

# GEOMETRY HOMEWORK 7

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**Problem 2.** 若  $F(x, y, z) = 0$  定義一 *surface*, 證明  $\nabla F \neq 0$  的地方 Gauss curvature  $K = \frac{\nabla F^t A \nabla F}{\|\nabla F\|^4}$ . 其中  $A$  為  $\partial^2 F = \begin{pmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{pmatrix}$  的 *adjoint Matrix*, i.e.  $A = \det(\partial^2 F)(\partial^2 F)^{-1}$

*Proof.* 因為  $K$  為局部性質, 而在  $\nabla F \neq 0$  的地方我們可以使用隱函數定理將其中一維表示為另兩維的函數, WLOG 不妨設  $z = z(x, y)$  在某點附近。

$$\begin{aligned} F(x, y, z(x, y)) &= 0 \\ \mathbb{X}(x, y) &= (x, y, z(x, y)) \\ \rightarrow \mathbb{X}_x &= (1, 0, z_x) \\ \mathbb{X}_y &= (0, 1, z_y) \\ \rightarrow N &= \frac{\mathbb{X}_x \times \mathbb{X}_y}{|\mathbb{X}_x \times \mathbb{X}_y|} \\ &= \frac{(-z_x, -z_y, 1)}{\sqrt{1 + z_x^2 + z_y^2}} \end{aligned}$$

$$\begin{aligned} E &= \langle \mathbb{X}_x, \mathbb{X}_x \rangle \\ &= 1 + z_x^2 \\ F &= \langle \mathbb{X}_x, \mathbb{X}_y \rangle \\ &= z_x z_y \\ G &= \langle \mathbb{X}_y, \mathbb{X}_y \rangle \\ &= 1 + z_y^2 \end{aligned}$$

$$\begin{aligned}
\mathbb{X}_{xx} &= (0, 0, z_{xx}) \\
\mathbb{X}_{xy} &= (0, 0, z_{xy}) \\
\mathbb{X}_{yy} &= (0, 0, z_{yy}) \\
\rightarrow e &= \langle N, \mathbb{X}_{xx} \rangle \\
&= \frac{z_{xx}}{\sqrt{1 + z_x^2 + z_y^2}} \\
f &= \langle N, \mathbb{X}_{xy} \rangle \\
&= \frac{z_{xy}}{\sqrt{1 + z_x^2 + z_y^2}} \\
g &= \langle N, \mathbb{X}_{yy} \rangle \\
&= \frac{z_{yy}}{\sqrt{1 + z_x^2 + z_y^2}} \\
\rightarrow K &= \det([-dN]) \\
&= \det \left( \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \right) \\
&= \det \left( \begin{bmatrix} E & F \\ F & G \end{bmatrix} \right)^{-1} \det \left( \begin{bmatrix} e & f \\ f & g \end{bmatrix} \right) \\
&= \frac{z_{xx}z_{yy} - z_{xy}^2}{(1 + z_x^2 + z_y^2)^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial F(x, y, z)}{\partial x} &= 0 \\
&= F_x(x, y, z) + F_z(x, y, z)z_x \\
\rightarrow z_x &= -\frac{F_x}{F_z} \\
\frac{\partial F(x, y, z)}{\partial y} &= 0 \\
&= F_y(x, y, z) + F_z(x, y, z)z_y \\
\rightarrow z_y &= -\frac{F_y}{F_z}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 F(x, y, z)}{\partial x^2} &= 0 \\
&= \frac{\partial}{\partial x} (F_x(x, y, z) + F_z(x, y, z)z_x) \\
&= F_{xx}(x, y, z) + 2F_{xz}(x, y, z)z_x + F_{zz}(x, y, z)z_x^2 + F_z(x, y, z)z_{xx} \\
\rightarrow z_{xx} &= -\frac{F_{xx} - 2F_{xz}\frac{F_x}{F_z} + F_{zz}\left(\frac{F_x}{F_z}\right)^2}{F_z}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 F(x, y, z)}{\partial y^2} &= 0 \\
&= \frac{\partial}{\partial y} (F_y(x, y, z) + F_z(x, y, z)z_y) \\
&= F_{yy}(x, y, z) + 2F_{yz}(x, y, z)z_y + F_{zz}(x, y, z)z_y^2 + F_z(x, y, z)z_{yy} \\
\rightarrow z_{yy} &= -\frac{F_{yy} - 2F_{yz}\frac{F_y}{F_z} + F_{zz}\left(\frac{F_y}{F_z}\right)^2}{F_z}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 F(x, y, z)}{\partial x \partial y} &= 0 \\
&= \frac{\partial}{\partial x} (F_y(x, y, z) + F_z(x, y, z)z_y) \\
&= F_{xy}(x, y, z) + F_{yz}(x, y, z)z_x + F_{xz}(x, y, z)z_y + F_{zz}(x, y, z)z_x z_y + F_z(x, y, z)z_{xy} \\
\rightarrow z_{xy} &= -\frac{F_{xy} - F_{yz}\frac{F_x}{F_z} - F_{xz}\frac{F_y}{F_z} + F_{zz}\frac{F_y F_x}{F_z^2}}{F_z}
\end{aligned}$$

