

GEOMETRY HOMEWORK 1

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Problem 3 (P7: 4). Let $\alpha : (0, \pi) \rightarrow \mathbf{R}^2$ be given by

$$\alpha(t) = \left(\sin t, \cos t + \log \tan \frac{t}{2} \right),$$

where t is the angle that the y axis makes with the vector $\alpha(t)$. The trace of α is called the tractrix (Fig. 1-9). Show that

- (a) α is a differentiable parametrized curve, regular except at $t = \pi/2$.
- (b) The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

Proof.

- (a) Let $x(t) = \sin t$, $y(t) = \cos t + \log \tan \frac{t}{2}$, then

$$x'(t) = \cos t; \quad y'(t) = -\sin t + \frac{1}{\sin t}.$$

It's trivial that both $x'(t)$ and $y'(t)$ are infinitely differentiable in $(0, \pi)$, so α is a differentiable parametrized curve.

$x'(t) = 0, y'(t) = 0 \iff t = \frac{\pi}{2}$, so α is regular except at $t = \pi/2$.

- (b) The intersection of y axis and the tangent of the tractrix is $\left(0, y(t) - \frac{y'(t)}{x'(t)}x(t) \right)$.

The length of the segment of the tangent of the tractrix between the point

of tangency and the y axis is $\sqrt{x(t)^2 + \left(\frac{y'(t)}{x'(t)}x(t) \right)^2}$

$$\begin{aligned} x(t)^2 + \left(\frac{y'(t)}{x'(t)}x(t) \right)^2 &= \sin^2 t \left(1 + \left(\frac{y'(t)}{x'(t)} \right)^2 \right) \\ &= \sin^2 t \left(1 + \left(\frac{-\sin t + \frac{1}{\sin t}}{\cos t} \right)^2 \right) \\ &= \sin^2 t \left(1 + \left(\frac{1 - \sin^2 t}{\sin t \cos t} \right)^2 \right) \\ &= \sin^2 t \left(\frac{1}{\sin^2 t} \right) \\ &= 1 \end{aligned}$$

So the length of the segment of the tangent of the tractrix between the point of tangency and the y axis $= \sqrt{x(t)^2 + \left(\frac{y'(t)}{x'(t)}x(t)\right)^2} = 1$.

□

Problem 5 (P47: 6). Let $\alpha(s), s \in [0, l]$ be a closed convex plane curve positively oriented. The curve

$$\beta(s) = \alpha(s) - rn(s),$$

where r is a positive constant and n is the normal vector, is called a parallel curve to α (Fig. 1-37). Show that

(a) Length of β = length of α + $2\pi r$.

(b) $A(\beta) = A(\alpha) + rl + \pi r^2$.

(c) $\kappa_\beta(s) = \kappa_\alpha(s)/(1 + r\kappa_\alpha(s))$.

For (a)-(c), $A(\cdot)$ denotes the area bounded by the corresponding curve, and $\kappa_\alpha, \kappa_\beta$ are the curvatures of α and β , respectively.

Proof.

(a) Since α is a closed convex plane curve, by the theorem of turning tangents (P.37), we have

$$\int_0^l \kappa(s) ds = 2\pi$$

Moreover, $\kappa(s)$ and r are both positive by definition, so $r\kappa(s)/\|\alpha'(s)\|$ is always non-negative. Therefore, length of β is

$$\int_0^l \|\beta'(s)\| ds = \int_0^l \|\alpha'(s) - rn'(s)\| ds.$$

By definition of normal vector n , $n'(s) = -\kappa(s)\alpha'(s)/\|\alpha'(s)\|$. Therefore, the length of β equals

$$\begin{aligned} \int_0^l \|\beta'(s)\| ds &= \int_0^l \left\| \alpha'(s) + r\kappa(s) \frac{\alpha'(s)}{\|\alpha'(s)\|} \right\| ds \\ &= \int_0^l \left\| \left(1 + \frac{r\kappa(s)}{\|\alpha'(s)\|}\right) \alpha'(s) \right\| ds \\ &= \int_0^l \left(1 + \frac{r\kappa(s)}{\|\alpha'(s)\|}\right) \cdot \|\alpha'(s)\| ds \\ &= \int_0^l \|\alpha'(s)\| ds + r \int_0^l \kappa(s) ds \\ &= \text{length of } \alpha + 2\pi r. \end{aligned}$$

- (b) By using the result of (c), we knew that κ_β is always positive, so β is a convex closed curve and hence simple.

