

# Geometry Homework 1

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**Problem 3** (P7: 4). Let  $\alpha : (0, \pi) \rightarrow \mathbf{R}^2$  be given by

$$\alpha(t) = \left( \sin t, \cos t + \log \tan \frac{t}{2} \right),$$

where  $t$  is the angle that the  $y$  axis makes with the vector  $\alpha(t)$ . The trace of  $\alpha$  is called the tractrix (Fig. 1-9). Show that

- (a)  $\alpha$  is a differentiable parametrized curve, regular except at  $t = \pi/2$ .
- (b) The length of the segment of the tangent of the tractrix between the point of tangency and the  $y$  axis is constantly equal to 1.

*Proof.*

- (a) Let  $x(t) = \sin t$ ,  $y(t) = \cos t + \log \tan \frac{t}{2}$ , then

$$x'(t) = \cos t; \quad y'(t) = -\sin t + \frac{1}{\sin t}.$$

It's trivial that both  $x'(t)$  and  $y'(t)$  are infinitely differentiable in  $(0, \pi)$ , so  $\alpha$  is a differentiable parametrized curve.

$x'(t) = 0, y'(t) = 0 \iff t = \frac{\pi}{2}$ , so  $\alpha$  is regular except at  $t = \pi/2$ .

- (b) The intersection of  $y$  axis and the tangent of the tractrix is  $\left(0, y(t) - \frac{y'(t)}{x'(t)}x(t)\right)$ .  
The length of the segment of the tangent of the tractrix between the point of tangency and the  $y$  axis is  $\sqrt{x(t)^2 + \left(\frac{y'(t)}{x'(t)}x(t)\right)^2}$

$$\begin{aligned}
x(t)^2 + \left( \frac{y'(t)}{x'(t)} x(t) \right)^2 &= \sin^2 t \left( 1 + \left( \frac{y'(t)}{x'(t)} \right)^2 \right) \\
&= \sin^2 t \left( 1 + \left( \frac{-\sin t + \frac{1}{\sin t}}{\cos t} \right)^2 \right) \\
&= \sin^2 t \left( 1 + \left( \frac{1 - \sin^2 t}{\sin t \cos t} \right)^2 \right) \\
&= \sin^2 t \left( \frac{1}{\sin^2 t} \right) \\
&= 1
\end{aligned}$$

So the length of the segment of the tangent of the tractrix between the point of tangency and the  $y$  axis  $= \sqrt{x(t)^2 + \left( \frac{y'(t)}{x'(t)} x(t) \right)^2} = 1$ .

□

**Problem 5** (P47: 6). Let  $\alpha(s)$ ,  $s \in [0, l]$  be a closed convex plane curve positively oriented. The curve

$$\beta(s) = \alpha(s) - rn(s),$$

where  $r$  is a positive constant and  $n$  is the normal vector, is called a parallel curve to  $\alpha$  (Fig. 1-37). Show that

(a) Length of  $\beta$  = length of  $\alpha$  +  $2\pi r$ .

(b)  $A(\beta) = A(\alpha) + rl + \pi r^2$ .

(c)  $\kappa_\beta(s) = \kappa_\alpha(s)/(1 + r\kappa_\alpha(s))$ .

For (a)-(c),  $A(\cdot)$  denotes the area bounded by the corresponding curve, and  $\kappa_\alpha, \kappa_\beta$  are the curvatures of  $\alpha$  and  $\beta$ , respectively.

*Proof.*

(a) Length of  $\beta$  is

$$\int_0^l \|\beta'(s)\| ds = \int_0^l \|\alpha'(s) - rn'(s)\| ds.$$

By definition of normal vector  $n$ ,  $n'(s) = -\kappa(s)\alpha'(s)/\|\alpha'(s)\|$ . Therefore, the length of  $\beta$  equals

$$\begin{aligned}
\int_0^l \|\beta'(s)\| ds &= \int_0^l \|\alpha'(s) + r\kappa(s) \frac{\alpha'(s)}{\|\alpha'(s)\|}\| ds \\
&= \int_0^l \|(1 + \frac{r\kappa(s)}{\|\alpha'(s)\|})\alpha'(s)\| ds \\
&= \int_0^l (1 + \frac{r\kappa(s)}{\|\alpha'(s)\|}) \cdot \|\alpha'(s)\| ds \\
&= \int_0^l \|\alpha'(s)\| ds + r \int_0^l \kappa(s) ds \\
&= \text{length of } \alpha + 2\pi r.
\end{aligned}$$

