Geometry Homework 1

B96201044 黃上恩, *B98901182* 時丕勳, *K0020100x* 劉士瑋

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Problem 3 (P7: 4). *Let* $\alpha : (0, \pi) \to \mathbf{R}^2$ *be given by*

$$\alpha(t) = \left(\sin t, \cos t + \log \tan \frac{t}{2}\right),$$

where t *is the angle that the* y *axis makes with the vector* $\alpha(t)$*. The trace of* α *is called the* tractrix *(Fig. 1-9). Show that*

(a) α *is a differentiable parametrized curve, regular except at* $t = \pi/2$*.*

(b) The length of the segment of the tangent of the tractrix between the point of tangency and the y *axis is constantly equal to* 1*.*

Proof.

(a) Let $x(t) = \sin t$, $y(t) = \cos t + \log \tan \frac{t}{2}$, then

$$x'(t) = \cos t; \quad y'(t) = -\sin t + \frac{1}{\sin t}.$$

It's trivial that both $x'(t)$ and $y'(t)$ are infinitely differentiable in $(0, \pi)$, so α is a differentiable parametrized curve.
$x'(t) = 0, y'(t) = 0 \iff t = \frac{\pi}{2}$, so α is regular except at $t = \pi/2$.

(b) The intersection of y axis and the tangent of the tractrix is $\left(0, y(t) - \frac{y'(t)}{x'(t)}x(t)\right)$.
The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is $\sqrt{x(t)^2 + \left(\frac{y'(t)}{x'(t)}x(t)\right)^2}$

$$\begin{aligned}
x(t)^2 + \left(\frac{y'(t)}{x'(t)}x(t)\right)^2 &= \sin^2 t \left(1 + \left(\frac{y'(t)}{x'(t)}\right)^2\right) \\
&= \sin^2 t \left(1 + \left(\frac{-\sin t + \frac{1}{\sin t}}{\cos t}\right)^2\right) \\
&= \sin^2 t \left(1 + \left(\frac{1 - \sin^2 t}{\sin t \cos t}\right)^2\right) \\
&= \sin^2 t \left(\frac{1}{\sin^2 t}\right) \\
&= 1
\end{aligned}$$

So the length of the segment of the tangent of the tractrix between the point of tangency and the y axis $= \sqrt{x(t)^2 + \left(\frac{y'(t)}{x'(t)}x(t)\right)^2} = 1.$

$\square$

Problem 5 (P47: 6). *Let $\alpha(s), s \in [0,l]$ be a closed convex plane curve positively oriented. The curve*

$$\beta(s) = \alpha(s) - rn(s),$$

where r is a positive constant and n is the normal vector, is called a parallel *curve to α (Fig. 1-37). Show that*

(a) Length of $\beta =$ length of $\alpha + 2\pi r$.

(b) $A(\beta) = A(\alpha) + rl + \pi r^2$.

(c) $\kappa_\beta(s) = \kappa_\alpha(s)/(1 + r\kappa_\alpha(s))$.

For (a)-(c), $A(\cdot)$ denotes the area bounded by the corresponding curve, and $\kappa_\alpha, \kappa_\beta$ are the curvatures of α and β, respectively.

Proof.

(a) Since α is a closed convex plane curve, by the theorem of turning tangents (P.37), we have

$$\int_0^l \kappa(s)ds = 2\pi$$

Moreover, $\kappa(s)$ and r are both positive by definition, so $r\kappa(s)/||\alpha'(s)||$ is always non-negative. Therefore, length of β is

$$\int_0^l ||\beta'(s)||ds = \int_0^l ||\alpha'(s) - rn'(s)||ds.$$

By definition of normal vector n, $n'(s) = -\kappa(s)\alpha'(s)/||\alpha'(s)||$. Therefore, the length of β equals

$$\begin{aligned}
\int_0^l ||\beta'(s)||ds &= \int_0^l ||\alpha'(s) + r\kappa(s)\frac{\alpha'(s)}{||\alpha'(s)||}||ds \\
&= \int_0^l ||(1 + \frac{r\kappa(s)}{||\alpha'(s)||})\alpha'(s)||ds \\
&= \int_0^l (1 + \frac{r\kappa(s)}{||\alpha'(s)||}) \cdot ||\alpha'(s)||ds \\
&= \int_0^l ||\alpha'(s)||ds + r\int_0^l \kappa(s)ds \\
&= \text{length of } \alpha + 2\pi r.
\end{aligned}$$

(b) By using the result of (c), we knew that κ_β is always positive, so β is a convex closed curve and hence simple.

