Geometry Homework 12

B96201044 黃上恩, *B98901182* 時丕勳, *K0020100x* 劉士瑋

December 23, 2011

Problem 3 (Ex p294 3.). *If p is a point of a regular surface S, prove that*

$$K(p) = \lim_{r \to 0} \frac{12}{\pi} \frac{\pi r^2 - A}{r^4},$$

where $K(p)$ is the Gaussian curvature of S at p, r is the radius of a geodesic circle $S_r(p)$ centered in p, and A is the area of the region bounded by $S_r(p)$.

Proof.

$$\begin{aligned}
A_R &= \int_0^R \int_0^{2\pi} \sqrt{EG - F^2} d\theta dr \\
&= \int_0^R \int_0^{2\pi} \sqrt{G} d\theta dr \\
&\approx \int_0^R \int_0^{2\pi} r - \frac{K}{6} r^3 d\theta dr \\
&= \int_0^{2\pi} \frac{1}{2} R^2 - \frac{K}{24} R^4 d\theta \\
&= \pi R^2 - \frac{R^4}{24} \int_0^{2\pi} K d\theta \\
\to \frac{1}{2\pi} \int_0^{2\pi} K d\theta &= \frac{12}{r^4} (r^2 - \frac{1}{\pi} A_r) \\
\to K(p) &= \lim_{r \to 0} \frac{12}{r^4} (r^2 - \frac{1}{\pi} A_r) \\
&= \lim_{r \to 0} \frac{12}{\pi} \frac{\pi r^2 - A_r}{r^4}
\end{aligned}$$

□

Problem 4 (Ex p295 4.). *Show that in a system of normal coordinates centered in p, all the Christoffel symbols are zero at p.*

Proof. Let (u, v) be normal coordinate centered at p, (r, θ) be the geodesic polar coordinate centered at p.

Let $\hat{E}, \hat{F}, \hat{G}$ be the first fundamental form of the coordinate (r, θ), E, F, G be the first fundamental form of the coordinate (u, v),

$$\hat{E} = 1, \hat{F} = 0$$
$$\lim_{r \to 0} \hat{G} = 0, \lim_{r \to 0} \sqrt{\hat{G}}_r = 1$$
$$\to \hat{G} = r^2 + o(r^3)$$

$$\begin{aligned}
r &= \sqrt{u^2 + v^2} \\
\theta &= \tan^{-1} \frac{v}{u} \\
\mathbb{X}_u &= \frac{u}{r}\mathbb{X}_r - \frac{v}{r^2}\mathbb{X}_\theta \\
\mathbb{X}_v &= \frac{v}{r}\mathbb{X}_r + \frac{u}{r^2}\mathbb{X}_\theta \\
\to E &= \frac{u^2}{r^2} + \frac{v^2}{r^4}\hat{G} \\
F &= \frac{uv}{r^2} - \frac{uv}{r^4}\hat{G} \\
G &= \frac{v^2}{r^2} + \frac{u^2}{r^4}\hat{G}
\end{aligned}$$

When $r \to 0$:

$$\begin{aligned}
\hat{G} &\to r^2 \\
\hat{G}_u &\to 2u \\
\hat{G}_v &\to 2v
\end{aligned}$$

$$\begin{aligned}
E_u &= \frac{2uv^2}{r^4} - \frac{4uv^2}{r^6}\hat{G} + \frac{v^2}{r^4}\hat{G}_u \\
&= \frac{2uv^2}{r^4} - \frac{4uv^2}{r^4} + \frac{2uv^2}{r^4} = 0 \\
E_v &= -\frac{2u^2v}{r^4} + \frac{2v(u^2 - v^2)}{r^6}\hat{G} + \frac{v^2}{r^4}\hat{G}_v \\
&= -\frac{2u^2v}{r^4} + \frac{2v(u^2 - v^2)}{r^4} + \frac{2v^3}{r^4} = 0 \\
F_u &= \frac{v^3 - vu^2}{r^4} - \frac{v^3 - 3u^2v}{r^6}\hat{G} - \frac{uv}{r^4}\hat{G}_u \\
&= \frac{v^3 - vu^2}{r^4} - \frac{v^3 - 3u^2v}{r^4} - \frac{2u^2v}{r^4} = 0 \\
F_v &= \frac{u^3 - uv^2}{r^4} - \frac{u^3 - 3v^2u}{r^6}\hat{G} - \frac{uv}{r^4}\hat{G}_v \\
&= \frac{u^3 - uv^2}{r^4} - \frac{u^3 - 3v^2u}{r^4} - \frac{2uv^2}{r^4} = 0
\end{aligned}$$

