HGEN 48800 HW4

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1.

$$\text{Condition Number} = \frac{|\Delta y / y|}{|\Delta x / x|} = \frac{|x f'(x)|}{|f(x)|} = \left| \frac{x \cos x}{\sin x} \right| = |x \cdot \cot(x)|$$

when $x = n\pi$, the condition number might go to infinity.

2. (a).

$$P = (St)^{n_s} \cdot (rt)^{n_c}$$

$$\log P = n_s \cdot \log(St) + n_c \cdot \log rt$$

$$= n_s \cdot \log 3\left(\frac{1}{4} - \frac{1}{4} e^{-\frac{4}{3}t}\right) + n_c \cdot \log\left(\frac{1}{4} + \frac{3}{4} e^{-\frac{4}{3}t}\right)$$

$$= n_s \cdot \log 3 + n_s \cdot \log\left(\frac{1}{4} - \frac{1}{4} e^{-\frac{4}{3}t}\right) + n_c \cdot \log\left(\frac{1}{4} + \frac{3}{4} e^{-\frac{4}{3}t}\right)$$

(b) When $n_s = 5$, $n_c = 25$

$$\log P = 5 \cdot \log 3 + 5 \cdot \log\left(\frac{1}{4} - \frac{1}{4} e^{-\frac{4}{3}t}\right) + 25 \cdot \log\left(\frac{1}{4} + \frac{3}{4} e^{-\frac{4}{3}t}\right)$$

$$\frac{\partial \log P}{\partial t} = 5 \cdot \left(\frac{1}{4} - \frac{1}{4} e^{-\frac{4}{3}t}\right)^{-1} \cdot \frac{1}{4} \cdot e^{-\frac{4}{3}t} \cdot \left(-\frac{4}{3}\right) + 25 \cdot \left(\frac{1}{4} + \frac{3}{4} e^{-\frac{4}{3}t}\right)^{-1} \cdot \frac{3}{4} \cdot e^{-\frac{4}{3}t} \cdot -\frac{4}{3}$$

Use Newton method on computer for the rest part.

as we can't solve $\frac{\partial \log P}{\partial t} = 0$ analytically.

3. (a)

$$\frac{\partial f}{\partial x} = 6x^2 - 6x - 6y\cdot(x-y-1) - 6xy\cdot 1$$

$$= 6x^2 - 6x - 12xy + 6y^2 + 6y$$

$$\frac{\partial f}{\partial y} = -6x(x-y-1) - 6xy\cdot(-1)$$

$$= -6x^2 + 6xy + 6x + 6xy$$

$$= -6x^2 + 6x + 12xy$$

For critical points, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$$\begin{cases} 6x^2 - 6x - 12xy + 6y^2 + 6y = 0 \\ -6x^2 + 6x + 12xy = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \quad \begin{cases} x = 0 \\ y = -1 \end{cases} \quad \begin{cases} x = 1 \\ y = 0 \end{cases}$$

$$\begin{cases} x = -1 \\ y = -1 \end{cases}$$

(b).

$$\frac{\partial^2 f}{\partial x^2} = 12x - 6 - 12y \qquad \frac{\partial f}{\partial x \partial y} = -12x + 12y + 6$$

$$\frac{\partial^2 f}{\partial y^2} = 12x$$

$$Hf((0,0)) = \begin{pmatrix} -6 & 6 \\ 6 & 0 \end{pmatrix} \qquad Hf((0,-1)) = \begin{pmatrix} 6 & -6 \\ -6 & 0 \end{pmatrix} \qquad Hf((1,0)) = \begin{pmatrix} 6 & -6 \\ -6 & 12 \end{pmatrix}$$

$$Hf((-1,-1)) = \begin{pmatrix} -6 & 6 \\ 6 & -12 \end{pmatrix}$$

4. As A is $n \times n$ real symmetric matrix, A has an orthonormal eigenbasis and all eigenvalues are real.

let $\lambda_1, \lambda_2, \cdots \lambda_n$ be A's eigenvalues, $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ and $v_1, \cdots v_n$ are their corresponding eigenvectors. and $\|v_i\|^2 = 1$ for $i = 1, 2, \cdots n$

