GEOMETRY HOMEWORK 5

B96201044 黃上恩, *B98901182* 時丕勳, *K0020100x* 劉士瑋

November 1, 2011

Problem 1 (Ex P151 2). *Show that if a surface is tangent to a plane along a curve, then the points of this curve are either parabolic or planar.*

Proof. Assume that the curve is $\gamma(s)$, then along this curve, $N(\gamma(s))$ is perpendicular to the plane, so it is constant.
At point $\gamma(s)$, $[dN](\gamma'(s)) = \left(\frac{dN(\gamma(t))}{dt}\right)_{t=s} = 0$, so $\gamma'(s)$ is one of the principal direction of the surface at $\gamma(s)$, and it's associated principal curvature is 0.
So the gaussian curvature of the surface at $\gamma(s)$ is $K = 0$, and this means that the point $\gamma(s)$ is either parabolic or planar. $\square$

Problem 3 (Ex P151 3).

(a) *Let $C \subset S$ be a regular curve on a surface S with Gaussian curvature $K > 0$. Show that the curvature κ of C at p satisfies*

$$\kappa \geq \min(|\kappa_1|, |\kappa_2|),$$

where κ_1, κ_2 are the principal curvatures of S at p.

(b) 為什麼上一小題需要 $\kappa > 0$ 的條件，$\kappa \geq 0$ 不可以嗎？

Proof. (a)

$$\begin{aligned}\kappa &\geq |\kappa_n| \\ &= |\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta| \\ &= |\kappa_1|\cos^2\theta + |\kappa_2|\sin^2\theta (\because \kappa_1, \kappa_2 \text{ has equal sign.}) \\ &\geq \min(|\kappa_1|, |\kappa_2|)(\cos^2\theta + \sin^2\theta) \\ &= \min(|\kappa_1|, |\kappa_2|)\end{aligned}$$

(b) 上一小題的證明中並沒有用到 $\kappa \neq 0$ 或是 $K \neq 0$.

$\square$

Problem 7.

(a) T_λ 是縮放 λ 倍的映射，$\lambda > 0$。$\mathbb{X} : \Omega \to \mathbb{R}^3$ *regular surface*。討論 $T_\lambda \circ \mathbb{X} : \Omega \to \mathbb{R}^3$ 上對應點 κ_n, H, K 的變化。

(b) $\mathbb{X}: \begin{array}{c}\Omega \\ (u,v)\end{array} \to \mathbb{R}^3$，若定義 $\overline{\mathbb{X}}(u,v)=\mathbb{X}(v,u)$(因此 N 轉向)。討論 $\overline{\mathbb{X}}(\Omega)$ 上相對應點的 κ_n, H, K 變化。

Proof. (a) 不妨設縮放中心為原點. 令 $\overline{\mathbb{X}}=T_\lambda\circ\mathbb{X}$, 所以 $\overline{\mathbb{X}}(u,v)=\lambda\mathbb{X}(u,v)$.

$$
\begin{aligned}
\overline{\mathbb{X}}_u &= \lambda\mathbb{X}_u \\
\overline{\mathbb{X}}_v &= \lambda\mathbb{X}_v \\
N_{\overline{\mathbb{X}}} &= \frac{\overline{\mathbb{X}}_u\times\overline{\mathbb{X}}_v}{|\overline{\mathbb{X}}_u\times\overline{\mathbb{X}}_v|} \\
&= \frac{\lambda^2\mathbb{X}_u\times\mathbb{X}_v}{\lambda^2|\mathbb{X}_u\times\mathbb{X}_v|} \\
&= \frac{\mathbb{X}_u\times\mathbb{X}_v}{|\mathbb{X}_u\times\mathbb{X}_v|} \\
&= N_{\mathbb{X}}
\end{aligned}
$$

$$
\begin{aligned}
E_{\overline{\mathbb{X}}} &= \langle\overline{\mathbb{X}}_u,\overline{\mathbb{X}}_u\rangle \\
&= \lambda^2\langle\mathbb{X}_u,\mathbb{X}_u\rangle \\
&= \lambda^2E_{\mathbb{X}} \\
F_{\overline{\mathbb{X}}} &= \langle\overline{\mathbb{X}}_u,\overline{\mathbb{X}}_v\rangle \\
&= \lambda^2\langle\mathbb{X}_u,\mathbb{X}_v\rangle \\
&= \lambda^2F_{\mathbb{X}} \\
G_{\overline{\mathbb{X}}} &= \langle\overline{\mathbb{X}}_v,\overline{\mathbb{X}}_v\rangle \\
&= \lambda^2\langle\mathbb{X}_v,\mathbb{X}_v\rangle \\
&= \lambda^2G_{\mathbb{X}}
\end{aligned}
$$

