Geometry Homework 5

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Problem 1 (參見 P67 Ex16). 考慮

$$\begin{array}{cccc} \mathbb{X}: & \mathbb{R}^2 & \to & S^2 \setminus \{N\} \\ & (u,v) & \mapsto & \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right) \end{array}, N = (0,0,1)$$

(a) 檢查這的確是 $S^2 \setminus \{N\}$ 的參數式

(b) 計算 E, F, G，$E = G$ 嗎？

(c) 計算 $\mathbb{X}_u, \mathbb{X}_v$

(d) 若 W_1, W_2 是 $\mathbb{R}^2$ 兩以 a 為起點的向量，說明 W_1, W_2 的夾角 $= d\mathbb{X}(W_1)$ 與 $d\mathbb{X}(W_2)$ 的夾角

Proof. □

Problem 2 (旋轉面). $\mathbb{X}(\theta, s) = (a(s)\cos\theta, a(s)\sin\theta, b(s))$，其中 $(a(s), b(s))$ 為長度參數之平面曲線。計算 E, F, G 並討論其 *regular* 的條件。

Proof.

$$\begin{aligned} \mathbb{X}_\theta &= (-a(s)\sin\theta, a(s)\cos\theta, 0) \\ \mathbb{X}_s &= (a'(s)\cos\theta, a'(s)\sin\theta, b'(s)) \\ \Rightarrow E &= a(s)^2\sin^2\theta + a(s)^2\cos^2\theta \\ &= a(s)^2 \\ F &= -a(s)a'(s)\sin\theta\cos\theta + a(s)a'(s)\cos\theta\sin\theta \\ &= 0 \\ G &= a'(s)^2\sin^2\theta + a'(s)^2\cos^2\theta \\ &= a'(s)^2 \\ &= 1 \\ |\mathbb{X}_\theta \times \mathbb{X}_s| &= \sqrt{EG - F^2} \\ &= \sqrt{a(s)^2} \\ &= |a(s)| \end{aligned}$$

So $\mathbb{X}$ is regular iff $a(s) \neq 0$. $\square$

Problem 3 (管面). 設空間曲線 $\gamma(s)$，s 長度參數，$\vec{t}, \vec{n}, \vec{b}$ 為 *Frenet frame*。令 $\mathbb{X}_l(s,\theta) = \gamma(s) + l\cos\theta\vec{n}(s) + l\sin\theta\vec{b}(s), l > 0$，計算 E, F, G 並討論其 *regular* 條件。

Proof.

$$
\begin{aligned}
\mathbb{X}_{ls} &= \gamma'(s) + l\cos\theta\vec{n}'(s) + l\sin\theta\vec{b}'(s) \\
&= \vec{t}(s) + l\cos\theta\left(-\kappa(s)\vec{t}(s) - \tau(s)\vec{b}(s)\right) + l\tau(s)\sin\theta\vec{n}(s) \\
&= (1 - l\kappa(s)\cos\theta)\vec{t}(s) + l\tau(s)\sin\theta\vec{n}(s) - l\tau(s)\cos\theta\vec{b}(s) \\
\mathbb{X}_{l\theta} &= -l\sin\theta\vec{n}(s) + l\cos\theta\vec{b}(s) \\
\Rightarrow E &= (1 - l\kappa(s)\cos\theta)^2 + (l\tau(s)\sin\theta)^2 + (l\tau(s)\cos\theta)^2 \\
&= (1 - l\kappa(s)\cos\theta)^2 + l^2\tau(s)^2 \\
F &= -l^2\tau(s)\sin^2\theta - l^2\tau(s)\cos^2\theta \\
&= -l^2\tau(s) \\
G &= l^2\sin^2\theta + l^2\cos^2\theta \\
&= l^2 \\
|\mathbb{X}_{ls} \times \mathbb{X}_{l\theta}| &= \sqrt{EG - F^2} \\
&= \sqrt{l^2\left((1 - l\kappa(s)\cos\theta)^2 + l^2\tau(s)^2\right) - l^4\tau(s)^2} \\
&= \sqrt{l^2(1 - l\kappa(s)\cos\theta)^2} \\
&= l\,|1 - l\kappa(s)\cos\theta|
\end{aligned}
$$

