Gelfand's Formula

Yuanshen Lin

July 14, 2023

Abstract

When we study the bounded linear map, we'd like to know what is the 'eigenvalues' of it. Then we define what is the inverse of a bounded operator. Based on that, we introduce the spectrum, resolvent set, spectral radius of a bounded operator. By definition, the resolvent set is open and the spectrum is closed, bounded and nonempty. By these properties, we inroduce the Gelfand's formula that represents a essential way to get the spectral radius using complex analysis.

Kewords: functional analysis, spectral theory, spectral radius

2010 Mathematics Subject Classification. 46B15, 47A10, 47A25

1 Introduction

In linear algebra, we study matrices which is a linear transformation A from $\mathbb{R}^n$ to $\mathbb{R}^m$. One of the most important theory is the eigenvalue theory of matrix. In functional analysis, we study the linear transformation T from a normed space X to another normed space Y. The eigenvalue theory of bounded linear operator is called the spectral theory.

In this paper, we will introduce the basic concepts of spectral theory and some important theorems. We will also give some examples to illustrate the theory.

Definition 1.1. *X be a noremed linear space, $M \in \mathcal{B}(X, X)$ has an inverse $N \in \mathcal{B}(X, X)$, then one has*

$$NM = MN = I$$

where I is the identity operator on X. In this case, M is called invertible.Denote $M^{-1} = N$.

Lemmma 1.1. *Suppose that X is a Banach space and $K \in \mathcal{B}(X, X)$ is invertible, then so are all elements of $\mathcal{B}(X, X)$ close enough to K. In particular, all elements of form $L = K - A$ are invertible, provided*

$$\|A\| < \frac{1}{\|K^{-1}\|}$$

Proof. Step1. Suppose that $K = I$. We claim that $I - B$ is invertible if $|B| < 1$, and

$$(I - B)^{-1} = \sum_{n=0}^{\infty} B^n$$

Since $\|B\| < 1$

$$S_k = \sum_{n=0}^{k} B^n$$

is a Cauchy sequence. Note that $\mathcal{B}(X, X)$ is complete. $\{S_k\}$ converges. One can see that

$$B \cdot \sum_{n=0}^{\infty} B^n = \sum_{n=1}^{\infty} B^n = \sum_{n=0}^{\infty} B^n - I = \left(\sum_{n=0}^{\infty} B^n \right) B$$

Hence one has

$$(I - B) \cdot \sum^{\infty} B^n = I = \sum^{\infty} B^n \cdot (I - B)$$

This implies

$$(I - B)^{-1} = \sum_{n=0}^{\infty} B^n$$

Step 2. General case. Suppose that K is invertible, one has $K - A = K\left(I - K^{-1}A\right)$. Denote $B = K^{-1}A$. Note that

$$\|B\| = \left\|K^{-1}A\right\| \le \left\|K^{-1}\right\| \cdot \|A\|.$$

Hence if

$$\|A\| < \frac{1}{\|K^{-1}\|}$$

$I - B$ is invertible so that $K - A$ is invertible. Furthermore,

$$(K - A)^{-1} = \left(I - K^{-1}A\right)^{-1} K^{-1} = \sum_{n=0}^{\infty} \left(K^{-1}A\right)^n K^{-1}.$$

This finishes the proof of the lemmma. □

[1]

Definition 1.2. *[2] Let X be a normed space and M be a bounded linear operator on X. The spectrum of M is defined as*

$$\sigma(M) = \{\lambda \in \mathbb{C} : M - \lambda I \text{ is not invertible}\}$$

The resolvent set of M is defined as

$$\rho(M) = \{\lambda \in \mathbb{C} : M - \lambda I \text{ is invertible}\}$$

The spectral radius of M is defined as

$$|\sigma(M)| = \sup\{|\lambda| : \lambda \in \sigma(T)\}$$

It looks like the eigenvalue theory of matrix. However, the spectrum of a bounded linear operator is more complicated than the eigenvalue of a matrix. Consider the matrix over $\mathbb{C}^n$, the spectrum of the matrix is the set of eigenvalues. In a general Banach space, the spectrum of a bounded linear operator is including the set of eigenvalues and the set of λ such that $M - \lambda I$ is injective but not surjective.

Lemmma 1.2. *Let X be a normed space and M be a bounded linear operator on X. Then $\rho(M)$ is an open set of $\mathbb{C}$.*

Proof. Suppose that $\lambda \in \rho(M)$. For any $h \in \mathbb{C}$, one has

$$(\lambda - h)I - M = (\lambda I - M - hI).$$

Applying Lemma 1.1for $K = \lambda I - M$ and $A = hI$ yields that if $|h| < \frac{1}{\|(\lambda I - M)^{-1}\|}$, then $(\lambda - h)I - M$ is invertible. Hence $\rho(M)$ is an open set. $\square$

Corollary 1.3. *$\sigma(M)$ is a closed, bounded, nonempty set in $\mathbb{C}$.*

Now we know some basic propositions of the spectrum of a bounded linear operator. We will introduce some important theorems in the next section.

2 Gelfand's Formula

The first thing to start with the theory is that we need to know how $(\lambda I - M)$ changes when λ changes.

