

## GEOMETRY HOMEWORK 5

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**Problem 1** (Ex P151 2). *Show that if a surface is tangent to a plane along a curve, then the points of this curve are either parabolic or planar.*

*Proof.* Assume that the curve is  $\gamma(s)$ , then along this curve,  $N(\gamma(s))$  is perpendicular to the plane, so it is constant.

At point  $\gamma(s)$ ,  $[dN](\gamma'(s)) = \left(\frac{dN(\gamma(t))}{dt}\right)_{t=s} = 0$ , so  $\gamma'(s)$  is one of the principal direction of the surface at  $\gamma(s)$ , and it's associated principal curvature is 0. So the gaussian curvature of the surface at  $\gamma(s)$  is  $K = 0$ , and this means that the point  $\gamma(s)$  is either parabolic or planar.  $\square$

**Problem 3** (Ex P151 3).

(a) *Let  $C \subset S$  be a regular curve on a surface  $S$  with Gaussian curvature  $K > 0$ . Show that the curvature  $\kappa$  of  $C$  at  $p$  satisfies*

$$\kappa \geq \min(|\kappa_1|, |\kappa_2|),$$

*where  $\kappa_1, \kappa_2$  are the principal curvatures of  $S$  at  $p$ .*

(b) 為什麼上一小題需要  $\kappa > 0$  的條件,  $\kappa \geq 0$  不可以嗎?

*Proof.* (a)

$$\begin{aligned} \kappa &\geq |\kappa_n| \\ &= |\kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta| \\ &= |\kappa_1| \cos^2 \theta + |\kappa_2| \sin^2 \theta (\because \kappa_1, \kappa_2 \text{ has equal sign.}) \\ &\geq \min(|\kappa_1|, |\kappa_2|)(\cos^2 \theta + \sin^2 \theta) \\ &= \min(|\kappa_1|, |\kappa_2|) \end{aligned}$$

(b)

$\square$

**Problem 7.**

(a)  $T_\lambda$  是縮放  $\lambda$  倍的映射,  $\lambda > 0$ .  $\mathbb{X} : \Omega \rightarrow \mathbb{R}^3$  regular surface. 討論  $T_\lambda \circ \mathbb{X} : \Omega \rightarrow \mathbb{R}^3$  上對應點  $\kappa_n, H, K$  的變化。

(b)  $\mathbb{X}: \begin{matrix} \Omega \\ (u, v) \end{matrix} \rightarrow \mathbb{R}^3$ , 若定義  $\bar{\mathbb{X}}(u, v) = \mathbb{X}(v, u)$  (因此  $N$  轉向)。討論  $\bar{\mathbb{X}}(\Omega)$  上相對應點的  $K_n, H, K$  變化。

*Proof.*

□

**Problem 9 (旋轉面).**  $\mathbb{X}(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$ ,  $f > 0$

(a) 計算其  $e, f, g, H, K$

(b) 討論其 *principal direction* 與 *principal curvature*  $K_1, K_2$ 。

*Proof.* To avoid the notational ambiguity, let  $\mathbb{X}(u, v) = (s(u) \cos v, s(u) \sin v, t(u))$ , and that  $s > 0$ .

(a) We have

$$\begin{aligned}
\mathbb{X}_u &= (s'(u) \cos v, s'(u) \sin v, t'(u)); \\
\mathbb{X}_v &= (-s(u) \sin v, s(u) \cos v, 0); \\
E &= \langle \mathbb{X}_u, \mathbb{X}_u \rangle = s'(u)^2 + t'(u)^2 \\
F &= \langle \mathbb{X}_u, \mathbb{X}_v \rangle = 0 \\
G &= \langle \mathbb{X}_v, \mathbb{X}_v \rangle = s(u)^2 \\
\mathbb{X}_{uu} &= (s''(u) \cos v, s''(u) \sin v, t''(u)); \\
\mathbb{X}_{uv} &= (-s'(u) \sin v, s'(u) \cos v, 0); \\
\mathbb{X}_{vv} &= (-s(u) \cos v, -s(u) \sin v, 0); \\
N &= \frac{\mathbb{X}_u \times \mathbb{X}_v}{|\mathbb{X}_u \times \mathbb{X}_v|} = \frac{(-t'(u)s(u) \cos v, -t'(u)s(u) \sin v, s'(u)s(u))}{\sqrt{t'(u)^2 s(u)^2 + s'(u)^2 s(u)^2}} \\
&= \frac{(-t'(u) \cos v, -t'(u) \sin v, s'(u))}{\sqrt{t'(u)^2 + s'(u)^2}}; \\
e &= \langle N, \mathbb{X}_{uu} \rangle = \frac{-s''(u)t'(u) + t''(u)s'(u)}{\sqrt{t'(u)^2 + s'(u)^2}} \\
f &= \langle N, \mathbb{X}_{uv} \rangle = 0 \\
g &= \langle N, \mathbb{X}_{vv} \rangle = \frac{s(u)t'(u)}{\sqrt{t'(u)^2 + s'(u)^2}} \\
-dN^T &= \begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} = \frac{1}{EG - F^2} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} = \begin{bmatrix} e/E & 0 \\ 0 & g/G \end{bmatrix} \\
K &= \det(-dN) = \frac{eg}{EG} \\
H &= \frac{1}{2} \text{tr}(-dN) = \frac{eG + gE}{2EG}
\end{aligned}$$

