HGEN 48800 HW4

Lin Yu

1.

$$\text{Condition Number} = \frac{|\Delta y / y|}{|\Delta x / x|} = \frac{|x f'(x)|}{|f(x)|} = \left| \frac{x \cos x}{\sin x} \right| = |x \cdot \cot(x)|$$

when $x = n\pi$, the condition number might go to infinity.

2. (a)

$$P = (S_t)^{n_s} \cdot (r_t)^{n_c}$$

$$\log P = n_s \cdot \log(S_t) + n_c \cdot \log r_t$$

$$= n_s \cdot \log 3\left(\tfrac{1}{4} - \tfrac{1}{4} e^{-\frac{4\alpha t}{3}}\right) + n_c \cdot \log\left(\tfrac{1}{4} + \tfrac{3}{4} e^{-\frac{4\alpha t}{3}}\right)$$

$$= n_s \cdot \log 3 + n_s \cdot \log\left(\tfrac{1}{4} - \tfrac{1}{4} e^{-\frac{4}{3}t}\right) + n_c \cdot \log\left(\tfrac{1}{4} + \tfrac{3}{4} e^{-\frac{4}{3}t}\right)$$

(b) when $n_s = 5$, $n_c = 25$

$$\log P = 5 \cdot \log 3 + 5 \cdot \log\left(\tfrac{1}{4} - \tfrac{1}{4} e^{-\frac{4}{3}t}\right) + 25 \cdot \log\left(\tfrac{1}{4} + \tfrac{3}{4} e^{-\frac{4}{3}t}\right)$$

$$\frac{\partial \log P}{\partial t} = 5 \cdot \left(\tfrac{1}{4} - \tfrac{1}{4} e^{-\frac{4}{3}t}\right)^{-1} \cdot \tfrac{1}{4} \cdot e^{-\frac{4}{3}t} \cdot \left(-\tfrac{4}{3}\right) + 25 \cdot \left(\tfrac{1}{4} + \tfrac{3}{4} e^{-\frac{4}{3}t}\right)^{-1} \cdot \tfrac{3}{4} \cdot e^{-\frac{4}{3}t} \cdot -\tfrac{4}{3}$$

Use Newton Method on computer for the rest part, as we can't solve $\frac{\partial \log P}{\partial t} = 0$ analytically.

3. (a)

$$\frac{\partial f}{\partial x} = 6x^2 - 6x - 6y\cdot(x-y-1) - 6xy\cdot 1$$

$$= 6x^2 - 6x - 12xy + 6y^2 + 6y$$

$$\frac{\partial f}{\partial y} = -6x(x-y-1) - 6xy\cdot(-1)$$

$$= -6x^2 + 6xy + 6x + 6xy$$

$$= -6x^2 + 6x + 12xy$$

For critical points, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$$\begin{cases} 6x^2 - 6x - 12xy + 6y^2 + 6y = 0 \\ -6x^2 + 6x + 12xy = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \quad \begin{cases} x = 0 \\ y = -1 \end{cases} \quad \begin{cases} x = 1 \\ y = 0 \end{cases}$$

$$\begin{cases} x = -1 \\ y = -1 \end{cases}$$

(b).

$$\frac{\partial^2 f}{\partial x^2} = 12x - 6 - 12y \qquad \frac{\partial f}{\partial x \partial y} = -12x + 12y + 6$$

$$\frac{\partial^2 f}{\partial y^2} = 12x$$

$$Hf((0,0)) = \begin{pmatrix} -6 & 6 \\ 6 & 0 \end{pmatrix} \qquad Hf((0,-1)) = \begin{pmatrix} 6 & -6 \\ -6 & 0 \end{pmatrix} \qquad Hf((1,0)) = \begin{pmatrix} 6 & -6 \\ -6 & 12 \end{pmatrix}$$

$$Hf((-1,-1)) = \begin{pmatrix} -6 & 6 \\ 6 & -12 \end{pmatrix}$$

4. As A is $n\times n$ real symmetric matrix, A has an orthonormal eigenbasis and all eigenvalues are real.

let $\lambda_1, \lambda_2, \cdots \lambda_n$ be A's eigenvalues, $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ and $v_1, \cdots v_n$ are their corresponding eigenvectors. and $\|v_i\|^2 = 1$ for $i = 1, 2, \dots n$

$$v_1^T A v_1 \geqslant \min_{\substack{x\in\mathbb{R}^n \\ \|x\|=1}} f(x), \qquad v_n^T A v_n \le \max_{\substack{x\in\mathbb{R}^n \\ \|x\|=1}} f(x) \Rightarrow \begin{array}{l}\lambda_1 \le \min f(x) \\ \lambda_n \geqslant \max f(x) \\ x\in\mathbb{R}^n, \|x\|=1\end{array}$$

Given $x \in \mathbb{R}^n$, $\|x\| = 1$, $x = \sum_{i=1}^n a_i v_i$ $\quad \sum_{i=1}^n a_i^2 = 1$.

$$f(x) = x^T A x = \langle x, Ax\rangle = \langle \sum_{i=1}^n a_i v_i, A\cdot \sum_{i=1}^n a_i v_i\rangle$$

$$= \langle \sum_{i=1}^n a_i v_i, \sum_{i=1}^n a_i \lambda_i v_i\rangle = \sum_{i=1}^n \lambda_i \cdot a_i^2$$

$f(x) \le \sum_{i=1}^n \lambda_n a_i^2 = \lambda_n$ for $\forall x \in \mathbb{R}^n$ $\|x\| = 1$.

$f(x) \geqslant \sum_{i=1}^n \lambda_1 a_i^2 = \lambda_1$, for any $x \in \mathbb{R}^n$, $\|x\| = 1$

Thus, $\min f(x) = \lambda_1$, $\max f(x) = \lambda_2$ for $x \in \mathbb{R}^n$, $\|x\| = 1$.