Geometry Homework 7

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Problem 2. 若 $F(x,y,z)=0$ 定義一 *surface*，證明 $\nabla F \neq 0$ 的地方 *Gauss curvature* $K = \frac{\nabla F^t A \nabla F}{\|\nabla F\|^4}$。其中 A 為 $\partial^2 F = \begin{pmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{pmatrix}$ 的 *adjoint Matrix, i.e.* $A = \det(\partial^2 F)(\partial^2 F)^{-1}$

Proof. 因為 K 為局部性質，而在 $\nabla F \neq 0$ 的地方我們可以使用隱函數定理將其中一維表示為另兩維的函數，WLOG 不妨設 $z = z(x,y)$ 在某點附近。

$$
\begin{aligned}
F(x,y,z(x,y)) &= 0 \\
\mathbb{X}(x,y) &= (x,y,z(x,y)) \\
\to \mathbb{X}_x &= (1,0,z_x) \\
\mathbb{X}_y &= (0,1,z_y) \\
\to N &= \frac{\mathbb{X}_x \times \mathbb{X}_y}{|\mathbb{X}_x \times \mathbb{X}_y|} \\
&= \frac{(-z_x,-z_y,1)}{\sqrt{1+z_x^2+z_y^2}}
\end{aligned}
$$

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\begin{aligned}
E &= \langle \mathbb{X}_x, \mathbb{X}_x \rangle \\
&= 1 + z_x^2 \\
F &= \langle \mathbb{X}_x, \mathbb{X}_y \rangle \\
&= z_x z_y \\
G &= \langle \mathbb{X}_y, \mathbb{X}_y \rangle \\
&= 1 + z_y^2
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{X}_{xx} &= (0, 0, z_{xx}) \\
\mathbb{X}_{xy} &= (0, 0, z_{xy}) \\
\mathbb{X}_{yy} &= (0, 0, z_{yy}) \\
\rightarrow e &= \langle N, \mathbb{X}_{xx} \rangle \\
&= \frac{z_{xx}}{\sqrt{1 + z_x^2 + z_y^2}} \\
f &= \langle N, \mathbb{X}_{xy} \rangle \\
&= \frac{z_{xy}}{\sqrt{1 + z_x^2 + z_y^2}} \\
g &= \langle N, \mathbb{X}_{yy} \rangle \\
&= \frac{z_{yy}}{\sqrt{1 + z_x^2 + z_y^2}} \\
\rightarrow K &= det([-dN]) \\
&= det\left(\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \right) \\
&= det\left(\begin{bmatrix} E & F \\ F & G \end{bmatrix} \right)^{-1} \det\left(\begin{bmatrix} e & f \\ f & g \end{bmatrix} \right) \\
&= \frac{z_{xx} z_{yy} - z_{xy}^2}{(1 + z_x^2 + z_y^2)^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial F(x, y, z(x, y))}{\partial x} &= 0 \\
&= F_x(x, y, z) + F_z(x, y, z) z_x \\
\rightarrow z_x &= -\frac{F_x}{F_z} \\
\frac{\partial F(x, y, z(x, y))}{\partial y} &= 0 \\
&= F_y(x, y, z) + F_z(x, y, z) z_y \\
\rightarrow z_y &= -\frac{F_y}{F_z} \\
\frac{\partial^2 F(x, y, z(x, y))}{\partial x^2} &= 0 \\
&= \frac{\partial}{\partial x}(F_x(x, y, z) + F_z(x, y, z) z_x) \\
&= F_{xx}(x, y, z) + 2F_{xz}(x, y, z) z_x + F_{zz}(x, y, z) z_x^2 + F_z(x, y, z) z_{xx} \\
\rightarrow z_{xx} &= -\frac{F_{xx} - 2F_{xz}\frac{F_x}{F_z} + F_{zz}\left(\frac{F_x}{F_z}\right)^2}{F_z}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 F(x,y,z(x,y))}{\partial y^2} &= 0 \\
&= \frac{\partial}{\partial y}(F_y(x,y,z) + F_z(x,y,z)z_y) \\
&= F_{yy}(x,y,z) + 2F_{yz}(x,y,z)z_y + F_{zz}(x,y,z)z_y^2 + F_z(x,y,z)z_{yy} \\
\to z_{yy} &= -\frac{F_{yy} - 2F_{yz}\frac{F_y}{F_z} + F_{zz}\left(\frac{F_y}{F_z}\right)^2}{F_z} \\
\frac{\partial^2 F(x,y,z(x,y))}{\partial x \partial y} &= 0 \\
&= \frac{\partial}{\partial x}(F_y(x,y,z) + F_z(x,y,z)z_y) \\
&= F_{xy}(x,y,z) + F_{yz}(x,y,z)z_x + F_{xz}(x,y,z)z_y + F_{zz}(x,y,z)z_x z_y + F_z(x,y,z)z_{xy} \\
\to z_{xy} &= -\frac{F_{xy} - F_{yz}\frac{F_x}{F_z} - F_{xz}\frac{F_y}{F_z} + F_{zz}\frac{F_y F_x}{F_z^2}}{F_z}
\end{aligned}
$$

