# Geometry Homework 12

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**Problem 3** (Ex p294 3.). *If $p$ is a point of a regular surface $S$, prove that*

$$K(p) = \lim_{r\to 0} \frac{12}{\pi} \frac{\pi r^2 - A}{r^4},$$

*where $K(p)$ is the Gaussian curvature of $S$ at $p$, $r$ is the radius of a geodesic circle $S_r(p)$ centered in $p$, and $A$ is the area of the region bounded by $S_r(p)$.*

*Proof.*

$$\begin{aligned}
A_R &= \int_0^R \int_0^{2\pi} \sqrt{EG - F^2} d\theta dr \\
&= \int_0^R \int_0^{2\pi} \sqrt{G} d\theta dr \\
&\approx \int_0^R \int_0^{2\pi} r - \frac{K}{6} r^3 d\theta dr \\
&= \int_0^{2\pi} \frac{1}{2} R^2 - \frac{K}{24} R^4 d\theta \\
&= \pi R^2 - \frac{R^4}{24} \int_0^{2\pi} K d\theta \\
\to \frac{1}{2\pi} \int_0^{2\pi} K d\theta &= \frac{12}{r^4}(r^2 - \frac{1}{\pi} A_r) \\
\to K(p) &= \lim_{r\to 0} \frac{12}{r^4}(r^2 - \frac{1}{\pi} A_r) \\
&= \lim_{r\to 0} \frac{12}{\pi} \frac{\pi r^2 - A_r}{r^4}
\end{aligned}$$

□

**Problem 4** (Ex p295 4.). *Show that in a system of normal coordinates centered in $p$, all the Christoffel symbols are zero at $p$.*

**Problem 5** (Ex p295 5.). *For which of the pair of surfaces given below does there exist a local isometry?*

*(a) Torus of revolution and cone.*

(b) *Cone and sphere.*

(c) *Cone and cylinder.*

**Problem 8.**

(a) 在半徑 $R$ 的球面上，計算 *geodesic circle* 的長度，並驗證 *P292* 課文中間 $K(p)$ 的公式。

(b) 用一樣的精神，檢驗 *P294 3.* 的公式。

*Proof.* (a) WLOG, let $p = (0,0,R)$, If $q \in T_p$ with $q = (l,\theta)$, then

$$\exp(q) = \left(R\sin\frac{l}{R}\cos\theta, R\sin\frac{l}{R}\sin\theta, R\cos\frac{l}{R}\right),$$

and thus the length of the image of the circle $\{q \in T_p : d(q,p) = l\}$ is

$$2\pi\left\langle \frac{\partial\exp(q)}{\partial\theta}, \frac{\partial\exp(q)}{\partial\theta}\right\rangle^{1/2} = 2\pi\left|R\sin\frac{l}{R}\right|$$

. When $l \to 0$, it is

$$2\pi R\sin\frac{l}{R}.$$

, which is the length of the geodesic circle. By the formula,

$$\begin{aligned}
K(p) &= \lim_{r\to 0}\frac{3}{\pi}\frac{2\pi r - L}{r^3} = \lim_{r\to 0}\frac{3}{\pi}\frac{2\pi r - 2\pi R\sin\frac{l}{R}}{r^3} \\
&\approx \lim_{r\to 0}\frac{3}{\pi}\frac{2\pi r - 2\pi R\left(r/R - \left(\frac{r}{R}\right)^3/6\right)}{r^3} \\
&= \lim_{r\to 0}\frac{3}{\pi}\frac{2\pi r\left(\frac{r}{R}\right)^3/6}{r^3} = \frac{1}{R^2}.
\end{aligned}$$

(b) The area bounded by the geodesic circle is

$$2\pi R\int_0^l \left|\sin\frac{r}{R}\right| dr.$$

When $l \to 0$, it is

$$2\pi R^2 - 2\pi R^2\cos\frac{l}{R}.$$

By the formula,

$$\begin{aligned}
K(p) &= \lim_{r\to 0}\frac{12}{\pi}\frac{\pi r^2 - A}{r^4} = \lim_{r\to 0}\frac{12}{\pi}\frac{\pi r^2 + 2\pi R^2\cos\frac{r}{R} - 2\pi R^2}{r^4} \\
&\approx \lim_{r\to 0}\frac{12}{\pi}\frac{\pi r^2 + 2\pi R^2\left(1 - \left(\frac{r}{R}\right)^2/2 + \left(\frac{r}{R}\right)^4/24\right) - 2\pi R^2}{r^4} \\
&= \lim_{r\to 0}\frac{12}{\pi}\frac{\pi R^2\left(\frac{r}{R}\right)^4/12}{r^4} = \frac{1}{R^2}
\end{aligned}$$

□