$$\begin{aligned}
\rightarrow K &= \frac{z_{xx}z_{yy} - z_{xy}^2}{(1 + z_x^2 + z_y^2)^2} \\
&= \frac{\left( \frac{F_{xx} - 2F_{xz}\frac{F_x}{F_z} + F_{zz}\left(\frac{F_x}{F_z}\right)^2}{F_z} \right) \left( \frac{F_{yy} - 2F_{yz}\frac{F_y}{F_z} + F_{zz}\left(\frac{F_y}{F_z}\right)^2}{F_z} \right) - \left( \frac{F_{xy} - F_{yz}\frac{F_x}{F_z} - F_{xz}\frac{F_y}{F_z} + F_{zz}\frac{F_x F_y}{F_z^2}}{F_z} \right)^2}{\left( 1 + \left( \frac{F_x}{F_z} \right)^2 + \left( \frac{F_y}{F_z} \right)^2 \right)^2} \\
&= \frac{1}{F_z^2(F_x^2 + F_y^2 + F_z^2)^2} \left( (F_{xx}F_z^2 - 2F_{xz}F_xF_z + F_{zz}F_x^2)(F_{yy}F_z^2 - 2F_{yz}F_yF_z + F_{zz}F_y^2) \right. \\
&\quad \left. - (F_{xy}F_z^2 - F_{yz}F_xF_z - F_{xz}F_yF_z + F_{zz}F_xF_y)^2 \right) \\
&= \frac{1}{F_z^2(F_x^2 + F_y^2 + F_z^2)^2} \left( F_{xx}F_{yy}F_z^4 - 2F_{xz}F_{yy}F_xF_z^3 + F_{zz}F_{yy}F_z^2F_x^2 - 2F_{xx}F_{yz}F_yF_z^3 \right. \\
&\quad + 4F_{xz}F_{yz}F_xF_yF_z^2 - 2F_{zz}F_{yz}F_yF_zF_x^2 + F_{xx}F_{zz}F_y^2F_z^2 - 2F_{xz}F_{zz}F_y^2F_xF_z + F_{zz}^2F_y^2F_x^2 \\
&\quad - F_{xy}^2F_z^4 - F_{yz}^2F_x^2F_z^2 - F_{xz}^2F_y^2F_z^2 - F_{zz}^2F_x^2F_y^2 + 2F_{xy}F_{yz}F_xF_z^3 + 2F_{xy}F_{xz}F_yF_z^3 \\
&\quad \left. - 2F_{xy}F_{zz}F_xF_yF_z^2 - 2F_{yz}F_{xz}F_yF_xF_z^2 + 2F_{yz}F_{zz}F_yF_x^2F_z + 2F_{xz}F_{zz}F_xF_y^2F_z \right) \\
&= \frac{1}{(F_x^2 + F_y^2 + F_z^2)^2} \left( F_{xx}F_{yy}F_z^2 - 2F_{xz}F_{yy}F_xF_z + F_{zz}F_{yy}F_x^2 - 2F_{xx}F_{yz}F_yF_z \right. \\
&\quad + 2F_{xz}F_{yz}F_xF_y + F_{xx}F_{zz}F_y^2 - F_{xy}^2F_z^2 - F_{yz}^2F_x^2 - F_{xz}^2F_y^2 + 2F_{xy}F_{yz}F_xF_z \\
&\quad \left. + 2F_{xy}F_{xz}F_yF_z - 2F_{xy}F_{zz}F_xF_y \right) \\
&= \frac{1}{(F_x^2 + F_y^2 + F_z^2)^2} \left( (F_{zz}F_{yy} - F_{yz}^2)F_x^2 + (F_{xx}F_{zz} - F_{xz}^2)F_y^2 + (F_{xx}F_{yy} - F_{xy}^2)F_z^2 \right. \\
&\quad + 2(F_{xy}F_{yz} - F_{xz}F_{yy})F_xF_z + 2(F_{xy}F_{xz} - F_{xx}F_{yz})F_yF_z \\
&\quad \left. + 2(F_{xz}F_{yz} - F_{xy}F_{zz})F_xF_y \right) \\
&= \frac{\nabla F^t A \nabla F}{\|\nabla F\|^4}
\end{aligned}$$

