Geometry Homework 11

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Problem 4 (Ex p261 8.). *Show that if all the geodesics of a connected surface are plane curves, then the surface is contained in a plane or a sphere.*

Proof. Assume that the surface is $\mathbb{X}(u,v)$, and $p \in \mathbb{X}$ is a point, $v \subset T_p\mathbb{X}$ is an unit vector. By solving the geodesic equation we can get an geodesic $\gamma(s)$ such that $\gamma(0) = p, \gamma'(0) = v$, and s is arc-length parameter. Since all geodesics are plane curves, assume that the plane which contains γ is P. So γ'' is in P. Since γ is geodesic, γ'' is parallel to N of the surface along γ, so N is also in P along γ. So $\frac{\partial N(\gamma(s))}{\partial s}$ is in P too.
But since $\frac{\partial N(\gamma(s))}{\partial s} \perp N(s)$, and $\gamma'(s) \perp N(s)$, $\gamma'(s) \parallel \frac{\partial N(\gamma(s))}{\partial s}$. So $v \parallel \left.\frac{\partial N(\gamma(s))}{\partial s}\right|_{s=0}$, and v is an eigenvector of dN_p. Since this is true for every v, we know that p is an umbilical point. So every point on the surface is an umbilical point.
Assume that the gaussian curvature of $\mathbb{X}(u,v)$ is $\kappa(u,v)^2$ with $\kappa(u,v) \geq 0$ (Since every point is an umbilical point, $K \geq 0$ at every point). So $[-dN](u,v) = \begin{bmatrix} \kappa(u,v) & 0 \\ 0 & \kappa(u,v) \end{bmatrix}$, and $N_u = \kappa(u,v)\mathbb{X}_u$, $N_v = \kappa(u,v)\mathbb{X}_v$. So $N_{uv} = \kappa_v\mathbb{X}_u + \kappa\mathbb{X}_{uv} = N_{vu} = \kappa_u\mathbb{X}_v + \kappa\mathbb{X}_{uv}$, so $\kappa_v\mathbb{X}_u = \kappa_u\mathbb{X}_v$. But since $\mathbb{X}_u$ and $\mathbb{X}_v$ are linearly independent, $\kappa_u = \kappa_v = 0$, thus κ is constant on the surface, and $K = \kappa^2$ is also constant on the surface.
If $K > 0$, then by Liebmann's theorem, the surface is contained in a sphere.
If $K = 0$, then since $N_u = \kappa(u,v)\mathbb{X}_u$, $N_v = \kappa(u,v)\mathbb{X}_v$, and $\kappa = 0$, N is constant on the surface, so the surface is a plane. □

Problem 5 (Ex p262 17.). *Let $\alpha : I \to S$ be a curve parametrized by arc length s, with nonzero curvature. Consider the parametrized surface*

$$\mathrm{x}(s,v) = \alpha(s) + vb(s), \quad s \in I, -\epsilon < v < \epsilon, \epsilon > 0,$$

where b is the binormal vector of α. Prove that if ϵ is small, $\mathrm{x}(I \times (-\epsilon,\epsilon)) = S$ is a regular surface over which $\alpha(I)$ is geodesic. (thus, every curve is a geodesic on the surface generated by its binormals).

Proof.

$$\begin{aligned}
\mathbf{x}_s &= \alpha'(s) + vb'(s) \\
&= t(s) + v\tau(s)n(s) \\
\mathbf{x}_v &= b(s) \\
\to \mathbf{x}_s \times \mathbf{x}_v &= -n(s) + v\tau(s)t(s) \\
&\neq 0
\end{aligned}$$

So $\mathbf{x}$ is a regualr surface.
Since $\alpha''(s) = n(s)$, and at $v = 0$, $N \parallel \mathbf{x}_s \times \mathbf{x}_v = -n(s)$. So $\alpha'(s) \parallel N$, and $\kappa_g = 0$. So $\alpha(I)$ is geodesic. □

Problem 8. 用 *(A)* 表示在座標變換下不變、用 *(B)* 表示在 *isometry* 下不變 *(*保 E, F, G*)* 下的性質

	line of curvature	*geodesic*	*asymptotic curve*	Γ_{ij}^k	H	K
(A)	$Yes_{(1)}$	$Yes_{(1)}$	$Yes_{(1)}$	$No_{(2)}$	$Yes_{(1)}$	$Yes_{(1)}$
(B)	$No_{(6)}$	$Yes_{problem\ 9(a)}$	$No_{(6)}$	$Yes_{(3)}$	$No_{(5)}$	$Yes_{(4)}$

Proof. (1) Since curves, surface, T,A,N, t,n,b are all geometry objects, κ_n, κ_g, τ_g are geometry objects too. So line of curvature, geodesic, asymptotic curve are also geometry objects. Since principal direction and principal curvature are geometry objects too, H and K are geometry objects.

(2) Consider an surface $\mathbb{X}(u, v)$ and $\hat{\mathbb{X}}(u, v) = \mathbb{X}(v, u)$, it's trivial that $\hat{\Gamma}_{11}^2 = \Gamma_{22}^1$, so $\hat{\Gamma}_{11}^2 \neq \Gamma_{11}^2$ when $\Gamma_{11}^2 \neq \Gamma_{22}^1$, and it's trivial to find a surface with $\Gamma_{11}^2 \neq \Gamma_{22}^1$ (For example, $\mathbb{X}(u, v) = (u\cos v, u\sin v, 0)$. As shown in HW 10, $\Gamma_{22}^1 = -u \neq \Gamma 11^2 = 0$).

(3) Since $\Gamma_{ij}^k = g^{kl}[i, j, l]$, and both g^{kl} and $[i, j, l]$ only depends on g_{ij}, Γ_{ij}^k is same in isometry.

(4) Gauss Theorema Egregium.

(5) Consider a plane $(u, v, 0)$ and a cone $(u, \cos v, \sin v)$. They are isometric but have different H.

(6) Consider a plane $(u, v, 0)$ and a cone $(u, \cos v, \sin v)$. Since every line in the plane is line of curvature, but there are only two line of curvatures passing one point in cone, some line of curvatures in the plane is not a line of curvature in cone. $u = 0$ is asymptotic curve of the plane, but it's not an asymtotic curve of the cone.

□

Problem 9. 考慮 *p221, p222* 中 *helicoid* Y 和 *catenoid* X 的 *parametrization*。
$X(u, v) = (a\cosh v\cos u, a\cosh v\sin u, av)$, $Y(u, v) = (a\sinh v\cos u, a\sinh v\sin u, au)$

(a) X 中的 *geodesics* 相對應映到 Y 中也是 *geodesics* 嗎？

(b) 已知 X 的經線 *(*$u = $ *const)* 與 $v = 0$ 都是 *geodesics*。描述他們在 Y 中的對應曲線？他們都是 *geodesics* 嗎？

Proof. (a) Since X and Y are isometry, they have the same Γ_{ij}^k, and have the same geodesic equation. So geodesic in X is also geodesic in Y.

(b) when $u = \text{const}$, $Y(u, v)$ is a line $(a\cos C\sinh v, a\sin C\sinh v, aC)$, and is a geodesic of Y.
when $v = 0$, $Y(u, v)$ is a line $(0, 0, au)$, and is a geodesic of Y.

□