## **GEOMETRY HOMEWORK 3**

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**Problem 3** (P26: 16). Show that the knowledge of the vector function n = n(s) (normal vector) of a curve  $\alpha$ , with nonzero torsion everywhere, determines the curvature  $\kappa(s)$  and the torsion  $\tau(s)$  of  $\alpha$ . ( $\vec{n}$  能決定曲線嗎? 說明題目錯誤並找反例。)

Proof. Consider the helix  $\alpha(s)=\left(a\cos\frac{s}{\sqrt{a^2+b^2}},a\sin\frac{s}{\sqrt{a^2+b^2}},\frac{bs}{\sqrt{a^2+b^2}}\right)$ Then  $n(s)=\left(-\cos\frac{s}{\sqrt{a^2+b^2}},-\sin\frac{s}{\sqrt{a^2+b^2}},0\right)$ .

So if two helix has the same  $a^2 + b^2$  (e.g.  $\alpha_1(s) = (\frac{1}{2}\cos s, \frac{1}{2}\sin s, \frac{\sqrt{3}}{2}s), \alpha_2(s) = (\frac{\sqrt{3}}{2}\cos s, \frac{\sqrt{3}}{2}\sin s, \frac{1}{2}s))$ , then they have same n(s), but they're not the same curve.

**Problem 4** (P26: 17, 另一種描述 Helix 的方式). In general, a curve  $\alpha$  is called a helix if the tangent lines of  $\alpha$  make a constant angle with a fixed direction. Assume that  $\tau(s) \neq 0$ ,  $s \in I$ , and prove that:

- (a)  $\alpha$  is a helix if and only if  $\kappa/\tau = constant$ .
- (b)  $\alpha$  is a helix if and only if the lines containing N(s) and passing through  $\alpha(s)$  are parallel to a fixed plane.
- (c)  $\alpha$  is a helix if and only if the lines containing B(s) and passing through  $\alpha(s)$  make a constant angle with a fixed direction.
- (d) The curve

$$lpha(s) = \left(rac{a}{c}\int\sin heta(s)ds,rac{a}{c}\int\cos heta(s)ds,rac{b}{c}s
ight),$$

where  $c^2 = a^2 + b^2$ , is a helix, and that  $\kappa/\tau = a/b$ .

*Proof.* WLOG, assume that s is arc-length parameter, V is the fixed direction. Let  $T(s) = \alpha'(s)$ ,  $T'(s) = \kappa N(s)$  and  $B = T \times N$ .

(a)

 $T \perp N, V \perp N \rightarrow V \times T = \pm |V \times T| N$ 

T make a constant angle with V,  $|V \times T|$  is a constant.  $V = \mp \frac{|V \times T|}{C}$ , but because  $K = \pi$  are continuous,  $K = \pi$  is constant.

Conversely, let  $\kappa/\tau \equiv c$  be a constant. Define vector V by V(s) = T(s) - cB(s). We claim V is a constant since  $V'(s) = T'(s) - cB'(s) = \kappa(s)N(s) - c\tau(s)N(s) = (\kappa(s) - c\tau(s))N(s) = 0$ . Now  $\langle V,T \rangle$  is constant because  $\langle V,T \rangle' = \kappa \langle V,N \rangle = \kappa \langle T-cB,N \rangle = 0$ . This implies T make a constant angle with V.

(b)  $\langle V,T\rangle\equiv c$  implies  $0=\langle V,T\rangle'=\kappa\,\langle V,N\rangle$ , but  $\kappa=\|\alpha'\|\neq 0$ , so  $\langle V,N\rangle=0$ . Therefore N(s) is always perpendicular to V. Consider any fixed plane P with V be the normal vector, then N(s) is parallel to P.

Conversely, if N(s) parallel to a fixed plane P, define V be a normal vector of P. This implies  $N(s) \perp V$ , therefore  $\langle V, T \rangle' = \kappa \langle V, N \rangle = 0$ . So T makes a constant angle with V.

(c) From the proof above, we have  $\langle V, N \rangle = 0$ , so  $V = \langle V, T \rangle T + \langle V, B \rangle B$ . Since  $V, \langle V, T \rangle$  are both constant and the orientation between T, B is fixed,  $\langle V, B \rangle$  is a constant. This implies B makes a constant angle with V.

Conversely, let V be the fixed direction, then  $0 = \langle V, B \rangle = -\tau \langle V, N \rangle$ . Since  $\tau \neq 0$ , so  $V \perp N$  hence by (b), T makes a constant angle with V.

(d)

Problem 6.  $\gamma(s)$  長度參數。若將 T(s) 寫成  $(\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)$ ,  $\phi, \theta$  是 s 的函數。說明  $\kappa(s) = \sqrt{\phi'^2 + \theta'^2 \sin^2 \phi}$ 

Proof.

$$\begin{split} T'(s) &= (\phi'\cos\phi\cos\theta - \theta'\sin\phi\sin\theta, \phi'\cos\phi\sin\theta + \theta'\sin\phi\cos\theta, -\phi'\sin\phi) \\ &\to \kappa(s) = |T'(s)| \\ &= \sqrt{\phi'^2\cos^2\phi\cos^2\theta + \theta'^2\sin^2\phi\sin^2\theta + \phi'^2\cos^2\phi\sin^2\theta + \theta'^2\sin^2\phi\cos^2\theta + \phi'^2\sin^2\phi} \\ &= \sqrt{\phi'^2 + \theta'^2\sin^2\phi} \end{split}$$

