Geometry Homework 4

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Problem 3.

(a) 假設 $\kappa(s) \neq 0, \tau(s) \neq 0$，由四點決定一球，討論空間曲線 $\gamma(s)$ 的密切球，並決定球心與半徑。

(b) 討論螺線 $(a\cos t, a\sin t, bt)$ 的密切球，$a > 0$。

Proof. (a) Assume that the sphere is $|X - C| = R$

$$\begin{aligned}
\to \langle X - C, X - C\rangle &= R^2 \\
\to \langle X - C, X - C\rangle' &= 0 \\
&= 2\langle X - C, T\rangle \\
\to \langle X - C, T\rangle' &= 0 \\
&= \langle T, T\rangle + \langle X - C, T'\rangle \\
&= 1 + \kappa\langle X - C, N\rangle \\
\to (\kappa\langle X - C, N\rangle)' &= 0 \\
&= \kappa'\langle X - C, N\rangle + \kappa\langle T, N\rangle + \kappa\langle X - C, N'\rangle \\
&= \kappa'\langle X - C, N\rangle + \kappa\langle X - C, -\kappa T - \tau B\rangle \\
&= \kappa'\langle X - C, N\rangle - \kappa\tau\langle X - C, B\rangle \\
\to \langle X - C, T\rangle &= 0 \\
\langle X - C, N\rangle &= -\frac{1}{\kappa} \\
\langle X - C, B\rangle &= -\frac{\kappa'}{\kappa^2\tau} \\
\to X - C &= -\frac{1}{\kappa}N - \frac{\kappa'}{\kappa^2\tau}B \\
\to C &= X + \frac{1}{\kappa}N + \frac{\kappa'}{\kappa^2\tau}B \\
R &= |X - C| \\
&= \sqrt{\frac{1}{\kappa^2} + \frac{\kappa'^2}{\kappa^4\tau^2}}
\end{aligned}$$

□

Problem 4. $\kappa \neq 0, \tau \neq 0$ 為兩常數，請決定 $\kappa(s) = \kappa, \tau(s) = \tau$ 的曲線方程式。*(*長度參數 *s)*

Proof. Upto translations and rotations, all space curves $\alpha(s)$ satisfying the condition are of the following form

$$\alpha(s) = (\frac{\kappa}{\kappa^2+\tau^2}\sin\sqrt{\kappa^2+\tau^2}s, \frac{\kappa}{\kappa^2+\tau^2}\cos\sqrt{\kappa^2+\tau^2}s, \frac{\tau}{\sqrt{\kappa^2+\tau^2}}s)$$
$$T(s) = (\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\cos\sqrt{\kappa^2+\tau^2}s, -\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\sin\sqrt{\kappa^2+\tau^2}s, \frac{\tau}{\sqrt{\kappa^2+\tau^2}})$$
$$\|T(s)\| = 1 \quad \text{(arc-length)}$$
$$T'(s) = (-\kappa\sin\sqrt{\kappa^2+\tau^2}s), -\kappa\cos\sqrt{\kappa^2+\tau^2}s, 0)$$
$$\kappa(s) = \|T(s)\| = \kappa$$
$$N(s) = (-\sin\sqrt{\kappa^2+\tau^2}s), -\cos\sqrt{\kappa^2+\tau^2}s, 0)$$
$$B(s) = T(s)\times N(s) = (\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\cos\sqrt{\kappa^2+\tau^2}s, -\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\sin\sqrt{\kappa^2+\tau^2}s, -\frac{\kappa}{\sqrt{\kappa^2+\tau^2}})$$
$$B'(s) = (-\tau\sin\sqrt{\kappa^2+\tau^2}s, -\tau\cos\sqrt{\kappa^2+\tau^2}s, 0)$$
$$\tau(s) = B'(s)/N(s) = \tau$$

□

Problem 5 (Darboux vector)**.** $\gamma(s)$ *arc length*

(a) 說明 $\exists$ *vector* $\omega(s)$ *(called* Darboux vector*) such that*

$$\begin{cases} T' &= \omega\times T \\ N' &= \omega\times N \\ B' &= \omega\times B \end{cases}$$

(b) $V(s)$ *is a vector along* $\gamma(s)$ 且 *w.r.t* (T,N,B), $V(s) = (v_1(s), v_2(s), v_3(s))$ $\Rightarrow V' = (v_1', v_2', v_3') + \omega\times V$

(c) 說明 $\omega = \frac{1}{2}(T\times T' + N\times N' + B\times B')$

Problem 8.

(a) 令函數 $x_i : \mathbb{R}^n \to \mathbb{R}.(x_1,\cdots,x_n) \mapsto x_i$。計算 $[dx_i]$，在不同的 $a \in \mathbb{R}^n$，dx_i 如何隨 a 變化。

(b) 由上題微分式 $df = \frac{\partial f}{\partial x_1}dx_1 + \cdots + \frac{\partial f}{\partial x_n}dx_n$ 與映射 df 結合起來。

(c) $f : \mathbb{R}^n \to \mathbb{R}^m$，怎麼利用上題幫你計算 df