□

**Problem 3** (Ex P168 4). *Determine the asymptotic curves and the lines of curvature of  $z = xy$ .*

*Proof.*

$$\begin{aligned}
\mathbb{X} &= (u, v, uv) \\
\rightarrow \mathbb{X}_u &= (1, 0, v) \\
\mathbb{X}_v &= (0, 1, u) \\
E &= \langle \mathbb{X}_u, \mathbb{X}_u \rangle = 1 + v^2 \\
F &= \langle \mathbb{X}_u, \mathbb{X}_v \rangle = uv \\
G &= \langle \mathbb{X}_v, \mathbb{X}_v \rangle = 1 + u^2
\end{aligned}$$

$$\begin{aligned}
N &= \frac{\mathbb{X}_u \times \mathbb{X}_v}{|\mathbb{X}_u \times \mathbb{X}_v|} \\
&= \frac{1}{\sqrt{1+u^2+v^2}}(-v, -u, 1) \\
\mathbb{X}_{uu} &= (0, 0, 0) \\
\mathbb{X}_{uv} &= (0, 0, 1) \\
\mathbb{X}_{vv} &= (0, 0, 0) \\
\rightarrow e &= \langle N, \mathbb{X}_{uu} \rangle = 0 \\
f &= \langle N, \mathbb{X}_{uv} \rangle = \frac{1}{\sqrt{1+u^2+v^2}} \\
g &= \langle N, \mathbb{X}_{vv} \rangle = 0
\end{aligned}$$

Asymptotic curves:

$$\begin{aligned}
eu'^2 + 2fu'v' + gv'^2 &= 0 \\
\rightarrow u'v' &= 0 \\
\rightarrow u = \text{const or } v = \text{const}
\end{aligned}$$

line of curvature:

$$\begin{aligned}
&\begin{vmatrix} v'^2 & -u'v' & u'^2 \\ 1+v^2 & uv & 1+u^2 \\ 0 & f & 0 \end{vmatrix} = 0 \\
&\rightarrow (1+u^2)v'^2 = (1+v^2)u'^2 \\
&\rightarrow \int \frac{1}{\sqrt{1+v^2}} dv = \pm \int \frac{1}{\sqrt{1+u^2}} du \\
&\rightarrow \sinh^{-1} v = \pm \sinh^{-1} u + C \\
&\rightarrow v = \pm u \cosh C + \sqrt{1+u^2} \sinh C
\end{aligned}$$

□

**Problem 4.** 已知  $\mathbb{X}(u, v)$  為一 surface  $\subset \mathbb{R}^3$  且  $E = G = (1+u^2+v^2)^2, F = 0$  而且  $e = 1, f = \sqrt{3}, g = -1$

(a) 求在  $\mathbb{X}(1, 1)$  的  $K$  與  $H$

(b) 如何決定過  $\mathbb{X}(1, 1)$  的 line of curvature 與 asymptotic curve (如果有的話)

*Proof.* (a) at  $(1, 1)$ ,  $E = G = 9, F = 0$ .

$$\begin{aligned}
[-dN] &= \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \\
&= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{9} & \frac{1}{3\sqrt{3}} \\ \frac{1}{3\sqrt{3}} & -\frac{1}{9} \end{bmatrix}
\end{aligned}$$

So  $K = \det([-dN]) = -\frac{4}{81}$ ,  $H = \text{tr}([-dN]) = 0$ .