$v_1^T A v_1 \ge \min_{\substack{x \in \mathbb{R}^n \\ \|x\|=1}} f(x)$, $\quad v_n^T A v_n \le \max_{\substack{x \in \mathbb{R}^n \\ \|x\|=1}} f(x) \Rightarrow \lambda_1 \le \min f(x)$

$\lambda_n \ge \max_{x \in \mathbb{R}^n,\ \|x\|=1} f(x)$

Given $x \in \mathbb{R}^n$, $\|x\| = 1$, $x = \sum_{i=1}^n a_i v_i$ $\quad \sum_{i=1}^n a_i^2 = 1$.

$$f(x) = x^T A x = \langle x, Ax \rangle = \langle \sum_{i=1}^n a_i v_i, A \cdot \sum_{i=1}^n a_i v_i \rangle$$

$$= \langle \sum_{i=1}^n a_i v_i, \sum_{i=1}^n a_i \lambda_i v_i \rangle = \sum_{i=1}^n \lambda_i \cdot a_i^2$$

$f(x) \le \sum_{i=1}^n \lambda_n a_i^2 = \lambda_n$ for $\forall x \in \mathbb{R}^n$ $\|x\| = 1$.

$f(x) \ge \sum_{i=1}^n \lambda_1 a_i^2 = \lambda_1$, for any $x \in \mathbb{R}^n$, $\|x\| = 1$

Thus, $\min f(x) = \lambda_1$, $\max f(x) = \lambda_2$ for $x \in \mathbb{R}^n$, $\|x\| = 1$.

Lin_Yu HW4

April 21, 2024

[1]:
```
import numpy as np
from scipy.optimize import newton
import matplotlib.pyplot as plt
```

1 Problem 2

Part B

[2]:
```
#(b)

# Log-likelihood derivatives
def log_likelihood_prime(t, nS=5, nC=25):
    """Calculates the first derivative of the log-likelihood function with␣
 ↪respect to t."""
    # Compute st and rt
    st = 3 * (0.25 - 0.25 * np.exp(-4 * t / 3))
    rt = 0.25 + 0.75 * np.exp(-4 * t / 3)

    # Compute derivatives of st and rt with respect to t
    dst_dt = 1 * np.exp(-4 * t / 3)  # Corrected to match the explanation above
    drt_dt = -1 * np.exp(-4 * t / 3)  # As you originally had, aligning with␣
 ↪the negative exponential derivative

    # Log-likelihood derivative calculation
    return nS * (dst_dt / st) + nC * (drt_dt / rt)

def log_likelihood_double_prime(t, nS=5, nC=25):
    """Calculates the second derivative of the log-likelihood function with␣
 ↪respect to t."""
    # Compute st and rt
    st = 3 * (0.25 - 0.25 * np.exp(-4 * t / 3))
    rt = 0.25 + 0.75 * np.exp(-4 * t / 3)

    # Compute derivatives of st and rt with respect to t
    dst_dt = 1 * np.exp(-4 * t / 3)
    drt_dt = -1 * np.exp(-4 * t / 3)
```

```
    # Compute second derivatives of st and rt with respect to t
    d2st_dt2 = -4/3 * dst_dt
    d2rt_dt2 = -4/3 * drt_dt

    # Log-likelihood second derivative calculation
    return nS * (-dst_dt**2 / st**2 + d2st_dt2 / st) + nC * (-drt_dt**2 / rt**2␣
 ↪+ d2rt_dt2 / rt)
```

