Geometry Homework 9

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Problem 4 (Ex p.101 14). *(Gradient on Surfaces.) The gradient of a differentiable function $f : S \mapsto \mathbb{R}$ is a differentiable map grad $f : S \mapsto \mathbb{R}^3$ which assigns to each point $p \in S$ a vector grad $f(p) \in T_p(S) \subset \mathbb{R}^3$ such that*

$$\langle \textit{grad } f(p), v\rangle_p = df_p(v) \qquad \textit{for all } v \in T_p(S)$$

Show that

(a) *If E, F, G are the coefficients of the first fundamental form in a parametrization $\mathbf{X} : U \subset \mathbb{R}^2 \mapsto S$, then grad f on $\mathbf{X}(U)$ is given by*

$$\textit{grad } f = \frac{f_u G - f_v F}{EG - F^2}\mathbf{X}_u + \frac{f_v E - f_u F}{EG - F^2}\mathbf{X}_v$$

In particular, if $S = \mathbb{R}^2$ with coordinates x, y,

$$\textit{grad } f = f_x e_1 + f_y e_2$$

where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$ (thus, the definition agrees with the usual definition of gradient in the plane)

(b) 為什麼不直接將 *gradient f* 定義成 $f_u\mathbb{X}_u + f_v\mathbb{X}_v$，這有什麼缺點 *(*例如座標變換*)*

Proof. (a) First,

$$\langle \text{grad } f(p), \mathbf{X}_u\rangle_p = df_p(\mathbf{X}_u) = f_u$$
$$\langle \text{grad } f(p), \mathbf{X}_v\rangle_p = df_p(\mathbf{X}_v) = f_v$$

Let grad $f = q\mathbf{X}_u + r\mathbf{X}_v$. Then

$$\langle \text{grad } f(p), \mathbf{X}_u\rangle = Eq + Fr = f_u$$
$$\langle \text{grad } f(p), \mathbf{X}_v\rangle = Fq + Gr = f_v$$

Therefore, solve the linear equations and get

$$q = \frac{f_u G - f_u F}{EG - F^2};$$
$$r = \frac{f_v E - f_u F}{EG - F^2}$$

Then the two results follow immediately.

(b) If we define the gradient in that way, let $S = \mathbb{R}^2$ be the surface and $\mathbf{X}(u, v) = (u, v)$, $\mathbf{Y}(s, t) = (s, s + t)$ be its two parametrizations. If $f(u, v) = v$, then $f(s, t) = s + t$ and therefore grad $f = \mathbf{X}_v = \mathbf{Y}_s + \mathbf{Y}_t$. But clearly $\mathbf{X}_v = (0, 1) \neq (1, 2) = \mathbf{Y}_s + \mathbf{Y}_t$, which is a contradiction.

□

Problem 7. 計算下列 *surface* 的 Γ_{ij}^k *(*共有六項*)*

(b) $(x(t), y(t)\cos\theta, y(t)\sin\theta)$

(c) $E = G = \lambda^2, F = 0$

Proof. (b) Let $u = t, v = \theta$.

$$\begin{aligned}
&\mathbb{X}_u = (x_u, y_u \cos v, y_u \sin v), \mathbb{X}_v = (0, -y \sin v, y \cos v) \\
&\to E = x_u^2 + y_u^2, F = 0, G = y^2 \\
&[1,1,1] = \frac{E_u}{2}, [1,1,2] = -\frac{E_v}{2}, [1,2,1] = \frac{E_v}{2} \\
&[1,2,2] = \frac{G_u}{2}, [2,2,1] = -\frac{G_u}{2}, [2,2,2] = \frac{G_v}{2} \\
&\Gamma_{11}^1 = \frac{E_u}{2E} = \frac{x_u x_{uu} + y_u y_{uu}}{x_u^2 + y_u^2} \\
&\Gamma_{11}^2 = -\frac{E_v}{2G} = 0 \\
&\Gamma_{12}^1 = \frac{E_v}{2E} = 0 \\
&\Gamma_{12}^2 = \frac{G_u}{2G} = \frac{y_u}{y} \\
&\Gamma_{22}^1 = -\frac{G_u}{2E} = \frac{y y_u}{x_u^2 + y_u^2} \\
&\Gamma_{22}^2 = \frac{G_v}{2G} = 0
\end{aligned}$$

(c)

$$
\begin{aligned}
&g^{11} = \frac{1}{\lambda^2}, g^{22} = \frac{1}{\lambda^2}, g^{12} = g^{21} = 0 \\
&[1,1,1] = \frac{E_u}{2}, [1,1,2] = -\frac{E_v}{2}, [1,2,1] = \frac{E_v}{2} \\
&[1,2,2] = \frac{G_u}{2}, [2,2,1] = -\frac{G_u}{2}, [2,2,2] = \frac{G_v}{2} \\
&\Gamma^1_{11} = \frac{E_u}{2E} = \frac{\lambda_u}{\lambda} \\
&\Gamma^2_{11} = -\frac{E_v}{2G} = \frac{\lambda_v}{\lambda} \\
&\Gamma^1_{12} = \frac{E_v}{2E} = \frac{\lambda_v}{\lambda} \\
&\Gamma^2_{12} = \frac{G_u}{2G} = \frac{\lambda_u}{\lambda} \\
&\Gamma^1_{22} = -\frac{G_u}{2E} = \frac{\lambda_u}{\lambda} \\
&\Gamma^2_{22} = \frac{G_v}{2G} = \frac{\lambda_v}{\lambda}
\end{aligned}
$$