$$
\begin{aligned}
\to K &= \frac{z_{xx}z_{yy} - z_{xy}^2}{(1 + z_x^2 + z_y^2)^2} \\
&= \frac{\left(\frac{F_{xx} - 2F_{xz}\frac{F_x}{F_z} + F_{zz}\left(\frac{F_x}{F_z}\right)^2}{F_z}\right)\left(\frac{F_{yy} - 2F_{yz}\frac{F_y}{F_z} + F_{zz}\left(\frac{F_y}{F_z}\right)^2}{F_z}\right) - \left(\frac{F_{xy} - F_{yz}\frac{F_x}{F_z} - F_{xz}\frac{F_y}{F_z} + F_{zz}\frac{F_y F_x}{F_z^2}}{F_z}\right)^2}{(1 + \left(\frac{F_x}{F_z}\right)^2 + \left(\frac{F_y}{F_z}\right)^2)^2} \\
&= \frac{1}{F_z^2(F_x^2 + F_y^2 + F_z^2)^2}(\left(F_{xx}F_z^2 - 2F_{xz}F_xF_z + F_{zz}F_x^2\right)\left(F_{yy}F_z^2 - 2F_{yz}F_yF_z + F_{zz}F_y^2\right) \\
&\quad - \left(F_{xy}F_z^2 - F_{yz}F_xF_z - F_{xz}F_yF_z + F_{zz}F_xF_y\right)^2) \\
&= \frac{1}{F_z^2(F_x^2 + F_y^2 + F_z^2)^2}(F_{xx}F_{yy}F_z^4 - 2F_{xz}F_{yy}F_xF_z^3 + F_{zz}F_{yy}F_z^2F_x^2 - 2F_{xx}F_{yz}F_yF_z^3 \\
&\quad + 4F_{xz}F_{yz}F_xF_yF_z^2 - 2F_{zz}F_{yz}F_yF_zF_x^2 + F_{xx}F_{zz}F_y^2F_z^2 - 2F_{xz}F_{zz}F_y^2F_xF_z + F_{zz}^2F_y^2F_x^2 \\
&\quad - F_{xy}^2F_z^4 - F_{yz}^2F_x^2F_z^2 - F_{xz}^2F_y^2F_z^2 - F_{zz}^2F_x^2F_y^2 + 2F_{xy}F_{yz}F_xF_z^3 + 2F_{xy}F_{xz}F_yF_z^3 \\
&\quad - 2F_{xy}F_{zz}F_xF_yF_z^2 - 2F_{yz}F_{xz}F_yF_xF_z^2 + 2F_{yz}F_{zz}F_yF_x^2F_z + 2F_{xz}F_{zz}F_xF_y^2F_z) \\
&= \frac{1}{(F_x^2 + F_y^2 + F_z^2)^2}(F_{xx}F_{yy}F_z^2 - 2F_{xz}F_{yy}F_xF_z + F_{zz}F_{yy}F_x^2 - 2F_{xx}F_{yz}F_yF_z \\
&\quad + 2F_{xz}F_{yz}F_xF_y + F_{xx}F_{zz}F_y^2 - F_{xy}^2F_z^2 - F_{yz}^2F_x^2 - F_{xz}^2F_y^2 + 2F_{xy}F_{yz}F_xF_z \\
&\quad + 2F_{xy}F_{xz}F_yF_z - 2F_{xy}F_{zz}F_xF_y) \\
&= \frac{1}{(F_x^2 + F_y^2 + F_z^2)^2}((F_{zz}F_{yy} - F_{yz}^2)F_x^2 + (F_{xx}F_{zz} - F_{xz}^2)F_y^2 + (F_{xx}F_{yy} - F_{xy}^2)F_z^2 \\
&\quad + 2(F_{xy}F_{yz} - F_{xz}F_{yy})F_xF_z + 2(F_{xy}F_{xz} - F_{xx}F_{yz})F_yF_z \\
&\quad + 2(F_{xz}F_{yz} - F_{xy}F_{zz})F_xF_y) \\
&= \frac{\nabla F^t A \nabla F}{\|\nabla F\|^4}
\end{aligned}
$$