[3]:
```
initial_t = 0.5

# Use scipy.optimize.newton to find the root of the first derivative (extremum␣
 ↪of the log-likelihood)
t_estimated = newton(log_likelihood_prime, initial_t,␣
 ↪fprime=log_likelihood_double_prime, args=(5, 25))

print(f"Given ns=5, nc=25, the estimated divergence time t: {t_estimated:.4f}")
```

```
/var/folders/t2/7mtp11ks3nnd0l6dwsngdl680000gn/T/ipykernel_5337/3037943102.py:7:
RuntimeWarning: overflow encountered in exp
  st = 3 * (0.25 - 0.25 * np.exp(-4 * t / 3))
/var/folders/t2/7mtp11ks3nnd0l6dwsngdl680000gn/T/ipykernel_5337/3037943102.py:8:
RuntimeWarning: overflow encountered in exp
  rt = 0.25 + 0.75 * np.exp(-4 * t / 3)
/var/folders/t2/7mtp11ks3nnd0l6dwsngdl680000gn/T/ipykernel_5337/3037943102.py:11
: RuntimeWarning: overflow encountered in exp
  dst_dt = 1 * np.exp(-4 * t / 3)  # Corrected to match the explanation above
/var/folders/t2/7mtp11ks3nnd0l6dwsngdl680000gn/T/ipykernel_5337/3037943102.py:12
: RuntimeWarning: overflow encountered in exp
  drt_dt = -1 * np.exp(-4 * t / 3)  # As you originally had, aligning with the
negative exponential derivative
/var/folders/t2/7mtp11ks3nnd0l6dwsngdl680000gn/T/ipykernel_5337/3037943102.py:15
: RuntimeWarning: invalid value encountered in scalar divide
  return nS * (dst_dt / st) + nC * (drt_dt / rt)
/var/folders/t2/7mtp11ks3nnd0l6dwsngdl680000gn/T/ipykernel_5337/3037943102.py:20
: RuntimeWarning: overflow encountered in exp
  st = 3 * (0.25 - 0.25 * np.exp(-4 * t / 3))
/var/folders/t2/7mtp11ks3nnd0l6dwsngdl680000gn/T/ipykernel_5337/3037943102.py:21
: RuntimeWarning: overflow encountered in exp
  rt = 0.25 + 0.75 * np.exp(-4 * t / 3)
/var/folders/t2/7mtp11ks3nnd0l6dwsngdl680000gn/T/ipykernel_5337/3037943102.py:24
: RuntimeWarning: overflow encountered in exp
  dst_dt = 1 * np.exp(-4 * t / 3)
/var/folders/t2/7mtp11ks3nnd0l6dwsngdl680000gn/T/ipykernel_5337/3037943102.py:25
: RuntimeWarning: overflow encountered in exp
  drt_dt = -1 * np.exp(-4 * t / 3)
/var/folders/t2/7mtp11ks3nnd0l6dwsngdl680000gn/T/ipykernel_5337/3037943102.py:32
: RuntimeWarning: invalid value encountered in scalar divide
  return nS * (-dst_dt**2 / st**2 + d2st_dt2 / st) + nC * (-drt_dt**2 / rt**2 +
```

```
d2rt_dt2 / rt)
```

```
---------------------------------------------------------------------------
RuntimeError                              Traceback (most recent call last)
Cell In[3], line 4
      1 initial_t = 0.5
      3 # Use scipy.optimize.newton to find the root of the first derivative␣
 ↪(extremum of the log-likelihood)
----> 4 t_estimated =␣
 ↪newton(log_likelihood_prime, initial_t, fprime=log_likelihood_double_prime, a
 ↪gs=(5, 25))
      6 print(f"Given ns=5, nc=25, the estimated divergence time t: {t_estimate
 ↪.4f}")

File ~/opt/anaconda3/lib/python3.9/site-packages/scipy/optimize/_zeros_py.py:
 ↪381, in newton(func, x0, fprime, args, tol, maxiter, fprime2, x1, rtol,␣
 ↪full_output, disp)
    378 if disp:
    379     msg = ("Failed to converge after %d iterations, value is %s."
    380            % (itr + 1, p))
--> 381     raise RuntimeError(msg)
    383 return _results_select(full_output, (p, funcalls, itr + 1, _ECONVERR))

RuntimeError: Failed to converge after 50 iterations, value is nan.
```

