Geometry Homework 3

B96201044 黃上恩, *B98901182* 時丕勳, *K0020100x* 劉士瑋

October 5, 2011

Problem 3 (P26: 16). *Show that the knowledge of the vector function $n = n(s)$ (normal vector) of a curve α, with nonzero torsion everywhere, determines the curvature $\kappa(s)$ and the torsion $\tau(s)$ of α.* ($\vec{n}$ 能決定曲線嗎？說明題目錯誤並找反例。)

Proof. Consider the helix $\alpha(s) = (a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}})$

Then $n(s) = (-\cos\frac{s}{\sqrt{a^2+b^2}}, -\sin\frac{s}{\sqrt{a^2+b^2}}, 0)$.

So if two helix has the same $a^2 + b^2$ (e.g. $\alpha_1(s) = (\frac{1}{2}\cos s, \frac{1}{2}\sin s, \frac{\sqrt{3}}{2}s)$, $\alpha_2(s) = (\frac{\sqrt{3}}{2}\cos s, \frac{\sqrt{3}}{2}\sin s, \frac{1}{2}s)$), then they have same $n(s)$, but they're not the same curve (because $\kappa = \frac{a}{a^2+b^2}$, so they have different κ). $\square$

Problem 4 (P26: 17, 另一種描述 Helix 的方式). *In general, a curve α is called a* helix *if the tangent lines of α make a constant angle with a fixed direction. Assume that $\tau(s) \neq 0$, $s \in I$, and prove that:*

(a) *α is a helix if and only if $\kappa/\tau =$ constant.*

(b) *α is a helix if and only if the lines containing $N(s)$ and passing through $\alpha(s)$ are parallel to a fixed plane.*

(c) *α is a helix if and only if the lines containing $B(s)$ and passing through $\alpha(s)$ make a constant angle with a fixed direction.*

(d) *The curve*

$$\alpha(s) = \left(\frac{a}{c}\int \sin\theta(s)ds, \frac{a}{c}\int \cos\theta(s)ds, \frac{b}{c}s\right),$$

where $c^2 = a^2 + b^2$, is a helix, and that $\kappa/\tau = a/b$.

Proof. WLOG, assume that s is arc-length parameter, V is the fixed direction. Let $T(s) = \alpha'(s)$, $T'(s) = \kappa N(s)$ and $B = T \times N$.

(a)

$$\langle T, V \rangle = C$$
$$\rightarrow \langle T', V \rangle = 0$$
$$= \langle \kappa N, V \rangle$$
$$\rightarrow \langle N, V \rangle = 0$$
$$\rightarrow \langle N', V \rangle = 0$$
$$= \langle -\kappa T - \tau B, V \rangle$$
$$= -\kappa C - \tau \langle B, V \rangle$$
$$= -\kappa C - \tau \langle T \times N, V \rangle$$
$$= -\kappa C - \tau \langle V \times T, N \rangle$$
$$V = \langle V, B \rangle B + \langle V, T \rangle T + \langle V, N \rangle N$$
$$= \langle V, B \rangle B + CT$$
$$\langle V, B \rangle' = \langle V, \tau N \rangle = 0$$
$$\rightarrow \langle V, B \rangle = constant$$
$$\rightarrow \langle V \times T, N \rangle = \langle (\langle V, B \rangle B + CT) \times T, N \rangle$$
$$= \langle \langle V, B \rangle B \times T, N \rangle$$
$$= \langle V, B \rangle$$
$$\rightarrow 0 = -\kappa C - \tau \langle V \times T, N \rangle$$
$$= -\kappa C - \tau \langle V, B \rangle$$
$$\rightarrow \kappa/\tau = -\frac{\langle V, B \rangle}{C}$$

So κ/τ is constant.

Conversely, let $\kappa/\tau \equiv c$ be a constant. Define vector V by $V(s) = T(s) - cB(s)$. We claim V is a constant since $V'(s) = T'(s) - cB'(s) = \kappa(s)N(s) - c\tau(s)N(s) = (\kappa(s) - c\tau(s))N(s) = 0$. Now $\langle V, T \rangle$ is constant because $\langle V, T \rangle' = \kappa \langle V, N \rangle = \kappa \langle T - cB, N \rangle = 0$. This implies T make a constant angle with V.

