Geometry Homework 10

B96201044 黃上恩, *B98901182* 時丕勳, *K0020100x* 劉士瑋

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Problem 2. 若 $E=1, F=0, G=1, f=0$，假設再加入函數 e,g 後是某 *surface* 的 *1st&2nd fundamental form*。

(a) 說明 e,g 中至少有一為 *0*

(b) 說明若 $e=g=0$ 則此曲面為平面

(c) 說明若 $e \neq 0$，則此曲面為特別的 *ruled surface*，並討論 e 的意義。

Proof. (a) Since $E=1, F=0, G=1$, we know that the surface has $K=0$, so $eg-f^2=0$. So $eg=0$, and one of e and g is zero.

(b) If $e=g=0$, then

$$\begin{aligned}\mathbb{X}_{uu} &= [1,1,1]\mathbb{X}_u + [1,1,2]\mathbb{X}_v + eN = 0\\ \mathbb{X}_{uv} &= [1,2,1]\mathbb{X}_u + [1,2,2]\mathbb{X}_v + fN = 0\\ \mathbb{X}_{vv} &= [2,2,1]\mathbb{X}_u + [2,2,2]\mathbb{X}_v + gN = 0\end{aligned}$$

So $\mathbb{X}_u$ and $\mathbb{X}_v$ are constant, and the surface is a plane.

(c) if $e \neq 0$, then $g=0$, and $\mathbb{X}_{vu} = \mathbb{X}_{vv} = 0$, so $\mathbb{X}_v$ is constant.
So $\mathbb{X}(u,v)$ is a line when we fix u, thus $\mathbb{X}$ is a ruled surface.
Let $\gamma(u) = \mathbb{X}(u,0)$, then $\|\gamma'(u)\| = \|\mathbb{X}_u\| = 1$, so u is arc-length parameter for γ.
$\gamma''(u) = \mathbb{X}_{uu}(u,0) = eN$, so $\operatorname{sign}(e)N$ is also the n for γ, and γ has curvature $|e|$.

□

Problem 4 (Ex p237 8.). *Compute the Cristoffel symbols for an open set of the plane*

(a) In cartesian coordinates.

(b) In polar coordinates.

Use the Gauss formula to compute K in both cases.

Problem 6. 有一 *surface* $\mathbf{X}(u,v)$，令 $\hat{\mathbf{X}}(u,v) = \lambda\mathbf{X}(u,v), \lambda > 0$。

(a) 討論 $\hat{\Gamma}_{ij}^k$ 和 Γ_{ij}^k 的關係

(b) 從 *Gauss equation(GTE)* 討論 $\hat{K}$ 和 K 的關係

Problem 9. 舉一個例子說明有可能 $F: M \to N$ 是 *conformal map*，且相應點 $K_M > 0, K_N = 0$ *(想想曾經討論的例子)*

Proof. 取 M 為單位球 $x^2 + y^2 + z^2 = 1$, N 為平面 $z = 0$, 則顯然 $K_M > 0, K_N = 0$.
取 map $f: M \mapsto N$, $f(x,y,z) = (\frac{x}{1-z}, \frac{y}{1-z}, 0)$ 為 stereographic projection.
因為若 $f(x,y,z) = (u,v,w)$, 則

$$\begin{aligned}
du^2 + dv^2 + dw^2 &= \left(\frac{(1-z)dx + xdz}{(1-z)^2}\right)^2 + \left(\frac{(1-z)dy + ydz}{(1-z)^2}\right)^2 \\
&= \frac{1}{(1-z)^4}\left((1-z)^2dx^2 + (1-z)^2dy^2 + (x^2+y^2)dz^2 + (xdx+ydy)(1-z)dz\right) \\
&= \frac{1}{(1-z)^4}\left((1-z)^2dx^2 + (1-z)^2dy^2 + (-z^2+1)dz^2 + (-zdz)(1-z)dz\right) \\
&= \frac{1}{(1-z)^4}\left((1-z)^2dx^2 + (1-z)^2dy^2 + (1-z)dz^2\right)
\end{aligned}$$

□