GEOMETRY HOMEWORK 7

B96201044 黃上恩, B98901182 時丕勳, K0020100x 劉士瑋

November 1, 2011

Problem 2. 若 F(x,y,z)=0 定義一 surface, 證明 $\nabla F \neq 0$ 的地方 Gauss curvature $K=\frac{\nabla F^t A \nabla F}{\|\nabla F\|^4}$ 。其中 A 為 $\partial^2 F=\begin{pmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{pmatrix}$ 的 adjoint Matrix, i.e. $A=\det(\partial^2 F)(\partial^2 F)^{-1}$

Proof. 因為 K 為局部性質,而在 $\nabla F \neq 0$ 的地方我們可以使用隱函數定理將其中一維表示為另兩維的函數,WLOG 不妨設 z = z(x, y) 在某點附近。

$$F(x, y, z(x, y)) = 0$$

$$\mathbb{X}(x, y) = (x, y, z(x, y))$$

$$\to \mathbb{X}_x = (1, 0, z_x)$$

$$\mathbb{X}_y = (0, 1, z_y)$$

$$\to N = \frac{\mathbb{X}_x \times \mathbb{X}_y}{|\mathbb{X}_x \times \mathbb{X}_y|}$$

$$= \frac{(-z_x, -z_y, 1)}{\sqrt{1 + z_x^2 + z_y^2}}$$

$$\begin{split} E &= \langle \mathbb{X}_x, \mathbb{X}_x \rangle \\ &= 1 + z_x^2 \\ F &= \langle \mathbb{X}_x, \mathbb{X}_y \rangle \\ &= z_x z_y \\ G &= \langle \mathbb{X}_y, \mathbb{X}_y \rangle \\ &= 1 + z_y^2 \end{split}$$

$$\begin{split} &\mathbb{X}_{xy} = (0,0,z_{xy}) \\ &\mathbb{X}_{yy} = (0,0,z_{yy}) \\ &\to e = \langle N, \mathbb{X}_{xx} \rangle \\ &= \frac{z_{xx}}{\sqrt{1+z_x^2+z_y^2}} \\ &f = \langle N, \mathbb{X}_{xy} \rangle \\ &= \frac{z_{xy}}{\sqrt{1+z_x^2+z_y^2}} \\ &g = \langle N, \mathbb{X}_{yy} \rangle \\ &= \frac{z_{yy}}{\sqrt{1+z_x^2+z_y^2}} \\ &\to K = \det([-dN]) \\ &= \det\left(\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} e & f \\ f & g \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} E & F \\ F & G \end{bmatrix}\right)^{-1} \det\left(\begin{bmatrix} e & f \\ f & g \end{bmatrix}\right) \\ &= \frac{z_{xx}z_{yy}-z_{xy}^2}{(1+z_x^2+z_y^2)^2} \\ &\frac{\partial F(x,y,z)}{\partial x} = 0 \\ &= F_x(x,y,z) + F_z(x,y,z)z_x \\ &\to z_x = -\frac{F_x}{F_z} \\ &\frac{\partial F(x,y,z)}{\partial y} = 0 \\ &= F_y(x,y,z) + F_z(x,y,z)z_y \\ &\to z_y = -\frac{F_y}{F_z} \end{split}$$

$$\begin{split} \frac{\partial^2 F(x,y,z)}{\partial x^2} &= 0 \\ &= \frac{\partial}{\partial x} (F_x(x,y,z) + F_z(x,y,z) z_x) \\ &= F_{xx}(x,y,z) + 2 F_{xz}(x,y,z) z_x + F_{zz}(x,y,z) z_x + F_z(x,y,z) z_x \\ &\to z_{xx} = -\frac{F_{xx} - (2 F_{xz} + F_{zz}) \frac{F_x}{F_z}}{F_z} \\ \frac{\partial^2 F(x,y,z)}{\partial y^2} &= 0 \\ &= \frac{\partial}{\partial y} (F_y(x,y,z) + F_z(x,y,z) z_y) \\ &= F_{yy}(x,y,z) + 2 F_{yz}(x,y,z) z_y + F_{zz}(x,y,z) z_y + F_z(x,y,z) z_y \\ &\to z_{yy} = -\frac{F_{yy} - (2 F_{yz} + F_{zz}) \frac{F_y}{F_z}}{F_z}}{F_z} \\ \frac{\partial^2 F(x,y,z)}{\partial x \partial y} &= 0 \\ &= \frac{\partial}{\partial x} (F_y(x,y,z) + F_z(x,y,z) z_y) \\ &= F_{xy}(x,y,z) + F_{yz}(x,y,z) z_x + F_{xz}(x,y,z) z_y + F_{zz}(x,y,z) z_x z_y + F_z(x,y,z) z_{xy} \\ &\to z_{xy} = -\frac{F_{xy} - F_{yz} \frac{F_z}{F_z} - F_{xz} \frac{F_y}{F_z}}{F_z} \end{split}$$

$$\begin{split} \rightarrow K &= \frac{z_{xx}z_{yy} - z_{xy}^2}{(1 + z_x^2 + z_y^2)^2} \\ &= \frac{\left(\frac{F_{xx} - (2F_{xz} + F_{zz})\frac{F_x}{F_z}}{F_z}\right)\left(\frac{F_{yy} - (2F_{yz} + F_{zz})\frac{F_y}{F_z}}{F_z}\right) - \left(\frac{F_{xy} - F_{yz}\frac{F_x}{F_z} - F_{xz}\frac{F_y}{F_z}}{F_z}\right)^2}{(1 + \left(\frac{F_x}{F_z}\right)^2 + \left(\frac{F_y}{F_z}\right)^2)^2} \\ &= \frac{(F_{xx}F_z - (2F_{xz} + F_{zz})F_x)\left(F_{yy}F_z - (2F_{yz} + F_{zz})F_y\right) - (F_{xy}F_z - F_{yz}F_x - F_{xz}F_y)^2}{(F_z^2 + F_z^2 + F_z^2)^2} \end{split}$$

Problem 3 (Ex P168 4). Determine the asymptotic curves and the lines of curvature of z = xy.

Problem 4. 已知 $\mathbb{X}(u,v)$ 為一 $surface \subset \mathbb{R}^3$ 且 $E=G=(1+u^2+v^2)^2, F=0$ 而且 $e=1, f=\sqrt{3}, g=-1$

(a) 求在 X(1,1) 的 K 與 H

(b) 如何決定過 X(1,1) 的 line of curvature 與 asymptotic curve (如果有的 話)

Proof. (a) at (1,1), E = G = 9, F = 0.

$$[-dN] = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} e & f \\ f & g \end{bmatrix}$$
$$= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{9} & \frac{1}{3\sqrt{3}} \\ \frac{1}{3\sqrt{3}} & -\frac{1}{9} \end{bmatrix}$$

So $K = \det([-dN]) = -\frac{4}{81}, \, H = \operatorname{tr}([-dN]) = 0.$

Problem 5. $\mathbb{X}(u,v) = (v\cos u, v\sin u, u)$, $\Leftrightarrow \gamma(t) = \mathbb{X}(t,1)$

- (a) 求 $\gamma(t)$ 的 $\kappa_n, \kappa_g, \tau_g$
- (b) 與 $\gamma(t)$ 的 κ, τ 有何關係

Proof.

Problem 6. 令 $(x(t), y(t)) = (t - \tanh t, \operatorname{sech} t)$ 這基本就是 p7(4) 的 tratrix

- (a) 將此曲線化作長度參數
- (b) 利用上小題,求此曲線繞 x 軸旋轉的旋轉體的 K

Proof.