# Geometry Homework 5

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**Problem 1** (Ex P151 2). *Show that if a surface is tangent to a plane along a curve, then the points of this curve are either parabolic or planar.*

*Proof.* Assume that the curve is $\gamma(s)$, then along this curve, $N(\gamma(s))$ is perpendicular to the plane, so it is constant.
At point $\gamma(s)$, $[dN](\gamma'(s)) = \left(\frac{dN(\gamma(t))}{dt}\right)_{t=s} = 0$, so $\gamma'(s)$ is one of the principal direction of the surface at $\gamma(s)$, and it's associated principal curvature is 0.
So the gaussian curvature of the surface at $\gamma(s)$ is $K = 0$, and this means that the point $\gamma(s)$ is either parabolic or planar. □

**Problem 3** (Ex P151 3).

(a) *Let $C \subset S$ be a regular curve on a surface $S$ with Gaussian curvature $K > 0$. Show that the curvature $\kappa$ of $C$ at $p$ satisfies*

$$\kappa \geq \min(|\kappa_1|, |\kappa_2|),$$

*where $\kappa_1, \kappa_2$ are the principal curvatures of $S$ at $p$.*

(b) 為什麼上一小題需要 $\kappa > 0$ 的條件，$\kappa \geq 0$ 不可以嗎？

*Proof.* (a)

$$\begin{aligned}\kappa &\geq |\kappa_n| \\ &= |\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta| \\ &= |\kappa_1|\cos^2\theta + |\kappa_2|\sin^2\theta (\because \kappa_1, \kappa_2 \text{ has equal sign.}) \\ &\geq \min(|\kappa_1|, |\kappa_2|)(\cos^2\theta + \sin^2\theta) \\ &= \min(|\kappa_1|, |\kappa_2|)\end{aligned}$$

(b)

□

**Problem 7.**

(a) $T_\lambda$ 是縮放 $\lambda$ 倍的映射，$\lambda > 0$。$\mathbb{X} : \Omega \to \mathbb{R}^3$ *regular surface*。討論 $T_\lambda \circ \mathbb{X} : \Omega \to \mathbb{R}^3$ 上對應點 $\kappa_n, H, K$ 的變化。

*(b)* $\mathbb{X}: \begin{matrix}\Omega \\ (u,v)\end{matrix} \to \mathbb{R}^3$，若定義 $\overline{\mathbb{X}}(u,v)=\mathbb{X}(v,u)$*(*因此 $N$ 轉向*)*。討論 $\overline{\mathbb{X}}(\Omega)$ 上相對應點的 $K_n, H, K$ 變化。

*Proof.* □

**Problem 9** (旋轉面)**.** $\mathbb{X}(u,v)=(f(u)\cos v, f(u)\sin v, g(u))$，$f>0$

*(a)* 計算其 $e, f, g, H, K$

*(b)* 討論其 *principal direction* 與 *principal curvature* $K_1, K_2$。

*Proof.* To avoid the notational ambiguity, let $\mathbb{X}(u,v)=(s(u)\cos v, s(u)\sin v, t(u))$, and that $s>0$.

(a) We have

$$\begin{aligned}
\mathbb{X}_u =&(s'(u)\cos v, s'(u)\sin v, t'(u));\\
\mathbb{X}_v =&(-s(u)\sin v, s(u)\cos v, 0);\\
E =&\langle \mathbb{X}_u, \mathbb{X}_u\rangle = s'(u)^2+t'(u)^2\\
F =&\langle \mathbb{X}_u, \mathbb{X}_v\rangle = 0\\
G =&\langle \mathbb{X}_v, \mathbb{X}_v\rangle = s(u)^2\\
\mathbb{X}_{uu} =&(s''(u)\cos v, s''(u)\sin v, t''(u));\\
\mathbb{X}_{uv} =&(-s'(u)\sin v, s'(u)\cos v, 0);\\
\mathbb{X}_{vv} =&(-s(u)\cos v, -s(u)\sin v, 0);\\
N =&\frac{\mathbb{X}_u\times\mathbb{X}_v}{|\mathbb{X}_u\times\mathbb{X}_v|} = \frac{(-t'(u)s(u)\cos v, -t'(u)s(u)\sin v, s'(u)s(u))}{\sqrt{t'(u)^2s(u)^2+s'(u)^2s(u)^2}}\\
=&\frac{(-t'(u)\cos v, -t'(u)\sin v, s'(u))}{\sqrt{t'(u)^2+s'(u)^2}};\\
e =&\langle N, \mathbb{X}_{uu}\rangle = \frac{-s''(u)t'(u)+t''(u)s'(u)}{\sqrt{t'(u)^2+s'(u)^2}}\\
f =&\langle N, \mathbb{X}_{uv}\rangle = 0\\
g =&\langle N, \mathbb{X}_{vv}\rangle = \frac{s(u)t'(u)}{\sqrt{t'(u)^2+s'(u)^2}}\\
-dN^T =& \begin{bmatrix} e & f\\ f & g\end{bmatrix}\begin{bmatrix} E & F\\ F & G\end{bmatrix}^{-1} = \frac{1}{EG-F^2}\begin{bmatrix} e & f\\ f & g\end{bmatrix}\begin{bmatrix} G & -F\\ -F & E\end{bmatrix} = \begin{bmatrix} e/E & 0\\ 0 & g/G\end{bmatrix}\\
K =&\det(-dN) = \frac{eg}{EG}\\
H =&\frac{1}{2}\mathrm{tr}(-dN) = \frac{eG+gE}{2EG}
\end{aligned}$$