(b) Let  $D$  and  $D'$  denote the region bounded by  $\alpha$  and  $\beta'$ . By Green's theorem,

$$\iint_D dx dy = \frac{1}{2} \left\| \oint_0^l \alpha \times \alpha' ds \right\|_z ; \iint_{D'} dx dy = \frac{1}{2} \left\| \oint_0^l \beta \times \beta' ds \right\|_z$$

, where  $\|\cdot\|_z$  denotes the  $z$ -component of a vector, and then

$$\begin{aligned}
\oint_0^l \beta \times \beta' ds &= \oint_0^l (\alpha - rn) \times (\alpha - rn)' ds \\
&= \oint_0^l \alpha \times \alpha' ds + r^2 \oint_0^l n \times n' ds + r \oint_0^l (\alpha' \times n - \alpha \times n') ds \\
&= \oint_0^l \alpha \times \alpha' ds - r^2 \oint_0^l n \times (-\kappa \alpha' / \|\alpha'\|) ds \\
&\quad - r \oint_0^l (\alpha \times n)' ds + 2r \oint_0^l \alpha' \times n ds \\
&= \oint_0^l \alpha \times \alpha' ds - r^2 \oint_0^l n \times (-\kappa \alpha' / \|\alpha'\|) ds \\
&\quad - (\alpha \times n|_{s=0}^l) + 2r \oint_0^l \alpha' \times n ds.
\end{aligned}$$

Since the third term of the last value is zero and that  $\alpha' \times n = \|\alpha'\|e_z$ ,

$$\begin{aligned}
A(\beta) &= \left\| \frac{1}{2} \oint_0^l \alpha \times \alpha' ds - \frac{r^2}{2} \oint_0^l n \times (-\kappa \alpha' / \|\alpha'\|) ds + r \oint_0^l \alpha' \times n ds \right\|_z \\
&= A(\alpha) + \frac{r^2}{2} \oint_0^l \kappa ds + r \oint_0^l \|\alpha\| ds = A(\alpha) + \pi r^2 + rl.
\end{aligned}$$

□

**Problem 8** (Curvature is a geometric object I.).  $X(s) = (x(s), y(s))$ , where  $s$  is the arc-length parameter.

$$M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, M^t = M^{-1}, \text{ i.e. } M \text{ is orthogonal.}$$

Let  $\bar{X}(s) = M \cdot \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ ,  $\alpha, \beta \in \mathbf{R}$ . What is the relation between  $\kappa_X(s)$  and  $\kappa_{\bar{X}}(s)$ ?

*Proof.* □

**Problem 9** (Curvature is a geometric object II.).  $X(t) = (x(t), y(t))$  be a regular curve. Let

$$\kappa(x(t), y(t)) \equiv \kappa(t) = \frac{\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}}{(x'^2 + y'^2)^{\frac{3}{2}}}$$

Let  $Y(u) = X(t(u))$ ,  $t'(u) \neq 0$ . Discuss the relation of  $\kappa(x(t), y(t))$  and  $\kappa(x(t(u)), y(t(u)))$  at the corresponding points.

*Proof.* □

**Problem 10.** Let  $F(x, y) = c$  defines a plane curve. Prove that the curvature of the curve satisfies

$$|\kappa| = \left| \frac{\begin{bmatrix} F_y & -F_x \end{bmatrix} \begin{bmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \end{bmatrix} \begin{bmatrix} F_y \\ -F_x \end{bmatrix}}{(F_x^2 + F_y^2)^{\frac{3}{2}}} \right|$$

Where  $F_x^2 + F_y^2 \neq 0$ .