[4]:
```
initial_t = 10

# Use scipy.optimize.newton to find the root of the first derivative (extremum␣
 ↪of the log-likelihood)
t_estimated = newton(log_likelihood_prime, initial_t,␣
 ↪fprime=log_likelihood_double_prime, args=(5, 25))

print(f"Given ns=5, nc=25, the estimated divergence time t: {t_estimated:.4f}")
```

```
---------------------------------------------------------------------------
RuntimeError                              Traceback (most recent call last)
Cell In[4], line 4
      1 initial_t = 10
      3 # Use scipy.optimize.newton to find the root of the first derivative␣
 ↪(extremum of the log-likelihood)
----> 4 t_estimated =␣
 ↪newton(log_likelihood_prime, initial_t, fprime=log_likelihood_double_prime, a
 ↪gs=(5, 25))
      6 print(f"Given ns=5, nc=25, the estimated divergence time t: {t_estimate
 ↪.4f}")

File ~/opt/anaconda3/lib/python3.9/site-packages/scipy/optimize/_zeros_py.py:
 ↪381, in newton(func, x0, fprime, args, tol, maxiter, fprime2, x1, rtol,␣
 ↪full_output, disp)
```

```
    378 if disp:
    379     msg = ("Failed to converge after %d iterations, value is %s."
    380            % (itr + 1, p))
--> 381     raise RuntimeError(msg)
    383 return _results_select(full_output, (p, funcalls, itr + 1, _ECONVERR))

RuntimeError: Failed to converge after 50 iterations, value is 47.5000063138963
```

[5]:
```
# Initial guess for t
initial_t = 0.25
# Use scipy.optimize.newton to find the root of the first derivative (extremum
 ↪of the log-likelihood)
t_estimated = newton(log_likelihood_prime, initial_t,
 ↪fprime=log_likelihood_double_prime, args=(5, 25))

print(f"Given ns=5, nc=25, the estimated divergence time t: {t_estimated:.4f}")
```

```
Given ns=5, nc=25, the estimated divergence time t: 0.1885
```

Part C

[6]:
```
#(c)
print(f"The second derivative of the log_likelihood function at time t:
 ↪{t_estimated} is "
      f"{log_likelihood_double_prime(t_estimated, nS=5, nC=25)}")
```

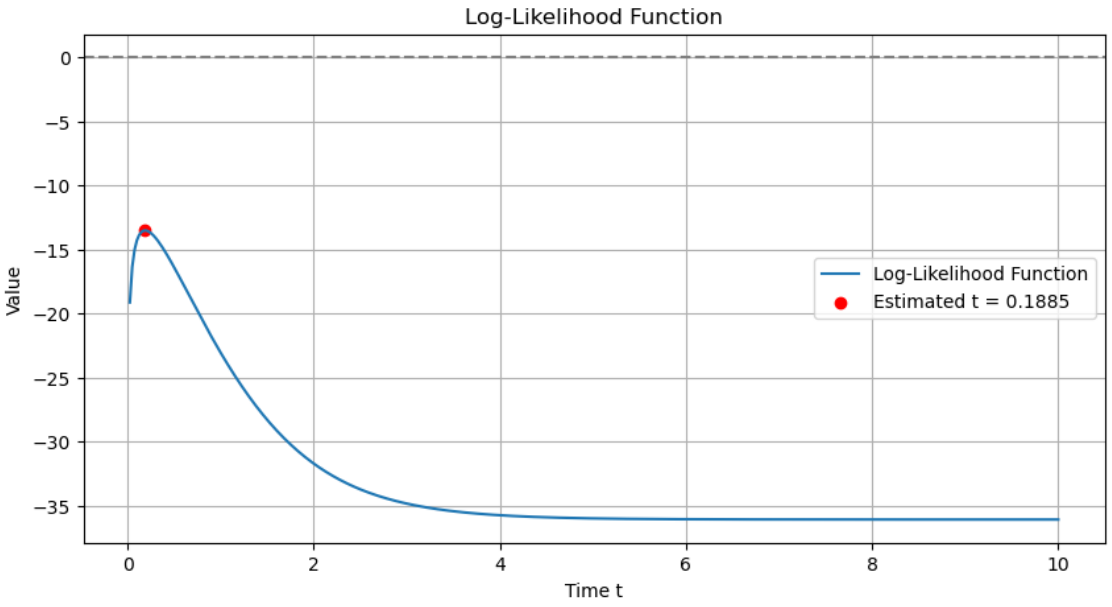
```
The second derivative of the log_likelihood function at time t:
0.18848582121067955 is -130.66666666666666
```