Theorem 2.1. *Let X be a Banach space and $M \in \mathcal{B}(X, X)$. The resolvent of M, which is defined on $\rho(M)$ as $(\zeta I - M)^{-1}$ and is abbreviated by $(\zeta - M)^{-1}$, is an analytic function of ζ on $\rho(M)$ [3].*

Proof. For any fixed $\lambda \in \rho(M)$, if $|h| < \left\|(\lambda - M)^{-1}\right\|^{-1}$, one has

$$\begin{aligned} ((\lambda - h) - M)^{-1} &= ((\lambda - M) - h)^{-1} = (\lambda - M)^{-1}\left(I - h(\lambda - M)^{-1}\right)^{-1} \\ &= \sum_{n=0}^{\infty}\left((\lambda - M)^{-1}h\right)^n \cdot (\lambda - M)^{-1} \\ &= \sum_{n=0}^{\infty}(\lambda - M)^{-n-1}h^n \end{aligned}$$

This shows that the resolvent can be expressed as a power series around each point $\lambda \in \rho(M)$, convergent for $|h| < \left\|(\lambda - M)^{-1}\right\|^{-1}$. Hence the resolvent is analytic with respect to ζ. $\square$

From the theorem above, we can know that the resolvent of M is a analytic function on $\rho(M)$.Using the Cauchy integral formula, we can get the connection between the multiple powers of M and the resolvent set of M. Consider that, we can introduce the following theorem.

Theorem 2.2 (Gelfand)**.** *[4] [5] Let X be a Banach space and $M \in \mathcal{B}(X,X)$, then*

$$|\sigma(M)| = \lim_{k\to\infty} \left\|M^k\right\|^{\frac{1}{k}}$$

Proof. Let $k \in \mathbb{N}$. Then $n = kq + r, 0 \le r < k$. $M^n = M^{kq+r} = \left(M^k\right)^q M^r$. Hence one has

$$\|M^n\| \le \|M^r\| \cdot \left\|M^k\right\|^q$$

Note that

$$\left\|\sum_{n=0}^{\infty} M^n \zeta^{n-1}\right\| \le \sum_{n=0}^{\infty} \frac{\|M^n\|}{|\zeta|^{n+1}} \le \sum_{r=0}^{k-1} \frac{\|M^r\|}{|\zeta|^{r+1}} \cdot \sum_{q=0}^{\infty} \left(\frac{\left\|M^k\right\|}{|\zeta|^k}\right)^q$$

Therefore, for any ζ satisfying

$$\frac{\left\|M^k\right\|}{|\zeta|^k} < 1$$

i.e., $|\zeta| > \left\|M^k\right\|^{\frac{1}{k}}$, one has

$$(\zeta I - M)^{-1} = \sum_{n=0}^{\infty} M^n \zeta^{-n-1}$$

Therefore, $\zeta \in \rho(M)$. Thus if $\lambda \in \sigma(M)$, then $|\lambda| \le \left\|M^k\right\|^{\frac{1}{k}}$ for any $k \in \mathbb{N}$. Therefore

$$|\sigma(M)| \le \liminf_{k\to\infty} \left\|M^k\right\|^{\frac{1}{k}} \tag{1}$$

On the other hand,

$$\frac{1}{2\pi i} \oint_C (\zeta - M)^{-1} \zeta^n d\zeta = M^n$$

where C is the anti-clockwise contour $|\zeta| = |\sigma(M)| + \delta$ with $\delta > 0$. Therefore, one has

$$\|M^n\| \le a(|\sigma(M)| + \delta)^{n+1} \text{ with } a = \max_{|\zeta|=|\sigma(M)|+\delta} \left\|(\zeta I - M)^{-1}\right\|.$$

This implies

$$\|M^n\|^{\frac{1}{n}} \le a^{\frac{1}{n}} \cdot (|\sigma(M)| + \delta)^{1+\frac{1}{n}}$$

Hence one has

$$\limsup_{n\to\infty} \|M^n\|^{\frac{1}{n}} \le |\sigma(M)| + \delta$$

Since δ is arbitrary, we have

$$\limsup_{n\to\infty} \|M^n\|^{\frac{1}{n}} \le |\sigma(M)| \tag{2}$$

Combining (1) and (2), we have

$$\lim_{n\to\infty} \|M^n\|^{\frac{1}{n}} = |\sigma(M)|$$

$\square$

Gelfand gave us an essential way to calculate the spectral radius. It doesn't relate to other operator like $\lambda I - M$ but the operator M itself. We can also know that the $\frac{1}{k}$ power of the norm of M^k has a limit $|\sigma(M)|$ when k tends to infty.

3 Acknowledgements

Over the course of my researching and writing this paper, I would like to express my thanks to all those who have helped me.

First, I would like express my gratitude to all those who helped me during the writing of this thesis. A special acknowledgement should be shown to Professor Sergei Kalmykov, from whose lectures I benefited greatly, I am particularly indebted to TA Xianyong Dai gave me kind encouragement and useful instruction all through my writing.

Sincere gratitude should also go to all my learned Professors and warm-hearted teachers who have greatly helped me in my study as well as in my life.

And my warm gratitude also goes to my friends and family who gave me much encouragement and financial support respectively. Moreover, I wish to extend my thanks to the library and the electronic reading room for their providing much useful information for my thesis.

References

[1] Peter D Lax. *Functional analysis*, volume 55. John Wiley & Sons, 2002.

[2] Alberto Bressan. Lecture notes on functional analysis. *Graduate studies in mathematics*, 143, 2012.

[3] Victor Kozyakin. On accuracy of approximation of the spectral radius by the gelfand formula. *Linear Algebra and its Applications*, 431(11):2134–2141, 2009.

[4] Kōsaku Yosida. *Functional analysis*. Springer Science & Business Media, 2012.

[5] John B Conway. *A course in functional analysis*, volume 96. Springer, 2019.