 說明若 $e \neq 0$，則此曲面為特別的 *ruled surface*，並討論 e 的意義。

Proof. (a) Since $E = 1, F = 0, G = 1$, we know that the surface has $K = 0$, so $eg - f^2 = 0$. So $eg = 0$, and one of e and g is zero.

(b) If $e = g = 0$, then

$$\begin{aligned}
\mathbb{X}_{uu} &= [1,1,1]\mathbb{X}_u + [1,1,2]\mathbb{X}_v + eN = 0 \\
\mathbb{X}_{uv} &= [1,2,1]\mathbb{X}_u + [1,2,2]\mathbb{X}_v + fN = 0 \\
\mathbb{X}_{vv} &= [2,2,1]\mathbb{X}_u + [2,2,2]\mathbb{X}_v + gN = 0
\end{aligned}$$

So $\mathbb{X}_u$ and $\mathbb{X}_v$ are constant, and the surface is a plane.

(c) if $e \neq 0$, then $g = 0$, and $\mathbb{X}_{vu} = \mathbb{X}_{vv} = 0$, so $\mathbb{X}_v$ is constant.
So $\mathbb{X}(u, v)$ is a line when we fix u, thus $\mathbb{X}$ is a ruled surface.
Let $\gamma(u) = \mathbb{X}(u, 0)$, then $\|\gamma'(u)\| = \|\mathbb{X}_u\| = 1$, so u is arc-length parameter for γ.
$\gamma''(u) = \mathbb{X}_{uu}(u, 0) = eN$, so $\mathrm{sign}(e)N$ is also the n for γ, and γ has curvature $|e|$.

□

Problem 4 (Ex p237 8.). *Compute the Cristoffel symbols for an open set of the plane*

(a) In cartesian coordinates.

(b) In polar coordinates.

Use the Gauss formula to compute K in both cases.

Proof. (a) Let $\mathbb{X}(u,v) = (u,v,0)$, then $\mathbb{X}_{uu} = \mathbb{X}_{uv} = \mathbb{X}_{vv} = 0$, so $\Gamma_{ij}^k = 0 \forall i,j,k = 1,2$.
So $R_{1212} = 0$ and $K = 0$.

(b) Let $\mathbb{X}(u,v) = (u\cos v, u\sin v, 0)$, then:

$$\begin{aligned}
\mathbb{X}_u &= (\cos v, \sin v, 0) \\
\mathbb{X}_v &= (-u\sin v, u\cos v, 0) \\
N &= (0,0,1) \\
E = 1, F &= 0, G = u^2 \\
\mathbb{X}_{uu} &= (0,0,0) \\
\mathbb{X}_{uv} &= (-\sin v, \cos v, 0) \\
\mathbb{X}_{vv} &= (-u\cos v, -u\sin v, 0) \\
\begin{bmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 & \Gamma_{22}^1 \\ \Gamma_{11}^2 & \Gamma_{12}^2 & \Gamma_{22}^2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & u^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & -u \\ 0 & -u & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & -u \\ 0 & -\frac{1}{u} & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
R_{112}^2 &= \Gamma_{11,2}^2 - \Gamma_{12,1}^2 + \Gamma_{11}^1\Gamma_{21}^2 + \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{12}^1\Gamma_{11}^2 - \Gamma_{12}^2\Gamma_{12}^2 \\
&= \frac{1}{u^2} - \frac{1}{u^2} = 0 \\
\rightarrow K &= \frac{R_{1212}}{EG} \\
&= \frac{R_{112}^2}{E} = 0
\end{aligned}$$

□

Problem 6. 有一 *surface* $\mathbf{X}(u,v)$，令 $\hat{\mathbf{X}}(u,v) = \lambda\mathbf{X}(u,v), \lambda > 0$。

(a) 討論 $\hat{\Gamma}_{ij}^k$ 和 Γ_{ij}^k 的關係

(b) 從 *Gauss equation(GTE)* 討論 $\hat{K}$ 和 K 的關係

Proof. (a)

$$\begin{aligned}
\hat{g}_{ij} &= \langle \hat{\mathbf{X}}_i, \hat{\mathbf{X}}_j \rangle \\
&= \lambda^2 \langle \mathbf{X}_i, \mathbf{X}_j \rangle \\
&= \lambda^2 g_{ij} \\
\rightarrow \hat{g}^{ij} &= \frac{1}{\lambda^2} g^{ij} \\
\langle \hat{\mathbf{X}}_{ij}, \hat{\mathbf{X}}_k \rangle &= \langle \lambda\mathbf{X}_{ij}, \lambda\mathbf{X}_k \rangle \\
&= \lambda^2 \langle \mathbf{X}_{ij}, \mathbf{X}_k \rangle \\
\rightarrow \hat{\Gamma}_{ij}^k &= \hat{g}^{kl} \langle \hat{\mathbf{X}}_{ij}, \hat{\mathbf{X}}_l \rangle \\
&= g^{kl} \langle \mathbf{X}_{ij}, \mathbf{X}_l \rangle \\
&= \Gamma_{ij}^k
\end{aligned}$$

(b)

Since $\hat{\Gamma}_{ij}^k = \Gamma_{ij}^k$, $\hat{R}_{ijk}^l = R_{ijk}^l$.

$$
\begin{aligned}
\hat{R}_{imjk} &= \hat{g}_{ml}\hat{R}_{ijk}^l \\
&= \lambda^2 g_{ml} R_{ijk}^l \\
&= \lambda^2 R_{imjk} \\
\to \hat{K} &= \frac{\hat{R}_{1212}}{\hat{E}\hat{G} - \hat{F}^2} \\
&= \frac{1}{\lambda^2}\frac{R_{1212}}{EG - F^2} \\
&= \frac{1}{\lambda^2} K
\end{aligned}
$$

□

Problem 9. 舉一個例子說明有可能 $F : M \to N$ 是 *conformal map*，且相應點 $K_M > 0, K_N = 0$ *(*想想曾經討論的例子*)*

Proof. 取 M 為單位球 $x^2 + y^2 + z^2 = 1$, N 為平面 $z = 0$, 則顯然 $K_M > 0, K_N = 0$.
取 map $f : M \mapsto N$, $f(x, y, z) = (\frac{x}{1-z}, \frac{y}{1-z}, 0)$ 為 stereographic projection.
因為若 $f(x, y, z) = (u, v, w)$, 則

$$
\begin{aligned}
du^2 + dv^2 + dw^2 &= \left(\frac{(1-z)dx + xdz}{(1-z)^2}\right)^2 + \left(\frac{(1-z)dy + ydz}{(1-z)^2}\right)^2 \\
&= \frac{1}{(1-z)^4}\left((1-z)^2dx^2 + (1-z)^2dy^2 + (x^2+y^2)dz^2 + 2(xdx + ydy)(1-z)dz\right) \\
&= \frac{1}{(1-z)^4}\left((1-z)^2dx^2 + (1-z)^2dy^2 + (-z^2+1)dz^2 + (-2zdz)(1-z)dz\right) \\
&= \frac{1}{(1-z)^4}\left((1-z)^2dx^2 + (1-z)^2dy^2 + (1-z)^2dz^2\right) \\
&= \frac{1}{(1-z)^2}\left(dx^2 + dy^2 + dz^2\right)
\end{aligned}
$$

So f is a conformal mapping, but $K_M > 0, K_N = 0$. □