[7]:
```
#(c)
def log_likelihood(t, nS=5, nC=25):
    """Calculates the log-likelihood function of t."""
    # Compute st and rt,
    st = 3 * (0.25 - 0.25 * np.exp(-4 * t / 3))
    rt = 0.25 + 0.75 * np.exp(-4 * t / 3)
    if st <= 0 or rt <= 0:
        return np.nan  # Return NaN if st or rt is non-positive, to avoid
 ↪taking log of non-positive numbers
    return nS * np.log(st) + nC * np.log(rt)

t_values = np.linspace(0, 10, 400)
function_values = [log_likelihood(t, 5, 25) for t in t_values]
estimated_y = log_likelihood(t_estimated, 5, 25)  # Calculate log-likelihood at
 ↪estimated t


# Plot the log-likelihood function
plt.figure(figsize=(10, 5))
```

```
plt.plot(t_values, function_values, label="Log-Likelihood Function")
plt.axhline(0, color='gray', linestyle='--')
plt.scatter([t_estimated], [estimated_y], color='red', label=f"Estimated t =␣
 ↪{t_estimated:.4f}")
plt.title("Log-Likelihood Function")
plt.xlabel("Time t")
plt.ylabel("Value")
plt.legend()
plt.grid(True)
plt.show()
```



As demonstrated, the solution where the derivative of the log-likelihood function equals zero is **0.1885**. Importantly, the outcome of Newton's method is highly sensitive to the initial guess provided. Since the second derivative at this point is negative, we can conclude that it represents a local **maximum** for the parameters $n_{S} = 5 $ and $n_C = 25$. This conclusion is corroborated by the plot of the log-likelihood function.

2 Problem 3

[8]:
```
# (b)


def is_positive_definite_eigen(matrix):
    # Compute the eigenvalues
    eigenvalues = np.linalg.eigvals(matrix)
    # Check if all eigenvalues are greater than zero
```

```
    if (np.all(eigenvalues > 0)):
        return("The Hessian matrix is positive definite")
    elif (np.all(eigenvalues < 0)):
        return("The Hessian matrix is negative definite")
    else:
        return("The Hessian matrix is indefinite")

Hf = np.array([[-6, 6],
               [6, 0]])

print("Critial point: (0,0)")
is_positive_definite_eigen(Hf)
```

```
Critial point: (0,0)
```

[8]:
```
'The Hessian matrix is indefinite'
```

[9]:
```
# (-1,-1)
Hf = np.array([[-6, 6],
               [6, -12]])

print("Critial point: (-1,-1)")
is_positive_definite_eigen(Hf)
```

```
Critial point: (-1,-1)
```

[9]:
```
'The Hessian matrix is negative definite'
```

[10]:
```
# (0,-1)
print("Critial point: (0,-1)")
Hf = np.array([[6, -6],
               [-6, 0]])

is_positive_definite_eigen(Hf)
```

```
Critial point: (0,-1)
```

[10]:
```
'The Hessian matrix is indefinite'
```

[11]:
```
# (1,0)
print("Critial point: (1,0)")
Hf = np.array([[6, -6],
               [-6, 12]])

is_positive_definite_eigen(Hf)
```

```
Critial point: (1,0)
```

[11]:
```
'The Hessian matrix is positive definite'
```

After evaluating the Hessian matrices at three critical points—(0,0), (-1,-1), (0,-1), and (1,0)—we find that both (0,0) and (0,-1) are saddle points, while (1,0) is a local minimum and (-1,-1) is a local maximum for the function.

