

GEOMETRY HOMEWORK 4

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Problem 3.

(a) 假設 $\kappa(s) \neq 0, \tau(s) \neq 0$, 由四點決定一球, 討論空間曲線 $\gamma(s)$ 的密切球, 並決定球心與半徑。

(b) 討論螺線 $(a \cos t, a \sin t, bt)$ 的密切球, $a > 0$ 。

Proof. (a) Assume that the sphere is $|X - C| = R$

$$\begin{aligned}
 &\rightarrow \langle X - C, X - C \rangle = R^2 \\
 &\rightarrow \langle X - C, X - C \rangle' = 0 \\
 &\quad = 2\langle X - C, T \rangle \\
 &\rightarrow \langle X - C, T \rangle' = 0 \\
 &\quad = \langle T, T \rangle + \langle X - C, T' \rangle \\
 &\quad = 1 + \kappa \langle X - C, N \rangle \\
 &\rightarrow (\kappa \langle X - C, N \rangle)' = 0 \\
 &\quad = \kappa' \langle X - C, N \rangle + \kappa \langle T, N \rangle + \kappa \langle X - C, N' \rangle \\
 &\quad = \kappa' \langle X - C, N \rangle + \kappa \langle X - C, -\kappa T - \tau B \rangle \\
 &\quad = \kappa' \langle X - C, N \rangle - \kappa \tau \langle X - C, B \rangle \\
 &\rightarrow \langle X - C, T \rangle = 0 \\
 &\langle X - C, N \rangle = -\frac{1}{\kappa} \\
 &\langle X - C, B \rangle = -\frac{\kappa'}{\kappa^2 \tau} \\
 &\rightarrow X - C = -\frac{1}{\kappa} N - \frac{\kappa'}{\kappa^2 \tau} B \\
 &\rightarrow C = X + \frac{1}{\kappa} N + \frac{\kappa'}{\kappa^2 \tau} B \\
 &R = |X - C| \\
 &\quad = \sqrt{\frac{1}{\kappa^2} + \frac{\kappa'^2}{\kappa^4 \tau^2}}
 \end{aligned}$$

□

Problem 4. $\kappa \neq 0, \tau \neq 0$ 為兩常數, 請決定 $\kappa(s) = \kappa, \tau(s) = \tau$ 的曲線方程式。(長度參數 s)

Proof. Upto translations and rotations, all space curves $\alpha(s)$ satisfying the condition are of the following form

$$\begin{aligned}\alpha(s) &= \left(\frac{\kappa}{\kappa^2 + \tau^2} \sin \sqrt{\kappa^2 + \tau^2} s, \frac{\kappa}{\kappa^2 + \tau^2} \cos \sqrt{\kappa^2 + \tau^2} s, \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} s \right) \\ T(s) &= \left(\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \cos \sqrt{\kappa^2 + \tau^2} s, -\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \sin \sqrt{\kappa^2 + \tau^2} s, \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \right) \\ \|T(s)\| &= 1 \quad (\text{arc-length}) \\ T'(s) &= (-\kappa \sin \sqrt{\kappa^2 + \tau^2} s, -\kappa \cos \sqrt{\kappa^2 + \tau^2} s, 0) \\ \kappa(s) &= \|T'(s)\| = \kappa \\ N(s) &= (-\sin \sqrt{\kappa^2 + \tau^2} s, -\cos \sqrt{\kappa^2 + \tau^2} s, 0) \\ B(s) &= T(s) \times N(s) = \left(\frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \cos \sqrt{\kappa^2 + \tau^2} s, -\frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \sin \sqrt{\kappa^2 + \tau^2} s, -\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \right) \\ B'(s) &= (-\tau \sin \sqrt{\kappa^2 + \tau^2} s, -\tau \cos \sqrt{\kappa^2 + \tau^2} s, 0) \\ \tau(s) &= B'(s)/N(s) = \tau\end{aligned}$$

□

Problem 5 (Darboux vector). $\gamma(s)$ arc length

(a) 說明 \exists vector $\omega(s)$ (called Darboux vector) such that

$$\begin{cases} T' &= \omega \times T \\ N' &= \omega \times N \\ B' &= \omega \times B \end{cases}$$

(b) $V(s)$ is a vector along $\gamma(s)$ 且 w.r.t (T, N, B) , $V(s) = (v_1(s), v_2(s), v_3(s))$
 $\Rightarrow V' = (v'_1, v'_2, v'_3) + \omega \times V$

(c) 說明 $\omega = \frac{1}{2}(T \times T' + N \times N' + B \times B')$

Proof. (a) Let $\omega(s) = -\tau T + \kappa B$, then:

$$\begin{aligned}\omega \times T &= (-\tau T + \kappa B) \times T \\ &= \kappa N \\ &= T' \\ \omega \times N &= (-\tau T + \kappa B) \times N \\ &= -\kappa T - \tau B \\ &= N' \\ \omega \times B &= (-\tau T + \kappa B) \times B \\ &= \tau N \\ &= B'\end{aligned}$$

So $\omega(s)$ satisfy the conditions.

(b)

$$\begin{aligned}
V &= v_1 T + v_2 N + v_3 B \\
\rightarrow V' &= v'_1 T + v'_2 N + v'_3 B + v_1 T' + v_2 N' + v_3 B' \\
&= v'_1 T + v'_2 N + v'_3 B + v_1 \omega \times T + v_2 \omega \times N + v_3 \omega \times B \\
&= v'_1 T + v'_2 N + v'_3 B + \omega \times (v_1 T) + \omega \times (v_2 N) + \omega \times (v_3 B) \\
&= v'_1 T + v'_2 N + v'_3 B + \omega \times (v_1 T + v_2 N + v_3 B) \\
&= (v'_1, v'_2, v'_3) + \omega \times V
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{1}{2}(T \times T' + N \times N' + B \times B') &= \frac{1}{2}(T \times \kappa N + N \times (-\kappa T - \tau B) + B \times \tau N) \\
&= \frac{1}{2}(\kappa B + \kappa B - \tau T - \tau T) \\
&= -\tau T + \kappa B \\
&= \omega
\end{aligned}$$

□

Problem 8.

(a) 令函數 $x_i : \mathbb{R}^n \rightarrow \mathbb{R}$.
 $(x_1, \dots, x_n) \mapsto x_i$ 。計算 $[dx_i]$ ，在不同的 $a \in \mathbb{R}^n$ ， dx_i 如何隨 a 變化。

(b) 由上題將微分式 $df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n$ 與映射 df 結合起來。

(c) $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ，怎麼利用上題幫你計算 df

Proof. (a) $[dx_i] = \begin{bmatrix} \frac{\partial x_i}{\partial x_1} & \dots & \frac{\partial x_i}{\partial x_j} & \dots & \frac{\partial x_i}{\partial x_n} \end{bmatrix} = e_i^T$
 dx_i 不隨 a 變化。

(b)

$$\begin{aligned}
df &= \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n \\
&= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_m}{\partial x_1} \end{bmatrix} e_1^T + \dots + \begin{bmatrix} \frac{\partial f_1}{\partial x_n} \\ \vdots \\ \frac{\partial f_m}{\partial x_n} \end{bmatrix} e_n^T \\
&= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}
\end{aligned}$$

(c)

□