Geometry Homework 13

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Problem 4. *Helicoid* $\mathbb{X}(u,v) = (v\cos u, v\sin u, u)$, $\gamma(t) = \mathbb{X}(t,1)$, $p = \mathbb{X}(0,1) = (1,0,0)$, $V(0) = \gamma'(0)$ 求解平行向量場 $V(t)$ *along* $\gamma(t)$

Proof.

$$\begin{aligned}
\mathbb{X}_u &= (-v\sin u, v\cos u, 1)\\
\mathbb{X}_v &= (\cos u, \sin u, 0)\\
\rightarrow E &= v^2+1, F = 0, G = 1\\
[1,1,1] &= \frac{E_u}{2}, [1,1,2] = -\frac{E_v}{2}, [1,2,1] = \frac{E_v}{2}\\
[1,2,2] &= \frac{G_u}{2}, [2,2,1] = -\frac{G_u}{2}, [2,2,2] = \frac{G_v}{2}\\
\begin{bmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 & \Gamma_{22}^1 \\ \Gamma_{11}^2 & \Gamma_{12}^2 & \Gamma_{22}^2 \end{bmatrix} &= \begin{bmatrix} v^2+1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & v & 0 \\ -v & 0 & 0 \end{bmatrix}\\
&= \begin{bmatrix} 0 & \frac{v}{v^2+1} & 0 \\ -v & 0 & 0 \end{bmatrix}
\end{aligned}$$

On γ, $v = 1$, so the parallel equation is:

$$\begin{aligned}
a' + \frac{b}{2} &= 0\\
b' - a &= 0\\
\rightarrow a &= \frac{1}{\sqrt{2}}(-c_1 \sin\frac{1}{\sqrt{2}}t + c_2\cos\frac{1}{\sqrt{2}}t)\\
b &= c_1\cos\frac{1}{\sqrt{2}}t + c_2\sin\frac{1}{\sqrt{2}}t\\
V(0) &= (0,1,1) = \mathbb{X}_u(p)\\
\rightarrow a(0) &= 1, b(0) = 0\\
\rightarrow c_1 &= 0, c_2 = \sqrt{2}\\
\rightarrow a(t) &= \cos\frac{1}{\sqrt{2}}t, b(t) = \sqrt{2}\sin\frac{1}{\sqrt{2}}t\\
\rightarrow V(t) &= a(t)\mathbb{X}_u + b(t)\mathbb{X}_v\\
&= \cos\frac{1}{\sqrt{2}}t(-\sin t, \cos t, 1) + \sqrt{2}\sin\frac{1}{\sqrt{2}}t(\cos t, \sin t, 0)
\end{aligned}$$

$\square$

Problem 6. 如圖考慮一旋轉體上的緯圈 γ，已知其 *generating curve*(經線) 切線與中心軸夾角為 θ。

(a) 求一向量沿 γ 平行移動，繞一圈後與原向量的夾角 (不妨假設起始向量與緯圈同向)

(b) 將該 *surface* 放大或縮小，相對應問題的夾角有何變化

(c) 計算此緯圈之 $\oint_\gamma \kappa_g ds$，值與 *surface* 的縮放有關嗎？

Proof. (a) 不妨設緯圈的參數為 $\gamma(t) = (x, r\cos t, r\sin t)$, 則其有 $\kappa_g = \frac{1}{r}\sin\theta$.

$$\begin{aligned}\text{繞一圈後與原向量的夾角} &= 2\pi - \int_0^l \kappa_g ds \\ &= 2\pi - 2\pi r\frac{1}{r}\sin\theta \\ &= 2\pi(1-\sin\theta)\end{aligned}$$

(b) 由上式可知, 夾角和 r 無關，而 θ 並不會因為放大縮小而改變，故 surface 放大縮小並不會造成夾角的變化。

(c) 由以上計算可知, $\oint_\gamma \kappa_g ds = 2\pi\sin\theta$, 不與 surface 的縮放有關。

$\square$

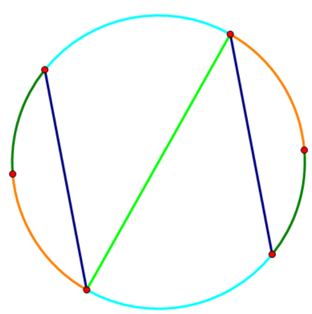
Problem 10 (Ex P282 4.).

(a) *Compute the Euler-Poincaré characteristic of (1) An ellipsoid. (2) The surface* $S = \{(x,y,z)\in\mathbb{R}^3; x^2+y^{10}+z^6 = 1\}$.