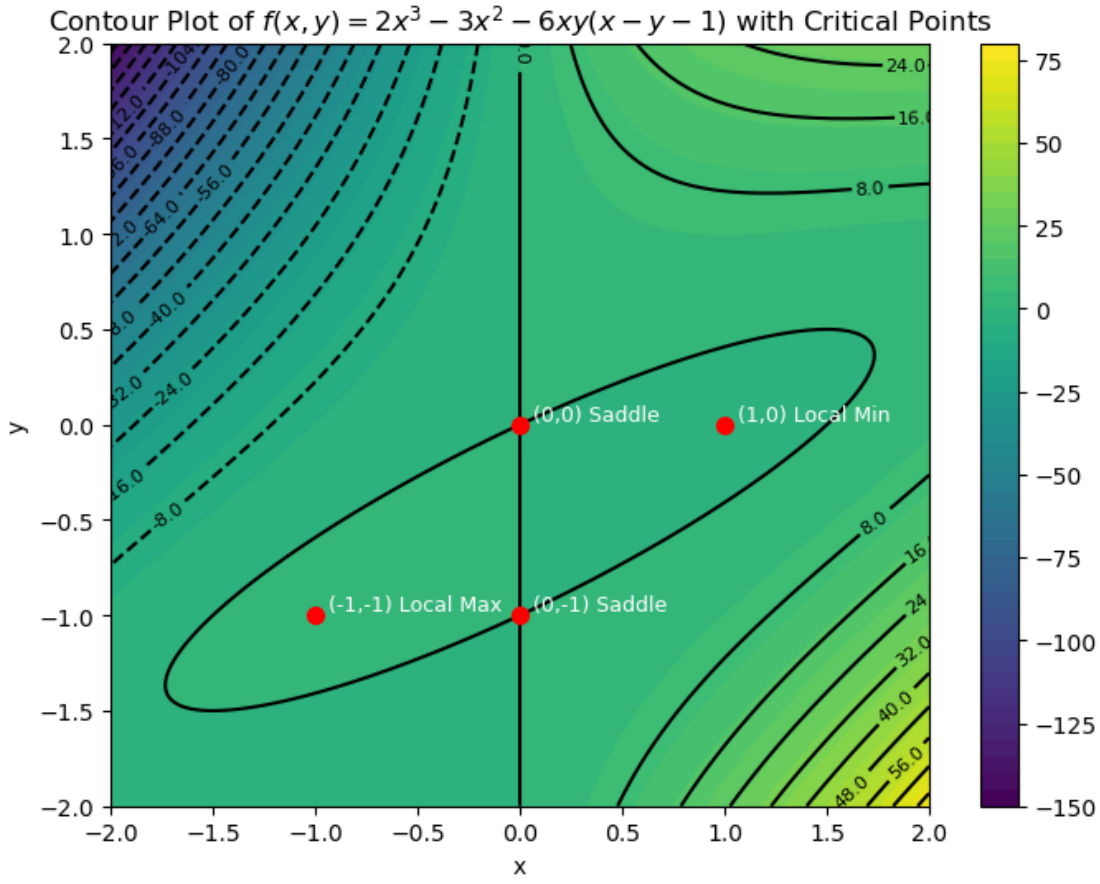
[12]:
```
# (c)
# Define the function f(x, y)
def f(x, y):
    return 2*x**3 - 3*x**2 - 6*x*y*(x - y - 1)

# Create a grid of x and y values
x = np.linspace(-2, 2, 1000)
y = np.linspace(-2, 2, 1000)
X, Y = np.meshgrid(x, y)
Z = f(X, Y)

# Create a contour plot with labels for better visibility of critical points
plt.figure(figsize=(8, 6))
contours = plt.contour(X, Y, Z, levels=30, colors='black')
plt.clabel(contours, inline=True, fontsize=8, fmt='%1.1f')
contourf = plt.contourf(X, Y, Z, levels=50, cmap='viridis')
plt.colorbar(contourf)

# Annotating the critical points
critical_points = [(0, 0), (0, -1), (1, 0), (-1,-1)]
cp_labels = ['(0,0) Saddle', '(0,-1) Saddle', '(1,0) Local Min', '(-1,-1) Local␣
 ↪Max']
for i, (x, y) in enumerate(critical_points):
    plt.scatter(x, y, color='red', s=50, zorder=5)  # Mark critical points
    plt.text(x, y, '  ' + cp_labels[i], color='white', fontsize=9, ha='left',␣
 ↪va='bottom')

plt.title('Contour Plot of $f(x, y) = 2x^3 - 3x^2 - 6xy(x - y - 1)$ with␣
 ↪Critical Points')
plt.xlabel('x')
plt.ylabel('y')
plt.show()
```