$$\begin{aligned}
\overline{\mathbb{X}}_{uu} &= \lambda \mathbb{X}_{uu} \\
\overline{\mathbb{X}}_{uv} &= \lambda \mathbb{X}_{uv} \\
\overline{\mathbb{X}}_{vv} &= \lambda \mathbb{X}_{vv} \\
e_{\overline{\mathbb{X}}} &= \langle N_{\overline{\mathbb{X}}}, \overline{\mathbb{X}}_{uu} \rangle \\
&= \lambda \langle N_{\mathbb{X}}, \mathbb{X}_{uu} \rangle \\
&= \lambda e_{\mathbb{X}} \\
f_{\overline{\mathbb{X}}} &= \langle N_{\overline{\mathbb{X}}}, \overline{\mathbb{X}}_{uv} \rangle \\
&= \lambda \langle N_{\mathbb{X}}, \mathbb{X}_{uv} \rangle \\
&= \lambda f_{\mathbb{X}} \\
g_{\overline{\mathbb{X}}} &= \langle N_{\overline{\mathbb{X}}}, \overline{\mathbb{X}}_{vv} \rangle \\
&= \lambda \langle N_{\mathbb{X}}, \mathbb{X}_{vv} \rangle \\
&= \lambda g_{\mathbb{X}}
\end{aligned}$$

$$\begin{aligned}
[-dN]_{\overline{\mathbb{X}}} &= \begin{bmatrix} E_{\overline{\mathbb{X}}} & F_{\overline{\mathbb{X}}} \\ F_{\overline{\mathbb{X}}} & G_{\overline{\mathbb{X}}} \end{bmatrix}^{-1} \begin{bmatrix} e_{\overline{\mathbb{X}}} & f_{\overline{\mathbb{X}}} \\ f_{\overline{\mathbb{X}}} & g_{\overline{\mathbb{X}}} \end{bmatrix} \\
&= \frac{1}{\lambda^2} \begin{bmatrix} E_{\mathbb{X}} & F_{\mathbb{X}} \\ F_{\mathbb{X}} & G_{\mathbb{X}} \end{bmatrix}^{-1} \begin{bmatrix} e_{\mathbb{X}} & f_{\mathbb{X}} \\ f_{\mathbb{X}} & g_{\mathbb{X}} \end{bmatrix} \\
&= \frac{1}{\lambda^2} [-dN]_{\mathbb{X}} \\
\to \kappa_{n\overline{\mathbb{X}}} &= \frac{1}{\lambda^2} \kappa_{n\mathbb{X}} \\
K_{\overline{\mathbb{X}}} &= \frac{1}{\lambda^4} K_{\mathbb{X}} \\
H_{\overline{\mathbb{X}}} &= \frac{1}{\lambda^2} H_{\mathbb{X}}
\end{aligned}$$

(b)

$$\begin{aligned}
\overline{\mathbb{X}}_u &= \mathbb{X}_v \\
\overline{\mathbb{X}}_v &= \mathbb{X}_u \\
N_{\overline{\mathbb{X}}} &= \frac{\overline{\mathbb{X}}_u \times \overline{\mathbb{X}}_v}{|\overline{\mathbb{X}}_u \times \overline{\mathbb{X}}_v|} \\
&= \frac{\mathbb{X}_v \times \mathbb{X}_u}{|\mathbb{X}_u \times \mathbb{X}_v|} \\
&= -N_{\mathbb{X}}
\end{aligned}$$

$$
\begin{aligned}
E_{\overline{\mathbb{X}}} &= \langle \overline{\mathbb{X}}_u, \overline{\mathbb{X}}_u \rangle \\
&= \langle \mathbb{X}_v, \mathbb{X}_v \rangle \\
&= G_{\mathbb{X}} \\
F_{\overline{\mathbb{X}}} &= \langle \overline{\mathbb{X}}_u, \overline{\mathbb{X}}_v \rangle \\
&= \langle \mathbb{X}_v, \mathbb{X}_u \rangle \\
&= F_{\mathbb{X}} \\
G_{\overline{\mathbb{X}}} &= \langle \overline{\mathbb{X}}_v, \overline{\mathbb{X}}_v \rangle \\
&= \langle \mathbb{X}_u, \mathbb{X}_u \rangle \\
&= E_{\mathbb{X}}
\end{aligned}
$$