$\square$

Problem 3 (Ex P168 4). *Determine the asymptotic curves and the lines of curvature of $z = xy$.*

Proof.

$$\begin{aligned}
\mathbb{X} &= (u, v, uv) \\
\to \mathbb{X}_u &= (1, 0, v) \\
\mathbb{X}_v &= (0, 1, u) \\
E &= \langle \mathbb{X}_u, \mathbb{X}_u \rangle = 1 + v^2 \\
F &= \langle \mathbb{X}_u, \mathbb{X}_v \rangle = uv \\
G &= \langle \mathbb{X}_v, \mathbb{X}_v \rangle = 1 + u^2
\end{aligned}$$

$$\begin{aligned}
N &= \frac{\mathbb{X}_u \times \mathbb{X}_v}{|\mathbb{X}_u \times \mathbb{X}_v|} \\
&= \frac{1}{\sqrt{1 + u^2 + v^2}}(-v, -u, 1) \\
\mathbb{X}_{uu} &= (0, 0, 0) \\
\mathbb{X}_{uv} &= (0, 0, 1) \\
\mathbb{X}_{vv} &= (0, 0, 0) \\
\to e &= \langle N, \mathbb{X}_{uu} \rangle = 0 \\
f &= \langle N, \mathbb{X}_{uv} \rangle = \frac{1}{\sqrt{1 + u^2 + v^2}} \\
g &= \langle N, \mathbb{X}_{vv} \rangle = 0
\end{aligned}$$

Asymptotic curves:

$$\begin{aligned}
eu'^2 + 2fu'v' + gv'^2 &= 0 \\
\to u'v' &= 0 \\
\to u = \text{const or } v = \text{const}
\end{aligned}$$

line of curvature:

$$\begin{aligned}
\begin{vmatrix} v'^2 & -u'v' & u'^2 \\ 1 + v^2 & uv & 1 + u^2 \\ 0 & f & 0 \end{vmatrix} &= 0 \\
\to (1 + u^2)v'^2 &= (1 + v^2)u'^2 \\
\to \int \frac{1}{\sqrt{1 + v^2}} dv &= \pm \int \frac{1}{\sqrt{1 + u^2}} du \\
\to \sinh^{-1} v &= \pm \sinh^{-1} u + C \\
\to v &= \pm u \cosh C + \sqrt{1 + u^2} \sinh C
\end{aligned}$$

$\square$

Problem 4. 已知 $\mathbb{X}(u, v)$ 為一 *surface* $\subset \mathbb{R}^3$ 且 $E = G = (1+u^2+v^2)^2, F = 0$ 而且 $e = 1, f = \sqrt{3}, g = -1$