[13]:
```
#(d)
from scipy.optimize import minimize
import numpy as np

# Define the function and its negative
def f(x):
    return 2*x[0]**3 - 3*x[0]**2 - 6*x[0]*x[1]*(x[0] - x[1] - 1)

def neg_f(x):
    return -f(x)

# Define different starting points
starting_points = [(0, 0), (1, 0), (-1, 1), (0.5, -0.5), (-0.5, 0.5)]

# Find the minimum of f and maximum of -f for each starting point using for␣
 ↪loops
results_f = []
results_neg_f = []
```

```python
for start in starting_points:
    result_f = minimize(f, start)
    result_neg_f = minimize(neg_f, start)
    results_f.append((start, result_f.fun, result_f.x))
    results_neg_f.append((start, -result_neg_f.fun, result_neg_f.x))

# Display results
for start, min_f, point_f in results_f:
    print(f"Starting Point: {start} -> Minimum of f: {min_f}, at point:␣
 ↪{point_f}")
    print( )

for start, min_neg_f, point_neg_f in results_neg_f:
    print(f"Starting Point: {start} -> Minimum of -f: {min_neg_f}, at point:␣
 ↪{point_neg_f}")
    print( )
```

```
Starting Point: (0, 0) -> Minimum of f: 0.0, at point: [0. 0.]

Starting Point: (1, 0) -> Minimum of f: -1.0, at point: [1. 0.]

Starting Point: (-1, 1) -> Minimum of f: -5550808007.577093, at point:
[-861.53969642  574.69313094]

Starting Point: (0.5, -0.5) -> Minimum of f: -0.9999999999999212, at point: [
9.99999800e-01 -4.47909316e-08]

Starting Point: (-0.5, 0.5) -> Minimum of f: -5476568621.81335, at point:
[-877.53376092  548.64611174]

Starting Point: (0, 0) -> Minimum of -f: 0.0, at point: [0. 0.]

Starting Point: (1, 0) -> Minimum of -f: -1.0, at point: [1. 0.]

Starting Point: (-1, 1) -> Minimum of -f: 5606573018.859791, at point:
[1228.29292677 1410.53354554]

Starting Point: (0.5, -0.5) -> Minimum of -f: 3210706.0018725474, at point: [
5.00030823e-01 -1.03474000e+03]

Starting Point: (-0.5, 0.5) -> Minimum of -f: 14994354.946734795, at point: [
37.13467489 277.26325098]
```

As seen from the results above, different starting points yield varying minima for f and −f. This variation is attributed to the nature of the algorithms used in scipy.optimize.minimize (such as 'BFGS', 'Nelder-Mead', 'L-BFGS-B', etc.), which are local optimization methods. These algorithms

efficiently find a local minimum near the initial starting point but do not guarantee the discovery of the global minimum unless the function is convex, which is not applicable in this case.

3 Dynamic Programming

[14]:
```
from Bio.Align.substitution_matrices import load
# Load the BLOSUM62 substitution matrix
blosum62 = load("BLOSUM62")
```

[15]:
```
import numpy as np

def initialize_score_matrix(seq1, seq2, gap_penalty):
    '''Initialize the first row and column of score matrix'''
    m, n = len(seq1), len(seq2)
    score_matrix = np.zeros((m+1, n+1))

    # Initialize first row and column
    for i in range(1, m+1):
        score_matrix[i][0] = i * gap_penalty
    for j in range(1, n+1):
        score_matrix[0][j] = j * gap_penalty

    return score_matrix
```

