

GEOMETRY HOMEWORK 2

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Problem 3 (P47: 5). *If a closed plane curve C is contained inside a disk of radius r , prove that there exists a point $p \in C$ such that the curvature κ of C at p satisfies $|\kappa| \geq 1/r$.*

Proof. Let $X(s)$ denote the curve C , where $s \in [0, l]$ is an arc-length parameter, that is, $\|X'(s)\| \equiv 1$. Since C is contained inside a disk of radius r , let A be the centre of the disk. So we have

$$\|X(s) - A\| \leq r \quad (1)$$

Consider $f(s) = \langle X(s) - A, X(s) - A \rangle$. Since $[0, l]$ is compact, the maximum exists, denoting by $f(s') = \max_{s \in [0, l]} f(s)$. Therefore, we have $f'(s') = 0$ and $f''(s') \leq 0$. Now

$$f''(s) = 2 (\|X'(s)\|^2 + \kappa(s) \langle X(s) - A, N(s) \rangle), \quad (2)$$

where $X''(s) = \kappa(s)N(s)$ and $N(s)$ is the normal vector. Take $s = s'$ in (2) we have $f''(s') \leq 0$ and hence

$$\kappa(s') \langle X(s) - A, N(s) \rangle \leq -1 \quad (3)$$

This implies

$$|\kappa(s') \langle X(s) - A, N(s) \rangle| \geq 1 \quad (4)$$

By (1), $|\langle X(s) - A, N(s) \rangle| \leq \|X(s) - A\| \cdot \|N(s)\| \leq r$. We have $|\kappa(s')| \geq 1/r$ as desired. \square

Problem 4 (P23: 4, 僅討論平面情形). *Assume that all normals of a parametrized curve pass through a fixed point. Prove that the trace of the curve is contained in a circle.*

Problem 5. 以 $t = 0$ 開始將曲線 (t^2, t^3) 化成長度參數。並討論 $t = 0$ 時的曲率。

Problem 6.

- (a) 以原點為中心，將 $y = f(x)$ 的圖形縮放 λ 倍，並說明新的圖形是 $y = \lambda f(\frac{x}{\lambda})$ 的函數圖形。
- (b) 討論曲率的變化。

Problem 7. 如圖，有一橢圓，其焦點為 O_1 和 O_2 ，設 L 切橢圓於 P ，且 L 與 $\overline{O_2P}$ 之夾角為 θ 。以 θ 為參數，說明曲率 $\kappa \propto \sin^3 \theta$

Problem 9. 如圖，有 *regular curve* $\gamma(t)$ ， $\gamma_0 = \gamma(0)$ ， $N_0 = N(0)$ ， $L_0 = \{\gamma_0 + vN_0\}$ 。現考慮直線 $L_t = \{\gamma(t) + uN(t)\}$ ，令 $P(t) = L_t \cap L_0$ 證明

$$\kappa(0) \neq 0 \Rightarrow \lim_{t \rightarrow 0} P(t) = \gamma_0 + \frac{1}{\kappa(0)} N_0$$