Let D and D' denote the region bounded by α and β. By Green's theorem,

$$\iint_D dxdy = \frac{1}{2}\left\|\oint_0^l \alpha\times\alpha' ds\right\|_z ; \iint_{D'} dxdy = \frac{1}{2}\left\|\oint_0^l \beta\times\beta' ds\right\|_z$$

, where $\|\cdot\|_z$ denotes the z-component of a vector , and then

$$\begin{aligned}
\oint_0^l \beta\times\beta' ds &= \oint_0^l (\alpha - rn)\times(\alpha - rn)' ds \\
&= \oint_0^l \alpha\times\alpha' ds + r^2\oint_0^l n\times n' ds + r\oint_0^l (\alpha'\times n - \alpha\times n')ds \\
&= \oint_0^l \alpha\times\alpha' ds - r^2\oint_0^l n\times(-\kappa\alpha'/\|\alpha'\|)ds \\
&\quad - r\oint_0^l (\alpha\times n)' ds + 2r\oint_0^l \alpha'\times n ds \\
&= \oint_0^l \alpha\times\alpha' ds - r^2\oint_0^l n\times(-\kappa\alpha'/\|\alpha'\|)ds \\
&\quad - (\alpha\times n|_{s=0}^l) + 2r\oint_0^l \alpha'\times n ds.
\end{aligned}$$

Since the third term of the last value is zero and that $\alpha'\times n = \|\alpha'\| e_z$,

$$\begin{aligned}
A(\beta) &= \left\|\frac{1}{2}\oint_0^l \alpha\times\alpha' ds - \frac{r^2}{2}\oint_0^l n\times(-\kappa\alpha'/\|\alpha'\|)ds + r\oint_0^l \alpha'\times n ds\right\|_z \\
&= \left|A(\alpha) + \frac{r^2}{2}\oint_0^l \kappa ds + r\oint_0^l \|\alpha'\| ds\right| = A(\alpha) + \pi r^2 + rl.
\end{aligned}$$

(c)

$$\begin{aligned}
\beta'(s) &= \alpha'(s) - rn'(s) \\
&= t(s) + r\kappa_\alpha(s)t(s) \\
&= (1 + r\kappa_\alpha(s))t(s) \\
\beta''(s) &= (1 + r\kappa_\alpha(s))\kappa_\alpha n(s) + r\kappa_\alpha'(s)t(s) \\
\kappa_\beta(s) &= \frac{\begin{vmatrix} 1 + r\kappa_\alpha(s) & 0 \\ r\kappa_\alpha'(s) & (1 + r\kappa_\alpha(s))\kappa_\alpha \end{vmatrix}}{(1 + r\kappa_\alpha(s))^3} \\
&= \frac{(1 + r\kappa_\alpha(s))(1 + r\kappa_\alpha(s))\kappa_\alpha}{(1 + r\kappa_\alpha(s))^3} \\
&= \frac{\kappa_\alpha}{1 + r\kappa_\alpha(s)}
\end{aligned}$$

□

Problem 8 (Curvature is a geometric object I.). *$X(s) = (x(s), y(s))$, where s is the arc-length parameter.*

$$M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \; M^t = M^{-1}, \textit{i.e. } M \textit{ is orthogonal.}$$

Let $\bar{X}(s) = M \cdot \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, $\alpha, \beta \in \mathbf{R}$. What is the relation between $\kappa_X(s)$ and $\kappa_{\bar{X}}(s)$?

Proof. We first claim that s is also the arc-length parameter for $\bar{X}$. This is because $\|\bar{X}'(s)\| = \|(a_{11}x'(s) + a_{12}y'(s), a_{21}x'(s) + a_{22}y'(s))\| = (a_{11}^2 + a_{12}^2)(x'(s))^2 + (a_{21}^2 + a_{22}^2)(y'(s))^2$. Since M is orthogonal, we have $a_{11}^2 + a_{12}^2 = 1 = a_{21}^2 + a_{22}^2$ and since s is the arc-length parameter of X, $\|X'(s)\| = 1$. Therefore, $\|\bar{X}'(s)\| = (x'(s))^2 + (y'(s))^2 = \|X'(s)\| = 1$.

Now, $|\kappa_{\bar{X}}(s)|$ is simply $\|\bar{X}''(s)\| = \|X''(s)\| = |\kappa_X(s)|$. There might be a negation on $\kappa_{\bar{X}}(s)$ from $\kappa_X(s)$ due to the reflection of the curve. □

Problem 9 (Curvature is a geometric object II.). *$X(t) = (x(t), y(t))$ be a regular curve. Let*

$$\kappa(x(t), y(t)) \equiv \kappa(t) = \frac{\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}}{(x'^2 + y'^2)^{\frac{3}{2}}}$$

Let $Y(u) = X(t(u))$, $t'(u) \neq 0$. Discuss the relation of $\kappa(x(t), y(t))$ and $\kappa(x(t(u)), y(t(u)))$ at the corresponding points.