(b) line of curvature:

$$\begin{aligned}
&\begin{vmatrix} v'^2 & -u'v' & u'^2 \\ (1+u^2+v^2)^2 & 0 & (1+u^2+v^2)^2 \\ 1 & \sqrt{3} & -1 \end{vmatrix} = 0 \\
&\rightarrow (1+u^2+v^2)^2(-\sqrt{3}v'^2 - 2u'v' + \sqrt{3}u'^2) = 0 \\
&\quad \rightarrow (-\sqrt{3}v' + u')(v' + \sqrt{3}u') = 0 \\
&\quad \rightarrow -\sqrt{3}v + u = \text{const or } v + \sqrt{3}u = \text{const} \rightarrow -\sqrt{3}v + u = 1 - \sqrt{3} \text{ or } v = \sqrt{3}u - 1
\end{aligned}$$

asymptotic curve:

$$\begin{aligned}
&u'^2 + 2\sqrt{3}u'v' - v'^2 = 0 \\
&\rightarrow (u' + (\sqrt{3} - 2)v')(u' + (\sqrt{3} + 2)v') = 0 \\
&\quad \rightarrow u' + (\sqrt{3} - 2)v' = 0 \text{ or } u' + (\sqrt{3} + 2)v' = 0 \\
&\quad \rightarrow u + ((\sqrt{3} - 2)v = \text{const or } u + (\sqrt{3} + 2)v = \text{const} \\
&\quad \rightarrow u + ((\sqrt{3} - 2)v = \sqrt{3} - 1 \text{ or } u + (\sqrt{3} + 2)v = \sqrt{3} + 3
\end{aligned}$$

□

**Problem 5.**  $\mathbb{X}(u, v) = (v \cos u, v \sin u, u)$ , 令  $\gamma(t) = \mathbb{X}(t, 1)$

(a) 求  $\gamma(t)$  的  $\kappa_n, \kappa_g, \tau_g$

(b) 與  $\gamma(t)$  的  $\kappa, \tau$  有何關係

*Proof.* (a)

$$\begin{aligned}
\gamma(t) &= (\cos t, \sin t, t) \\
\rightarrow \gamma'(t) &= (-\sin t, \cos t, 1) \\
T(t) &= \frac{1}{2}(-\sin t, \cos t, 1) \\
T'(t) &= \frac{1}{2}(-\cos t, -\sin t, 0) \\
\rightarrow n(t) &= (-\cos t, -\sin t, 0) \\
\kappa &= \frac{1}{2} \\
X_u &= (-v \sin u, v \cos u, 1) \\
X_v &= (\cos u, \sin u, 0) \\
\rightarrow N &= \frac{1}{1+v^2}(-\sin u, \cos u, -v) \\
\kappa_n &= \langle N, T' \rangle \\
&= 0 \\
\kappa_g &= \kappa = \frac{1}{2} \\
A &= n \\
A' &= (\sin t, -\cos t, 0) \\
\tau_g &= -\langle A', N \rangle \\
&= -\frac{1}{2}
\end{aligned}$$

(b)

$$\begin{aligned}
b &= T \times n \\
&= \frac{1}{2}(\sin t, -\cos t, 1) \\
\tau &= -\langle n', b \rangle \\
&= -\frac{1}{2} \\
\rightarrow \kappa_g &= \kappa \\
\tau_g &= \tau
\end{aligned}$$