Problem 7.  $\gamma: \mathbb{R} \to \mathbb{R}^3$ ,不妨假設是長度參數。

(b) 若  $M^tM=I$ ,  $\det(M)=-1$  且  $\overline{\gamma}=M\gamma$ , 討論  $\kappa, \tau$  變化。

$$(c)$$
  $\overline{\gamma}(s) = \gamma(-s)$ ,說明  $\kappa, \tau$  變化。

Proof. (b)

$$\begin{split} |\overline{\gamma}'| &= \sqrt{\overline{\gamma}'^T \overline{\gamma}'} \\ &= \sqrt{\gamma'^T M^T M \gamma'} \\ &= \sqrt{\gamma'^T \gamma'} \\ &= |\gamma'| \\ &= 1 \end{split}$$

So s is arc-length parameter for  $\overline{\gamma}$  too.

$$\begin{split} \kappa_{\overline{\gamma}} &= |\overline{\gamma}''| \\ &= \sqrt{\overline{\gamma}''^T \overline{\gamma}''} \\ &= \sqrt{\gamma''^T M^T M \gamma''} \\ &= \sqrt{\gamma''^T \gamma''} \\ &= |\gamma''| \\ &= \kappa_{\gamma} \end{split}$$

So  $\kappa$  remains the same.

(c)

$$\begin{aligned} |\overline{\gamma}'(s)| &= \sqrt{\overline{\gamma}'(s)^T \overline{\gamma}'(s)} \\ &= \sqrt{(-\gamma'^T(-s))(-\gamma'(-s))} \\ &= \sqrt{\gamma'(-s)^T \gamma'(-s)} \\ &= |\gamma'(-s)| \\ &= 1 \end{aligned}$$

So s is arc-length parameter for  $\overline{\gamma}$  too.

$$\kappa_{\overline{\gamma}}(s) = |\overline{\gamma}''(s)|$$

$$= \sqrt{\overline{\gamma}''(s)^T \overline{\gamma}''(s)}$$

$$= \sqrt{\gamma''(-s)^T \gamma''(-s)}$$

$$= |\gamma''(-s)|$$

$$= \kappa_{\gamma}(-s)$$

So  $\kappa_{\overline{\gamma}}(s) = \kappa_{\gamma}(-s)$ .

$$\begin{split} \tau_{\overline{\gamma}}(s) &= \frac{\left|\overline{\gamma}'(s)\ \overline{\gamma}''(s)\ \overline{\gamma}'''(s)\right|^2}{\left|\overline{\gamma}'(s)\times\overline{\gamma}''(s)\right|^2} \\ &= \frac{\left|-\gamma'(-s)\ \gamma''(-s)\ -\gamma'''(-s)\right|}{\left|-\gamma'(-s)\times\gamma''(-s)\right|^2} \\ &= \frac{\left|\gamma'(-s)\ \gamma''(-s)\ \gamma'''(-s)\right|}{\left|\gamma'(-s)\times\gamma''(-s)\right|^2} \\ &= \tau_{\gamma}(-s) \end{split}$$

So  $au_{\overline{\gamma}}(s) = au_{\gamma}(-s)$ .

Problem 8. 說明  $\overline{\gamma}(u) = \gamma(t(u))$  時,在對應點

$$\frac{\det(\overline{\gamma}',\overline{\gamma}'',\overline{\gamma}''')}{|\overline{\gamma}'\times\overline{\gamma}''|^2}(u)=\frac{\det(\gamma',\gamma'',\gamma''')}{|\gamma'\times\gamma''|^2}(t)$$

再用 chain rule 直接說明。

Proof.

$$\overline{\gamma}'(u) = \gamma'(t(u))t'(u) 
\overline{\gamma}''(u) = \gamma''(t(u))t'(u)^2 + \gamma'(t(u))t''(u) 
\overline{\gamma}'''(u) = \gamma'''(t(u))t'(u)^3 + 3\gamma''(t(u))t'(u)t''(u) + \gamma'(t(u))t'''(u)$$

$$\begin{split} \to \det(\overline{\gamma}',\overline{\gamma}'',\overline{\gamma}''')(u) &= \det(\gamma'(t(u))t'(u),\gamma''(t(u))t'(u)^2 + \gamma'(t(u))t''(u), \\ \gamma'''(t(u))t'(u)^3 + 3\gamma''(t(u))t'(u)t''(u) + \gamma'(t(u))t'''(u)) \\ &= \det(\gamma'(t(u))t'(u),\gamma''(t(u))t'(u)^2,\gamma'''(t(u))t'(u)^3 + 3\gamma''(t(u))t'(u)t''(u)) \\ &= \det(\gamma'(t(u))t'(u),\gamma''(t(u))t'(u)^2,\gamma'''(t(u))t'(u)^3) \\ &= t'(u)^6 \det(\gamma'(t(u)),\gamma''(t(u)),\gamma'''(t(u))) \end{split}$$

$$\begin{split} |\overline{\gamma}' \times \overline{\gamma}''|^2(u) &= |\left(\gamma'(t(u))t'(u)\right) \times \left(\gamma''(t(u))t'(u)^2 + \gamma'(t(u))t''(u)\right)|^2 \\ &= |\left(\gamma'(t(u))t'(u)\right) \times \left(\gamma''(t(u))t'(u)^2\right)|^2 \\ &= t'(u)^6 |\gamma'(t(u)) \times \gamma''(t(u))|^2 \end{split}$$

$$egin{aligned} & 
ightarrow rac{\det(\overline{\gamma}',\overline{\gamma}'',\overline{\gamma}''')}{|\overline{\gamma}' imes\overline{\gamma}''|^2}(u) = rac{t'(u)^6\det(\gamma',\gamma'',\gamma''')}{t'(u)^6|\gamma' imes\gamma''|^2}(t) \ & = rac{\det(\gamma',\gamma'',\gamma''')}{|\gamma' imes\gamma''|^2}(t) \end{aligned}$$