Geometry Homework 1

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Problem 3 (P7: 4)**.** *Let* $\alpha : (0, \pi) \to \mathbf{R}^2$ *be given by*

$$\alpha(t) = \left(\sin t, \cos t + \log \tan \frac{t}{2}\right),$$

where t *is the angle that the* y *axis makes with the vector* $\alpha(t)$*. The trace of* α *is called the* tractrix *(Fig. 1-9). Show that*

(a) α *is a differentiable parametrized curve, regular except at* $t = \pi/2$*.*

(b) The length of the segment of the tangent of the tractrix between the point of tangency and the y *axis is constantly equal to* 1*.*

Proof. (a) Let $x(t) = \sin t$, $y(t) = \cos t + \log \tan \frac{t}{2}$, then $x'(t) = \cos t$, $y'(t) = -\sin t + \frac{1}{\sin t}$.

It's trivial that both $x'(t)$ and $y'(t)$ are infinitely differentiable in $(0, \pi)$, so α is a differentiable parametrized curve.

$x'(t) = 0, y'(t) = 0 \Leftrightarrow t = \frac{\pi}{2}$, so α is regular except at $t = \pi/2$.

(b) The intersection of y axis and the tangent of the tractrix is $\left(0, y(t) - \frac{y'(t)}{x'(t)}x(t)\right)$.

The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is $\sqrt{x(t)^2 + \left(\frac{y'(t)}{x'(t)}x(t)\right)^2}$

$$
\begin{aligned}
x(t)^2 + \left(\frac{y'(t)}{x'(t)}x(t)\right)^2 &= \sin^2 t\left(1 + \left(\frac{y'(t)}{x'(t)}\right)^2\right) \\
&= \sin^2 t\left(1 + \left(\frac{-\sin t + \frac{1}{\sin t}}{\cos t}\right)^2\right) \\
&= \sin^2 t\left(1 + \left(\frac{1 - \sin^2 t}{\sin t \cos t}\right)^2\right) \\
&= \sin^2 t\left(\frac{1}{\sin^2 t}\right) \\
&= 1
\end{aligned}
$$

So the length of the segment of the tangent of the tractrix between the point of tangency and the y axis $= \sqrt{x(t)^2 + \left(\frac{y'(t)}{x'(t)}x(t)\right)^2} = 1$.

□

Problem 5 (P47: 6). *If a closed plane curve C is contained inside a disk of radius r, prove that there exists a point $p \in C$ such that the curvature κ of C at p satisfies $|\kappa| \geq 1/r$.*

Proof. Let $p \in C$ is the farthest point from the center of the disk on the curve. We want to prove that $\kappa(p) \geq 1/r$.

WLOG, we can rotate and translate the curve so that p is at the origin and the tangent of C at p is the x axis. □

Problem 8 (Curvature is a geometric object I.). $X(s) = (x(s), y(s))$, *where s is the arc-length parameter.*

$$
M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, M^t = M^{-1}, i.e.\ M \text{ is orthogonal.}
$$

Let $\bar{X}(s) = M \cdot \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, $\alpha, \beta \in \mathbf{R}$. *What is the relation between* $\kappa_X(s)$ *and* $\kappa_{\bar{X}}(s)$*?*

Proof. □

Problem 9 (Curvature is a geometric object II.). $X(t) = (x(t), y(t))$ *be a regular curve. Let*

$$
\kappa(x(t), y(t)) \equiv \kappa(t) = \frac{\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}}{(x'^2 + y'^2)^{\frac{3}{2}}}
$$

Let $Y(u) = X(t(u))$, $t'(u) \neq 0$. Discuss the relation of $\kappa(x(t), y(t))$ and $\kappa(x(t(u)), y(t(u)))$ at the corresponding points.

Proof. □

Problem 10. *Let $F(x, y) = c$ defines a plane curve. Prove that the curvature of the curve satisfies*

$$|\kappa| = \left| \frac{\begin{bmatrix} F_y, & -F_x \end{bmatrix} \begin{bmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \end{bmatrix} \begin{bmatrix} F_y \\ -F_x \end{bmatrix}}{(F_x^2 + F_y^2)^{\frac{3}{2}}} \right|$$

Where $F_x^2 + F_y^2 \neq 0$.

Proof. □