(b) Since  $-dN$  is already a diagonal matrix, clearly,

$$K_1 = e/E;$$

$$K_2 = g/G;$$

$$V_1 = \mathbb{X}_u;$$

$$V_2 = \mathbb{X}_v;$$

□

**Problem 10 (管面).**  $\mathbb{X}(s, \theta) = \gamma(s) + \cos \theta \vec{n}(s) + \sin \theta \vec{b}(s)$ ,  $0 < \kappa < 1$

(a) 計算其  $e, f, g, H, K$

(b) 討論曲面上  $K$  的分佈。

*Proof.* (a) Let  $\vec{t}(s), \vec{n}(s), \vec{b}(s)$  be the basis, and  $\gamma(0)$  be the origin.

$$\mathbb{X}_s = (1 - \kappa \cos \theta, \tau \sin \theta, -\tau \cos \theta);$$

$$\mathbb{X}_\theta = (0, -\sin \theta, \cos \theta);$$

$$N = \frac{\mathbb{X}_s \times \mathbb{X}_\theta}{|\mathbb{X}_s \times \mathbb{X}_\theta|} = \frac{(0, (\kappa \cos \theta - 1) \cos \theta, (\kappa \cos \theta - 1) \sin \theta)}{|\kappa \cos \theta - 1|}$$

$$= (0, -\cos \theta, -\sin \theta); \quad (\text{since } \kappa < 1)$$

$$\mathbb{X}_{ss} = (-\sin \theta \kappa \tau - \kappa' \cos \theta, \kappa + \cos \theta(-\kappa^2 - \tau^2) + \tau' \sin \theta, -\sin \theta \tau^2 - \tau' \cos \theta);$$

$$\mathbb{X}_{s\theta} = (1 + \kappa \sin \theta, \tau \cos \theta, \tau \sin \theta);$$

$$\mathbb{X}_{\theta\theta} = (0, -\cos \theta, -\sin \theta);$$

$$e = -\kappa \cos \theta + \cos^2 \theta (\kappa^2 + \tau^2) + \tau^2 \sin^2 \theta = -\kappa \cos \theta + \kappa^2 \cos^2 \theta + \tau^2;$$

$$f = -\tau;$$

$$g = 1;$$

$$dN(\mathbb{X}_s) = N_s = (\kappa \cos \theta, -\tau \sin \theta, \tau \cos \theta) = \mathbb{X}_s \kappa / (1 - \kappa) + \frac{\tau}{1 - \kappa} \mathbb{X}_\theta;$$

$$dN(\mathbb{X}_\theta) = N_\theta = (0, \sin \theta, -\cos \theta) = -\mathbb{X}_\theta;$$

$$[-dN] = \begin{bmatrix} \frac{-\kappa}{1-\kappa} & \frac{-\tau}{1-\kappa} \\ 0 & 1 \end{bmatrix}$$

$$\kappa_1 = \frac{\kappa}{1 - \kappa};$$

$$\kappa_2 = 1;$$

$$K = \frac{\kappa}{1 - \kappa};$$

$$H = \frac{1}{2 - 2\kappa}.$$

(b)

$$K = \frac{\kappa}{1 - \kappa}; \quad (\text{how to discuss?})$$