$$
\begin{aligned}
\overline{\mathbb{X}}_{uu} &= \mathbb{X}_{vv} \\
\overline{\mathbb{X}}_{uv} &= \mathbb{X}_{uv} \\
\overline{\mathbb{X}}_{vv} &= \mathbb{X}_{uu} \\
e_{\overline{\mathbb{X}}} &= \langle N_{\overline{\mathbb{X}}}, \overline{\mathbb{X}}_{uu} \rangle \\
&= \langle -N_{\mathbb{X}}, \mathbb{X}_{vv} \rangle \\
&= -g_{\mathbb{X}} \\
f_{\overline{\mathbb{X}}} &= \langle N_{\overline{\mathbb{X}}}, \overline{\mathbb{X}}_{uv} \rangle \\
&= \langle -N_{\mathbb{X}}, \mathbb{X}_{uv} \rangle \\
&= -f_{\mathbb{X}} \\
g_{\overline{\mathbb{X}}} &= \langle N_{\overline{\mathbb{X}}}, \overline{\mathbb{X}}_{vv} \rangle \\
&= \langle -N_{\mathbb{X}}, \mathbb{X}_{uu} \rangle \\
&= -e_{\mathbb{X}}
\end{aligned}
$$

$$
\begin{aligned}
[-dN]_{\overline{\mathbb{X}}} &= \begin{bmatrix} E_{\overline{\mathbb{X}}} & F_{\overline{\mathbb{X}}} \\ F_{\overline{\mathbb{X}}} & G_{\overline{\mathbb{X}}} \end{bmatrix}^{-1} \begin{bmatrix} e_{\overline{\mathbb{X}}} & f_{\overline{\mathbb{X}}} \\ f_{\overline{\mathbb{X}}} & g_{\overline{\mathbb{X}}} \end{bmatrix} \\
&= - \begin{bmatrix} G_{\mathbb{X}} & F_{\mathbb{X}} \\ F_{\mathbb{X}} & E_{\mathbb{X}} \end{bmatrix}^{-1} \begin{bmatrix} g_{\mathbb{X}} & f_{\mathbb{X}} \\ f_{\mathbb{X}} & e_{\mathbb{X}} \end{bmatrix} \\
&= - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} [-dN]_{\mathbb{X}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
\to \kappa_{n\overline{\mathbb{X}}} &= -\kappa_{n\mathbb{X}} \\
K_{\overline{\mathbb{X}}} &= K_{\mathbb{X}} \\
H_{\overline{\mathbb{X}}} &= -H_{\mathbb{X}}
\end{aligned}
$$

□

Problem 9 (旋轉面)**.** $\mathbb{X}(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$， $f > 0$

(a) 計算其 e, f, g, H, K

(b) 討論其 *principal direction* 與 *principal curvature* K_1, K_2。

Proof. To avoid the notational ambiguity, let $\mathbb{X}(u,v) = (s(u)\cos v, s(u)\sin v, t(u))$, and that $s > 0$.

(a) We have

$$\begin{aligned}
\mathbb{X}_u &= (s'(u)\cos v, s'(u)\sin v, t'(u));\\
\mathbb{X}_v &= (-s(u)\sin v, s(u)\cos v, 0);\\
E &= \langle \mathbb{X}_u, \mathbb{X}_u\rangle = s'(u)^2 + t'(u)^2\\
F &= \langle \mathbb{X}_u, \mathbb{X}_v\rangle = 0\\
G &= \langle \mathbb{X}_v, \mathbb{X}_v\rangle = s(u)^2\\
\mathbb{X}_{uu} &= (s''(u)\cos v, s''(u)\sin v, t''(u));\\
\mathbb{X}_{uv} &= (-s'(u)\sin v, s'(u)\cos v, 0);\\
\mathbb{X}_{vv} &= (-s(u)\cos v, -s(u)\sin v, 0);\\
N &= \frac{\mathbb{X}_u \times \mathbb{X}_v}{|\mathbb{X}_u \times \mathbb{X}_v|} = \frac{(-t'(u)s(u)\cos v, -t'(u)s(u)\sin v, s'(u)s(u))}{\sqrt{t'(u)^2 s(u)^2 + s'(u)^2 s(u)^2}}\\
&= \frac{(-t'(u)\cos v, -t'(u)\sin v, s'(u))}{\sqrt{t'(u)^2 + s'(u)^2}};\\
e &= \langle N, \mathbb{X}_{uu}\rangle = \frac{-s''(u)t'(u) + t''(u)s'(u)}{\sqrt{t'(u)^2 + s'(u)^2}}\\
f &= \langle N, \mathbb{X}_{uv}\rangle = 0\\
g &= \langle N, \mathbb{X}_{vv}\rangle = \frac{s(u)t'(u)}{\sqrt{t'(u)^2 + s'(u)^2}}\\
-dN^T &= \begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} = \frac{1}{EG - F^2} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} = \begin{bmatrix} e/E & 0 \\ 0 & g/G \end{bmatrix}\\
K &= \det(-dN) = \frac{eg}{EG}\\
H &= \frac{1}{2}\mathrm{tr}(-dN) = \frac{eG + gE}{2EG}
\end{aligned}$$