(a) 求在 $\mathbb{X}(1,1)$ 的 K 與 H

(b) 如何決定過 $\mathbb{X}(1,1)$ 的 *line of curvature* 與 *asymptotic curve*（如果有的話）

Proof. (a) at $(1, 1)$, $E = G = 9, F = 0$.

$$\begin{aligned}
[-dN] &= \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \\
&= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{9} & \frac{1}{3\sqrt{3}} \\ \frac{1}{3\sqrt{3}} & -\frac{1}{9} \end{bmatrix}
\end{aligned}$$

So $K = \det([-dN]) = -\frac{4}{81}$, $H = \operatorname{tr}([-dN]) = 0$.

(b) line of curvature:

$$\begin{aligned}
&\begin{vmatrix} v'^2 & -u'v' & u'^2 \\ (1+u^2+v^2)^2 & 0 & (1+u^2+v^2)^2 \\ 1 & \sqrt{3} & -1 \end{vmatrix} = 0 \\
&\to (1+u^2+v^2)^2(-\sqrt{3}v'^2 - 2u'v' + \sqrt{3}u'^2) = 0 \\
&\qquad \to (-\sqrt{3}v' + u')(v' + \sqrt{3}u') = 0 \\
&\qquad\qquad \to -\sqrt{3}v + u = \text{const or } v + \sqrt{3}u = \text{const} \to -\sqrt{3}v + u \quad = 1 - \sqrt{3} \text{ or } v -
\end{aligned}$$

asymptotic curve:

$$\begin{aligned}
&u'^2 + 2\sqrt{3}u'v' - v'^2 = 0 \\
&\to (u' + (\sqrt{3} - 2)v')(u' + (\sqrt{3} + 2)v') = 0 \\
&\qquad \to u' + (\sqrt{3} - 2)v' = 0 \text{ or } u' + (\sqrt{3} + 2)v' = 0 \\
&\qquad \to u + ((\sqrt{3} - 2)v = \text{const or } u + (\sqrt{3} + 2)v = \text{const} \\
&\qquad \to u + ((\sqrt{3} - 2)v = \sqrt{3} - 1 \text{ or } u + (\sqrt{3} + 2)v = \sqrt{3} + 3
\end{aligned}$$

$\square$

Problem 5. $\mathbb{X}(u, v) = (v \cos u, v \sin u, u)$，令 $\gamma(t) = \mathbb{X}(t, 1)$

(a) 求 $\gamma(t)$ 的 $\kappa_n, \kappa_g, \tau_g$

(b) 與 $\gamma(t)$ 的 κ, τ 有何關係

Proof. (a)

$$
\begin{aligned}
\gamma(t) &= (\cos t, \sin t, t) \\
\to \gamma'(t) &= (-\sin t, \cos t, 1) \\
T(t) &= \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1) \\
T'(t) &= \frac{1}{\sqrt{2}}(-\cos t, -\sin t, 0) \\
\to n(t) &= (-\cos t, -\sin t, 0)
\end{aligned}
$$

$$
\kappa = \frac{1}{\sqrt{2}}
$$

$$
\begin{aligned}
\mathbb{X}_u &= (-v\sin u, v\cos u, 1) \\
\mathbb{X}_v &= (\cos u, \sin u, 0) \\
\to N &= \frac{1}{\sqrt{1+v^2}}(-\sin u, \cos u, -v) \\
\kappa_n &= \langle N, T' \rangle \\
&= 0 \\
\kappa_g &= \kappa = \frac{1}{\sqrt{2}} \\
A &= n \\
A' &= (\sin t, -\cos t, 0) \\
\tau_g &= -\langle A', N \rangle \\
&= -\frac{1}{\sqrt{2}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
b &= T \times n \\
&= \frac{1}{\sqrt{2}}(\sin t, -\cos t, 1) \\
\tau &= -\langle n', b \rangle \\
&= -\frac{1}{\sqrt{2}} \\
\to \kappa_g &= \kappa \\
\tau_g &= \tau
\end{aligned}
$$

□

Problem 6. 令 $(x(t), y(t)) = (t - \tanh\ t, \text{sech}\ t)$ 這基本就是 *p7(4)* 的 *tratrix*

(a) 將此曲線化作長度參數

(b) 利用上小題，求此曲線繞 x 軸旋轉的旋轉體的 K

Proof.