Proof. We denote $\frac{dx}{dt}, \frac{d^2x}{dt^2}, \frac{dy}{dt}, \frac{d^2y}{dt^2}$ by x', x'', y', y'' respectively:

$$\begin{aligned} \kappa(x(t(u)), y(t(u))) = & \kappa(u) = \frac{\begin{vmatrix} x'\frac{dt}{du} & y'\frac{dt}{du} \\ x''\left(\frac{dt}{du}\right)^2 + x'\frac{d^2t}{du^2} & y''\left(\frac{dt}{du}\right)^2 + y'\frac{d^2t}{du^2} \end{vmatrix}}{\left(\left(x'\frac{dt}{du}\right)^2 + \left(y'\frac{dt}{du}\right)^2\right)^{\frac{3}{2}}} \\ & = \frac{\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}(dt/du)^3}{(x'^2 + y'^2)^{\frac{3}{2}}(dt/du)^3} = \kappa(t) \end{aligned}$$

This means that the curvature is never changed at corresponding points when in change of variables. □

Problem 10. *Let $F(x, y) = c$ define a plane curve. Prove that the curvature of the curve satisfies*

$$|\kappa| = \left| \frac{\begin{bmatrix} F_y, & -F_x \end{bmatrix} \begin{bmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \end{bmatrix} \begin{bmatrix} F_y \\ -F_x \end{bmatrix}}{(F_x^2 + F_y^2)^{\frac{3}{2}}} \right|$$

Where $F_x^2 + F_y^2 \neq 0$.

Proof. Let α be a point in the plane such that $F(\alpha) = c$. Consider the circle of curvature passing through α. If we observed the intersection of two line respectively perpendicular to the lines tangent to $F = c$ and passing respectively through α and another point $\alpha' \in F = c$, the intersection approaches the centre of the circle o as $\alpha' \to \alpha$. Thus,

$$|\alpha' - \alpha| = r \sin\theta,$$

where θ is the angle between the vectors $o - \alpha$ and $o - \alpha'$. By the formula of exterior product,

$$\sin\theta = \frac{|(o-\alpha) \times (o-\alpha')|}{|o-\alpha||o-\alpha'|}$$

Let n denote $\nabla F/|\nabla F|$ rotated counterclockwise by $\pi/2$. Since $\alpha' - \alpha$ is perpendicular to $\dfrac{\nabla F(\alpha)}{|\nabla F(\alpha)|}$, and since $o - \alpha$ and $o - \alpha'$ are respectively parallel to $\dfrac{\nabla F(\alpha)}{|\nabla F(\alpha)|}$ and $\dfrac{\nabla F(\alpha')}{|\nabla F(\alpha')|}$, we obtained

$$\begin{aligned}
|\kappa| = 1/r &= \lim_{\alpha' \to \alpha} \frac{|(o-\alpha) \times (o-\alpha')|}{|\alpha'-\alpha||o-\alpha||o-\alpha'|} = \lim_{\alpha' \to \alpha} \frac{|\nabla F(\alpha) \times \nabla F(\alpha')|}{|\alpha'-\alpha||\nabla F(\alpha)||\nabla F(\alpha')|} \\
&= \lim_{t \to 0} \frac{|\nabla F(\alpha) \times \nabla F(\alpha+tn)|}{tn|\nabla F(\alpha)||\nabla F(\alpha+tn)|} \\
&= \lim_{t \to 0} \frac{|(F_x(\alpha)\vec{i} + F_y(\alpha)\vec{j}) \times (F_x(\alpha+tn)\vec{i} + F_y(\alpha+tn)\vec{j})|}{tn|\nabla F(\alpha)|^2} \\
&= \lim_{t \to 0} \frac{|F_x(\alpha)F_y(\alpha+tn) - F_x(\alpha+tn)F_y(\alpha)|}{tn|\nabla F(\alpha)|^2} \\
&= \lim_{t \to 0} \frac{|F_x(\alpha)[F_y(\alpha+tn) - F_y(\alpha)] - [F_x(\alpha+tn) - F_x(\alpha)]F_y(\alpha)|}{tn|\nabla F(\alpha)|^2} \\
&= \frac{|F_x(F_{yx}\vec{i} + F_{yy}\vec{j}) \cdot n - F_y(F_{xx}\vec{i} + F_{xy}\vec{j}) \cdot n|}{|\nabla F|^2} \\
&= \frac{|[(F_xF_{yx} - F_yF_{xx})\vec{i} + (F_xF_{yy} - F_yF_{xy})\vec{j}] \cdot n|}{|\nabla F|^2} \\
&= \frac{|[(F_xF_{yx} - F_yF_{xx})\vec{i} + (F_xF_{yy} - F_yF_{xy})\vec{j}] \cdot (F_y\vec{i} - F_x\vec{j})/|\nabla F||}{|\nabla F|^2} \\
&= \frac{\left| \begin{bmatrix} F_y & -F_x \end{bmatrix} \begin{bmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \end{bmatrix} \begin{bmatrix} F_y \\ -F_x \end{bmatrix} \right|}{|\nabla F|^3} \\
&= \left| \frac{\begin{bmatrix} F_y, & -F_x \end{bmatrix} \begin{bmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \end{bmatrix} \begin{bmatrix} F_y \\ -F_x \end{bmatrix}}{(F_x^2 + F_y^2)^{\frac{3}{2}}} \right|
\end{aligned}$$

$\square$