Topology From the Differentiable Viewpoint [1]

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Abstract

This report examines the connections between smooth maps, homotopy classes, and cobordism in differential topology, focusing on the mod 2 degree of mappings and Pontryagin manifolds. We explore how the topology of smooth manifolds interacts with algebraic properties through regular values and smooth homotopy, highlighting the theoretical underpinnings that guide our understanding of manifold structures and their mathematical implications.

Kewords: Smooth Manifolds, regular value, Pontryagin Construction

1 Preliminary

Let $f : M \to N$ be a smooth map between smooth manifolds of the same dimension. We say that $x \in M$ is a **regular point** of f if the differential $df_x : T_x M \to T_{f(x)} N$ is nonsingular. In this case it follows from the inverse function theorem that f is a local diffeomorphism near x. The point $y = f(x) \in N$ is called a **regular value** of f if $f^{-1}(y)$ contains only regular points.

If df_x is singular, then x is called a **critical point** of f. and the image $f(x)$ is called a **critical value** of f. Thus each $y \in N$ is either a regular value or a critical value of f.

For a smooth $f : M \to N$, with M compact, and a regular value $y \in N$, we define the $\sharp f^{-1}(y)$(**degree** of f at y) to be the number of points in $f^{-1}(y)$.

2 The Degree Modulo 2 of A Mapping

Consider a smooth map $f : S^n \to S^m$. We will prove that the **residue calss modulo 2** of $\sharp f^{-1}(y)$ does not depend on the choice of regular value y. This residue calss is called the mod 2 degree of f. More generally this same definition works for aany smooth map

$$f : M \to N$$

where M is compact without boundary, N is connected and both manifolds have the same dimension.

2.1 Smooth Homotopy and Smooth Isotopy

Given $X \subset R^k$, let $X \times [0,1]$ denote the subset of R^{k+1} consisting of all (x,t) with $x \in X$ and $0 \leq t \leq 1$. Two mappings

$$f, g : X \to Y$$

are called smoothly homotopic (abbreviated $f \sim g$) if there exists a smooth map $F : X \times [0,1] \to Y$ with

$$F(x,0) = f(x), \qquad F(x,1) = g(x)$$

for all $x \in X$. This map F is called a **smooth homotopy** between f and g.

Note that the relation of smooth homotopy is an equivalence relation.

Definition 2.1. *The diffeomorphism f is smoothly isotopic to g if there exists a smooth homotopy $F : X \times [0,1] \to Y$ from f to g so that, for each $t \in [0,1]$, the correspondence $x \to F(x,t)$ maps X diffeomorphically onto Y.*

It will turn out that the mod 2 degree of a map depends only on its smooth homotopy class:

2.2 Homotopy Lemma

Lemmma 2.1. *Let $f, g : M \to N$ be smoothly homotopic maps between manifolds of the same dimension, where M is compact and without boundary. If $y \in N$ is a regular value for both f and g, then*

$$\sharp f^{-1}(y) = \sharp g^{-1}(y)(mod 2)$$

Proof. Let $F : M \times [0,1] \to N$ be a smooth homotopy between f and g. First suppose that y is also a regular value for F. Then $F^{-1}(y)$ is a compact 1-manifold, with boundary equal to

$$F^{-1}(y) \cap (M \times \{0\} \cup M \times \{1\}) = f^{-1}(y) \times \{0\} \cup g^{-1}(y) \times \{1\}$$

Thus the total number of boundary points of $F^{-1}(y)$ is equal to

$$\sharp f^{-1}(y) + \sharp g^{-1}(y)$$

From a lemma, a compact 1-manifold always has an even number of boundary points. Thus $\sharp f^{-1}(y) + \sharp g^{-1}(y)$ is even. Therefore

$$\sharp f^{-1}(y) \equiv \sharp g^{-1}(y)(mod2)$$

Now suppose that y is not a regular value for F. Obiviously, $\sharp f^{-1}(y), \sharp g^{-1}(y)$ are locally constant functions of y. Thus there is a neighborhood U of y in N, $\sharp f^{-1}(y') = \sharp f^{-1}(y)$ for all $y' \in U$. And there is an analogous neighborhood V of y in N, $\sharp g^{-1}(y') = \sharp g^{-1}(y)$ for all $y' \in V$. Choose a regular value z of F with $U \cap V$. Then

$$\sharp f^{-1}(y) = \sharp f^{-1}(z) = \sharp g^{-1}(z) = \sharp g^{-1}(y)$$

which completes the proof. □

3 Homogeneity Lemma

Lemmma 3.1. *Let y and z be arbitrary interior points of the smooth, connected manifold N. Then there exists a diffeomorphism $h : N \to N$ with $h(y) = z$ that is smoothly isotopic to the identity map.*

Proof. First, we construct a smooth isotopy from R^n to itself which

- (1) leaves all points outside of the unit ball fixed
- (2) slides the origin to any desired point of the open unit ball.

