Geometry Homework 7

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Problem 2. 若 $F(x, y, z) = 0$ 定義一 *surface*，證明 $\nabla F \neq 0$ 的地方 *Gauss curvature* $K = \frac{\nabla F^t A \nabla F}{\|\nabla F\|^4}$。其中 A 為 $\partial^2 F = \begin{pmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{pmatrix}$ 的 *adjoint Matrix, i.e.* $A = \det(\partial^2 F)(\partial^2 F)^{-1}$

Proof. 因為 K 為局部性質，而在 $\nabla F \neq 0$ 的地方我們可以使用隱函數定理將其中一維表示為另兩維的函數，WLOG 不妨設 $z = z(x, y)$ 在某點附近。

$$\begin{aligned}
F(x, y, z(x, y)) &= 0 \\
\mathbb{X}(x, y) &= (x, y, z(x, y)) \\
\to \mathbb{X}_x &= (1, 0, z_x) \\
\mathbb{X}_y &= (0, 1, z_y) \\
\to N &= \frac{\mathbb{X}_x \times \mathbb{X}_y}{|\mathbb{X}_x \times \mathbb{X}_y|} \\
&= \frac{(-z_x, -z_y, 1)}{\sqrt{1 + z_x^2 + z_y^2}}
\end{aligned}$$

$$\begin{aligned}
E &= \langle \mathbb{X}_x, \mathbb{X}_x \rangle \\
&= 1 + z_x^2 \\
F &= \langle \mathbb{X}_x, \mathbb{X}_y \rangle \\
&= z_x z_y \\
G &= \langle \mathbb{X}_y, \mathbb{X}_y \rangle \\
&= 1 + z_y^2
\end{aligned}$$

$$
\begin{aligned}
\mathbb{X}_{xx} &= (0, 0, z_{xx}) \\
\mathbb{X}_{xy} &= (0, 0, z_{xy}) \\
\mathbb{X}_{yy} &= (0, 0, z_{yy}) \\
\rightarrow e &= \langle N, \mathbb{X}_{xx} \rangle \\
&= \frac{z_{xx}}{\sqrt{1 + z_x^2 + z_y^2}} \\
f &= \langle N, \mathbb{X}_{xy} \rangle \\
&= \frac{z_{xy}}{\sqrt{1 + z_x^2 + z_y^2}} \\
g &= \langle N, \mathbb{X}_{yy} \rangle \\
&= \frac{z_{yy}}{\sqrt{1 + z_x^2 + z_y^2}} \\
\rightarrow K &= det([-dN]) \\
&= det\left(\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \right) \\
&= det\left(\begin{bmatrix} E & F \\ F & G \end{bmatrix} \right)^{-1} \det\left(\begin{bmatrix} e & f \\ f & g \end{bmatrix} \right) \\
&= \frac{z_{xx} z_{yy} - z_{xy}^2}{(1 + z_x^2 + z_y^2)^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial F(x, y, z)}{\partial x} &= 0 \\
&= F_x(x, y, z) + F_z(x, y, z) z_x \\
\rightarrow z_x &= -\frac{F_x}{F_z} \\
\frac{\partial F(x, y, z)}{\partial y} &= 0 \\
&= F_y(x, y, z) + F_z(x, y, z) z_y \\
\rightarrow z_y &= -\frac{F_y}{F_z}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 F(x,y,z)}{\partial x^2} &= 0 \\
&= \frac{\partial}{\partial x}(F_x(x,y,z) + F_z(x,y,z)z_x) \\
&= F_{xx}(x,y,z) + 2F_{xz}(x,y,z)z_x + F_{zz}(x,y,z)z_x^2 + F_z(x,y,z)z_{xx} \\
\to z_{xx} &= -\frac{F_{xx} - 2F_{xz}\frac{F_x}{F_z} + F_{zz}\left(\frac{F_x}{F_z}\right)^2}{F_z} \\
\frac{\partial^2 F(x,y,z)}{\partial y^2} &= 0 \\
&= \frac{\partial}{\partial y}(F_y(x,y,z) + F_z(x,y,z)z_y) \\
&= F_{yy}(x,y,z) + 2F_{yz}(x,y,z)z_y + F_{zz}(x,y,z)z_y^2 + F_z(x,y,z)z_{yy} \\
\to z_{yy} &= -\frac{F_{yy} - 2F_{yz}\frac{F_y}{F_z} + F_{zz}\left(\frac{F_y}{F_z}\right)^2}{F_z} \\
\frac{\partial^2 F(x,y,z)}{\partial x \partial y} &= 0 \\
&= \frac{\partial}{\partial x}(F_y(x,y,z) + F_z(x,y,z)z_y) \\
&= F_{xy}(x,y,z) + F_{yz}(x,y,z)z_x + F_{xz}(x,y,z)z_y + F_{zz}(x,y,z)z_x z_y + F_z(x,y,z)z_{xy} \\
\to z_{xy} &= -\frac{F_{xy} - F_{yz}\frac{F_x}{F_z} - F_{xz}\frac{F_y}{F_z} + F_{zz}\frac{F_y F_x}{F_z^2}}{F_z}
\end{aligned}
$$