Let D and D' denote the region bounded by α and β . By Green's theorem,

$$\iint_D dx dy = \frac{1}{2} \left\| \oint_0^l \alpha \times \alpha' ds \right\|_z; \quad \iint_{D'} dx dy = \frac{1}{2} \left\| \oint_0^l \beta \times \beta' ds \right\|_z$$

, where $\|\cdot\|_z$ denotes the z -component of a vector, and then

$$\begin{aligned} \oint_0^l \beta \times \beta' ds &= \oint_0^l (\alpha - rn) \times (\alpha - rn)' ds \\ &= \oint_0^l \alpha \times \alpha' ds + r^2 \oint_0^l n \times n' ds + r \oint_0^l (\alpha' \times n - \alpha \times n') ds \\ &= \oint_0^l \alpha \times \alpha' ds - r^2 \oint_0^l n \times (-\kappa \alpha' / \|\alpha'\|) ds \\ &\quad - r \oint_0^l (\alpha \times n)' ds + 2r \oint_0^l \alpha' \times n ds \\ &= \oint_0^l \alpha \times \alpha' ds - r^2 \oint_0^l n \times (-\kappa \alpha' / \|\alpha'\|) ds \\ &\quad - (\alpha \times n)|_{s=0}^l + 2r \oint_0^l \alpha' \times n ds. \end{aligned}$$

Since the third term of the last value is zero and that $\alpha' \times n = \|\alpha'\| e_z$,

$$\begin{aligned} A(\beta) &= \left\| \frac{1}{2} \oint_0^l \alpha \times \alpha' ds - \frac{r^2}{2} \oint_0^l n \times (-\kappa \alpha' / \|\alpha'\|) ds + r \oint_0^l \alpha' \times n ds \right\|_z \\ &= \left| A(\alpha) + \frac{r^2}{2} \oint_0^l \kappa ds + r \oint_0^l \|\alpha'\| ds \right| = A(\alpha) + \pi r^2 + rl. \end{aligned}$$

(c)

$$\begin{aligned} \beta'(s) &= \alpha'(s) - rn'(s) \\ &= t(s) + r\kappa_\alpha(s)t(s) \\ &= (1 + r\kappa_\alpha(s))t(s) \\ \beta''(s) &= (1 + r\kappa_\alpha(s))\kappa_\alpha n(s) + r\kappa'_\alpha(s)t(s) \\ \kappa_\beta(s) &= \frac{\begin{vmatrix} 1 + r\kappa_\alpha(s) & 0 \\ r\kappa'_\alpha(s) & (1 + r\kappa_\alpha(s))\kappa_\alpha \end{vmatrix}}{(1 + r\kappa_\alpha(s))^3} \\ &= \frac{(1 + r\kappa_\alpha(s))(1 + r\kappa_\alpha(s))\kappa_\alpha}{(1 + r\kappa_\alpha(s))^3} \\ &= \frac{\kappa_\alpha}{1 + r\kappa_\alpha(s)} \end{aligned}$$

□

Problem 8 (Curvature is a geometric object I.). $X(s) = (x(s), y(s))$, where s is the arc-length parameter.

$$M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, M^t = M^{-1}, \text{ i.e. } M \text{ is orthogonal.}$$

Let $\bar{X}(s) = M \cdot \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, $\alpha, \beta \in \mathbf{R}$. What is the relation between $\kappa_X(s)$ and $\kappa_{\bar{X}}(s)$?

Proof. We first claim that s is also the arc-length parameter for \bar{X} . This is because $\|\bar{X}'(s)\| = \|(a_{11}x'(s) + a_{12}y'(s), a_{21}x'(s) + a_{22}y'(s))\| = (a_{11}^2 + a_{12}^2)(x'(s))^2 + (a_{21}^2 + a_{22}^2)(y'(s))^2$. Since M is orthogonal, we have $a_{11}^2 + a_{12}^2 = 1 = a_{21}^2 + a_{22}^2$ and since s is the arc-length parameter of X , $\|X'(s)\| = 1$. Therefore, $\|\bar{X}'(s)\| = (x'(s))^2 + (y'(s))^2 = \|X'(s)\| = 1$.

Now, $|\kappa_{\bar{X}}(s)|$ is simply $\|\bar{X}''(s)\| = \|X''(s)\| = |\kappa_X(s)|$. There might be a negation on $\kappa_{\bar{X}}(s)$ from $\kappa_X(s)$ due to the reflection of the curve. \square

Problem 9 (Curvature is a geometric object II.). $X(t) = (x(t), y(t))$ be a regular curve. Let

$$\kappa(x(t), y(t)) \equiv \kappa(t) = \frac{\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}}{(x'^2 + y'^2)^{\frac{3}{2}}}$$

Let $Y(u) = X(t(u))$, $t'(u) \neq 0$. Discuss the relation of $\kappa(x(t), y(t))$ and $\kappa(x(t(u)), y(t(u)))$ at the corresponding points.

