

GEOMETRY HOMEWORK 9

B96201044 黃上恩, B98901182 時丕勳, K0020100x 劉士璋

December 1, 2011

Problem 4 (Ex p.101 14). (*Gradient on Surfaces.*) The gradient of a differentiable function $f : S \rightarrow \mathbb{R}$ is a differentiable map $\text{grad } f : S \rightarrow \mathbb{R}^3$ which assigns to each point $p \in S$ a vector $\text{grad } f(p) \in T_p(S) \subset \mathbb{R}^3$ such that

$$\langle \text{grad } f(p), v \rangle_p = df_p(v) \quad \text{for all } v \in T_p(S)$$

Show that

- (a) If E, F, G are the coefficients of the first fundamental form in a parametrization $\mathbf{X} : U \subset \mathbb{R}^2 \rightarrow S$, then $\text{grad } f$ on $\mathbf{X}(U)$ is given by

$$\text{grad } f = \frac{f_u G - f_v F}{EG - F^2} \mathbf{X}_u + \frac{f_v E - f_u F}{EG - F^2} \mathbf{X}_v$$

In particular, if $S = \mathbb{R}^2$ with coordinates x, y ,

$$\text{grad } f = f_x e_1 + f_y e_2$$

where $\{e_1, e_2\}$ is the canonical basis of \mathbb{R}^2 (thus, the definition agrees with the usual definition of gradient in the plane)

- (b) 為什麼不直接將 $\text{gradient } f$ 定義成 $f_u \mathbf{X}_u + f_v \mathbf{X}_v$, 這有什麼缺點 (例如座標變換)

Proof.

□

Problem 7. 計算下列 surface 的 Γ_{ij}^k (共有六項)

(b) $(x(t), y(t) \cos \theta, y(t) \sin \theta)$

(c) $E = G = \lambda^2, F = 0$

Proof. (b) Let $u = t, v = \theta$.

$$\mathbf{X}_u = (x_u, y_u \cos v, y_u \sin v)$$

$$\mathbf{X}_v = (0, -y \sin v, y \cos v)$$

$$\rightarrow E = x_u^2 + y_u^2$$

$$F = 0$$

$$G = y^2$$

$$\begin{aligned}
[1, 1, 1] &= \frac{E_u}{2} \\
[1, 1, 2] &= -\frac{E_v}{2} \\
[1, 2, 1] &= \frac{E_v}{2} \\
[1, 2, 2] &= \frac{G_u}{2} \\
[2, 2, 1] &= -\frac{G_u}{2} \\
[2, 2, 2] &= \frac{G_v}{2} \\
\Gamma_{11}^1 &= \frac{E_u}{2E} \\
&= \frac{x_u x_{uu} + y_u y_{uu}}{x_u^2 + y_u^2} \\
\Gamma_{11}^2 &= -\frac{E_v}{2G} \\
&= 0 \\
\Gamma_{12}^1 &= \frac{E_v}{2E} \\
&= 0 \\
\Gamma_{12}^2 &= \frac{G_u}{2G} \\
&= \frac{y_u}{y} \\
\Gamma_{22}^1 &= -\frac{G_u}{2E} \\
&= \frac{yy_u}{x_u^2 + y_u^2} \\
\Gamma_{22}^2 &= \frac{G_v}{2G} \\
&= 0
\end{aligned}$$

(c)

$$\begin{aligned}
g^{11} &= \frac{1}{\lambda^2} \\
g^{22} &= \frac{1}{\lambda^2} \\
g^{12} &= g^{21} = 0
\end{aligned}$$

$$\begin{aligned}
[1, 1, 1] &= \frac{E_u}{2} \\
[1, 1, 2] &= -\frac{E_v}{2} \\
[1, 2, 1] &= \frac{E_v}{2} \\
[1, 2, 2] &= \frac{G_u}{2} \\
[2, 2, 1] &= -\frac{G_u}{2} \\
[2, 2, 2] &= \frac{G_v}{2} \\
\Gamma_{11}^1 &= \frac{E_u}{2E} \\
&= \frac{\lambda_u}{\lambda} \\
\Gamma_{11}^2 &= -\frac{E_v}{2G} \\
&= \frac{\lambda_v}{\lambda} \\
\Gamma_{12}^1 &= \frac{E_v}{2E} \\
&= \frac{\lambda_v}{\lambda} \\
\Gamma_{12}^2 &= \frac{G_u}{2G} \\
&= \frac{\lambda_u}{\lambda} \\
\Gamma_{22}^1 &= -\frac{G_u}{2E} \\
&= \frac{\lambda_u}{\lambda} \\
\Gamma_{22}^2 &= \frac{G_v}{2G} \\
&= \frac{\lambda_v}{\lambda}
\end{aligned}$$

□

Problem 8 (Ex p.237 1, 2). (a) Show that if \mathbf{X} is an orthogonal parametrization, that is, $F = 0$, then

$$K = -\frac{1}{2\sqrt{EG}} \left\{ \left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right\}$$

(b) Show that if \mathbf{X} is an isothermal parametrization, that is, $E = G = \lambda(u, v)$ and $F = 0$, then

$$K = -\frac{1}{2\lambda} \Delta(\log \lambda)$$

where $\Delta\phi$ denotes the Laplacian $(\partial^2\phi/\partial u^2)+(\partial^2\phi/\partial v^2)$ of the function ϕ . Conclude that when $E = G = (u^2 + v^2 + c)^{-2}$ and $F = 0$, then $K = \text{const} = 4c$.

