Geometry Homework 2

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Problem 3 (P47: 5)**.** *If a closed plane curve C is contained inside a disk of radius r, prove that there exists a point $p \in C$ such that the curvature κ of C at p satifies $|\kappa| \geq 1/r$.*

Proof. Let $X(s)$ denote the curve C, where $s \in [0, l]$ is an arc-length parameter, that is, $\|X'(s)\| \equiv 1$. Since C is contained inside a disk of radius r, let A be the centre of the disk. So we have

$$\|X(s) - A\| \leq r \tag{1}$$

Consider $f(s) = \langle X(s) - A, X(s) - A \rangle$. Since $[0, l]$ is compact, the maximum exists, denoting by $f(s') = \max_{s \in [0,l]} f(s)$. Therefore, we have $f'(s') = 0$ and $f''(s') \leq 0$. Now

$$f''(s) = 2\left(\|X'(s)\|^2 + \kappa(s) \langle X(s) - A, N(s) \rangle\right), \tag{2}$$

where $X''(s) = \kappa(s)N(s)$ and $N(s)$ is the normal vector. Take $s = s'$ in (2) we have $f''(s') \leq 0$ and hence

$$\kappa(s') \langle X(s) - A, N(s) \rangle \leq -1 \tag{3}$$

This implies

$$|\kappa(s') \langle X(s) - A, N(s) \rangle| \geq 1 \tag{4}$$

By (1), $|\langle X(s) - A, N(s) \rangle| \leq \|X(s) - A\| \cdot \|N(s)\| \leq r$. We have $|\kappa(s')| \geq 1/r$ as desired. □

Problem 4 (P23: 4, 僅討論平面情形)**.** *Assume that all normals of a parametrized curve pass through a fixed point. Prove that the trace of the curve is contained in a circle.*

Proof. Let P be the fixed point, and let $X(s)$ be this curve. Then from description, $\langle X(s) - P, X'(s) \rangle \equiv 0$ for all s. Let $f(s) = \|X(s) - P\|^2$, then we have $f'(s) = 2 \langle X(s) - P, X'(s) \rangle = 0$ for all s. This implies the trace of the curve is contained in a circle centered at point P with radius $\sqrt{f(s_0)}$ for some s_0. □

Problem 5. 以 $t = 0$ 開始將曲線 (t^2, t^3) 化成長度參數。並討論 $t = 0$ 時的曲率。

Proof. Consider $t > 0$, the length of the curve of t is

$$\int_0^t 3t\sqrt{(4/9)+t^2}\; dt = \int_0^t \frac{3}{2}\sqrt{(4/9)+t^2}\; dt^2 = \left(\frac{4}{9}+t^2\right)^{3/2} - \frac{8}{27} \tag{5}$$

Let $s > 0$ be the arc-length parameter, note that $s > 0$ equivalent to $t > 0$, so we have $s = (4/9 + t^2)^{3/2} - 8/27$ and hence

$$t = \sqrt{\left(s + \frac{8}{27}\right)^{2/3} - \frac{4}{9}} \tag{6}$$

Therefore the curve with arc-length parameter $s > 0$ is

$$\left((s+8/27))^{2/3} - 4/9, \left((s+8/27)^{2/3} - 4/9\right)^{3/2}\right) \tag{7}$$

By symmetry, for the case $s < 0$ the corresponding curve is

$$\left((8/27-s))^{2/3} - 4/9, -\left((8/27-s)^{2/3} - 4/9\right)^{3/2}\right) \tag{8}$$

We can write them together to get the result,

$$\left((8/27+|s|))^{2/3} - 4/9, sign(s)\cdot\left((8/27+|s|)^{2/3} - 4/9\right)^{3/2}\right) \tag{9}$$

For $t \neq 0$, $X'(t) = (2t, 3t^2) \neq 0$, so the curvature is $\kappa(t) = ((2t)(6t) - (3t^2)(2))/\sqrt{4t^2+9t^4} = 6t/\sqrt{4+9t^2} \to 0$ as $t \to 0$. So when $t = 0$, the curvature can be defined to be 0 so that $\kappa(t)$ at $t = 0$ is continuous. □

Problem 6.

(a) 以原點為中心，將 $y = f(x)$ 的圖形縮放 λ 倍，並說明新的圖形是 $y = \lambda f(\frac{x}{\lambda})$ 的函數圖形。

(b) 討論曲率的變化。

Proof.

(a) 原本圖形上的點 $(x, f(x))$ 經過縮放後會到 $(\lambda x, \lambda f(x)) = (\lambda x, \lambda f(\frac{\lambda x}{\lambda}))$，所以新的函數圖形就是 $y = \lambda f(\frac{x}{\lambda})$。

(b) 原本的曲率是

$$\kappa = \frac{\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}}{(x'^2+y'^2)^{3/2}} = \frac{\begin{vmatrix} 1 & f' \\ 0 & f'' \end{vmatrix}}{(1+f'^2)^{3/2}} = \frac{f''}{(1+f'^2)^{3/2}} \tag{10}$$

新的曲率是

$$\kappa_{\text{new}} = \frac{\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}}{(x'^2+y'^2)^{3/2}} = \frac{\begin{vmatrix} 1 & \lambda f'\cdot\frac{1}{\lambda} \\ 0 & \frac{1}{\lambda}\cdot f'' \end{vmatrix}}{(1+f'^2)^{3/2}} = \frac{\frac{1}{\lambda}f''}{(1+f'^2)^{3/2}} \tag{11}$$