(a) Let $\alpha(t) = (x(t), y(t))$. Then $\alpha'(t) = (1-\text{sech}^2 t, \text{sech}\ \tanh t) = (\tanh^2 t, \text{sech}\ \tanh t)$. So the length of the curve s when $t \geq 0$ is given by

$$\begin{aligned} s &= \int \|\alpha'(t)\| dt \\ &= \int \tanh t \sqrt{\tanh^2 t + \text{sech}^2 t}\ dt \\ &= \int \tanh t dt \\ &= \ln(\cosh t) + C_0 \end{aligned}$$

We wish s to be arc-length parameter according to $t = 0$, so when $t = 0$ we need $s = 0$. This implies $C_0 = 0$.

Therefore, we have the arc-length parameter $s \geq 0$ with $t = \cosh^{-1}(e^s)$. Bring it back and by symmetry we would get

$$\alpha(s) = \Big(sign(s)(\cosh^{-1}(e^{|s|}) - \tanh(\cosh^{-1}(e^{|s|}))), \text{sech}(\cosh^{-1}(e^{|s|})) \Big)$$

To check s is the arc-length parameter we can simply check its derivatives.

(b) Let $\mathbb{X}(u, v) = (x(u), y(u) \cos v, y(u) \sin v)$, where $u \in \mathbb{R}$ and $0 < v < 2\pi$ and u is exactly s, the arc-length parameter described in (a). Then,

$$\begin{aligned} \mathbb{X}_u &= (x'(u), y'(u) \cos v, y'(u) \sin v) \\ \mathbb{X}_v &= (0, -y(u) \sin v, y(u) \cos v) \\ E &= \langle \mathbb{X}_u, \mathbb{X}_u \rangle = x'^2(u) + y'^2(u) = 1 \\ F &= \langle \mathbb{X}_u, \mathbb{X}_v \rangle = 0 \\ G &= \langle \mathbb{X}_v, \mathbb{X}_v \rangle = y^2(u) \\ N &= \frac{\mathbb{X}_u \times \mathbb{X}_v}{\|\mathbb{X}_u \times \mathbb{X}_v\|} = (y'(u), -x'(u) \cos v, -x'(u) \sin v) \\ \mathbb{X}_{uu} &= (x''(u), y''(u) \cos v, y''(u) \sin v) \\ \mathbb{X}_{uv} &= (0, -y'(u) \sin v, y'(u) \cos v) \\ \mathbb{X}_{vv} &= (0, -y(u) \cos v, -y(u) \sin v) \\ e &= \langle N, \mathbb{X}_{uu} \rangle = x''(u)y'(u) - x'(u)y''(u) \\ f &= \langle N, \mathbb{X}_{uv} \rangle = 0 \\ g &= \langle N, \mathbb{X}_{vv} \rangle = x'(u)y(u) \end{aligned}$$

Hence the Gaussian curvature K equals

$$K = \frac{eg - f^2}{EG - F^2} = \frac{x'(u)x''(u)y'(u) - x'^2(u)y''(u)}{y(u)}$$

Now, we have when $u \geq 0$,

$$\begin{aligned}
x'(u) &= (e^u - e^{-u})/\sqrt{e^{2u}-1} \\
x''(u) &= (e^u + \frac{1}{2}e^{-u})/\sqrt{e^{2u}-1} \\
y(u) &= \operatorname{sech}(\cosh^{-1} e^u) = e^{-u} \\
y'(u) &= -e^{-u}
\end{aligned}$$

Therefore, by symmetry, finally we get

$$K = \frac{(x''(u)y'(u) - x'(u)y''(u))x'(u)}{y(u)} = 2 - \frac{1}{2}e^{-2|u|}$$

□