$\sum_{s=1}^{n} z_s$

6. (a). $\mathbb{E}(f(x)\mid f) = \mathbb{E}(f(x)) + \text{cov}(f(x), f)\cdot \text{cov}(f, f)^{-1}\cdot(f - \mathbb{E}(f))$

$= 0 + k_x^T \cdot k^{-1}\cdot f = k_x^T\cdot k^{-1}\cdot f$

$\text{Var}(f(x)\mid f) = \text{Var}(f(x)) - \text{Cov}(y, f)\cdot \text{cov}(f, f)^{-1}\cdot \text{cov}(f, y)$

$= k_x - k_x^T k^{-1}\cdot k_x$

(b). $\text{cov}(y_i, y_j) = \mathbb{E}[(y_i - \mathbb{E}(y_i))\cdot(y_j - \mathbb{E}(y_j))]$

$= \mathbb{E}[(f(x_i) + y_i - E(f(x_i) + y_i))\cdot(y_j - \mathbb{E}(f(x_j) + y_j))]$

$= \mathbb{E}[(f(x_i) + y_i - E(f(x_i)))\cdot(f(x_j) + y_j - \mathbb{E}(f(x_j)))]$

$= \mathbb{E}[f(x_i)\cdot f(x_j) + f(x_i)\cdot y_j - f(x_i)\cdot E(f(x_j)) + y_i f(x_j)$

$+ y_i y_j - y_i\cdot \mathbb{E}(f(x_j)) - E(f(x_i))\cdot f(x_j) + y_j\cdot E(f(x_i))$

$- \mathbb{E}(f(x_i))\cdot\mathbb{E}(f(x_j))]$

$= \mathbb{E}[f(x_i)\cdot f(x_j)] - \mathbb{E}[f(x_i)]\cdot\mathbb{E}[f(x_j)] + \mathbb{E}(y_i y_j)$

If $i = j$, $\text{cov}(y_i, y_j) = \mathbb{E}[f(x_i)\cdot f(x_i)] - \mathbb{E}^2[f(x_i)] + \mathbb{E}(y_i^2)$

$= \text{cov}(f(x_i), f(x_j)) + \sigma^2$

If $i \neq j$, $\text{cov}(y_i, y_j) = \mathbb{E}[f(x_i) f(x_j)] - \mathbb{E}[f(x_i)]\cdot\mathbb{E}[f(x_j)] + 0$

$= \text{cov}(f(x_i), f(x_j))$

Thus, covariance matrix of $(y_1, \dots y_n)$ is $K + \sigma^2 I$.

$\text{cov}(f(x), y_i) = \text{cov}(f(x), f(x_i) + y_i) = \text{cov}(f(x), f(x_i)) + \text{cov}(f(x), y_i)$

$= \text{cov}(f(x), f(x_i)) = (k_x)_i$

Thus, covariance matrix of $(y_1, \dots y_n, f(x))$ is

$$\tilde{K} = \begin{bmatrix} K + \sigma^2 I & k_x \\ k_x & k_x \end{bmatrix}$$

By part (a), we know, let $\bar{y} = (y_1, \cdots y_n)$

$$\operatorname{Var}(f(x) \mid \bar{y}) = k_x - k_x^T (K+\sigma^2 I)^{-1} \cdot k_x$$

$$\mathbb{E}(f(x) \mid \bar{y}) = k_x^T \cdot (K^2+\sigma^2 I)^{-1} \cdot \bar{y}$$

Thus, $f(x) \mid \bar{y} \sim N(k_x^T (K+\sigma^2 I)^{-1} y \,,\, k_x - k_x^T (K+\sigma^2 I)^{-1} k_x)$