$$
\begin{aligned}
\to K &= \frac{z_{xx}z_{yy} - z_{xy}^2}{(1 + z_x^2 + z_y^2)^2} \\
&= \frac{\left(\frac{F_{xx} - 2F_{xz}\frac{F_x}{F_z} + F_{zz}\left(\frac{F_x}{F_z}\right)^2}{F_z}\right)\left(\frac{F_{yy} - 2F_{yz}\frac{F_y}{F_z} + F_{zz}\left(\frac{F_y}{F_z}\right)^2}{F_z}\right) - \left(\frac{F_{xy} - F_{yz}\frac{F_x}{F_z} - F_{xz}\frac{F_y}{F_z} + F_{zz}\frac{F_yF_x}{F_z^2}}{F_z}\right)^2}{(1 + \left(\frac{F_x}{F_z}\right)^2 + \left(\frac{F_y}{F_z}\right)^2)^2} \\
&= \frac{1}{F_z^2(F_x^2 + F_y^2 + F_z^2)^2}(\left(F_{xx}F_z^2 - 2F_{xz}F_xF_z + F_{zz}F_x^2\right)\left(F_{yy}F_z^2 - 2F_{yz}F_yF_z + F_{zz}F_y^2\right) \\
&\quad - \left(F_{xy}F_z^2 - F_{yz}F_xF_z - F_{xz}F_yF_z + F_{zz}F_xF_y\right)^2) \\
&= \frac{1}{F_z^2(F_x^2 + F_y^2 + F_z^2)^2}(F_{xx}F_{yy}F_z^4 - 2F_{xz}F_{yy}F_xF_z^3 + F_{zz}F_{yy}F_z^2F_x^2 - 2F_{xx}F_{yz}F_yF_z^3 \\
&\quad + 4F_{xz}F_{yz}F_xF_yF_z^2 - 2F_{zz}F_{yz}F_yF_zF_x^2 + F_{xx}F_{zz}F_y^2F_z^2 - 2F_{xz}F_{zz}F_y^2F_xF_z + F_{zz}^2F_y^2F_x^2 \\
&\quad - F_{xy}^2F_z^4 - F_{yz}^2F_x^2F_z^2 - F_{xz}^2F_y^2F_z^2 - F_{zz}^2F_x^2F_y^2 + 2F_{xy}F_{yz}F_xF_z^3 + 2F_{xy}F_{xz}F_yF_z^3 \\
&\quad - 2F_{xy}F_{zz}F_xF_yF_z^2 - 2F_{yz}F_{xz}F_yF_xF_z^2 + 2F_{yz}F_{zz}F_yF_x^2F_z + 2F_{xz}F_{zz}F_xF_y^2F_z) \\
&= \frac{1}{(F_x^2 + F_y^2 + F_z^2)^2}(F_{xx}F_{yy}F_z^2 - 2F_{xz}F_{yy}F_xF_z + F_{zz}F_{yy}F_x^2 - 2F_{xx}F_{yz}F_yF_z \\
&\quad + 2F_{xz}F_{yz}F_xF_y + F_{xx}F_{zz}F_y^2 - F_{xy}^2F_z^2 - F_{yz}^2F_x^2 - F_{xz}^2F_y^2 + 2F_{xy}F_{yz}F_xF_z \\
&\quad + 2F_{xy}F_{xz}F_yF_z - 2F_{xy}F_{zz}F_xF_y) \\
&= \frac{1}{(F_x^2 + F_y^2 + F_z^2)^2}((F_{zz}F_{yy} - F_{yz}^2)F_x^2 + (F_{xx}F_{zz} - F_{xz}^2)F_y^2 + (F_{xx}F_{yy} - F_{xy}^2)F_z^2 \\
&\quad + 2(F_{xy}F_{yz} - F_{xz}F_{yy})F_xF_z + 2(F_{xy}F_{xz} - F_{xx}F_{yz})F_yF_z \\
&\quad + 2(F_{xz}F_{yz} - F_{xy}F_{zz})F_xF_y) \\
&= \frac{\nabla F^t A \nabla F}{\|\nabla F\|^4}
\end{aligned}
$$

□

Problem 3 (Ex P168 4). *Determine the asymptotic curves and the lines of curvature of $z = xy$.*

Proof.

