Geometry Homework 3

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Problem 3 (P26: 16)**.** *Show that the knowledge of the vector function $n = n(s)$ (normal vector) of a curve α, with nonzero torsion everywhere, determines the curvature $\kappa(s)$ and the torsion $\tau(s)$ of α. ($\vec{n}$ 能決定曲線嗎？說明題目錯誤並找反例。)*

Proof. Consider the helix $\alpha(s) = (a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}})$
Then $n(s) = (-\cos\frac{s}{\sqrt{a^2+b^2}}, -\sin\frac{s}{\sqrt{a^2+b^2}}, 0)$.
So if two helix has the same a^2+b^2 (e.g. $\alpha_1(s) = (\frac{1}{2}\cos s, \frac{1}{2}\sin s, \frac{\sqrt{3}}{2}s)$, $\alpha_2(s) = (\frac{\sqrt{3}}{2}\cos s, \frac{\sqrt{3}}{2}\sin s, \frac{1}{2}s)$), then they have same $n(s)$, but they're not the same curve (because $\kappa = \frac{a}{a^2+b^2}$, so they have different κ). □

Problem 4 (P26: 17，另一種描述 Helix 的方式)**.** *In general, a curve α is called a* helix *if the tangent lines of α make a constant angle with a fixed direction. Assume that $\tau(s) \neq 0$, $s \in I$, and prove that:*

(a) α is a helix if and only if κ/τ =constant.

(b) α is a helix if and only if the lines containing $N(s)$ and passing through $\alpha(s)$ are parallel to a fixed plane.

(c) α is a helix if and only if the lines containing $B(s)$ and passing through $\alpha(s)$ make a constant angle with a fixed direction.

(d) The curve

$$\alpha(s) = \left(\frac{a}{c}\int \sin\theta(s)ds, \frac{a}{c}\int \cos\theta(s)ds, \frac{b}{c}s\right),$$

where $c^2 = a^2 + b^2$, is a helix, and that $\kappa/\tau = a/b$.

Proof. WLOG, assume that s is arc-length parameter, V is the fixed direction. Let $T(s) = \alpha'(s)$, $T'(s) = \kappa N(s)$ and $B = T \times N$.

(a)

$$\begin{aligned}
\langle T, V\rangle &= C \\
\to \langle T', V\rangle &= 0 \\
&= \langle \kappa N, V\rangle \\
\to \langle N, V\rangle &= 0 \\
\to \langle N', V\rangle &= 0 \\
&= \langle -\kappa T - \tau B, V\rangle \\
&= -\kappa C - \tau\langle B, V\rangle \\
&= -\kappa C - \tau\langle T \times N, V\rangle \\
&= -\kappa C - \tau\langle V \times T, N\rangle \\
V &= \langle V, B\rangle B + \langle V, T\rangle T + \langle V, N\rangle N \\
&= \langle V, B\rangle B + CT \\
\langle V, B\rangle' &= \langle V, \tau N\rangle = 0 \\
\to \langle V, B\rangle &= constant \\
\to \langle V \times T, N\rangle &= \langle(\langle V, B\rangle B + CT) \times T, N\rangle \\
&= \langle\langle V, B\rangle B \times T, N\rangle \\
&= \langle V, B\rangle \\
\to 0 &= -\kappa C - \tau\langle V \times T, N\rangle \\
&= -\kappa C - \tau\langle V, B\rangle
\end{aligned}$$

$$\to \kappa/\tau = -\frac{\langle V, B\rangle}{C}$$

So κ/τ is constant.

Conversely, let $\kappa/\tau \equiv c$ be a constant. Define vector V by $V(s) = T(s) - cB(s)$. We claim V is a constant since $V'(s) = T'(s) - cB'(s) = \kappa(s)N(s) - c\tau(s)N(s) = (\kappa(s) - c\tau(s))N(s) = 0$. Now $\langle V, T\rangle$ is constant because $\langle V, T\rangle' = \kappa \langle V, N\rangle = \kappa \langle T - cB, N\rangle = 0$. This implies T make a constant angle with V.

(b) $\langle V, T\rangle \equiv c$ implies $0 = \langle V, T\rangle' = \kappa \langle V, N\rangle$, but $\kappa = \|\alpha'\| \neq 0$, so $\langle V, N\rangle = 0$. Therefore $N(s)$ is always perpendicular to V. Consider any fixed plane P with V be the normal vector, then $N(s)$ is parallel to P.

