Geometry Homework 9

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Problem 4 (Ex p.101 14). *(Gradient on Surfaces.) The gradient of a differentiable function $f : S \mapsto \mathbb{R}$ is a differentiable map grad $f : S \mapsto \mathbb{R}^3$ which assigns to each point $p \in S$ a vector grad $f(p) \in T_p(S) \subset \mathbb{R}^3$ such that*

$$\langle \textit{grad } f(p), v \rangle_p = df_p(v) \qquad \textit{for all } v \in T_p(S)$$

Show that

(a) *If E, F, G are the coefficients of the first fundamental form in a parametrization $\mathbf{X} : U \subset \mathbb{R}^2 \mapsto S$, then grad f on $\mathbf{X}(U)$ is given by*

$$\textit{grad } f = \frac{f_u G - f_v F}{EG - F^2}\mathbf{X}_u + \frac{f_v E - f_u F}{EG - F^2}\mathbf{X}_v$$

In particular, if $S = \mathbb{R}^2$ with coordinates x, y,

$$\textit{grad } f = f_x e_1 + f_y e_2$$

where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$ (thus, the definition agrees with the usual definition of gradient in the plane)

(b) 為什麼不直接將 *gradient f* 定義成 $f_u\mathbb{X}_u + f_v\mathbb{X}_v$，這有什麼缺點 *(*例如座標變換*)*

Proof. (a) First,

$$\langle \text{grad } f(p), \mathbf{X}_u \rangle_p = df_p(\mathbf{X}_u) = f_u$$
$$\langle \text{grad } f(p), \mathbf{X}_v \rangle_p = df_p(\mathbf{X}_v) = f_v$$

Let grad $f = q\mathbf{X}_u + r\mathbf{X}_v$. Then

$$\langle \text{grad } f(p), \mathbf{X}_u \rangle = Eq + Fr = f_u$$
$$\langle \text{grad } f(p), \mathbf{X}_v \rangle = Fq + Gr = f_v$$

Therefore, solve the linear equations and get

$$q = \frac{f_u G - f_u F}{EG - F^2};$$
$$r = \frac{f_v E - f_u F}{EG - F^2}$$

Then the two results follow immediately.

(b) If we define the gradient in that way, let $S = \mathbb{R}^2$ be the surface and $\mathbf{X}(u, v) = (u, v)$, $\mathbf{Y}(s, t) = (s, s + t)$ be its two parametrizations. If $f(u, v) = v$, then $f(s, t) = s + t$ and therefore grad $f = \mathbf{X}_v = \mathbf{Y}_s + \mathbf{Y}_t$. But clearly $\mathbf{X}_v = (0, 1) \neq (1, 2) = \mathbf{Y}_s + \mathbf{Y}_t$, which is a contradiction.

$\square$

Problem 7. 計算下列 *surface* 的 Γ_{ij}^k *(*共有六項*)*

(b) $(x(t), y(t)\cos\theta, y(t)\sin\theta)$

(c) $E = G = \lambda^2, F = 0$

Proof. $\square$

Problem 8 (Ex p.237 1, 2)**.** *(a) Show that if* $\mathbf{X}$ *is an orthogonal parametrization, that is,* $F = 0$*, then*

$$K = -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\}$$

(b) Show that if $\mathbf{X}$ *is an isothermal parametrization, that is,* $E = G = \lambda(u, v)$ *and* $F = 0$*, then*

$$K = -\frac{1}{2\lambda}\Delta(\log\lambda)$$

where $\Delta\phi$ *denotes the Laplacian* $(\partial^2\phi/\partial u^2) + (\partial^2\phi/\partial v^2)$ *of the function* ϕ*. Conclude that when* $E = G = (u^2 + v^2 + c)^{-2}$ *and* $F = 0$*, then* $K = $ *const.* $= 4c$*.*

Proof.

$\square$