GEOMETRY HOMEWORK 5

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Problem 1 (Ex P151 2). *Show that if a surface is tangent to a plane along a curve, then the points of this curve are either parabolic or planar.*

Proof. Assume that the curve is $\gamma(s)$, then along this curve, $N(\gamma(s))$ is perpendicular to the plane, so it is constant.
At point $\gamma(s)$, $[dN](\gamma'(s)) = \left(\frac{dN(\gamma(t))}{dt}\right)_{t=s} = 0$, so $\gamma'(s)$ is one of the principal direction of the surface at $\gamma(s)$, and it's associated principal curvature is 0.
So the gaussian curvature of the surface at $\gamma(s)$ is $K = 0$, and this means that the point $\gamma(s)$ is either parabolic or planar. □

Problem 3 (Ex P151 3).

(a) *Let $C \subset S$ be a regular curve on a surface S with Gaussian curvature $K > 0$. Show that the curvature κ of C at p satisfies*

$$\kappa \geq \min(|\kappa_1|, |\kappa_2|),$$

where κ_1, κ_2 are the principal curvatures of S at p.

(b) 為什麼上一小題需要 $\kappa > 0$ 的條件，$\kappa \geq 0$ 不可以嗎？

Proof. (a)

$$\begin{aligned}
\kappa &\geq |\kappa_n| \\
&= |\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta| \\
&= |\kappa_1|\cos^2\theta + |\kappa_2|\sin^2\theta (\because \kappa_1, \kappa_2 \text{ has equal sign.}) \\
&\geq \min(|\kappa_1|, |\kappa_2|)(\cos^2\theta + \sin^2\theta) \\
&= \min(|\kappa_1|, |\kappa_2|)
\end{aligned}$$

(b)

□

Problem 7.

(a) T_λ 是縮放 λ 倍的映射，$\lambda > 0$。$\mathbb{X} : \Omega \to \mathbb{R}^3$ *regular surface*。討論 $T_\lambda \circ \mathbb{X} : \Omega \to \mathbb{R}^3$ 上對應點 κ_n, H, K 的變化。

(b) $\mathbb{X}: \begin{array}{c}\Omega \\ (u,v)\end{array} \to \mathbb{R}^3$，若定義 $\overline{\mathbb{X}}(u,v) = \mathbb{X}(v,u)$(因此 N 轉向)。討論 $\overline{\mathbb{X}}(\Omega)$ 上相對應點的 K_n, H, K 變化。

Proof. $\square$

Problem 9 (旋轉面)**.** $\mathbb{X}(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$，$f > 0$

(a) 計算其 e, f, g, H, K

(b) 討論其 *principal direction* 與 *principal curvature* K_1, K_2。

Proof. To avoid the notational ambiguity, let $\mathbb{X}(u,v) = (s(u)\cos v, s(u)\sin v, t(u))$, and that $s > 0$.

(a) We have

$$\begin{aligned}
\mathbb{X}_u =&(s'(u)\cos v, s'(u)\sin v, t'(u));\\
\mathbb{X}_v =&(-s(u)\sin v, s(u)\cos v, 0);\\
E =&\langle \mathbb{X}_u, \mathbb{X}_u\rangle = s'(u)^2 + t'(u)^2\\
F =&\langle \mathbb{X}_u, \mathbb{X}_v\rangle = 0\\
G =&\langle \mathbb{X}_v, \mathbb{X}_v\rangle = s(u)^2\\
\mathbb{X}_{uu} =&(s''(u)\cos v, s''(u)\sin v, t''(u));\\
\mathbb{X}_{uv} =&(-s'(u)\sin v, s'(u)\cos v, 0);\\
\mathbb{X}_{vv} =&(-s(u)\cos v, -s(u)\sin v, 0);\\
N =&\frac{\mathbb{X}_u \times \mathbb{X}_v}{|\mathbb{X}_u \times \mathbb{X}_v|} = \frac{(-t'(u)s(u)\cos v, -t'(u)s(u)\sin v, s'(u)s(u))}{\sqrt{t'(u)^2 s(u)^2 + s'(u)^2 s(u)^2}}\\
=&\frac{(-t'(u)\cos v, -t'(u)\sin v, s'(u))}{\sqrt{t'(u)^2 + s'(u)^2}};\\
e =&\langle N, \mathbb{X}_{uu}\rangle = \frac{-s''(u)t'(u) + t''(u)s'(u)}{\sqrt{t'(u)^2 + s'(u)^2}}\\
f =&\langle N, \mathbb{X}_{uv}\rangle = 0\\
g =&\langle N, \mathbb{X}_{vv}\rangle = \frac{s(u)t'(u)}{\sqrt{t'(u)^2 + s'(u)^2}}\\
-dN =&\begin{bmatrix} e & f \\ f & g\end{bmatrix}\begin{bmatrix} E & F \\ F & G\end{bmatrix}^{-1} = \frac{1}{EG - F^2}\begin{bmatrix} e & f \\ f & g\end{bmatrix}\begin{bmatrix} G & -F \\ -F & E\end{bmatrix} = \begin{bmatrix} e/E & 0 \\ 0 & g/G\end{bmatrix}\\
K =&\det(-dN) = \frac{eg}{EG}\\
H =&\frac{1}{2}\mathrm{tr}(-dN) = \frac{eG + gE}{2EG}
\end{aligned}$$

(b) Since $-dN$ is already a diagonal matrix, clearly,

$$\begin{aligned}K_1 &= e/E;\\ K_2 &= g/G;\\ V_1 &= \mathbb{X}_u;\\ V_2 &= \mathbb{X}_v;\end{aligned}$$

□

Problem 10 (管面). $\mathbb{X}(s,\theta) = \gamma(s) + \cos\theta\vec{n}(s) + \sin\theta\vec{b}(s),\ 0 < \kappa < 1$

(a) 計算其 e, f, g, H, K

(b) 討論曲面上 K 的分佈。

Proof. □