Proof. (a)

$$\begin{aligned} g^{11} &= \frac{1}{E} \\ g^{22} &= \frac{1}{G} \\ g^{12} &= g^{21} = 0 \end{aligned}$$

$$\begin{aligned} [1, 1, 1] &= \frac{E_u}{2} \\ [1, 1, 2] &= -\frac{E_v}{2} \\ [1, 2, 1] &= \frac{E_v}{2} \\ [1, 2, 2] &= \frac{G_u}{2} \\ [2, 2, 1] &= -\frac{G_u}{2} \\ [2, 2, 2] &= \frac{G_v}{2} \\ \Gamma_{11}^1 &= \frac{E_u}{2E} \\ \Gamma_{11}^2 &= -\frac{E_v}{2G} \\ \Gamma_{12}^1 &= \frac{E_v}{2E} \\ \Gamma_{12}^2 &= \frac{G_u}{2G} \\ \Gamma_{22}^1 &= -\frac{G_u}{2E} \\ \Gamma_{22}^2 &= \frac{G_v}{2G} \end{aligned}$$

$$\begin{aligned}
R_{112}^2 &= \Gamma_{11,2}^2 - \Gamma_{12,1}^2 + \Gamma_{11}^1 \Gamma_{21}^2 + \Gamma_{11}^2 \Gamma_{22}^2 - \Gamma_{12}^1 \Gamma_{11}^2 - \Gamma_{12}^2 \Gamma_{12}^2 \\
&= -\left(\frac{E_v}{2G}\right)_v - \left(\frac{G_u}{2G}\right)_u + \frac{E_u}{2E} \frac{G_u}{2G} - \frac{E_v}{2G} \frac{G_v}{2G} + \frac{E_v}{2E} \frac{E_v}{2G} - \frac{G_u}{2G} \frac{G_u}{2G} \\
&= -\frac{GE_{vv} - G_v E_v}{2G^2} - \frac{GG_{uu} - G_u G_u}{2G^2} + \frac{E_u}{2E} \frac{G_u}{2G} - \frac{E_v}{2G} \frac{G_v}{2G} + \frac{E_v}{2E} \frac{E_v}{2G} - \frac{G_u}{2G} \frac{G_u}{2G} \\
&= \frac{1}{4G^2} \left(-2GE_{vv} + G_v E_v - 2GG_{uu} + G_u G_u + \frac{GE_u}{E} G_u + \frac{GE_v}{E} E_v \right) \\
K &= \frac{R_{1212}}{EG} \\
&= \frac{GR_{112}^2}{EG} \\
&= \frac{R_{112}^2}{E} \\
&= \frac{1}{4EG^2} \left(-2GE_{vv} + G_v E_v - 2GG_{uu} + G_u G_u + \frac{GE_u}{E} G_u + \frac{GE_v}{E} E_v \right) \\
&= \frac{1}{4} \left(-2\frac{E_{vv}}{EG} - 2\frac{G_{uu}}{EG} + \frac{E_v G_v}{EG^2} + \frac{E_u G_u}{E^2 G} + \frac{G_u G_u}{EG^2} + \frac{E_v E_v}{E^2 G} \right) \\
&\quad - \frac{1}{2\sqrt{EG}} \left\{ \left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right\} \\
&= -\frac{1}{2\sqrt{EG}} \left\{ \frac{\sqrt{EG} E_{vv} - \frac{E_v G + EG_v}{2\sqrt{EG}} E_v}{EG} + \frac{\sqrt{EG} G_{uu} - \frac{E_u G + EG_u}{2\sqrt{EG}} G_u}{EG} \right\} \\
&= -\frac{1}{2} \left\{ \frac{E_{vv} - \frac{E_v G + EG_v}{2EG} E_v}{EG} + \frac{G_{uu} - \frac{E_u G + EG_u}{2EG} G_u}{EG} \right\} \\
&= \frac{1}{4} \left(-2\frac{E_{vv}}{EG} - 2\frac{G_{uu}}{EG} + \frac{E_v G_v}{EG^2} + \frac{E_u G_u}{E^2 G} + \frac{G_u G_u}{EG^2} + \frac{E_v E_v}{E^2 G} \right)
\end{aligned}$$

$$\text{So } K = -\frac{1}{2\sqrt{EG}} \left\{ \left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right\}.$$

(b)

$$\begin{aligned}
K &= -\frac{1}{2\sqrt{EG}} \left\{ \left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right\} \\
&= -\frac{1}{2\lambda} \left\{ \left(\frac{\lambda_v}{\lambda} \right)_v + \left(\frac{\lambda_u}{\lambda} \right)_u \right\} \\
\Delta(\log \lambda) &= (\log \lambda)_{uu} + (\log \lambda)_{vv} \\
&= \left(\frac{\lambda_u}{\lambda} \right)_u + \left(\frac{\lambda_v}{\lambda} \right)_v \\
&\rightarrow K = -\frac{1}{2\lambda} \Delta(\log \lambda)
\end{aligned}$$

□