$$
\begin{aligned}
\mathbb{X} &= (u, v, uv) \\
\to \mathbb{X}_u &= (1, 0, v) \\
\mathbb{X}_v &= (0, 1, u) \\
E &= \langle \mathbb{X}_u, \mathbb{X}_u \rangle = 1 + v^2 \\
F &= \langle \mathbb{X}_u, \mathbb{X}_v \rangle = uv \\
G &= \langle \mathbb{X}_v, \mathbb{X}_v \rangle = 1 + u^2
\end{aligned}
$$

$$
\begin{aligned}
N &= \frac{\mathbb{X}_u \times \mathbb{X}_v}{|\mathbb{X}_u \times \mathbb{X}_v|} \\
&= \frac{1}{\sqrt{1+u^2+v^2}}(-v,-u,1) \\
\mathbb{X}_{uu} &= (0,0,0) \\
\mathbb{X}_{uv} &= (0,0,1) \\
\mathbb{X}_{vv} &= (0,0,0) \\
\to e &= \langle N, \mathbb{X}_{uu} \rangle = 0 \\
f &= \langle N, \mathbb{X}_{uv} \rangle = \frac{1}{\sqrt{1+u^2+v^2}} \\
g &= \langle N, \mathbb{X}_{vv} \rangle = 0
\end{aligned}
$$

Asymptotic curves:

$$
\begin{aligned}
eu'^2 + 2fu'v' + gv'^2 &= 0 \\
\to u'v' &= 0 \\
\to u = \text{const or } v &= \text{const}
\end{aligned}
$$

line of curvature:

$$
\begin{aligned}
\begin{vmatrix} v'^2 & -u'v' & u'^2 \\ 1+v^2 & uv & 1+u^2 \\ 0 & f & 0 \end{vmatrix} &= 0 \\
\to (1+u^2)v'^2 &= (1+v^2)u'^2 \\
\to \int \frac{1}{\sqrt{1+v^2}} dv &= \pm \int \frac{1}{\sqrt{1+u^2}} du \\
\to \sinh^{-1} v &= \pm \sinh^{-1} u + C \\
\to v &= \pm u \cosh C + \sqrt{1+u^2} \sinh C
\end{aligned}
$$

□

Problem 4. 已知 $\mathbb{X}(u,v)$ 為一 *surface* $\subset \mathbb{R}^3$ 且 $E = G = (1+u^2+v^2)^2, F = 0$ 而且 $e = 1, f = \sqrt{3}, g = -1$

(a) 求在 $\mathbb{X}(1,1)$ 的 K 與 H

(b) 如何決定過 $\mathbb{X}(1,1)$ 的 *line of curvature* 與 *asymptotic curve*（如果有的話）

Proof. (a) at $(1,1)$, $E = G = 9, F = 0$.

$$
\begin{aligned}
[-dN] &= \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \\
&= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{9} & \frac{1}{3\sqrt{3}} \\ \frac{1}{3\sqrt{3}} & -\frac{1}{9} \end{bmatrix}
\end{aligned}
$$

So $K = \det([-dN]) = -\frac{4}{81}$, $H = \operatorname{tr}([-dN]) = 0$.

(b) line of curvature:

$$
\begin{vmatrix} v'^2 & -u'v' & u'^2 \\ (1+u^2+v^2)^2 & 0 & (1+u^2+v^2)^2 \\ 1 & \sqrt{3} & -1 \end{vmatrix} = 0
$$

$$
\begin{aligned}
&\to (1+u^2+v^2)^2(-\sqrt{3}v'^2 - 2u'v' + \sqrt{3}u'^2) = 0 \\
&\to (-\sqrt{3}v' + u')(v' + \sqrt{3}u') = 0 \\
&\to -\sqrt{3}v + u = \text{const or } v + \sqrt{3}u = \text{const} \to -\sqrt{3}v + u \quad = 1 - \sqrt{3} \text{ or } v -
\end{aligned}
$$

asymptotic curve:

$$
\begin{aligned}
&u'^2 + 2\sqrt{3}u'v' - v'^2 = 0 \\
&\to (u' + (\sqrt{3}-2)v')(u' + (\sqrt{3}+2)v') = 0 \\
&\to u' + (\sqrt{3}-2)v' = 0 \text{ or } u' + (\sqrt{3}+2)v' = 0 \\
&\to u + ((\sqrt{3}-2)v = \text{const or } u + (\sqrt{3}+2)v = \text{const} \\
&\to u + ((\sqrt{3}-2)v = \sqrt{3} - 1 \text{ or } u + (\sqrt{3}+2)v = \sqrt{3} + 3
\end{aligned}
$$

□

Problem 5. $\mathbb{X}(u,v) = (v\cos u, v\sin u, u)$，令 $\gamma(t) = \mathbb{X}(t,1)$

(a) 求 $\gamma(t)$ 的 $\kappa_n, \kappa_g, \tau_g$

(b) 與 $\gamma(t)$ 的 κ, τ 有何關係

Proof. □

Problem 6. 令 $(x(t), y(t)) = (t - \tanh\, t, \operatorname{sech}\, t)$ 這基本就是 *p7(4)* 的 *tratrix*

(a) 將此曲線化作長度參數

(b) 利用上小題，求此曲線繞 x 軸旋轉的旋轉體的 K

Proof. □