(b) $\langle V, T \rangle \equiv c$ implies $0 = \langle V, T \rangle' = \kappa \langle V, N \rangle$, but $\kappa = \|\alpha'\| \neq 0$, so $\langle V, N \rangle = 0$. Therefore $N(s)$ is always perpendicular to V. Consider any fixed plane P with V be the normal vector, then $N(s)$ is parallel to P.

Conversely, if $N(s)$ parallel to a fixed plane P, define V be a normal vector of P. This implies $N(s) \perp V$, therefore $\langle V, T \rangle' = \kappa \langle V, N \rangle = 0$. So T makes a constant angle with V.

(c) From the proof above, we have $\langle V, N \rangle = 0$, so $V = \langle V, T \rangle T + \langle V, B \rangle B$. Since $V, \langle V, T \rangle$ are both constant and the orientation between T, B is fixed, $\langle V, B \rangle$ is a constant. This implies B makes a constant angle with V.

Conversely, let V be the fixed direction, then $0 = \langle V, B \rangle = -\tau \langle V, N \rangle$. Since $\tau \neq 0$, so $V \perp N$ hence by (b), T makes a constant angle with V.

(d)

$\square$

Problem 6. $\gamma(s)$ 長度參數。若將 $T(s)$ 寫成 $(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$，$\phi, \theta$ 是 s 的函數。說明 $\kappa(s) = \sqrt{\phi'^2 + \theta'^2\sin^2\phi}$

Proof.

$T'(s) = (\phi'\cos\phi\cos\theta - \theta'\sin\phi\sin\theta, \phi'\cos\phi\sin\theta + \theta'\sin\phi\cos\theta, -\phi'\sin\phi)$

$\rightarrow \kappa(s) = |T'(s)|$

$\quad = \sqrt{\phi'^2\cos^2\phi\cos^2\theta + \theta'^2\sin^2\phi\sin^2\theta + \phi'^2\cos^2\phi\sin^2\theta + \theta'^2\sin^2\phi\cos^2\theta + \phi'^2\sin^2\phi}$

$\quad = \sqrt{\phi'^2 + \theta'^2\sin^2\phi}$

$\square$

Problem 7. $\gamma : \mathbb{R} \to \mathbb{R}^3$，不妨假設是長度參數。

(b) 若 $M^t M = I$，$\det(M) = -1$ 且 $\overline{\gamma} = M\gamma$，討論 κ, τ 變化。

(c) $\overline{\gamma}(s) = \gamma(-s)$，說明 κ, τ 變化。

Proof. (b)

$$\begin{aligned}
|\overline{\gamma}'| &= \sqrt{\overline{\gamma}'^T \overline{\gamma}'} \\
&= \sqrt{\gamma'^T M^T M \gamma'} \\
&= \sqrt{\gamma'^T \gamma'} \\
&= |\gamma'| \\
&= 1
\end{aligned}$$

So s is arc-length parameter for $\overline{\gamma}$ too.

$$\begin{aligned}
\kappa_{\overline{\gamma}} &= |\overline{\gamma}''| \\
&= \sqrt{\overline{\gamma}''^T \overline{\gamma}''} \\
&= \sqrt{\gamma''^T M^T M \gamma''} \\
&= \sqrt{\gamma''^T \gamma''} \\
&= |\gamma''| \\
&= \kappa_\gamma
\end{aligned}$$

So κ remains the same.