Let $\phi : R^n \to R$ be a smooth function which satisfies

$$\phi(x) > 0, \quad ||x|| < 1$$

$$\phi(x) = 0, \quad ||x|| \geq 1$$

Given any fixed unit vector $c \in S^{n-1}$, consider the differential equations

$$\frac{dx_i}{dt} = c_i \phi(x1, \ldots, x_n); \qquad i = 1, \ldots, n$$

For any $\bar{x} \in R^n$ these equations have a unique solution $x = x(t)$ with $x(0) = \bar{x}$. We will use the notation $x(t) = F_t(\bar{x})$ for this solution. Then clearly

- (1) $F_t(\bar{x})$ is defined for all t and $\bar{x}$ and depends smoothly on t and $\bar{x}$,
- (2) $F_0(\bar{x}) = \bar{x}$,
- (3) $F_{s+t}(\bar{x}) = F_s \circ F_t(\bar{x})$.

Therefore each F_t is a diffeomorphism from R^n onto itself. Letting t vary, we see that each F_t is a smooth isotopy to the identity map under an isotopy which leaves all points outside of unit ball fixed. But clearly, with suitable choice of c and t, the diffeomorphism F_t, will carry the origin to any desired point in the open unit ball.

Now consider a connected manifold N. Cll two points of N "isotopic" if there exists a smooth isotopy carrying one to the other. This is an equivalence relation. If y is an interior point, then it has a neighorhood diffeomorphic to R^n. Hence the above argument shows that every point sufficiently close to y is "isotopic" to y. In other words, each "isotopy class" of points in the interior of N is open. Since N is connected, hence there can be only one such isotopy class. This completes the proof. □

3.1 Main Theorem

Theorem 3.2. *M is compact and without boundary, and N is connected. Let $f : M \to N$ be a smooth map. If y and z are regular values of f, then*

$$\sharp f^{-1}(y) \equiv \sharp f^{-1}(z)(mod 2)$$

This common residue class, which is called the mod 2 degree of f, depends only on the smooth homotopy class of f.

Proof. Given regular values y and z, let h be a diffeomorphism from N to N with $h(y) = z$ that is smoothly isotopic to the identity map. Then z is a regular value of the composition $h \circ f$. By the homotopy lemma, since $h \circ f$ is smoothly homotopic to f, we have

$$\sharp f^{-1}(y) = \sharp(h \circ f)^{-1}(z) \equiv \sharp f^{-1}(z)(mod 2)$$

Call this common residue class $deg_2(f)$. Now suppose that f is smoothly homotopic to g. By Sard's Theorem, there exists an element $y \in N$ which is a regular value for both f and g. The congruence

$$deg_2(f) \equiv \sharp f^{-1}(y) \equiv \sharp g^{-1}(y) \equiv deg_2(g)(mod 2)$$

now shows that $deg_2(f)$ is a smooth homotopy invariant, and completes the proof. □

4 Framed Cobordism: The Pontryagin Construction

The degree of a mapping $M \to M'$ is defined only when the manifolds M and M' are oriented and have the same dimension. We will study a generalization, due to Pontryagin, which defined a smooth map

$$f : M \to S^p$$

from an arbitrary compact, boundaryless manifold to a sphere.

4.1 Pontryagin Manifold

Definition 4.1. *N is cobordant to N' within M if the subset*

$$N \times [0, \epsilon) \cup N' \times (1 - \epsilon, 1]$$

of $M \times [0, 1]$ can be extended to a compact manifold

$$X \subset M \times [0, 1]$$

so that

$$\partial X = N \times 0 \cup N' \times 1$$

and so that X does not intersect $M \times 0 \cup M \times 1$ except at the points of ∂X.

Clearly cobordism is an equivalence relation.

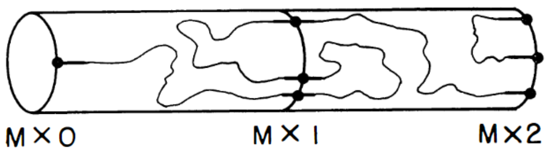

Figure 1: Cobordism

Definition 4.2. *A framing of the submanifold $N \subset M$ is a smooth function b which assigns to each $x \in N$ a basis*

$$b(x) = (v^1(x), \ldots v^{m-n}(x))$$

for the space $TN_x^\perp \subset TM_x$ of normal vectors to N in M at x. The pair (N, b) is called a framed submanifold of M. Two framed submanifolds (N, b) and (N', m) are framed cobordant if there exists a cobordism $X \subset M \times [0, 1]$ between N and N' and a framing u of X, so that

$$u^i(x, t) = (v^i(x), 0) \quad \textit{for} \quad (x, t) \in N \times [0, \epsilon]$$

$$u^i(x, t) = (w^i(x), 0) \quad \textit{for} \quad (x, t) \in N' \times (1 - \epsilon, 1).$$

