Geometry Homework 3

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Problem 3 (P26: 16)**.** *Show that the knowledge of the vector function $n = n(s)$ (normal vector) of a curve α, with nonzero torsion everywhere, determines the curvature $\kappa(s)$ and the torsion τ of α. ($\vec{n}$ 能決定曲線嗎？說明題目錯誤並找反例。)*

Proof. Consider the helix $\alpha(s) = (a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}})$
Then $n(s) = (-\cos\frac{s}{\sqrt{a^2+b^2}}, -\sin\frac{s}{\sqrt{a^2+b^2}}, 0)$.
So if two helix has the same a^2+b^2 (e.g. $\alpha_1(s) = (\frac{1}{2}\cos s, \frac{1}{2}\sin s, \frac{\sqrt{3}}{2}s)$, $\alpha_2(s) = (\frac{\sqrt{3}}{2}\cos s, \frac{\sqrt{3}}{2}\sin s, \frac{1}{2}s)$), then they have same $n(s)$, but they're not the same curve. □

Problem 4 (P26: 17，另一種描述 Helix 的方式)**.** *In general, a curve α is called a* helix *if the tangent lines of α make a constant angle with a fixed direction. Assume that $\tau(s) \neq 0$, $s \in I$, and prove that:*

(a) α is a helix if and only if $\kappa/\tau =$constant.

(b) α is a helix if and only if the lines containing $n(s)$ and passing through $\alpha(s)$ are parallel to a fixed plane.

(c) α is a helix if and only if the lines containing $b(s)$ and passing through $\alpha(s)$ make a constant angle with a fixed direction.

(d) The curve

$$\alpha(s) = \left(\frac{a}{c}\int \sin\theta(s)ds, \frac{a}{c}\int \cos\theta(s)ds, \frac{b}{c}s\right),$$

where $c^2 = a^2 + b^2$, is a helix, and that $\kappa/\tau = a/b$.

Proof. □

Problem 6. $\gamma(s)$ 長度參數。若將 $T(s)$ 寫成 $(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$，$\phi, \theta$ 是 s 的函數。說明 $\kappa(s) = \sqrt{\phi'^2 + \theta'^2\sin^2\phi}$

Proof.

$$\begin{aligned}
T'(s) &= (\phi' \cos\phi\cos\theta - \theta' \sin\phi\sin\theta, \phi' \cos\phi\sin\theta + \theta' \sin\phi\cos\theta, -\phi' \sin\phi) \\
\to \kappa(s) &= |T'(s)| \\
&= \sqrt{\phi'^2 \cos^2\phi\cos^2\theta + \theta'^2 \sin^2\phi\sin^2\theta + \phi'^2 \cos^2\phi\sin^2\theta + \theta'^2 \sin^2\phi\cos^2\theta + \phi'^2 \sin^2\phi} \\
&= \sqrt{\phi'^2 + \theta'^2 \sin^2\phi}
\end{aligned}$$

□

Problem 7. $\gamma : \mathbb{R} \to \mathbb{R}^3$，不妨假設是長度參數。

(b) 若 $M^t M = I$，$\det(M) = -1$ 且 $\overline{\gamma} = M\gamma$，討論 κ, τ 變化。

(c) $\overline{\gamma}(s) = \gamma(-s)$，說明 κ, τ 變化。

Proof. (b) $\overline{\gamma}' = M\gamma'$

$$\begin{aligned}
|\overline{\gamma}'| &= \sqrt{\overline{\gamma}'^T \overline{\gamma}'} \\
&= \sqrt{\gamma'^T M^T M \gamma'} \\
&= \sqrt{\gamma'^T \gamma'} \\
&= |\gamma'| \\
&= 1
\end{aligned}$$

So s is arc-length parameter for $\overline{\gamma}$ too.

$$\begin{aligned}
\kappa_{\overline{\gamma}} &= |\overline{\gamma}''| \\
&= \sqrt{\overline{\gamma}''^T \overline{\gamma}''} \\
&= \sqrt{\gamma''^T M^T M \gamma''} \\
&= \sqrt{\gamma''^T \gamma''} \\
&= |\gamma''| \\
&= \kappa_\gamma
\end{aligned}$$

(c)

□

Problem 8. 說明 $\overline{\gamma}(u) = \gamma(-s)$ 時，在對應點

$$\frac{\det(\overline{\gamma}', \overline{\gamma}'', \overline{\gamma}''')}{|\overline{\gamma}' \times \overline{\gamma}''|^2}(u) = \frac{\det(\gamma', \gamma'', \gamma''')}{|\gamma' \times \gamma''|^2}(t)$$

再用 *chain rule* 直接說明。

Proof. □