Geometry Homework 1

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September 19, 2011

Problem 3 (P7: 4). *Let* $\alpha : (0, \pi) \to \mathbf{R}^2$ *be given by*

$$\alpha(t) = \left(\cos t, \cos t + \log \tan \frac{t}{2}\right),$$

where t *is the angle that the* y *axis makes with the vector* $\alpha(t)$*. The trace of* α *is called the* tractrix *(Fig. 1-9). Show that*

(a) α *is a differentiable parametrized curve, regular except at* $t = \pi/2$*.*

(b) The length of the segment of the tangent of the tractrix between the point of tangency and the y *axis is constantly equal to* 1*.*

Proof. This is the proof. □

Problem 5 (P47: 6). *If a closed plane curve* C *is contained inside a disk of radius* r*, prove that there exists a point* $p \in C$ *such that the curvature* κ *of* C *at* p *satisfies* $|\kappa| \geq 1/r$*.*

Problem 8 (Curvature is a geometric object I.). $X(s) = (x(s), y(s))$*, where* s *is the arc-length parameter.*

$$M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, M^t = M^{-1}, i.e.\ M \text{ is orthogonal.}$$

Let $\bar{M}(s) = M \cdot \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$*,* $\alpha, \beta \in \mathbf{R}$*. What is the relation between* $\kappa_X(s)$ *and* $\kappa_{\bar{X}}(s)$*?*

Problem 9 (Curvature is a geometric object II.). *$X(t) = (x(t), y(t))$ be a regular curve. Let*

$$\kappa(x(t), y(t)) \equiv \kappa(t) = \frac{\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}}{(x'^2 + y'^2)^{\frac{3}{2}}}$$

Let $Y(u) = X(t(u))$, $t'(u) \neq 0$. Discuss the relation of $\kappa(x(t), y(t))$ and $\kappa(x(t(u)), y(t(u)))$ at the corresponding points.

Problem 10. *Let $F(x, y) = c$ defines a plane curve. Prove that the curvature of the curve satisfies*

$$|\kappa| = \left| \frac{\begin{bmatrix} F_y, & -F_x \end{bmatrix} \begin{bmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \end{bmatrix} \begin{bmatrix} F_y \\ -F_x \end{bmatrix}}{(F_x^2 + F_y^2)^{\frac{3}{2}}} \right|$$

Where $F_x^2 + F_y^2 \neq 0$.