Geometry Homework 5

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Problem 1 (Ex P151 2). *Show that if a surface is tangent to a plane along a curve, then the points of this curve are either parabolic or planar.*

Proof. Assume that the curve is $\gamma(s)$, then along this curve, $N(\gamma(s))$ is perpendicular to the plane, so it is constant.
At point $\gamma(s)$, $[dN](\gamma'(s)) = \left(\frac{dN(\gamma(t))}{dt}\right)_{t=s} = 0$, so $\gamma'(s)$ is one of the principal direction of the surface at $\gamma(s)$, and it's associated principal curvature is 0.
So the gaussian curvature of the surface at $\gamma(s)$ is $K = 0$, and this means that the point $\gamma(s)$ is either parabolic or planar. $\square$

Problem 3 (Ex P151 3).

(a) *Let $C \subset S$ be a regular curve on a surface S with Gaussian curvature $K > 0$. Show that the curvature κ of C at p satisfies*

$$\kappa \geq \min(|\kappa_1|, |\kappa_2|),$$

where κ_1, κ_2 are the principal curvatures of S at p.

(b) 為什麼上一小題需要 $\kappa > 0$ 的條件，$\kappa \geq 0$ 不可以嗎？

Proof. (a)

$$\begin{aligned}
\kappa &\geq |\kappa_n| \\
&= |\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta| \\
&= |\kappa_1| \cos^2\theta + |\kappa_2| \sin^2\theta (\because \kappa_1, \kappa_2 \text{ has equal sign.}) \\
&\geq \min(|\kappa_1|, |\kappa_2|)(\cos^2\theta + \sin^2\theta) \\
&= \min(|\kappa_1|, |\kappa_2|)
\end{aligned}$$

(b)

$\square$

Problem 7.

(a) T_λ 是縮放 λ 倍的映射，$\lambda > 0$。$\mathbb{X} : \Omega \to \mathbb{R}^3$ *regular surface*。討論 $T_\lambda \circ \mathbb{X} : \Omega \to \mathbb{R}^3$ 上對應點 κ_n, H, K 的變化。

(b) $\mathbb{X}:\ \begin{matrix}\Omega\\ (u,v)\end{matrix} \to \mathbb{R}^3$，若定義 $\overline{\mathbb{X}}(u,v)=\mathbb{X}(v,u)$*(*因此 N 轉向*)*。討論 $\overline{\mathbb{X}}(\Omega)$ 上相對應點的 K_n, H, K 變化。

Proof. □

Problem 9 (旋轉面)**.** $\mathbb{X}(u,v)=(f(u)\cos v, f(u)\sin v, g(u))$，$f>0$

(a) 計算其 e, f, g, H, K

(b) 討論其 *principal direction* 與 *principal curvature* K_1, K_2。

Proof. To avoid the notational ambiguity, let $\mathbb{X}(u,v)=(s(u)\cos v, s(u)\sin v, t(u))$, and that $s>0$.

(a) We have

$$\begin{aligned}
\mathbb{X}_u &= (s'(u)\cos v, s'(u)\sin v, t'(u));\\
\mathbb{X}_v &= (-s(u)\sin v, s(u)\cos v, 0);\\
E &= \langle \mathbb{X}_u, \mathbb{X}_u\rangle = s'(u)^2+t'(u)^2\\
F &= \langle \mathbb{X}_u, \mathbb{X}_v\rangle = 0\\
G &= \langle \mathbb{X}_v, \mathbb{X}_v\rangle = s(u)^2\\
\mathbb{X}_{uu} &= (s''(u)\cos v, s''(u)\sin v, t''(u));\\
\mathbb{X}_{uv} &= (-s'(u)\sin v, s'(u)\cos v, 0);\\
\mathbb{X}_{vv} &= (-s(u)\cos v, -s(u)\sin v, 0);\\
N &= \frac{\mathbb{X}_u\times\mathbb{X}_v}{|\mathbb{X}_u\times\mathbb{X}_v|} = \frac{(-t'(u)s(u)\cos v, -t'(u)s(u)\sin v, s'(u)s(u))}{\sqrt{t'(u)^2s(u)^2+s'(u)^2s(u)^2}}\\
&= \frac{(-t'(u)\cos v, -t'(u)\sin v, s'(u))}{\sqrt{t'(u)^2+s'(u)^2}};\\
e &= \langle N, \mathbb{X}_{uu}\rangle = \frac{-s''(u)t'(u)+t''(u)s'(u)}{\sqrt{t'(u)^2+s'(u)^2}}\\
f &= \langle N, \mathbb{X}_{uv}\rangle = 0\\
g &= \langle N, \mathbb{X}_{vv}\rangle = \frac{s(u)t'(u)}{\sqrt{t'(u)^2+s'(u)^2}}\\
-dN^T &= \begin{bmatrix} e & f\\ f & g\end{bmatrix}\begin{bmatrix} E & F\\ F & G\end{bmatrix}^{-1} = \frac{1}{EG-F^2}\begin{bmatrix} e & f\\ f & g\end{bmatrix}\begin{bmatrix} G & -F\\ -F & E\end{bmatrix} = \begin{bmatrix} e/E & 0\\ 0 & g/G\end{bmatrix}\\
K &= \det(-dN) = \frac{eg}{EG}\\
H &= \frac{1}{2}\mathrm{tr}(-dN) = \frac{eG+gE}{2EG}
\end{aligned}$$