$$
\begin{aligned}
G_u &= -\frac{2v^2u}{r^4} + \frac{2u(v^2-u^2)}{r^6}\hat{G} + \frac{u^2}{r^4}\hat{G}_u \\
&= -\frac{2v^2u}{r^4} + \frac{2u(v^2-u^2)}{r^4} + \frac{2u^3}{r^4} = 0 \\
G_v &= \frac{2vu^2}{r^4} - \frac{4vu^2}{r^6}\hat{G} + \frac{u^2}{r^4}\hat{G}_v \\
&= \frac{2vu^2}{r^4} - \frac{4vu^2}{r^4} + \frac{2vu^2}{r^4} = 0
\end{aligned}
$$

So $[i,j,k] = 0$ and $\Gamma_{ij}^k = 0$. □

Problem 5 (Ex p295 5.). *For which of the pair of surfaces given below does there exist a local isometry?*

(a) Torus of revolution and cone.

(b) Cone and sphere.

(c) Cone and cylinder.

Proof. (a) Since cone has constant Gaussian curvature, and torus of revolution does not have constant gaussian curvature for every neighborhood of a point, but isometry preserves Gaussian curvature, there doesn't exist a local isometry between cone and torus.

(b) Since cone has zero Gaussian curvature everywhere, and sphere has Gaussian curvature > 0 everywhere, but isometry preserves Gaussian curvature, there doesn't exist a local isometry between cone and sphere.

(c) By Example 1 in page 219 and Example 3 in page 223 of the notebook, we know that there exist a local isometry from a cone to a plane and from a plane to a cylinder for some neighborhood of every point on the cone/plane. So there exist a local isometry from a cone to a cylinder.

□

Problem 8.

(a) 在半徑 R 的球面上，計算 *geodesic circle* 的長度，並驗證 *P292* 課文中間 $K(p)$ 的公式。

(b) 用一樣的精神，檢驗 *P294 3.* 的公式。

Proof. (a) WLOG, let $p = (0,0,R)$, If $q \in T_p$ with $q = (l,\theta)$, then

$$
\exp(q) = \left(R\sin\frac{l}{R}\cos\theta, R\sin\frac{l}{R}\sin\theta, R\cos\frac{l}{R}\right),
$$

and thus the length of the image of the circle $\{q \in T_p : d(q,p) = l\}$ is

$$
2\pi\left\langle \frac{\partial\exp(q)}{\partial\theta}, \frac{\partial\exp(q)}{\partial\theta}\right\rangle^{1/2} = 2\pi\left|R\sin\frac{l}{R}\right|
$$

. When $l \to 0$, it is

$$2\pi R \sin \frac{l}{R}.$$

, which is the length of the geodesic circle. By the formula,

$$\begin{aligned}
K(p) &= \lim_{r\to 0} \frac{3}{\pi} \frac{2\pi r - L}{r^3} = \lim_{r\to 0} \frac{3}{\pi} \frac{2\pi r - 2\pi R \sin \frac{l}{R}}{r^3} \\
&\approx \lim_{r\to 0} \frac{3}{\pi} \frac{2\pi r - 2\pi R \left(r/R - \left(\frac{r}{R}\right)^3 /6 \right)}{r^3} \\
&= \lim_{r\to 0} \frac{3}{\pi} \frac{2\pi r \left(\frac{r}{R}\right)^3 /6}{r^3} = \frac{1}{R^2}.
\end{aligned}$$

(b) The area bounded by the geodesic circle is

$$2\pi R \int_0^l \left| \sin \frac{r}{R} \right| dr.$$

When $l \to 0$, it is

$$2\pi R^2 - 2\pi R^2 \cos \frac{l}{R}.$$

By the formula,

$$\begin{aligned}
K(p) &= \lim_{r\to 0} \frac{12}{\pi} \frac{\pi r^2 - A}{r^4} = \lim_{r\to 0} \frac{12}{\pi} \frac{\pi r^2 + 2\pi R^2 \cos \frac{r}{R} - 2\pi R^2}{r^4} \\
&\approx \lim_{r\to 0} \frac{12}{\pi} \frac{\pi r^2 + 2\pi R^2 \left(1 - \left(\frac{r}{R}\right)^2 /2 + \left(\frac{r}{R}\right)^4 /24\right) - 2\pi R^2}{r^4} \\
&= \lim_{r\to 0} \frac{12}{\pi} \frac{\pi R^2 \left(\frac{r}{R}\right)^4 /12}{r^4} = \frac{1}{R^2}
\end{aligned}$$

$\square$