So $\mathbb{X}$ is regualr iff $l\kappa(s)\cos\theta \neq 1 \forall s, \theta$
$\Leftrightarrow \frac{1}{l\kappa(s)} \neq \cos\theta$
$\Leftrightarrow \left|\frac{1}{l\kappa(s)}\right| > 1$
$\Leftrightarrow |\kappa(s)| < \frac{1}{l}$

$\square$

Problem 6 (Ex6, p100). *Show that*

$$\mathbf{x}(u, v) = (u\sin\alpha\cos v, u\sin\alpha\sin v, u\cos\alpha)$$

where $0 < u < \infty, 0 < v < 2\pi, \alpha = const.$, *is a parametrization of the cone with* 2α *as the angle of the vertex. In the corresponding coordinate neighborhood, prove that the curve*

$$\mathbf{x}(c\exp(v\sin\alpha\cot\beta), v), \quad c = const., \beta = const.,$$

intersects the generators of the cone ($v = const.$) *under the constant angle* β.

Proof.

$$d\mathbf{x} = \begin{bmatrix} \sin\alpha\cos v & \sin\alpha\sin v & \cos\alpha \\ -u\sin\alpha\sin v & u\sin\alpha\cos v & 0 \end{bmatrix}$$

We have $\mathbf{x}_u \times \mathbf{x}_v = (-,-,u) \neq 0$ whenever $u > 0$. To show that the angle is β, define

$$\begin{aligned} E &:= \langle \mathbf{x}_u, \mathbf{x}_u \rangle = \sin^2\alpha\cos^2 v + \sin^2\alpha\sin^2 v + \cos^2\alpha) = 1 \\ F &:= \langle \mathbf{x}_u, \mathbf{x}_v \rangle = -u\sin^2\alpha\cos v\sin v + u\sin^2\alpha\sin v\cos v + 0 = 0 \\ G &:= \langle \mathbf{x}_v, \mathbf{x}_v \rangle = u^2\sin^2\alpha\sin^2 v + u^2\sin^2\alpha\cos^2 v + 0 = u^2\sin^2\alpha \\ \langle\langle (a,b),(c,d) \rangle\rangle &:= \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \end{aligned}$$

to be the first fundamental form of $\mathbf{x}$. Then

$$\begin{aligned} &A := \partial(u,v)/\partial u = (1,0) \\ &B := \partial(c\exp(v\sin\alpha\cot\beta), v)/\partial v = (c\sin\alpha\cot\beta\exp(v\sin\alpha\cot\beta), 1). \end{aligned}$$

Fixing the intersection at $(c\exp(v\sin\alpha\cot\beta), v)$, we got

$$\begin{aligned} \langle\langle A,B \rangle\rangle &= \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & c^2\exp(2v\sin\alpha\cot\beta)\sin^2\alpha \end{bmatrix} \begin{bmatrix} c\sin\alpha\cot\beta\exp(v\sin\alpha\cot\beta) \\ 1 \end{bmatrix} \\ &= c\sin\alpha\cot\beta\exp(v\sin\alpha\cot\beta) \\ \langle\langle A,A \rangle\rangle &= \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & c^2\exp(2v\sin\alpha\cot\beta)\sin^2\alpha \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= 1 \\ \langle\langle B,B \rangle\rangle &= \begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \\ &= \begin{bmatrix} c\sin\alpha\cot\beta\exp(v\sin\alpha\cot\beta) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & c^2\exp(2v\sin\alpha\cot\beta)\sin^2\alpha \end{bmatrix} \\ &\quad \begin{bmatrix} c\sin\alpha\cot\beta\exp(v\sin\alpha\cot\beta) \\ 1 \end{bmatrix} \\ &= c^2\sin^2\alpha\cot^2\beta\exp(2v\sin\alpha\cot\beta) + c^2\sin^2\alpha\exp(2v\sin\alpha\cot\beta) \\ &= c^2\sin^2\alpha\csc^2\beta\exp(2v\sin\alpha\cot\beta) \\ \cos\theta &= \frac{\langle\langle A,B \rangle\rangle}{\sqrt{\langle\langle A,A \rangle\rangle\langle\langle B,B \rangle\rangle}} = \frac{c\sin\alpha\cot\beta\exp(v\sin\alpha\cot\beta)}{|c\sin\alpha\csc\beta\exp(v\sin\alpha\cot\beta)|} = \pm\cos\beta \end{aligned}$$

$\square$