Problem 6

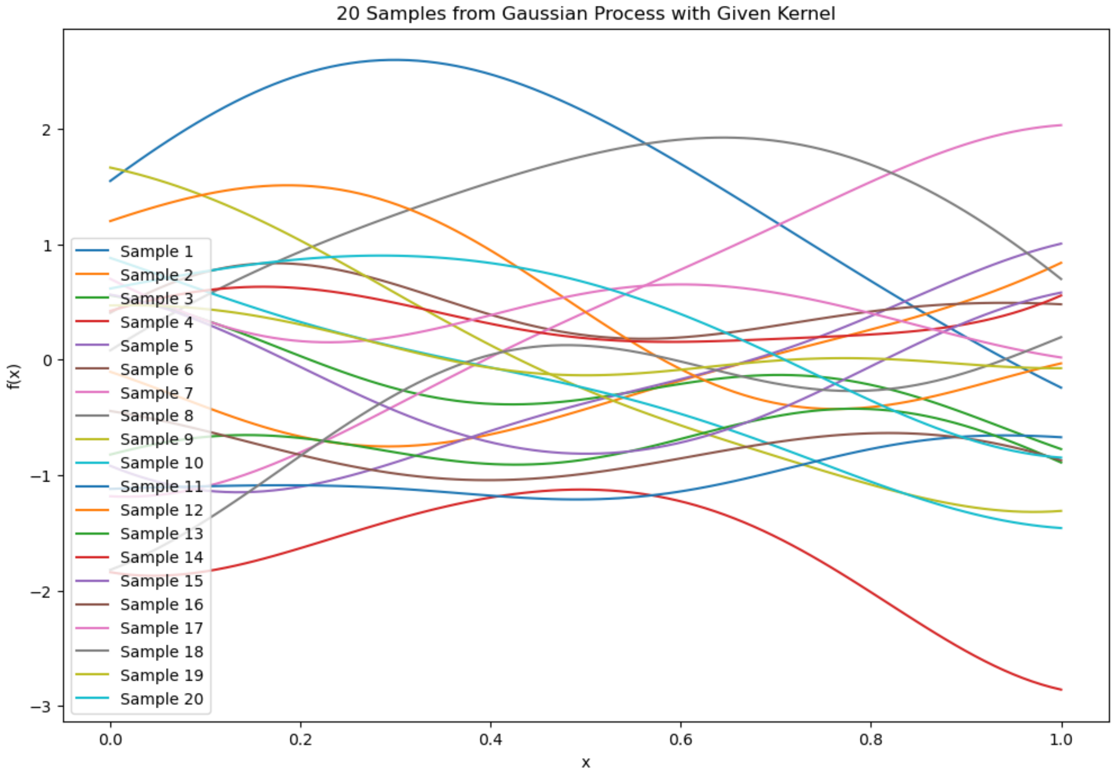
April 25, 2024

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
```

1 Part c

[3]:
```
# Define the kernel function
def kernel(x, x_prime, tau_squared=0.12):
    return np.exp(-(x - x_prime)**2 / (2 * tau_squared))
```

[27]:
```
# Number of samples and points in the domain
num_samples = 20
num_points = 100

# Equispaced points on [0, 1]
z = np.linspace(0, 1, num_points)

# Construct the covariance matrix based on the kernel function
K = np.zeros((num_points, num_points))
for i in range(num_points):
    for j in range(num_points):
        K[i, j] = kernel(z[i], z[j])

# Generate n samples from the multivariate normal distribution
samples = np.random.multivariate_normal(np.zeros(num_points), K, num_samples)

