Geometry Homework 1

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Problem 3 (P7: 4). *Let $\alpha : (0, \pi) \to \mathbf{R}^2$ be given by*

$$\alpha(t) = \left(\sin t, \cos t + \log \tan \frac{t}{2}\right),$$

where t is the angle that the y axis makes with the vector $\alpha(t)$. The trace of α is called the tractrix *(Fig. 1-9). Show that*

(a) α is a differentiable parametrized curve, regular except at $t = \pi/2$.

(b) The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

Proof.

(a) Let $x(t) = \sin t$, $y(t) = \cos t + \log \tan \frac{t}{2}$, then

$$x'(t) = \cos t; \quad y'(t) = -\sin t + \frac{1}{\sin t}.$$

It's trivial that both $x'(t)$ and $y'(t)$ are infinitely differentiable in $(0, \pi)$, so α is a differentiable parametrized curve.

$x'(t) = 0, y'(t) = 0 \Longleftrightarrow t = \frac{\pi}{2}$, so α is regular except at $t = \pi/2$.

(b) The intersection of y axis and the tangent of the tractrix is $\left(0, y(t) - \frac{y'(t)}{x'(t)} x(t)\right)$. The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is $\sqrt{x(t)^2 + \left(\frac{y'(t)}{x'(t)} x(t)\right)^2}$

$$
\begin{aligned}
x(t)^2 + \left(\frac{y'(t)}{x'(t)}x(t)\right)^2 &= \sin^2 t \left(1 + \left(\frac{y'(t)}{x'(t)}\right)^2\right) \\
&= \sin^2 t \left(1 + \left(\frac{-\sin t + \frac{1}{\sin t}}{\cos t}\right)^2\right) \\
&= \sin^2 t \left(1 + \left(\frac{1-\sin^2 t}{\sin t \cos t}\right)^2\right) \\
&= \sin^2 t \left(\frac{1}{\sin^2 t}\right) \\
&= 1
\end{aligned}
$$

So the length of the segment of the tangent of the tractrix between the point of tangency and the y axis $= \sqrt{x(t)^2 + \left(\frac{y'(t)}{x'(t)}x(t)\right)^2} = 1$.

$\square$

Problem 5 (P47: 6). *Let $\alpha(s), s \in [0,l]$ be a closed convex plane curve positively oriented. The curve*

$$\beta(s) = \alpha(s) - rn(s),$$

where r is a positive constant and n is the normal vector, is called a parallel *curve to α (Fig. 1-37). Show that*

(a) Length of $\beta =$ length of $\alpha + 2\pi r$.

(b) $A(\beta) = A(\alpha) + rl + \pi r^2$.

(c) $\kappa_\beta(s) = \kappa_\alpha(s)/(1 + r\kappa_\alpha(s))$.

Proof. $\square$

Problem 8 (Curvature is a geometric object I.). $X(s) = (x(s), y(s))$, *where s is the arc-length parameter.*

$$M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, M^t = M^{-1}, \textit{i.e. } M \textit{ is orthogonal.}$$

Let $\bar{X}(s) = M \cdot \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, $\alpha, \beta \in \mathbf{R}$. What is the relation between $\kappa_X(s)$ and $\kappa_{\bar{X}}(s)$?

Proof. □

Problem 9 (Curvature is a geometric object II.). *$X(t) = (x(t), y(t))$ be a regular curve. Let*

$$\kappa(x(t), y(t)) \equiv \kappa(t) = \frac{\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}}{(x'^2 + y'^2)^{\frac{3}{2}}}$$

Let $Y(u) = X(t(u))$, $t'(u) \neq 0$. Discuss the relation of $\kappa(x(t), y(t))$ and $\kappa(x(t(u)), y(t(u)))$ at the corresponding points.

Proof. □

Problem 10. *Let $F(x, y) = c$ defines a plane curve. Prove that the curvature of the curve satisfies*

$$|\kappa| = \left| \frac{\begin{bmatrix} F_y, & -F_x \end{bmatrix} \begin{bmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \end{bmatrix} \begin{bmatrix} F_y \\ -F_x \end{bmatrix}}{(F_x^2 + F_y^2)^{\frac{3}{2}}} \right|$$

Where $F_x^2 + F_y^2 \neq 0$.

Proof. □