

GEOMETRY HOMEWORK 5

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Problem 1 (Ex P151 2). *Show that if a surface is tangent to a plane along a curve, then the points of this curve are either parabolic or planar.*

Proof. Assume that the curve is $\gamma(s)$, then along this curve, $N(\gamma(s))$ is perpendicular to the plane, so it is constant.

At point $\gamma(s)$, $[dN](\gamma'(s)) = \left(\frac{dN(\gamma(t))}{dt} \right)_{t=s} = 0$, so $\gamma'(s)$ is one of the principal direction of the surface at $\gamma(s)$, and it's associated principal curvature is 0. So the gaussian curvature of the surface at $\gamma(s)$ is $K = 0$, and this means that the point $\gamma(s)$ is either parabolic or planar. \square

Problem 3 (Ex P151 3).

(a) *Let $C \subset S$ be a regular curve on a surface S with Gaussian curvature $K > 0$. Show that the curvature κ of C at p satisfies*

$$\kappa \geq \min(|\kappa_1|, |\kappa_2|),$$

where κ_1, κ_2 are the principal curvatures of S at p .

(b) 為什麼上一小題需要 $\kappa > 0$ 的條件, $\kappa \geq 0$ 不可以嗎?

Proof. (a)

$$\begin{aligned} \kappa &\geq |\kappa_n| \\ &= |\kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta| \\ &= |\kappa_1| \cos^2 \theta + |\kappa_2| \sin^2 \theta (\because \kappa_1, \kappa_2 \text{ has equal sign.}) \\ &\geq \min(|\kappa_1|, |\kappa_2|)(\cos^2 \theta + \sin^2 \theta) \\ &= \min(|\kappa_1|, |\kappa_2|) \end{aligned}$$

(b)

\square

Problem 7.

(a) T_λ 是縮放 λ 倍的映射, $\lambda > 0$. $\mathbb{X} : \Omega \rightarrow \mathbb{R}^3$ regular surface. 討論 $T_\lambda \circ \mathbb{X} : \Omega \rightarrow \mathbb{R}^3$ 上對應點 κ_n, H, K 的變化。

(b) $\mathbb{X}: \begin{matrix} \Omega \\ (u, v) \end{matrix} \rightarrow \mathbb{R}^3$, 若定義 $\bar{\mathbb{X}}(u, v) = \mathbb{X}(v, u)$ (因此 N 轉向)。討論 $\bar{\mathbb{X}}(\Omega)$ 上相對應點的 K_n, H, K 變化。

Proof.

□

Problem 9 (旋轉面). $\mathbb{X}(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$, $f > 0$

(a) 計算其 e, f, g, H, K

(b) 討論其 *principal direction* 與 *principal curvature* K_1, K_2 。

Proof. To avoid the notational ambiguity, let $\mathbb{X}(u, v) = (s(u) \cos v, s(u) \sin v, t(u))$, and that $s > 0$.

(a) We have

$$\begin{aligned}
\mathbb{X}_u &= (s'(u) \cos v, s'(u) \sin v, t'(u)); \\
\mathbb{X}_v &= (-s(u) \sin v, s(u) \cos v, 0); \\
E &= \langle \mathbb{X}_u, \mathbb{X}_u \rangle = s'(u)^2 + t'(u)^2 \\
F &= \langle \mathbb{X}_u, \mathbb{X}_v \rangle = 0 \\
G &= \langle \mathbb{X}_v, \mathbb{X}_v \rangle = s(u)^2 \\
\mathbb{X}_{uu} &= (s''(u) \cos v, s''(u) \sin v, t''(u)); \\
\mathbb{X}_{uv} &= (-s'(u) \sin v, s'(u) \cos v, 0); \\
\mathbb{X}_{vv} &= (-s(u) \cos v, -s(u) \sin v, 0); \\
N &= \frac{\mathbb{X}_u \times \mathbb{X}_v}{|\mathbb{X}_u \times \mathbb{X}_v|} = \frac{(-t'(u)s(u) \cos v, -t'(u)s(u) \sin v, s'(u)s(u))}{\sqrt{t'(u)^2 s(u)^2 + s'(u)^2 s(u)^2}} \\
&= \frac{(-t'(u) \cos v, -t'(u) \sin v, s'(u))}{\sqrt{t'(u)^2 + s'(u)^2}}; \\
e &= \langle N, \mathbb{X}_{uu} \rangle = \frac{-s''(u)t'(u) + t''(u)s'(u)}{\sqrt{t'(u)^2 + s'(u)^2}} \\
f &= \langle N, \mathbb{X}_{uv} \rangle = 0 \\
g &= \langle N, \mathbb{X}_{vv} \rangle = \frac{s(u)t'(u)}{\sqrt{t'(u)^2 + s'(u)^2}} \\
-dN^T &= \begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} = \frac{1}{EG - F^2} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} = \begin{bmatrix} e/E & 0 \\ 0 & g/G \end{bmatrix} \\
K &= \det(-dN) = \frac{eg}{EG} \\
H &= \frac{1}{2} \text{tr}(-dN) = \frac{eG + gE}{2EG}
\end{aligned}$$

(b) Since $-dN$ is already a diagonal matrix, clearly,

$$\begin{aligned} K_1 &= e/E; \\ K_2 &= g/G; \\ V_1 &= \mathbb{X}_u; \\ V_2 &= \mathbb{X}_v; \end{aligned}$$

□

Problem 10 (管面). $\mathbb{X}(s, \theta) = \gamma(s) + \cos \theta \vec{n}(s) + \sin \theta \vec{b}(s)$, $0 < \kappa < 1$

(a) 計算其 e, f, g, H, K

(b) 討論曲面上 K 的分佈。

Proof. (a) Let $\vec{t}(s), \vec{n}(s), \vec{b}(s)$ be the basis, and $\gamma(0)$ be the origin.

$$\mathbb{X}_s = (1 - \kappa \cos \theta, \tau \sin \theta, -\tau \cos \theta);$$

$$\mathbb{X}_\theta = (0, -\sin \theta, \cos \theta);$$

$$N = \frac{\mathbb{X}_s \times \mathbb{X}_\theta}{|\mathbb{X}_s \times \mathbb{X}_\theta|} = \frac{(0, (\kappa \cos \theta - 1) \cos \theta, (\kappa \cos \theta - 1) \sin \theta)}{|\kappa \cos \theta - 1|}$$

$$= (0, -\cos \theta, -\sin \theta); \quad (\text{since } \kappa < 1)$$

$$\mathbb{X}_{ss} = (-\sin \theta \kappa \tau, \kappa + \cos \theta (-\kappa^2 - \tau^2), -\sin \theta \tau^2);$$

$$\mathbb{X}_{s\theta} = (1 + \kappa \sin \theta, \tau \cos \theta, \tau \sin \theta);$$

$$\mathbb{X}_{\theta\theta} = (0, -\cos \theta, -\sin \theta);$$

$$e = -\kappa \cos \theta + \cos^2 \theta (\kappa^2 + \tau^2) + \tau^2 \sin^2 \theta = -\kappa \cos \theta + \kappa^2 \cos^2 \theta + \tau^2;$$

$$f = -\tau;$$

$$g = 1;$$

$$dN(\mathbb{X}_s) = N_s = (0, 0, 0);$$

$$dN(\mathbb{X}_\theta) = N_\theta = (0, \sin \theta, -\cos \theta) = -\mathbb{X}_\theta;$$

$$\kappa_1 = 0;$$

$$\kappa_2 = 1;$$

$$K = 0;$$

$$H = 1/2.$$

(b)

$$K = 0 \text{ everywhere}$$

□