# Plotting the samples
plt.figure(figsize=(12, 8))
for i in range(num_samples):
    plt.plot(z, samples[i], label=f'Sample {i+1}')
plt.title('20 Samples from Gaussian Process with Given Kernel')
plt.xlabel('x')
plt.ylabel('f(x)')
plt.legend()
plt.show()
```

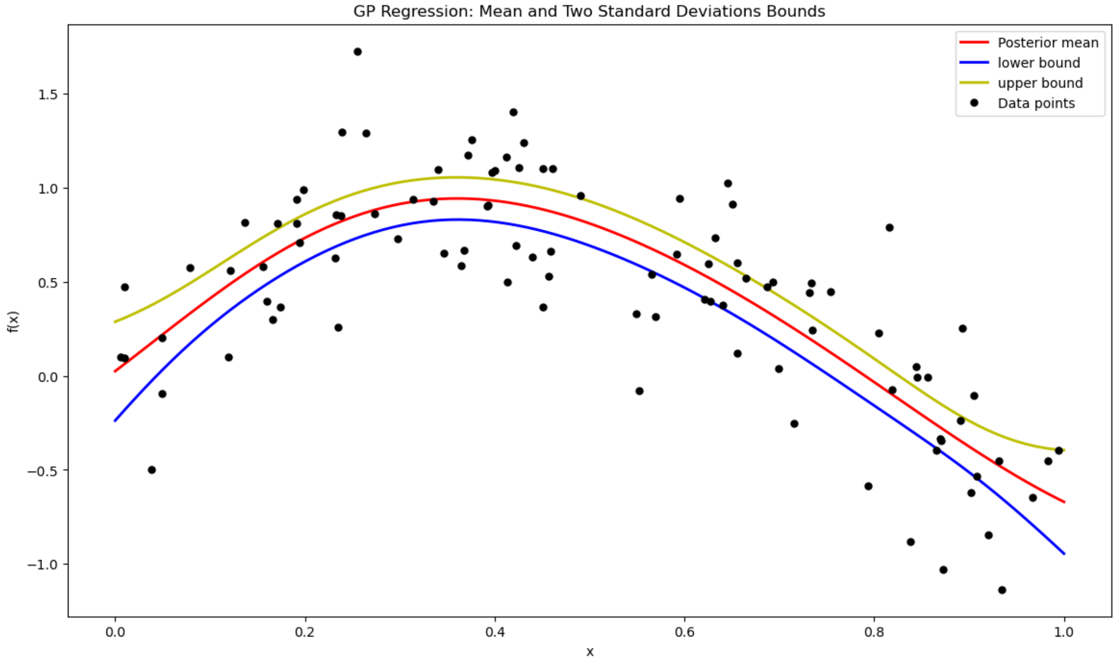


2 Part D

[28]:
```
# Apply GP regression to dataset gp.dat

# Load gp.dat
x_data, y_data = np.loadtxt('gp.dat', unpack=True)

# Compute the kernel matrix K for the data points from gp.dat
K = np.zeros((len(x_data), len(x_data)))
for i in range(len(x_data)):
    for j in range(len(x_data)):
        K[i, j] = kernel(x_data[i], x_data[j])