(b) Since $-dN$ is already a diagonal matrix, clearly,

$$\begin{aligned}
K_1 &= e/E;\\
K_2 &= g/G;\\
V_1 &= \mathbb{X}_u;\\
V_2 &= \mathbb{X}_v;
\end{aligned}$$

$\square$

Problem 10 (管面). $\mathbb{X}(s,\theta) = \gamma(s) + \cos\theta\vec{n}(s) + \sin\theta\vec{b}(s),\ 0 < \kappa < 1$

(a) 計算其 e, f, g, H, K

(b) 討論曲面上 K 的分佈。

Proof. (a) Let $\vec{t}(s), \vec{n}(s), \vec{b}(s)$ be the basis, and $\gamma(0)$ be the origin.

$$
\begin{aligned}
\mathbb{X}_s &= (1-\kappa\cos\theta, \tau\sin\theta, -\tau\cos\theta);\\
\mathbb{X}_\theta &= (0, -\sin\theta, \cos\theta);\\
N &= \frac{\mathbb{X}_s \times \mathbb{X}_\theta}{|\mathbb{X}_s \times \mathbb{X}_\theta|} = \frac{(0, (\kappa\cos\theta - 1)\cos\theta, (\kappa\cos\theta-1)\sin\theta)}{|\kappa\cos\theta - 1|}\\
&= (0, -\cos\theta, -\sin\theta); \quad (\text{since } \kappa < 1)\\
\mathbb{X}_{ss} &= (-\sin\theta\kappa\tau - \kappa'\cos\theta, \kappa + \cos\theta(-\kappa^2-\tau^2) + \tau'\sin\theta, -\sin\theta\tau^2 - \tau'\cos\theta);\\
\mathbb{X}_{s\theta} &= (1+\kappa\sin\theta, \tau\cos\theta, \tau\sin\theta);\\
\mathbb{X}_{\theta\theta} &= (0, -\cos\theta, -\sin\theta);\\
e &= -\kappa\cos\theta + \cos^2\theta(\kappa^2+\tau^2) + \tau^2\sin^2\theta = -\kappa\cos\theta + \kappa^2\cos^2\theta + \tau^2;\\
f &= -\tau;\\
g &= 1;\\
dN(\mathbb{X}_s) &= N_s = (\kappa\cos\theta, -\tau\sin\theta, \tau\cos\theta) = \frac{\kappa}{1-\kappa}\mathbb{X}_s + \frac{\tau}{1-\kappa}\mathbb{X}_\theta;\\
dN(\mathbb{X}_\theta) &= N_\theta = (0, \sin\theta, -\cos\theta) = -\mathbb{X}_\theta;\\
[-dN] &= \begin{bmatrix} \frac{-\kappa\cos\theta}{1-\kappa\cos\theta} & 0 \\ \frac{-\tau}{1-\kappa\cos\theta} & 1 \end{bmatrix}\\
\kappa_1 &= \frac{-\kappa\cos\theta}{1-\kappa\cos\theta};\\
\kappa_2 &= 1;\\
K &= \frac{-\kappa\cos\theta}{1-\kappa\cos\theta};\\
H &= \frac{1-2\kappa\cos\theta}{1-\kappa\cos\theta}.
\end{aligned}
$$

(b)

$$
\begin{aligned}
K &= \frac{-\kappa\cos\theta}{1-\kappa\cos\theta};\\
&= 1 - \frac{1}{1-\kappa\cos\theta};
\end{aligned}
$$

Since $0 < \kappa < 1$, K has a maximum of $\frac{\kappa}{1+\kappa}$ when $\cos\theta = -1$, i.e. $\theta = \pi$.
$K = 0$ when $\cos\theta = 0$, i.e. $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.
K has a minimum of $\frac{-\kappa}{1-\kappa}$ when $\cos\theta = 1$, i.e. $\theta = 0$.

□