(b) 如圖，將一圓盤的邊界如圖「黏」起來 (也可以想成將對稱點「黏」起來)，找一個三角分割，計算此 *projective space* 的 *Euler characteristic*。

Proof. (a) 這兩個的拓樸結構都和球一樣，故都有 Euler-Poincaré characteristic = 2.

(b) 在圓周上找三個" 點" a, b, c, 則可以切出:



由圖中可以看出 $V = 3, E = 6, F = 4$, 故 Euler-Poincaré characteristic $= 3 - 6 + 4 = 1$.

$\square$

Problem 12 (Ex P283 6.). *Show that* $(0,0)$ *is an isolated singular point and compute the index at* $(0,0)$ *of the following vector fields in the plane:*

(a) $v = (x, y)$.

(b) $v = (-x, y)$.

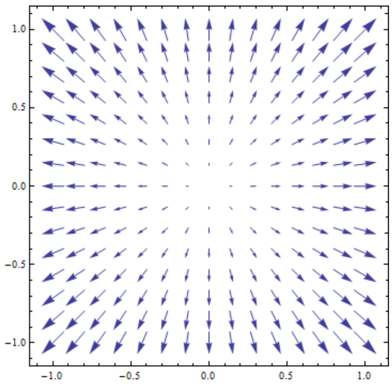
(c) $v = (x, -y)$.

(d) $v = (x^2 - y^2, -2xy)$.

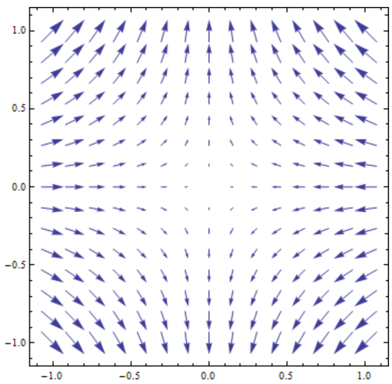
(e) $v = (x^3 - 3xy^2, y^3 - 3x^2y)$.

Proof. 我們觀察該向量場沿者單位圓繞著原點逆時針旋轉時的轉向，並數其指向 $+x$ 軸方向的次數來計算其 index。
在以下各題中, $v = 0$ 的解都只有 $(x, y) = (0, 0)$, 故 $(0, 0)$ 為 isolated singular point.

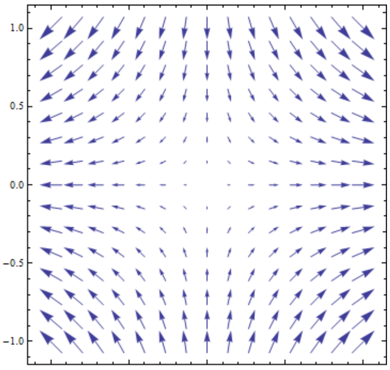
(a) 其轉向為持續逆時針方向且指向 $+x$ 軸方向只有在 $(x, y) = (1, 0)$ 時, 故 index=1.



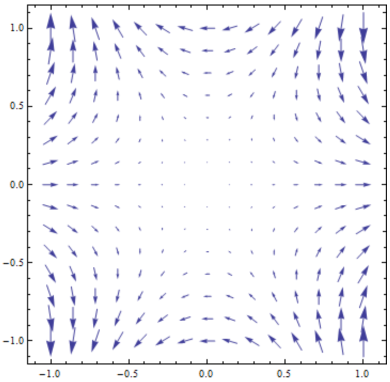
(b) 其轉向為持續順時針方向且指向 $+x$ 軸方向只有在 $(x, y) = (-1, 0)$ 時, 故 index=-1.



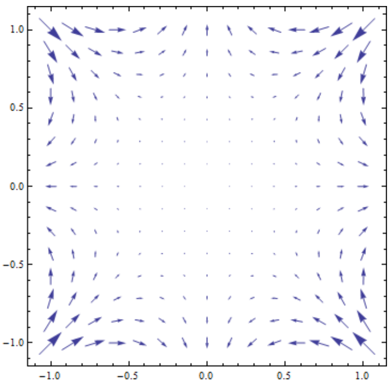
(c) 其轉向為持續順時針方向且指向 $+x$ 軸方向只有在 $(x,y)=(1,0)$ 時, 故 index=-1.



(d) 其轉向為持續順時針方向且指向 $+x$ 軸方向只有在 $(x,y)=(1,0),(-1,0)$ 時, 故 index=-2.



(e) 其轉向為持續順時針方向且指向 $+x$ 軸方向只有在 $(x,y)=(1,0),(-\frac{1}{2},\frac{\sqrt{3}}{2}),(-\frac{1}{2},-\frac{\sqrt{3}}{2})$ 時, 故 index=-3.



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