[16]:
```
def fill_score_matrix(seq1, seq2, score_matrix, gap_penalty):
    """Fill the rest of matrix from top to bottom and left to right with␣
 ↪blosum62 matrix"""
    m, n = len(seq1), len(seq2)
    for i in range(1, m+1):
        for j in range(1, n+1):
            match_score = blosum62.get((seq1[i-1], seq2[j-1]), blosum62.
 ↪get((seq2[j-1], seq1[i-1]), -1))
            # Calculate the match score between amino acids at current␣
 ↪positions i and j in seq1 and seq2
            # Fetch the score directly from the BLOSUM62 matrix using the tuple␣
 ↪(seq1[i-1], seq2[j-1]).
            # If the tuple is not found, try with the reversed tuple␣
 ↪(seq2[j-1], seq1[i-1]).
            # If neither tuple exists in the matrix, default to a penalty score␣
 ↪of -1.
            diagonal = score_matrix[i-1][j-1] + match_score
            delete = score_matrix[i-1][j] + gap_penalty
            insert = score_matrix[i][j-1] + gap_penalty
            score_matrix[i][j] = max(diagonal, delete, insert)

    return score_matrix
```

[17]:
```
def traceback(sequence1, sequence2, scoring_matrix, penalty_for_gap):
    """
    Perform the traceback step in sequence alignment using dynamic programming.
    Constructs the alignment by navigating from the bottom-right to the␣
↪top-left of the scoring matrix.

    sequence1: First sequence being aligned
    sequence2: Second sequence being aligned
    scoring_matrix: Matrix with computed scores for alignments
    penalty_for_gap: Penalty score used for gaps in the alignment (deletion or␣
↪insertion)

    """
    aligned_seq1, aligned_seq2 = "", ""
    index1, index2 = len(sequence1), len(sequence2)

    # Trace back from the bottom-right of the matrix to the top-left
    while index1 > 0 and index2 > 0:
        current_score = scoring_matrix[index1][index2]
        diagonal_score = scoring_matrix[index1-1][index2-1]
        left_score = scoring_matrix[index1][index2-1]
        up_score = scoring_matrix[index1-1][index2]

        # Check if the current cell was filled from the diagonal cell
        if current_score == diagonal_score + blosum62.get((sequence1[index1-1],␣
↪sequence2[index2-1]), blosum62.get((sequence2[index2-1],␣
↪sequence1[index1-1]), -1)):
            aligned_seq1 += sequence1[index1-1]
            aligned_seq2 += sequence2[index2-1]
            index1 -= 1
            index2 -= 1
        # Check if the current cell was filled from the left cell
        elif current_score == up_score + penalty_for_gap:
            aligned_seq1 += sequence1[index1-1]
            aligned_seq2 += '-'
            index1 -= 1
        # Otherwise, the current cell was filled from the upper cell
        else:
            aligned_seq1 += '-'
            aligned_seq2 += sequence2[index2-1]
            index2 -= 1

    # Finish tracing back to the top of the matrix if there are remaining␣
↪characters in sequence1
    while index1 > 0:
        aligned_seq1 += sequence1[index1-1]
        aligned_seq2 += '-'
```

```
        index1 -= 1
    # Finish tracing back to the left of the matrix if there are remaining␣
↪characters in sequence2
    while index2 > 0:
        aligned_seq1 += '-'
        aligned_seq2 += sequence2[index2-1]
        index2 -= 1

    # Return aligned sequences in the correct order (they are built backwards)
    return aligned_seq1[::-1], aligned_seq2[::-1]
```

[19]:
```
def align_sequences(seq1, seq2, gap_penalty=-2):
    """
    Aligns two biological sequences using dynamic programming with the BLOSUM62␣
↪substitution matrix.

    seq1: First sequence (string of amino acids) to align
    seq2: Second sequence (string of amino acids) to align
    gap_penalty: Penalty score for introducing a gap in alignment (default is␣
↪-2)
    """
    # Initialize the score matrix with base values for the first row and column
    score_matrix = initialize_score_matrix(seq1, seq2, gap_penalty)

    # Fill the rest of the score matrix based on sequence comparison and gap␣
↪penalties
    score_matrix = fill_score_matrix(seq1, seq2, score_matrix, gap_penalty)

    # Perform traceback to construct the optimal alignment from the score matrix
    print(traceback(seq1, seq2, score_matrix, gap_penalty))
```

[21]:
```
Sequence_1="PWAGAWHHEE"
Sequence_2="CAWDEAE"
align_sequences(Sequence_1, Sequence_2, gap_penalty=-2)
```

```
('PWAGAWHHE-E', '--C-AW-DEAE')
```