(c)

$$\begin{aligned}
|\overline{\gamma}'(s)| &= \sqrt{\overline{\gamma}'(s)^T \overline{\gamma}'(s)} \\
&= \sqrt{\left(-\gamma'^T(-s)\right)\left(-\gamma'(-s)\right)} \\
&= \sqrt{\gamma'(-s)^T \gamma'(-s)} \\
&= |\gamma'(-s)| \\
&= 1
\end{aligned}$$

So s is arc-length parameter for $\overline{\gamma}$ too.

$$\kappa_{\overline{\gamma}}(s) = |\overline{\gamma}''(s)|$$
$$= \sqrt{\overline{\gamma}''(s)^T \overline{\gamma}''(s)}$$
$$= \sqrt{\gamma''(-s)^T \gamma''(-s)}$$
$$= |\gamma''(-s)|$$
$$= \kappa_{\gamma}(-s)$$

So $\kappa_{\overline{\gamma}}(s) = \kappa_{\gamma}(-s)$.

$$\tau_{\overline{\gamma}}(s) = \frac{|\overline{\gamma}'(s)\ \overline{\gamma}''(s)\ \overline{\gamma}'''(s)|}{|\overline{\gamma}'(s) \times \overline{\gamma}''(s)|^2}$$
$$= \frac{|-\gamma'(-s)\ \gamma''(-s)\ -\gamma'''(-s)|}{|-\gamma'(-s) \times \gamma''(-s)|^2}$$
$$= \frac{|\gamma'(-s)\ \gamma''(-s)\ \gamma'''(-s)|}{|\gamma'(-s) \times \gamma''(-s)|^2}$$
$$= \tau_{\gamma}(-s)$$

So $\tau_{\overline{\gamma}}(s) = \tau_{\gamma}(-s)$.

$\square$

Problem 8. 說明 $\overline{\gamma}(u) = \gamma(t(u))$ 時，在對應點

$$\frac{\det(\overline{\gamma}', \overline{\gamma}'', \overline{\gamma}''')}{|\overline{\gamma}' \times \overline{\gamma}''|^2}(u) = \frac{\det(\gamma', \gamma'', \gamma''')}{|\gamma' \times \gamma''|^2}(t)$$

再用 *chain rule* 直接說明。

Proof.

$$\overline{\gamma}'(u) = \gamma'(t(u))t'(u)$$
$$\overline{\gamma}''(u) = \gamma''(t(u))t'(u)^2 + \gamma'(t(u))t''(u)$$
$$\overline{\gamma}'''(u) = \gamma'''(t(u))t'(u)^3 + 3\gamma''(t(u))t'(u)t''(u) + \gamma'(t(u))t'''(u)$$

$$\rightarrow \det(\overline{\gamma}', \overline{\gamma}'', \overline{\gamma}''')(u) = \det(\gamma'(t(u))t'(u), \gamma''(t(u))t'(u)^2 + \gamma'(t(u))t''(u),$$
$$\gamma'''(t(u))t'(u)^3 + 3\gamma''(t(u))t'(u)t''(u) + \gamma'(t(u))t'''(u))$$
$$= \det(\gamma'(t(u))t'(u), \gamma''(t(u))t'(u)^2, \gamma'''(t(u))t'(u)^3 + 3\gamma''(t(u))t'(u)t''(u))$$
$$= \det(\gamma'(t(u))t'(u), \gamma''(t(u))t'(u)^2, \gamma'''(t(u))t'(u)^3)$$
$$= t'(u)^6 \det(\gamma'(t(u)), \gamma''(t(u)), \gamma'''(t(u)))$$

$$|\overline{\gamma}' \times \overline{\gamma}''|^2(u) = |\left(\gamma'(t(u))t'(u)\right) \times \left(\gamma''(t(u))t'(u)^2 + \gamma'(t(u))t''(u)\right)|^2$$
$$= |\left(\gamma'(t(u))t'(u)\right) \times \left(\gamma''(t(u))t'(u)^2\right)|^2$$
$$= t'(u)^6 |\gamma'(t(u)) \times \gamma''(t(u))|^2$$

4

$$\to \frac{\det(\overline{\gamma}', \overline{\gamma}'', \overline{\gamma}''')}{|\overline{\gamma}' \times \overline{\gamma}''|^2}(u) = \frac{t'(u)^6 \det(\gamma', \gamma'', \gamma''')}{t'(u)^6 |\gamma' \times \gamma''|^2}(t)$$

$$= \frac{\det(\gamma', \gamma'', \gamma''')}{|\gamma' \times \gamma''|^2}(t)$$

$\square$