*Proof.* Let  $\alpha$  be a point in the plane such that  $F(\alpha) = c$ . Consider the circle of curvature passing through  $\alpha$ . If we observed the intersection of two line respectively perpendicular to the lines tangent to  $F = c$  and passing respectively through  $\alpha$  and another point  $\alpha' \in F = c$ , the intersection approaches the centre of the circle  $o$  as  $\alpha' \rightarrow \alpha$ . Thus,

$$|\alpha' - \alpha| = r \sin \theta,$$

where  $\theta$  is the angle between the vectors  $o - \alpha$  and  $o - \alpha'$ . By the formula of exterior product,

$$\sin \theta = \frac{|(o - \alpha) \times (o - \alpha')|}{|o - \alpha||o - \alpha'|}$$

Let  $n$  denote  $\nabla F/|\nabla F|$  rotated counterclockwise by  $\pi/2$ . Since  $\alpha' - \alpha$  is perpendicular to  $\frac{\nabla F(\alpha)}{|\nabla F(\alpha)|}$ , and since  $o - \alpha$  and  $o - \alpha'$  are respectively parallel to  $\frac{\nabla F(\alpha)}{|\nabla F(\alpha)|}$  and  $\frac{\nabla F(\alpha')}{|\nabla F(\alpha')|}$ , we obtained

$$\begin{aligned} |\kappa| = 1/r &= \lim_{\alpha' \rightarrow \alpha} \frac{|(o - \alpha) \times (o - \alpha')|}{|\alpha' - \alpha||o - \alpha||o - \alpha'|} = \lim_{\alpha' \rightarrow \alpha} \frac{|\nabla F(\alpha) \times \nabla F(\alpha')|}{|\alpha' - \alpha||\nabla F(\alpha)||\nabla F(\alpha')|} \\ &= \lim_{t \rightarrow 0} \frac{|\nabla F(\alpha) \times \nabla F(\alpha + tn)|}{tn|\nabla F(\alpha)||\nabla F(\alpha + tn)|} \\ &= \lim_{t \rightarrow 0} \frac{|(F_x(\alpha)\vec{i} + F_y(\alpha)\vec{j}) \times (F_x(\alpha + tn)\vec{i} + F_y(\alpha + tn)\vec{j})|}{tn|\nabla F(\alpha)|^2} \\ &= \lim_{t \rightarrow 0} \frac{|F_x(\alpha)F_y(\alpha + tn) - F_x(\alpha + tn)F_y(\alpha)|}{tn|\nabla F(\alpha)|^2} \\ &= \lim_{t \rightarrow 0} \frac{|F_x(\alpha)[F_y(\alpha + tn) - F_y(\alpha)] - [F_x(\alpha + tn) - F_x(\alpha)]F_y(\alpha)|}{tn|\nabla F(\alpha)|^2} \\ &= \frac{|F_x(F_{yx}\vec{i} + F_{yy}\vec{j}) \cdot n - F_y(F_{xx}\vec{i} + F_{xy}\vec{j}) \cdot n|}{|\nabla F|^2} \\ &= \frac{|[(F_x F_{yx} - F_y F_{xx})\vec{i} + (F_x F_{yy} - F_y F_{xy})\vec{j}] \cdot n|}{|\nabla F|^2} \\ &= \frac{|[(F_x F_{yx} - F_y F_{xx})\vec{i} + (F_x F_{yy} - F_y F_{xy})\vec{j}] \cdot (F_y\vec{i} - F_x\vec{j})|}{|\nabla F|^2} \\ &= \frac{\left| \begin{bmatrix} F_y & -F_x \end{bmatrix} \begin{bmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \end{bmatrix} \begin{bmatrix} F_y \\ -F_x \end{bmatrix} \right|}{|\nabla F|^3} \\ &= \frac{\left| \begin{bmatrix} F_y & -F_x \end{bmatrix} \begin{bmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \end{bmatrix} \begin{bmatrix} F_y \\ -F_x \end{bmatrix} \right|}{(F_x^2 + F_y^2)^{\frac{3}{2}}} \end{aligned}$$

□