是原本的 $1/\lambda$ 倍。

□

Problem 7. 如圖，有一橢圓，其焦點為 O_1 和 O_2，設 L 切橢圓於 P，且 L 與 $\overline{O_2P}$ 之夾角為 θ。以 θ 為參數，說明曲率 $\kappa \propto \sin^3\theta$

Proof. 不妨假設 O_1, O_2 皆落在 X 軸上，我們將此橢圓參數化為 $(a\cos t, b\sin t)$，其中 $t \in [0, 2\pi]$ 而且 $a > b$。於是可得橢圓之兩焦點座標分別是 $(c, 0), (-c, 0)$ 其中 $c = \sqrt{a^2 - b^2}$。計算此曲線的曲率為

$$\kappa(t) = \frac{\begin{vmatrix} -a\sin t & b\cos t \\ -a\cos t & -b\sin t \end{vmatrix}}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}} = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}} \tag{12}$$

現在來計算 $\sin\theta(t)$，其實 $\theta(t)$ 就是向量 $O_2P = (a\cos t - c, b\sin t)$ 與切向量 $(-a\sin t, b\cos t)$ 的有向夾角，所以

$$\begin{aligned}
\sin\theta(t) =& \frac{\begin{vmatrix} a\cos t - c & b\sin t \\ -a\sin t & b\cos t \end{vmatrix}}{\sqrt{(a\cos t - c)^2 + b^2\sin^2 t}\sqrt{a^2\sin^2 t + b^2\cos^2 t}} && (13)\\
=& \frac{ab - bc\cos t}{\sqrt{a^2\cos^2 t - 2ac\cos t + c^2 + b^2\sin^2 t}\sqrt{a^2\sin^2 t + b^2\cos^2 t}} && (14)\\
=& \frac{b(a - c\cos t)}{\sqrt{a^2\cos^2 t - 2ac\cos t + a^2 - b^2\cos^2 t}\sqrt{a^2\sin^2 t + b^2\cos^2 t}} && (15)\\
=& \frac{b(a - c\cos t)}{\sqrt{c^2\cos^2 t - 2ac\cos t + a^2}\sqrt{a^2\sin^2 t + b^2\cos^2 t}} && (16)\\
=& \frac{b(a - c\cos t)}{\sqrt{(a - c\cos t)^2}\sqrt{a^2\sin^2 t + b^2\cos^2 t}} && (17)\\
=& \frac{b}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}} && (18)
\end{aligned}$$

而從 (17) 推到 (18) 是因為 $c\cos t \le c < a$。於是

$$\kappa(t) = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}} = \frac{a}{b^2}\frac{b^3}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}} \propto \sin^3\theta \tag{19}$$

□

Problem 9. 如圖，有 *regular curve* $\gamma(t)$，$\gamma_0 = \gamma(0)$，$N_0 = N(0)$，$L_0 = \{\gamma_0 + vN_0\}$。現考慮直線 $L_t = \{\gamma(t) + uN(t)\}$，令 $P(t) = L_t \cap L_0$ 證明

$$\kappa(0) \neq 0 \Rightarrow \lim_{t\to 0} P(t) = \gamma_0 + \frac{1}{\kappa(0)}N_0$$

Proof. Let $P(t) = \gamma_0 + v(t)N_0$, then $\gamma_0 + v(t)N_0 = \gamma(t) + uN(t)$

$$v(t) = \frac{\begin{vmatrix} \gamma(t) - \gamma_0 & N(t) \end{vmatrix}}{\begin{vmatrix} N_0 & N(t) \end{vmatrix}}$$

Assume that t is arc-length parameter, then:

$$
\begin{aligned}
\lim_{t\to 0} P(t) &= \lim_{t\to 0} (\gamma_0 + v(t)N_0) && (20)\\
&= \gamma_0 + \lim_{t\to 0} v(t)N_0 && (21)\\
&= \gamma_0 + \lim_{t\to 0} \frac{\begin{vmatrix} \gamma(t)-\gamma_0 & N(t) \end{vmatrix}}{\begin{vmatrix} N_0 & N(t) \end{vmatrix}} N_0 && (22)\\
&= \gamma_0 + \lim_{t\to 0} \frac{\left(\begin{vmatrix} \gamma(t)-\gamma_0 & N(t) \end{vmatrix}\right)'}{\left(\begin{vmatrix} N_0 & N(t) \end{vmatrix}\right)'} N_0 && (23)\\
&= \gamma_0 + \lim_{t\to 0} \frac{\begin{vmatrix} \gamma'(t) & N(t) \end{vmatrix} + \begin{vmatrix} \gamma(t)-\gamma_0 & N'(t) \end{vmatrix}}{\begin{vmatrix} N_0 & N'(t) \end{vmatrix}} N_0 && (24)\\
&= \gamma_0 + \frac{\begin{vmatrix} T_0 & N_0 \end{vmatrix}}{\begin{vmatrix} N_0 & -\kappa_0 T_0 \end{vmatrix}} N_0 && (25)\\
&= \gamma_0 + \frac{1}{\kappa_0} N_0 && (26)
\end{aligned}
$$

□