(b) Since $-dN$ is already a diagonal matrix, clearly,

$$\begin{aligned} K_1 &= e/E; \\ K_2 &= g/G; \\ V_1 &= \mathbb{X}_u; \\ V_2 &= \mathbb{X}_v; \end{aligned}$$

□

Problem 10 (管面). $\mathbb{X}(s,\theta) = \gamma(s) + \cos\theta \vec{n}(s) + \sin\theta \vec{b}(s)$, $0 < \kappa < 1$

(a) 計算其 e, f, g, H, K

(b) 討論曲面上 K 的分佈。

Proof. (a) Let $\vec{t}(s), \vec{n}(s), \vec{b}(s)$ be the basis, and $\gamma(0)$ be the origin.

$$\begin{aligned} \mathbb{X}_s &= (1 - \kappa\cos\theta, \tau\sin\theta, -\tau\cos\theta); \\ \mathbb{X}_\theta &= (0, -\sin\theta, \cos\theta); \\ N &= \frac{\mathbb{X}_s \times \mathbb{X}_\theta}{|\mathbb{X}_s \times \mathbb{X}_\theta|} = \frac{(0, (\kappa\cos\theta - 1)\cos\theta, (\kappa\cos\theta - 1)\sin\theta)}{|\kappa\cos\theta - 1|} \\ &= (0, -\cos\theta, -\sin\theta); \quad (\text{since } \kappa < 1) \\ \mathbb{X}_{ss} &= (-\sin\theta\kappa\tau - \kappa'\cos\theta, \kappa + \cos\theta(-\kappa^2 - \tau^2) + \tau'\sin\theta, -\sin\theta\tau^2 - \tau'\cos\theta); \\ \mathbb{X}_{s\theta} &= (1 + \kappa\sin\theta, \tau\cos\theta, \tau\sin\theta); \\ \mathbb{X}_{\theta\theta} &= (0, -\cos\theta, -\sin\theta); \\ e &= -\kappa\cos\theta + \cos^2\theta(\kappa^2 + \tau^2) + \tau^2\sin^2\theta = -\kappa\cos\theta + \kappa^2\cos^2\theta + \tau^2; \\ f &= -\tau; \\ g &= 1; \\ dN(\mathbb{X}_s) &= N_s = (\kappa\cos\theta, -\tau\sin\theta, \tau\cos\theta) = \frac{\kappa}{1-\kappa}\mathbb{X}_s + \frac{\tau}{1-\kappa}\mathbb{X}_\theta; \\ dN(\mathbb{X}_\theta) &= N_\theta = (0, \sin\theta, -\cos\theta) = -\mathbb{X}_\theta; \\ [-dN] &= \begin{bmatrix} \frac{-\kappa\cos\theta}{1-\kappa\cos\theta} & 0 \\ \frac{-\tau}{1-\kappa\cos\theta} & 1 \end{bmatrix} \\ \kappa_1 &= \frac{-\kappa\cos\theta}{1 - \kappa\cos\theta}; \\ \kappa_2 &= 1; \\ K &= \frac{-\kappa\cos\theta}{1 - \kappa\cos\theta}; \\ H &= \frac{1 - 2\kappa\cos\theta}{1 - \kappa\cos\theta}. \end{aligned}$$

(b)

$$\begin{aligned} K &= \frac{-\kappa \cos\theta}{1 - \kappa\cos\theta}; \\ &= 1 + \frac{1}{1 - \kappa\cos\theta}; \end{aligned}$$

Since $0 < \kappa < 1$, K has a minimum of 2 when

$\square$