□

**Problem 6.** 令  $(x(t), y(t)) = (t - \tanh t, \operatorname{sech} t)$  這基本就是 p7(4) 的 *tratrix*

(a) 將此曲線化作長度參數

(b) 利用上小題，求此曲線繞  $x$  軸旋轉的旋轉體的  $K$

*Proof.*

- (a) Let  $\alpha(t) = (x(t), y(t))$ . Then  $\alpha'(t) = (1 - \operatorname{sech}^2 t, \operatorname{sech} t \tanh t) = (\tanh^2 t, \operatorname{sech} t \tanh t)$ .  
So the length of the curve  $s$  when  $t \geq 0$  is given by

$$\begin{aligned} s &= \int \|\alpha'(t)\| dt \\ &= \int \tanh t \sqrt{\tanh^2 t + \operatorname{sech}^2 t} dt \\ &= \int \tanh t dt \\ &= \ln(\cosh t) + C_0 \end{aligned}$$

We wish  $s$  to be arc-length parameter according to  $t = 0$ , so when  $t = 0$  we need  $s = 0$ . This implies  $C_0 = 0$ .

Therefore, we have the arc-length parameter  $s \geq 0$  with  $t = \cosh^{-1}(e^s)$ . Bring it back and by symmetry we would get

$$\alpha(s) = \left( \operatorname{sign}(s)(\cosh^{-1}(e^{|s|}) - \tanh(\cosh^{-1}(e^{|s|}))), \operatorname{sech}(\cosh^{-1}(e^{|s|})) \right)$$

To check  $s$  is the arc-length parameter we can simply check its derivatives.

- (b) Let  $\mathbb{X}(u, v) = (x(u), y(u) \cos v, y(u) \sin v)$ , where  $u \in \mathbb{R}$  and  $0 < v < 2\pi$  and  $u$  is exactly  $s$ , the arc-length parameter described in (a). Then,

$$\begin{aligned} \mathbb{X}_u &= (x'(u), y'(u) \cos v, y'(u) \sin v) \\ \mathbb{X}_v &= (0, -y(u) \sin v, y(u) \cos v) \\ E &= \langle \mathbb{X}_u, \mathbb{X}_u \rangle = x'^2(u) + y'^2(u) = 1 \\ F &= \langle \mathbb{X}_u, \mathbb{X}_v \rangle = 0 \\ G &= \langle \mathbb{X}_v, \mathbb{X}_v \rangle = y^2(u) \\ N &= \frac{\mathbb{X}_u \times \mathbb{X}_v}{\|\mathbb{X}_u \times \mathbb{X}_v\|} = (y'(u), -x'(u) \cos v, -x'(u) \sin v) \\ \mathbb{X}_{uu} &= (x''(u), y''(u) \cos v, y''(u) \sin v) \\ \mathbb{X}_{uv} &= (0, -y'(u) \sin v, y'(u) \cos v) \\ \mathbb{X}_{vv} &= (0, -y(u) \cos v, -y(u) \sin v) \\ e &= \langle N, \mathbb{X}_{uu} \rangle = x''(u)y'(u) - x'(u)y''(u) \\ f &= \langle N, \mathbb{X}_{uv} \rangle = 0 \\ g &= \langle N, \mathbb{X}_{vv} \rangle = x'(u)y(u) \end{aligned}$$

Hence the Gaussian curvature  $K$  equals

$$K = \frac{eg - f^2}{EG - F^2} = \frac{x'(u)x''(u)y'(u) - x'^2(u)y''(u)}{y(u)}$$

Now, we have when  $u \geq 0$ ,

$$\begin{aligned} x'(u) &= (e^u - e^{-u})/\sqrt{e^{2u} - 1} \\ x''(u) &= (e^u + \frac{1}{2}e^{-u})/\sqrt{e^{2u} - 1} \\ y(u) &= \operatorname{sech}(\cosh^{-1} e^u) = e^{-u} \\ y'(u) &= -e^{-u} \end{aligned}$$



Therefore, by symmetry, finally we get

$$K = \frac{(x''(u)y'(u) - x'(u)y''(u))x'(u)}{y(u)} = 2 - \frac{1}{2}e^{-2|u|}$$

□