

## GEOMETRY HOMEWORK 2

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**Problem 3** (P47: 5). *If a closed plane curve  $C$  is contained inside a disk of radius  $r$ , prove that there exists a point  $p \in C$  such that the curvature  $\kappa$  of  $C$  at  $p$  satisfies  $|\kappa| \geq 1/r$ .*

*Proof.* Let  $X(s)$  denote the curve  $C$ , where  $s \in [0, l]$  is an arc-length parameter, that is,  $\|X'(s)\| \equiv 1$ . Since  $C$  is contained inside a disk of radius  $r$ , let  $A$  be the centre of the disk. So we have

$$\|X(s) - A\| \leq r \quad (1)$$

Consider  $f(s) = \langle X(s) - A, X(s) - A \rangle$ . Since  $[0, l]$  is compact, the maximum exists, denoting by  $f(s') = \max_{s \in [0, l]} f(s)$ . Therefore, we have  $f'(s') = 0$  and  $f''(s') \leq 0$ . Now

$$f''(s) = 2 (\|X'(s)\|^2 + \kappa(s) \langle X(s) - A, N(s) \rangle), \quad (2)$$

where  $X''(s) = \kappa(s)N(s)$  and  $N(s)$  is the normal vector. Take  $s = s'$  in (2) we have  $f''(s') \leq 0$  and hence

$$\kappa(s') \langle X(s) - A, N(s) \rangle \leq -1 \quad (3)$$

This implies

$$|\kappa(s') \langle X(s) - A, N(s) \rangle| \geq 1 \quad (4)$$

By (1),  $|\langle X(s) - A, N(s) \rangle| \leq \|X(s) - A\| \cdot \|N(s)\| \leq r$ . We have  $|\kappa(s')| \geq 1/r$  as desired.  $\square$

**Problem 4** (P23: 4, 僅討論平面情形). *Assume that all normals of a parametrized curve pass through a fixed point. Prove that the trace of the curve is contained in a circle.*

*Proof.* Let  $P$  be the fixed point, and let  $X(s)$  be this curve. Then from description,  $\langle X(s) - P, X'(s) \rangle \equiv 0$  for all  $s$ . Let  $f(s) = \|X(s) - P\|^2$ , then we have  $f'(s) = 2 \langle X(s) - P, X'(s) \rangle = 0$  for all  $s$ . This implies the trace of the curve is contained in a circle centered at point  $P$  with radius  $\sqrt{f(s_0)}$  for some  $s_0$ .  $\square$

**Problem 5.** 以  $t = 0$  開始將曲線  $(t^2, t^3)$  化成長度參數。並討論  $t = 0$  時的曲率。

*Proof.* Consider  $t > 0$ , the length of the curve of  $t$  is

$$\int_0^t 3t\sqrt{(4/9)+t^2} dt = \int_0^t \frac{3}{2}\sqrt{(4/9)+t^2} dt^2 = \left(\frac{4}{9}+t^2\right)^{3/2} - \frac{8}{27} \quad (5)$$

Let  $s > 0$  be the arc-length parameter, note that  $s > 0$  equivalent to  $t > 0$ , so we have  $s = (4/9 + t^2)^{3/2} - 8/27$  and hence

$$t = \sqrt{\left(s + \frac{8}{27}\right)^{2/3} - \frac{4}{9}} \quad (6)$$

Therefore the curve with arc-length parameter  $s > 0$  is

$$\left((s + 8/27)^{2/3} - 4/9, \left((s + 8/27)^{2/3} - 4/9\right)^{3/2}\right) \quad (7)$$

By symmetry, for the case  $s < 0$  the corresponding curve is

$$\left((8/27 - s)^{2/3} - 4/9, -\left((8/27 - s)^{2/3} - 4/9\right)^{3/2}\right) \quad (8)$$

We can write them together to get the result,

$$\left((8/27 + |s|)^{2/3} - 4/9, \text{sign}(s) \cdot \left((8/27 + |s|)^{2/3} - 4/9\right)^{3/2}\right) \quad (9)$$

□

**Problem 6.**

(a) 以原點為中心，將  $y = f(x)$  的圖形縮放  $\lambda$  倍，並說明新的圖形是  $y = \lambda f(\frac{x}{\lambda})$  的函數圖形。

(b) 討論曲率的變化。

**Problem 7.** 如圖，有一橢圓，其焦點為  $O_1$  和  $O_2$ ，設  $L$  切橢圓於  $P$ ，且  $L$  與  $\overline{O_2P}$  之夾角為  $\theta$ 。以  $\theta$  為參數，說明曲率  $\kappa \propto \sin^3 \theta$

**Problem 9.** 如圖，有 *regular curve*  $\gamma(t)$ ， $\gamma_0 = \gamma(0)$ ， $N_0 = N(0)$ ， $L_0 = \{\gamma_0 + vN_0\}$ 。現考慮直線  $L_t = \{\gamma(t) + uN(t)\}$ ，令  $P(t) = L_t \cap L_0$  證明

$$\kappa(0) \neq 0 \Rightarrow \lim_{t \rightarrow 0} P(t) = \gamma_0 + \frac{1}{\kappa(0)} N_0$$