Conversely, if $N(s)$ parallel to a fixed plane P, define V be a normal vector of P. This implies $N(s) \perp V$, therefore $\langle V, T\rangle' = \kappa \langle V, N\rangle = 0$. So T makes a constant angle with V.

(c) From the proof above, we have $\langle V, N\rangle = 0$, so $V = \langle V, T\rangle T + \langle V, B\rangle B$. Since $V, \langle V, T\rangle$ are both constant and the orientation between T, B is fixed, $\langle V, B\rangle$ is a constant. This implies B makes a constant angle with V.

Conversely, let V be the fixed direction, then $0 = \langle V, B\rangle = -\tau \langle V, N\rangle$. Since $\tau \neq 0$, so $V \perp N$ hence by (b), T makes a constant angle with V.

(d)

$\square$

Problem 6. $\gamma(s)$ 長度參數。若將 $T(s)$ 寫成 $(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$，$\phi, \theta$ 是 s 的函數。說明 $\kappa(s) = \sqrt{\phi'^2 + \theta'^2 \sin^2\phi}$

Proof.

$$\begin{aligned}
T'(s) &= (\phi'\cos\phi\cos\theta - \theta'\sin\phi\sin\theta, \phi'\cos\phi\sin\theta + \theta'\sin\phi\cos\theta, -\phi'\sin\phi) \\
\rightarrow \kappa(s) &= |T'(s)| \\
&= \sqrt{\phi'^2\cos^2\phi\cos^2\theta + \theta'^2\sin^2\phi\sin^2\theta + \phi'^2\cos^2\phi\sin^2\theta + \theta'^2\sin^2\phi\cos^2\theta + \phi'^2\sin^2\phi} \\
&= \sqrt{\phi'^2 + \theta'^2\sin^2\phi}
\end{aligned}$$

□

Problem 7. $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$，不妨假設是長度參數。

(b) 若 $M^tM = I$，$\det(M) = -1$ 且 $\overline{\gamma} = M\gamma$，討論 κ, τ 變化。

(c) $\overline{\gamma}(s) = \gamma(-s)$，說明 κ, τ 變化。

Proof. (b)

$$\begin{aligned}
|\overline{\gamma}'| &= \sqrt{\overline{\gamma}'^T\overline{\gamma}'} \\
&= \sqrt{\gamma'^T M^T M \gamma'} \\
&= \sqrt{\gamma'^T\gamma'} \\
&= |\gamma'| \\
&= 1
\end{aligned}$$

So s is arc-length parameter for $\overline{\gamma}$ too.

$$\begin{aligned}
\kappa_{\overline{\gamma}} &= |\overline{\gamma}''| \\
&= \sqrt{\overline{\gamma}''^T\overline{\gamma}''} \\
&= \sqrt{\gamma''^T M^T M \gamma''} \\
&= \sqrt{\gamma''^T\gamma''} \\
&= |\gamma''| \\
&= \kappa_\gamma
\end{aligned}$$

So κ remains the same.

$$\begin{aligned}
N' &= -\kappa T - \tau B \\
\to \tau &= -\langle N', B\rangle \\
N_{\overline{\gamma}} &= \frac{\overline{\gamma}''}{\kappa_{\overline{\gamma}}} \\
&= M\frac{\gamma''}{\kappa_\gamma} \\
&= MN_\gamma \\
\to N'_{\overline{\gamma}} &= MN'_\gamma \\
\tau_{\overline{\gamma}} &= -\langle N'_{\overline{\gamma}}, B_{\overline{\gamma}}\rangle \\
&= -\langle MN'_\gamma, T_{\overline{\gamma}} \times N_{\overline{\gamma}}\rangle \\
&= -\langle MN'_\gamma, (MT_\gamma) \times (MN_\gamma)\rangle \\
&= -\det(M)\langle N'_\gamma, (T_\gamma) \times (N_\gamma)\rangle \\
&= -\det(M)\langle N'_\gamma, B_\gamma\rangle \\
&= \det(M)\tau_\gamma \\
&= -\tau_\gamma
\end{aligned}$$

So $\tau_{\overline{\gamma}} = -\tau_\gamma$.