(b) Since $-dN$ is already a diagonal matrix, clearly,

$$\begin{aligned} K_1 =& e/E; \\ K_2 =& g/G; \\ V_1 =& \mathbb{X}_u; \\ V_2 =& \mathbb{X}_v; \end{aligned}$$

$\square$

**Problem 10 (管面).** $\mathbb{X}(s,\theta) = \gamma(s) + \cos\theta\vec{n}(s) + \sin\theta\vec{b}(s)$, $0 < \kappa < 1$

*(a)* 計算其 $e, f, g, H, K$

*(b)* 討論曲面上 $K$ 的分佈。

*Proof.* (a) Let $\vec{t}(s), \vec{n}(s), \vec{b}(s)$ be the basis, and $\gamma(0)$ be the origin.

$$\begin{aligned} \mathbb{X}_s =& (1-\kappa\cos\theta, \tau\sin\theta, -\tau\cos\theta); \\ \mathbb{X}_\theta =& (0, -\sin\theta, \cos\theta); \\ N =& \frac{\mathbb{X}_s \times \mathbb{X}_\theta}{|\mathbb{X}_s \times \mathbb{X}_\theta|} = \frac{(0, (\kappa\cos\theta - 1)\cos\theta, (\kappa\cos\theta-1)\sin\theta)}{|\kappa\cos\theta - 1|} \\ =& (0, -\cos\theta, -\sin\theta); \quad \text{(since } \kappa < 1) \\ \mathbb{X}_{ss} =& (-\sin\theta\kappa\tau, \kappa + \cos\theta(-\kappa^2 - \tau^2), -\sin\theta\tau^2); \\ \mathbb{X}_{s\theta} =& (1+\kappa\sin\theta, \tau\cos\theta, \tau\sin\theta); \\ \mathbb{X}_{\theta\theta} =& (0, -\cos\theta, -\sin\theta); \\ e =& -\kappa\cos\theta + \cos^2\theta(\kappa^2+\tau^2) + \tau^2\sin^2\theta = -\kappa\cos\theta + \kappa^2\cos^2\theta + \tau^2; \\ f =& -\tau; \\ g =& 1; \\ dN(\mathbb{X}_s) =& N_s = (\kappa\cos\theta, -\tau\sin\theta, \tau\cos\theta) = \mathbb{X}_s\kappa/(1-\kappa) + \frac{\tau}{1-\kappa}\mathbb{X}_\theta; \\ dN(\mathbb{X}_\theta) =& N_\theta = (0, \sin\theta, -\cos\theta) = -\mathbb{X}_\theta; \\ [-dN] =& \begin{bmatrix} \frac{-\kappa}{1-\kappa} & \frac{-\tau}{1-\kappa} \\ 0 & 1 \end{bmatrix} \\ \kappa_1 =& \frac{\kappa}{1-\kappa}; \\ \kappa_2 =& 1; \\ K =& \frac{\kappa}{1-\kappa}; \\ H =& \frac{1}{2-2\kappa}. \end{aligned}$$

(b)

$$K = \frac{\kappa}{1-\kappa}; \quad \text{(how to discuss?)}$$

□