□

Problem 8 (Ex p.237 1, 2). *(a) Show that if* $\mathbf{X}$ *is an orthogonal parametrization, that is,* $F = 0$*, then*

$$K = -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\}$$

(b) Show that if $\mathbf{X}$ *is an isothermal parametrization, that is,* $E = G = \lambda(u,v)$ *and* $F = 0$*, then*

$$K = -\frac{1}{2\lambda}\Delta(\log\lambda)$$

where $\Delta\phi$ *denotes the Laplacian* $(\partial^2\phi/\partial u^2)+(\partial^2\phi/\partial v^2)$ *of the function* ϕ*. Conclude that when* $E = G = (u^2+v^2+c)^{-2}$ *and* $F = 0$*, then* $K = const. = 4c$*.*

Proof. (a)

$$
\begin{aligned}
&g^{11} = \frac{1}{E}, g^{22} = \frac{1}{G}, g^{12} = g^{21} = 0 \\
&[1,1,1] = \frac{E_u}{2}, [1,1,2] = -\frac{E_v}{2}, [1,2,1] = \frac{E_v}{2} \\
&[1,2,2] = \frac{G_u}{2}, [2,2,1] = -\frac{G_u}{2}, [2,2,2] = \frac{G_v}{2} \\
&\Gamma^1_{11} = \frac{E_u}{2E}, \Gamma^2_{11} = -\frac{E_v}{2G} \\
&\Gamma^1_{12} = \frac{E_v}{2E}, \Gamma^2_{12} = \frac{G_u}{2G} \\
&\Gamma^1_{22} = -\frac{G_u}{2E}, \Gamma^2_{22} = \frac{G_v}{2G}
\end{aligned}
$$

$$
\begin{aligned}
R^2_{112} &= \Gamma^2_{11,2} - \Gamma^2_{12,1} + \Gamma^1_{11}\Gamma^2_{21} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{12} \\
&= -\left(\frac{E_v}{2G}\right)_v - \left(\frac{G_u}{2G}\right)_u + \frac{E_u}{2E}\frac{G_u}{2G} - \frac{E_v}{2G}\frac{G_v}{2G} + \frac{E_v}{2E}\frac{E_v}{2G} - \frac{G_u}{2G}\frac{G_u}{2G} \\
&= -\frac{GE_{vv} - G_vE_v}{2G^2} - \frac{GG_{uu} - G_uG_u}{2G^2} + \frac{E_u}{2E}\frac{G_u}{2G} - \frac{E_v}{2G}\frac{G_v}{2G} + \frac{E_v}{2E}\frac{E_v}{2G} - \frac{G_u}{2G}\frac{G_u}{2G} \\
&= \frac{1}{4G^2}\left(-2GE_{vv} + G_vE_v - 2GG_{uu} + G_uG_u + \frac{GE_u}{E}G_u + \frac{GE_v}{E}E_v\right) \\
K &= \frac{R_{1212}}{EG} \\
&= \frac{R^2_{112}}{E} \\
&= \frac{1}{4EG^2}\left(-2GE_{vv} + G_vE_v - 2GG_{uu} + G_uG_u + \frac{GE_u}{E}G_u + \frac{GE_v}{E}E_v\right) \\
&= \frac{1}{4}\left(-2\frac{E_{vv}}{EG} - 2\frac{G_{uu}}{EG} + \frac{E_vG_v}{EG^2} + \frac{E_uG_u}{E^2G} + \frac{G_uG_u}{EG^2} + \frac{E_vE_v}{E^2G}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\} \\
&= -\frac{1}{2\sqrt{EG}}\left\{\frac{\sqrt{EG}E_{vv} - \frac{E_vG+EG_v}{2\sqrt{EG}}E_v}{EG} + \frac{\sqrt{EG}G_{uu} - \frac{E_uG+EG_u}{2\sqrt{EG}}G_u}{EG}\right\} \\
&= -\frac{1}{2}\left\{\frac{E_{vv} - \frac{E_vG+EG_v}{2EG}E_v}{EG} + \frac{G_{uu} - \frac{E_uG+EG_u}{2EG}G_u}{EG}\right\} \\
&= \frac{1}{4}\left(-2\frac{E_{vv}}{EG} - 2\frac{G_{uu}}{EG} + \frac{E_vG_v}{EG^2} + \frac{E_uG_u}{E^2G} + \frac{G_uG_u}{EG^2} + \frac{E_vE_v}{E^2G}\right)
\end{aligned}
$$

So $K = -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\}$.

(b)

$$
\begin{aligned}
K &= -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\} \\
&= -\frac{1}{2\lambda}\left\{\left(\frac{\lambda_v}{\lambda}\right)_v + \left(\frac{\lambda_u}{\lambda}\right)_u\right\} \\
\Delta(\log\lambda) &= (\log\lambda)_{uu} + (\log\lambda)_{vv} \\
&= \left(\frac{\lambda_u}{\lambda}\right)_u + \left(\frac{\lambda_v}{\lambda}\right)_v \\
\to K &= -\frac{1}{2\lambda}\Delta(\log\lambda)
\end{aligned}
$$

$\square$