Geometry Homework 11

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Problem 4 (Ex p261 8.)**.** *Show that if all the geodesics of a connected surface are plane curves, then the surface is contained in a plane or a sphere.*

Proof. □

Problem 5 (Ex p262 17.)**.** *Let $\alpha : I \to S$ be a curve parametrized by arc length s, with nonzero curvature. Consider the parametrized surface*

$$\mathbf{x}(s,v) = \alpha(s) + vb(s), \quad s \in I, -\epsilon < v < \epsilon, \epsilon > 0,$$

where b is the binormal vector of α. Prove that if ϵ is small, $\mathbf{x}(I \times (-\epsilon, \epsilon)) = S$ is a regular surface over which $\alpha(I)$ is geodesic. (thus, every curve is a geodesic on the surface generated by its binormals).

Proof.

$$\begin{aligned} \mathbf{x}_s &= \alpha'(s) + vb'(s) \\ &= t(s) + v\tau(s)n(s) \\ \mathbf{x}_v &= b(s) \\ \to \mathbf{x}_s \times \mathbf{x}_v &= -n(s) + v\tau(s)t(s) \\ &\neq 0 \end{aligned}$$

So $\mathbf{x}$ is a regualr surface.
Since $\alpha''(s) = n(s)$, and at $v = 0$, $N \parallel \mathbf{x}_s \times \mathbf{x}_v = -n(s)$. So $\alpha'(s) \parallel N$, and $\kappa_g = 0$. So $\alpha(I)$ is geodesic. □

Problem 8. 用 *(A)* 表示在座標變換下不變、用 *(B)* 表示在 *isometry* 下不變 *(*保 *E, F, G)* 下的性質

	line of curvature	*geodesic*	*asymptotic curve*	Γ_{ij}^k	H	K
(A)	$Yes_{(1)}$	$Yes_{(1)}$	$Yes_{(1)}$	$No_{(2)}$	$Yes_{(1)}$	$Yes_{(1)}$
(B)	*(2)*	$Yes_{problem\ 9(a)}$	*(6)*	$Yes_{(3)}$	No	$Yes_{(4)}$

Proof. (1) Since curves, surface, T,A,N, t,n,b are all geometry objects, κ_n, κ_g, τ_g are geometry objects too. So line of curvature, geodesic, asymptotic curve are also geometry objects. Since principal direction and principal curvature are geometry objects too, H and K are geometry objects.

(2) Consider an surface $\mathbb{X}(u,v)$ and $\hat{\mathbb{X}}(u,v) = \mathbb{X}(v,u)$, it's trivial that $\hat{\Gamma}^2_{11} = \Gamma^1_{22}$, so $\hat{\Gamma}^2_{11} \neq \Gamma^2_{11}$ when $\Gamma^2_{11} \neq \Gamma^1_{22}$, and it's trivial to find a surface with $\Gamma^2_{11} \neq \Gamma^1_{22}$ (For example, $\mathbb{X}(u,v) = (u\cos v, u\sin v, 0)$. As shown in HW 10, $\Gamma^1_{22} = -u \neq \Gamma 11^2 = 0$).

(3) Since $\Gamma^k_{ij} = g^{kl}[i,j,l]$, and both g^{kl} and $[i,j,l]$ only depends on g_{ij}, Γ^k_{ij} is same in isometry.

(4) Gauss Theorema Egregium.

$\square$

Problem 9. 考慮 *p221, p222* 中 *helicoid* Y 和 *catenoid* X 的 *parametrization*。
$X(u,v) = (a\cosh v\cos u, a\cosh v\sin u, av)$, $Y(u,v) = (a\sinh v\cos u, a\sinh v\sin u, au)$

(a) X 中的 *geodesics* 相對應映到 Y 中也是 *geodesics* 嗎？

(b) 已知 X 的經線 *(*$u =$ *const)* 與 $v = 0$ 都是 *geodesics*。描述他們在 Y 中的對應曲線？他們都是 *geodesics* 嗎？

Proof. (a) Since X and Y are isometry, they have the same Γ^k_{ij}, and have the same geodesic equation. So geodesic in X is also geodesic in Y.

(b) when $u =$ const, $Y(u,v)$ is a line $(a\cos C\sinh v, a\sin C\sinh v, aC)$, and is a geodesic of Y.
when $v = 0$, $Y(u,v)$ is a line $(0,0,au)$, and is a geodesic of Y.

$\square$