# Assume a noise level in the data
sigma_n_squared = 0.1
K_y = K + sigma_n_squared * np.eye(len(x_data))

# Inverse of K_y
K_y_inv = np.linalg.inv(K_y)

# Domain for test points
```

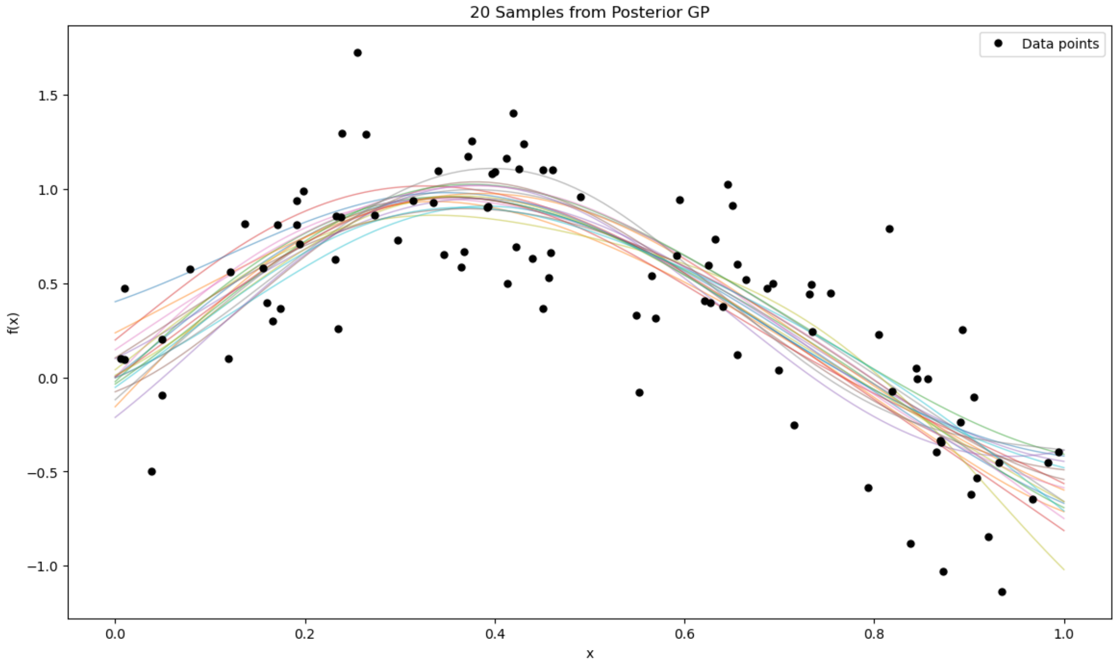
```python
x_pred = np.linspace(0, 1, 100)

# Compute the mean at our test points.
K_s = np.zeros((len(x_pred), len(x_data)))  # Cross-covariance
for i in range(len(x_pred)):
    for j in range(len(x_data)):
        K_s[i, j] = kernel(x_pred[i], x_data[j])

# Mean of the posterior
mu_s = K_s @ K_y_inv @ y_data

# Compute the covariance matrix for the test points
K_ss = np.zeros((len(x_pred), len(x_pred)))  # Covariance between test points
for i in range(len(x_pred)):
    for j in range(len(x_pred)):
        K_ss[i, j] = kernel(x_pred[i], x_pred[j])

# Covariance of the posterior
cov_s = K_ss - K_s @ K_y_inv @ K_s.T

# Standard deviation for plotting
std_s = np.sqrt(np.diag(cov_s))
std_bound1= mu_s - 2 * std_s
std_bound2= mu_s + 2 * std_s


# Plotting the mean and uncertainty
plt.figure(figsize=(14, 8))
plt.plot(x_pred, mu_s, 'r-', lw=2, label='Posterior mean')
plt.plot(x_pred, std_bound1, 'b-', lw=2, label='lower bound')
plt.plot(x_pred, std_bound2, 'y-', lw=2, label='upper bound')

plt.plot(x_data, y_data, 'ko', markersize=5, label='Data points')
plt.title('GP Regression: Mean and Two Standard Deviations Bounds')
plt.xlabel('x')
plt.ylabel('f(x)')
plt.legend()
```

```
[28]: <matplotlib.legend.Legend at 0x7fa149c8c940>
```



3 Part E

[24]:
```
# Draw samples from the posterior
num_samples = 20
posterior_samples = np.random.multivariate_normal(mu_s, cov_s, num_samples)

# Plot the samples from the posterior
plt.figure(figsize=(14, 8))
for i in range(num_samples):
    plt.plot(x_pred, posterior_samples[i], lw=1, alpha=0.5)
plt.plot(x_data, y_data, 'ko', markersize=5, label='Data points')
plt.title('20 Samples from Posterior GP')
plt.xlabel('x')
plt.ylabel('f(x)')
plt.legend()
plt.show()
```


20 Samples from Posterior GP
Data points
f(x)
x
1.5
1.0
0.5
0.0
−0.5
−1.0
0.0
0.2
0.4
0.6
0.8
1.0