

# GEOMETRY HOMEWORK 3

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**Problem 3** (P26: 16). *Show that the knowledge of the vector function  $n = n(s)$  (normal vector) of a curve  $\alpha$ , with nonzero torsion everywhere, determines the curvature  $\kappa(s)$  and the torsion  $\tau$  of  $\alpha$ . ( $\vec{n}$  能決定曲線嗎? 說明題目錯誤並找反例。)*

*Proof.* Consider the helix  $\alpha(s) = (a \cos \frac{s}{\sqrt{a^2+b^2}}, a \sin \frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}})$   
Then  $n(s) = (-\cos \frac{s}{\sqrt{a^2+b^2}}, -\sin \frac{s}{\sqrt{a^2+b^2}}, 0)$ .

So if two helix has the same  $a^2 + b^2$  (e.g.  $\alpha_1(s) = (\frac{1}{2} \cos s, \frac{1}{2} \sin s, \frac{\sqrt{3}}{2}s)$ ,  $\alpha_2(s) = (\frac{\sqrt{3}}{2} \cos s, \frac{\sqrt{3}}{2} \sin s, \frac{1}{2}s)$ ), then they have same  $n(s)$ , but they're not the same curve.  $\square$

**Problem 4** (P26: 17, 另一種描述 Helix 的方式). *In general, a curve  $\alpha$  is called a helix if the tangent lines of  $\alpha$  make a constant angle with a fixed direction. Assume that  $\tau(s) \neq 0$ ,  $s \in I$ , and prove that:*

- (a)  $\alpha$  is a helix if and only if  $\kappa/\tau = \text{constant}$ .
- (b)  $\alpha$  is a helix if and only if the lines containing  $n(s)$  and passing through  $\alpha(s)$  are parallel to a fixed plane.
- (c)  $\alpha$  is a helix if and only if the lines containing  $b(s)$  and passing through  $\alpha(s)$  make a constant angle with a fixed direction.
- (d) The curve

$$\alpha(s) = \left( \frac{a}{c} \int \sin \theta(s) ds, \frac{a}{c} \int \cos \theta(s) ds, \frac{b}{c} s \right),$$

where  $c^2 = a^2 + b^2$ , is a helix, and that  $\kappa/\tau = a/b$ .

*Proof.* (a) Assume that  $s$  is arc-length parameter,  $V$  is the fixed direction.

$$\begin{aligned}
\langle T, V \rangle &= C \\
\rightarrow \langle T', V \rangle &= 0 \\
&= \langle \kappa N, V \rangle \\
\rightarrow \langle N, V \rangle &= 0 \\
\rightarrow \langle N', V \rangle &= 0 \\
&= \langle -\kappa T - \tau B, V \rangle \\
&= -\kappa C - \tau \langle B, V \rangle \\
&= -\kappa C - \tau \langle T \times N, V \rangle \\
&= -\kappa C - \tau \langle V \times T, N \rangle
\end{aligned}$$

$$\because T \perp N, V \perp N \rightarrow V \times T = \pm |V \times T| N$$

$$\begin{aligned}
\rightarrow 0 &= -\kappa C - \tau \langle V \times T, N \rangle \\
&= -\kappa C \mp \tau |V \times T|
\end{aligned}$$

$\because T$  make a constant angle with  $V$ ,  $|V \times T|$  is a constant.  
 $\rightarrow \kappa/\tau = \mp \frac{|V \times T|}{C}$ , but because  $\kappa, \tau$  are continuous,  $\kappa/\tau$  is constant.

(b)

(c)

(d)

□

**Problem 6.**  $\gamma(s)$  長度參數。若將  $T(s)$  寫成  $(\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)$ ,  $\phi, \theta$  是  $s$  的函數。說明  $\kappa(s) = \sqrt{\phi'^2 + \theta'^2 \sin^2 \phi}$

*Proof.*

$$\begin{aligned}
T'(s) &= (\phi' \cos \phi \cos \theta - \theta' \sin \phi \sin \theta, \phi' \cos \phi \sin \theta + \theta' \sin \phi \cos \theta, -\phi' \sin \phi) \\
\rightarrow \kappa(s) &= |T'(s)| \\
&= \sqrt{\phi'^2 \cos^2 \phi \cos^2 \theta + \theta'^2 \sin^2 \phi \sin^2 \theta + \phi'^2 \cos^2 \phi \sin^2 \theta + \theta'^2 \sin^2 \phi \cos^2 \theta + \phi'^2 \sin^2 \phi} \\
&= \sqrt{\phi'^2 + \theta'^2 \sin^2 \phi}
\end{aligned}$$

□

**Problem 7.**  $\gamma: \mathbb{R} \rightarrow \mathbb{R}^3$ , 不妨假設是長度參數。

(b) 若  $M^t M = I$ ,  $\det(M) = -1$  且  $\bar{\gamma} = M\gamma$ , 討論  $\kappa, \tau$  變化。

(c)  $\bar{\gamma}(s) = \gamma(-s)$ , 說明  $\kappa, \tau$  變化。

*Proof.* (b)

$$\begin{aligned}
|\bar{\gamma}'| &= \sqrt{\bar{\gamma}'^T \bar{\gamma}'} \\
&= \sqrt{\gamma'^T M^T M \gamma'} \\
&= \sqrt{\gamma'^T \gamma'} \\
&= |\gamma'| \\
&= 1
\end{aligned}$$

So  $s$  is arc-length parameter for  $\bar{\gamma}$  too.

$$\begin{aligned}
\kappa_{\bar{\gamma}} &= |\bar{\gamma}''| \\
&= \sqrt{\bar{\gamma}''^T \bar{\gamma}''} \\
&= \sqrt{\gamma''^T M^T M \gamma''} \\
&= \sqrt{\gamma''^T \gamma''} \\
&= |\gamma''| \\
&= \kappa_{\gamma}
\end{aligned}$$

So  $\kappa$  remains the same.