(c)

$$\begin{aligned}
|\overline{\gamma}'(s)| &= \sqrt{\overline{\gamma}'(s)^T\overline{\gamma}'(s)} \\
&= \sqrt{(-\gamma'^T(-s))\,(-\gamma'(-s))} \\
&= \sqrt{\gamma'(-s)^T\gamma'(-s)} \\
&= |\gamma'(-s)| \\
&= 1
\end{aligned}$$

So s is arc-length parameter for $\overline{\gamma}$ too.

$$\begin{aligned}
\kappa_{\overline{\gamma}}(s) &= |\overline{\gamma}''(s)| \\
&= \sqrt{\overline{\gamma}''(s)^T\overline{\gamma}''(s)} \\
&= \sqrt{\gamma''(-s)^T\gamma''(-s)} \\
&= |\gamma''(-s)| \\
&= \kappa_\gamma(-s)
\end{aligned}$$

So $\kappa_{\overline{\gamma}}(s) = \kappa_{\gamma}(-s)$.

$$\begin{aligned}
\tau_{\overline{\gamma}}(s) &= \frac{|\overline{\gamma}'(s)\ \overline{\gamma}''(s)\ \overline{\gamma}'''(s)|}{|\overline{\gamma}'(s) \times \overline{\gamma}''(s)|^2} \\
&= \frac{|-\gamma'(-s)\ \gamma''(-s)\ -\gamma'''(-s)|}{|-\gamma'(-s) \times \gamma''(-s)|^2} \\
&= \frac{|\gamma'(-s)\ \gamma''(-s)\ \gamma'''(-s)|}{|\gamma'(-s) \times \gamma''(-s)|^2} \\
&= \tau_{\gamma}(-s)
\end{aligned}$$

So $\tau_{\overline{\gamma}}(s) = \tau_{\gamma}(-s)$.

□

Problem 8. 說明 $\overline{\gamma}(u) = \gamma(t(u))$ 時，在對應點

$$\frac{\det(\overline{\gamma}', \overline{\gamma}'', \overline{\gamma}''')}{|\overline{\gamma}' \times \overline{\gamma}''|^2}(u) = \frac{\det(\gamma', \gamma'', \gamma''')}{|\gamma' \times \gamma''|^2}(t)$$

再用 *chain rule* 直接說明。

Proof.

$$\begin{aligned}
\overline{\gamma}'(u) &= \gamma'(t(u))t'(u) \\
\overline{\gamma}''(u) &= \gamma''(t(u))t'(u)^2 + \gamma'(t(u))t''(u) \\
\overline{\gamma}'''(u) &= \gamma'''(t(u))t'(u)^3 + 3\gamma''(t(u))t'(u)t''(u) + \gamma'(t(u))t'''(u)
\end{aligned}$$

$$\begin{aligned}
\to \det(\overline{\gamma}', \overline{\gamma}'', \overline{\gamma}''')(u) &= \det(\gamma'(t(u))t'(u), \gamma''(t(u))t'(u)^2 + \gamma'(t(u))t''(u), \\
&\qquad \gamma'''(t(u))t'(u)^3 + 3\gamma''(t(u))t'(u)t''(u) + \gamma'(t(u))t'''(u)) \\
&= \det(\gamma'(t(u))t'(u), \gamma''(t(u))t'(u)^2, \gamma'''(t(u))t'(u)^3 + 3\gamma''(t(u))t'(u)t''(u)) \\
&= \det(\gamma'(t(u))t'(u), \gamma''(t(u))t'(u)^2, \gamma'''(t(u))t'(u)^3) \\
&= t'(u)^6 \det(\gamma'(t(u)), \gamma''(t(u)), \gamma'''(t(u)))
\end{aligned}$$

$$\begin{aligned}
|\overline{\gamma}' \times \overline{\gamma}''|^2(u) &= |\left(\gamma'(t(u))t'(u)\right) \times \left(\gamma''(t(u))t'(u)^2 + \gamma'(t(u))t''(u)\right)|^2 \\
&= |\left(\gamma'(t(u))t'(u)\right) \times \left(\gamma''(t(u))t'(u)^2\right)|^2 \\
&= t'(u)^6 |\gamma'(t(u)) \times \gamma''(t(u))|^2
\end{aligned}$$

$$\begin{aligned}
\to \frac{\det(\overline{\gamma}', \overline{\gamma}'', \overline{\gamma}''')}{|\overline{\gamma}' \times \overline{\gamma}''|^2}(u) &= \frac{t'(u)^6 \det(\gamma', \gamma'', \gamma''')}{t'(u)^6 |\gamma' \times \gamma''|^2}(t) \\
&= \frac{\det(\gamma', \gamma'', \gamma''')}{|\gamma' \times \gamma''|^2}(t)
\end{aligned}$$

□