Proof. We denote $\frac{dx}{dt}, \frac{d^2x}{dt^2}, \frac{dy}{dt}, \frac{d^2y}{dt^2}$ by x', x'', y', y'' respectively:

$$\begin{aligned} \kappa(x(t(u)), y(t(u))) &= \kappa(u) = \frac{\begin{vmatrix} x' \frac{dt}{du} & y' \frac{dt}{du} \\ x'' \left(\frac{dt}{du}\right)^2 + x' \frac{d^2t}{du^2} & y'' \left(\frac{dt}{du}\right)^2 + y' \frac{d^2t}{du^2} \end{vmatrix}}{\left(\left(x' \frac{dt}{du}\right)^2 + \left(y' \frac{dt}{du}\right)^2\right)^{\frac{3}{2}}} \\ &= \frac{\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix} (dt/du)^3}{(x'^2 + y'^2)^{\frac{3}{2}} (dt/du)^3} = \kappa(t) \end{aligned}$$

This means that the curvature is never changed at corresponding points when in change of variables. \square

Problem 10. Let $F(x, y) = c$ define a plane curve. Prove that the curvature of the curve satisfies

$$|\kappa| = \frac{\begin{vmatrix} F_y & -F_x \\ F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \end{vmatrix} \begin{bmatrix} F_y \\ -F_x \end{bmatrix}}{(F_x^2 + F_y^2)^{\frac{3}{2}}}$$

Where $F_x^2 + F_y^2 \neq 0$.

Proof. Let α be a point in the plane such that $F(\alpha) = c$. Consider the circle of curvature passing through α . If we observed the intersection of two line respectively perpendicular to the lines tangent to $F = c$ and passing respectively through α and another point $\alpha' \in F = c$, the intersection approaches the centre of the circle o as $\alpha' \rightarrow \alpha$. Thus,

$$|\alpha' - \alpha| = r \sin \theta,$$

where θ is the angle between the vectors $o - \alpha$ and $o - \alpha'$. By the formula of exterior product,

$$\sin \theta = \frac{|(o - \alpha) \times (o - \alpha')|}{|o - \alpha||o - \alpha'|}$$

Let n denote $\nabla F / |\nabla F|$ rotated counterclockwise by $\pi/2$. Since $\alpha' - \alpha$ is perpendicular to $\frac{\nabla F(\alpha)}{|\nabla F(\alpha)|}$, and since $o - \alpha$ and $o - \alpha'$ are respectively parallel to $\frac{\nabla F(\alpha)}{|\nabla F(\alpha)|}$ and $\frac{\nabla F(\alpha')}{|\nabla F(\alpha')|}$, we obtained

$$\begin{aligned} |\kappa| = 1/r &= \lim_{\alpha' \rightarrow \alpha} \frac{|(o - \alpha) \times (o - \alpha')|}{|\alpha' - \alpha||o - \alpha||o - \alpha'|} = \lim_{\alpha' \rightarrow \alpha} \frac{|\nabla F(\alpha) \times \nabla F(\alpha')|}{|\alpha' - \alpha||\nabla F(\alpha)||\nabla F(\alpha')|} \\ &= \lim_{t \rightarrow 0} \frac{|\nabla F(\alpha) \times \nabla F(\alpha + tn)|}{tn|\nabla F(\alpha)||\nabla F(\alpha + tn)|} \\ &= \lim_{t \rightarrow 0} \frac{|(F_x(\alpha)\vec{i} + F_y(\alpha)\vec{j}) \times (F_x(\alpha + tn)\vec{i} + F_y(\alpha + tn)\vec{j})|}{tn|\nabla F(\alpha)|^2} \\ &= \lim_{t \rightarrow 0} \frac{|F_x(\alpha)F_y(\alpha + tn) - F_x(\alpha + tn)F_y(\alpha)|}{tn|\nabla F(\alpha)|^2} \\ &= \lim_{t \rightarrow 0} \frac{|F_x(\alpha)[F_y(\alpha + tn) - F_y(\alpha)] - [F_x(\alpha + tn) - F_x(\alpha)]F_y(\alpha)|}{tn|\nabla F(\alpha)|^2} \\ &= \frac{|F_x(F_{yx}\vec{i} + F_{yy}\vec{j}) \cdot n - F_y(F_{xx}\vec{i} + F_{xy}\vec{j}) \cdot n|}{|\nabla F|^2} \\ &= \frac{|[(F_x F_{yx} - F_y F_{xx})\vec{i} + (F_x F_{yy} - F_y F_{xy})\vec{j}] \cdot n|}{|\nabla F|^2} \\ &= \frac{|[(F_x F_{yx} - F_y F_{xx})\vec{i} + (F_x F_{yy} - F_y F_{xy})\vec{j}] \cdot (F_y\vec{i} - F_x\vec{j})|}{|\nabla F|^2} \\ &= \frac{\left| \begin{bmatrix} F_y & -F_x \end{bmatrix} \begin{bmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \end{bmatrix} \begin{bmatrix} F_y \\ -F_x \end{bmatrix} \right|}{|\nabla F|^3} \\ &= \frac{\left| \begin{bmatrix} F_y & -F_x \end{bmatrix} \begin{bmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \end{bmatrix} \begin{bmatrix} F_y \\ -F_x \end{bmatrix} \right|}{(F_x^2 + F_y^2)^{\frac{3}{2}}} \end{aligned}$$

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