Again this is an equivalence relation. Now consider a smooth map $f : M \to S^p$ and a regular value $y \in S^o$. The map f induces a framing of the manifold $f^{-1}(y)$ as follows: Choose a positively oriented basis $b = (v^1, \dots, v^p)$ for the tangent space $T(S^p)_y$. For each $x \in f^{-1}(y)$ recall from page 12 that

$$df_x : TM_x \to T(S^p)_y,$$

maps the subspace $Tf^{-1}(y)_x$ to zero and maps its orthogonal complement $Tf^{-1}(y)_x^\perp$ isomorphically onto $T(S^p)_y$. Hence there is a unique vector

$$w^i(x) \in Tf^{-1}(y)_x^\perp \subset TM_x$$

that maps into v^i under df_x. It will be convenient to use the notation

$$\eta = f^* b$$

for the resulting framing $w^1(x), \dots, w^p(x)$ of $f^{-1}(y)$.

Definition 4.3. *This framed manifold $(f^{-1}(y), f^*b)$ will be called the Pontryagin manifold associated with f.*

Of course f has many Pontryagin manifolds, corresponding to different choices of y and b, but they all belong to a single framed cobordism class:

Theorem 4.1. *If y' is another regular value of f and b' is a positively oriented basis for $T(S^p)_{y'}$, then the framed manifold $(f^{-1}(y'), f^*b')$ is framed cobordant to $(f^{-1}(y), f^*b)$.*

Theorem 4.2. *Two mappings from M to S^p are smoothly homotopic if and only if the associated Pontryagin manifolds manifolds are framed coboradant.*

Theorem 4.3. *Any compact framed submanifold (N, η) of codimension p in M occurs as Pontryagin manifold for some smooth mapping $f : M \to S^n$.*

Thus the homotopy classes of maps are in one-one correspondence with the framed cobordism classes of submanifolds.

4.2 Product Neighborhood Theorem

Theorem 4.4. *Some neighborhood of N in M is diffeomorphic to the product $N \times \mathbb{R}^p$. Furthermore, the diffeomorphism can be chosen so that each $x \in N$ corresponds to $(x, 0) \in N \times \mathbb{R}^p$ and so that each normal frame $b(x)$ corresponds to the standard basis for $\mathbb{R}^p$.*

Proof. First suppose that M is the euclidean space $\mathbb{R}^{n+p}$. Consider the mapping $g : N \times \mathbb{R}^p \to M$, defined by

$$g(x; t_1, \dots, t_p) = x + t_1 v^1(x) + \cdots + t_p v^p(x).$$

Clearly $dg_{(x;0,\dots,0)}$ is nonsingular; hence g maps some neighborhood of $(x, 0) \in N \times \mathbb{R}^p$ diffeomorphically onto an open set. We will prove that g is one-one on the entire neighborhood $N \times U_\epsilon$ of $N \times 0$, providing that $\epsilon > 0$ is sufficiently small; where U_ϵ denotes the ϵ-neighborhood of 0 in $\mathbb{R}^p$. For otherwise there would exist pairs $(x, u) \neq (x', u')$ in $N \times \mathbb{R}^p$ with $\|u\|$ and $\|u'\|$ arbitrarily small and with

$$g(x, u) = g(x', u').$$

Since N is compact, we could choose a sequence of such pairs with x converging, say to x_0, with x' converging to x'_0, and with $u \to 0$ and $u' \to 0$. Then clearly $x_0 = x'_0$, and we have contradicted the statement that g is one-one in a neighborhood of $(x_0, 0)$.

Thus g maps $N \times U_\epsilon$ diffeomorphically onto an open set. But U_ϵ is diffeomorphic to the full euclidean space $\mathbb{R}^n$ under the correspondence

$$u \to u/(1 - \|u\|^2/\epsilon^2).$$

Since $g(x, 0) = x$, and since $dg_{(x,0)}$ does what is expected of it, this proves the Product Neighborhood Theorem for the special case $M = \mathbb{R}^{n+p}$.

For the general case it is necessary to replace straight lines in $\mathbb{R}^{n+p}$ by geodesics in M. More precisely, let $g(x; t_1, \dots, t_p)$ be the endpoint of the geodesic segment of length $\|t_1 v^1(x) + \ldots + t_p v^p(x)\|$ in M which starts at x with the initial velocity vector

$$t_1 v^1(x) + \ldots + t_p v^p(x) / \|t_1 v^1(x) + \ldots + t_p v^p(x)\|.$$

The reader who is familiar with geodesics will have no difficulty in checking that

$$g : N \times U_\epsilon \to M$$

is well defined and smooth, for ϵ sufficiently small. The remainder of the proof proceeds exactly as before. □

References

[1] John Willard Milnor and David W Weaver. *Topology from the Differentiable Viewpoint.* Princeton University Press, 1997.