# Geometry Homework 8

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**Problem 2.** 考慮直線族 $L_\lambda : \frac{x}{\lambda} + \frac{y}{1-\lambda} = 1$，令 *ruled surface* $\mathbb{X}$ 為 $(L_\lambda, \lambda) \subset \mathbb{R}^2 \times \mathbb{R}$

*(a)* 求出 *line of striction(*龍骨*)* $\beta(\lambda) \in \mathbb{R}^3$

*(b)* 令 $\gamma(\lambda)$ 為 $\beta(\lambda)$ 在 $\mathbb{R}^2$ 上的投影，說明 $L_\lambda$ 為 $\gamma(\lambda)$ 的切線

*(c)* $\gamma(\lambda)$ 是圓嗎？其方程式為何（以 $f(x,y)=c$ 的方式表示）？

*Proof.* $\mathbf{X}(t,u) = \alpha + uw(t)$,

where $\alpha(t) = (t, 0, t)$, $w(t) = (\frac{t}{\sqrt{2t^2-2t+1}}, -\frac{1-t}{\sqrt{2t^2-2t+1}}, 0)$.

$$\alpha' = (1, 0, 1);$$
$$w' = \left(\frac{\sqrt{2t^2-2t+1} - \frac{1}{2}\frac{t(4t-2)}{\sqrt{2t^2-2t+1}}}{2t^2-2t+1}, \frac{\sqrt{2t^2-2t+1} - \frac{1}{2}\frac{(t-1)(4t-2)}{\sqrt{2t^2-2t+1}}}{2t^2-2t+1}, 0\right);$$
$$= \left(\frac{-t+1}{(2t^2-2t+1)^{\frac{3}{2}}}, \frac{t}{(2t^2-2t+1)^{\frac{3}{2}}}, 0\right);$$
$$\frac{\langle \alpha', w' \rangle}{\langle w', w' \rangle} = (-t+1)\sqrt{2t^2-2t+1};$$
$$\Rightarrow \beta = \alpha - (-t+1)\sqrt{2t^2-2t+1}w.$$
$$= (t^2, (1-t)^2, t)$$

This yields (a).

$$\gamma(\lambda) = (\lambda^2, (1-\lambda)^2, 0)$$
$$\frac{\lambda^2}{\lambda} + \frac{(1-\lambda)^2}{1-\lambda} = 1$$
$$\to \gamma(\lambda) \in L_\lambda$$
$$\gamma'(\lambda) = (2\lambda, 2(1-\lambda), 0) \parallel L_\lambda$$

So the tangent line of $\gamma(\lambda)$ is $L_\lambda$, this yields (b).

For (c), it's obvious that the equation is $f(x,y) = \sqrt{x} + \sqrt{y} = 1$ and thus not a circle.

□

**Problem 4** (Ex p.210 6). *Let*

$$\mathbf{X}(t,v) = \alpha(t) + vw(t)$$

*be a developable surface. Prove that at a regular point we have*

$$\langle N_v, \mathbf{X}_v \rangle = \langle N_v, \mathbf{X}_t \rangle = 0.$$

*Conclude that* the tangent plane of a developable surface is constant along *(the regular points of)* a fixed ruling.

*Proof.*

$$\begin{aligned}
\mathbf{X}_{vv} =& 0 \Rightarrow g = \langle N, \mathbf{X}_{vv} \rangle = 0;\\
K =& \det(-dN) = 0 \Rightarrow eg = f^2 \Rightarrow f = 0;\\
N_v =& dN(\mathbf{X}_v) = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix};\\
\Rightarrow \langle N_v, \mathbf{X}_v \rangle =& \langle N_v, \mathbf{X}_t \rangle = 0.
\end{aligned}$$

Thus $N$, the normal vector of the tangent plane, is independent of $v$ and hence the conclusion follows. □

**Problem 5** (Ex p.210 8). *Show that if $C \subset S^2$ is a parallel of a unit sphere $S^2$, then the envelope of tangent planes of $S^2$ along $C$ is either a cylinder, if $C$ is an equator, or a cone, if $C$ is not an equator.*

*Proof.* WLOG, let the unit sphere's centre be the origin and let the plane on which the $C$ is be parallel to the xy-plane. If $C$ is an equator, that is, on the xy-plane, the tangent plane of each point is therefore parallel to the z-axis and thus the envelope form a cylinder. Hence consider that $C$ is not on the $xy$ plane. By the symmetry of $S$ and $C$, the intersection of the envelope and any plane containing z-axis is identical up to rotation along z-axis. Picking such a plane and observing that the intersection being a line should intersect z-axis at exactly one point since $\alpha \neq 0$, we conclude that each intersection passes through the very point in z-axis. Let the point in z-axis be the generator of the envelope. Since each ruler should pass through exactly one point in $C$, the envelope therefore forms a cone.

□