(c)

$$\begin{aligned}
|\bar{\gamma}'(s)| &= \sqrt{\bar{\gamma}'(s)^T \bar{\gamma}'(s)} \\
&= \sqrt{(-\gamma'^T(-s))(-\gamma'(-s))} \\
&= \sqrt{\gamma'^T(-s)\gamma'(-s)} \\
&= |\gamma'(-s)| \\
&= 1
\end{aligned}$$

So  $s$  is arc-length parameter for  $\bar{\gamma}$  too.

$$\begin{aligned}
\kappa_{\bar{\gamma}}(s) &= |\bar{\gamma}''(s)| \\
&= \sqrt{\bar{\gamma}''(s)^T \bar{\gamma}''(s)} \\
&= \sqrt{\gamma''(-s)^T \gamma''(-s)} \\
&= |\gamma''(-s)| \\
&= \kappa_{\gamma}(-s)
\end{aligned}$$

So  $\kappa_{\bar{\gamma}}(s) = \kappa_{\gamma}(-s)$ .

$$\begin{aligned}\tau_{\bar{\gamma}}(s) &= \frac{|\bar{\gamma}'(s) \bar{\gamma}''(s) \bar{\gamma}'''(s)|}{|\bar{\gamma}'(s) \times \bar{\gamma}''(s)|^2} \\ &= \frac{|-\gamma'(-s) \gamma''(-s) - \gamma'''(-s)|}{|-\gamma'(-s) \times \gamma''(-s)|^2} \\ &= \frac{|\gamma'(-s) \gamma''(-s) \gamma'''(-s)|}{|\gamma'(-s) \times \gamma''(-s)|^2} \\ &= \tau_{\gamma}(-s)\end{aligned}$$

So  $\tau_{\bar{\gamma}}(s) = \tau_{\gamma}(-s)$ .

□

**Problem 8.** 說明  $\bar{\gamma}(u) = \gamma(t(u))$  時，在對應點

$$\frac{\det(\bar{\gamma}', \bar{\gamma}'', \bar{\gamma}''')}{|\bar{\gamma}' \times \bar{\gamma}''|^2}(u) = \frac{\det(\gamma', \gamma'', \gamma''')}{|\gamma' \times \gamma''|^2}(t)$$

再用 *chain rule* 直接說明。

*Proof.*

$$\begin{aligned}\bar{\gamma}'(u) &= \gamma'(t(u))t'(u) \\ \bar{\gamma}''(u) &= \gamma''(t(u))t'(u)^2 + \gamma'(t(u))t''(u) \\ \bar{\gamma}'''(u) &= \gamma'''(t(u))t'(u)^3 + 3\gamma''(t(u))t'(u)t''(u) + \gamma'(t(u))t'''(u)\end{aligned}$$

$$\begin{aligned}\rightarrow \det(\bar{\gamma}', \bar{\gamma}'', \bar{\gamma}''')(u) &= \det(\gamma'(t(u))t'(u), \gamma''(t(u))t'(u)^2 + \gamma'(t(u))t''(u), \\ &\quad \gamma'''(t(u))t'(u)^3 + 3\gamma''(t(u))t'(u)t''(u) + \gamma'(t(u))t'''(u)) \\ &= \det(\gamma'(t(u))t'(u), \gamma''(t(u))t'(u)^2, \gamma'''(t(u))t'(u)^3 + 3\gamma''(t(u))t'(u)t''(u)) \\ &= \det(\gamma'(t(u))t'(u), \gamma''(t(u))t'(u)^2, \gamma'''(t(u))t'(u)^3) \\ &= t'(u)^6 \det(\gamma'(t(u)), \gamma''(t(u)), \gamma'''(t(u)))\end{aligned}$$

$$\begin{aligned}|\bar{\gamma}' \times \bar{\gamma}''|^2(u) &= |(\gamma'(t(u))t'(u)) \times (\gamma''(t(u))t'(u)^2 + \gamma'(t(u))t''(u))|^2 \\ &= |(\gamma'(t(u))t'(u)) \times (\gamma''(t(u))t'(u)^2)|^2 \\ &= t'(u)^6 |\gamma'(t(u)) \times \gamma''(t(u))|^2\end{aligned}$$

$$\begin{aligned}\rightarrow \frac{\det(\bar{\gamma}', \bar{\gamma}'', \bar{\gamma}''')}{|\bar{\gamma}' \times \bar{\gamma}''|^2}(u) &= \frac{t'(u)^6 \det(\gamma', \gamma'', \gamma''')}{t'(u)^6 |\gamma' \times \gamma''|^2}(t) \\ &= \frac{\det(\gamma', \gamma'', \gamma''')}{|\gamma' \times \gamma''|^2}(t)\end{aligned}$$

□