WORCESTER POLYTECHNIC INSTITUTE

PORTFOLIO VALUATION AND RISK MANAGEMENT

Capital Asset Pricing Model Project

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Chapter 1

Introduction

In this project, we choose S&P500 as our market portfolio, and then use different methods to estimate asset beta based on the CAPM model. We used traditional appraoch, least trimmed squares, shrinkage estimation,exponentially weighted moving average to do the estimation and then compared the results.

Chapter 2

Data Collection

Through Yahoo Finance[1], we download 1-years AAL, AAPL, BABA, BIDU, BURL, D, DAL, FB, FDX, FRT, GNC, GOOG, GPRO, L, NKE, O, S, SPWR, T, UPS and S&P 500 historical data from 10/24/16 to 10/24/17 and created a program which extracted all data. Python Pandas package helps us to read excel file into data frame. For stock, we extracted the adjust closed prices separately and transferred them into 1-dimensional arrays. They would be accessed in all problems of the project.

[1] **https://finance.yahoo.com/**.

Chapter 3

Methodology

3.1 Capital Asset Pricing Model

The capital asset pricing model (CAPM) is a standard model for asset returns, it's a theoretical justification of indexing, holding a diversified portfolio in some relative proportions (Ruppert, 2004).

The model based on these assumptions (Marcel, 2017):

- Market prices are in equilibrium for each asset supply equals demand.
- Every one has the same forecasts of expected returns & risk.
- All investors choose portfolio and the risk-free asset.
- The market rewards in investors for assuming unavoidable risk
- No reward for inefficient investing.

3.1.1 Capital Market Line

The capital market line (CML) relates the excess expected return on an efficient portfolio to its risk. "Excess expected return" means the amount by which the expected return of the portfolio exceeds the risk-free rate of return and is also called the risk premium. The CML is

$$\mu_R = \mu_f + \frac{\mu_M - \mu_f}{\sigma_M}\sigma_R, \tag{3.1}$$

where R is the return on a given efficient portfolio (mixture of the market portfolio and the risk-free asset), $\mu_R = E(R)$, μ_f is the rate of return on the risk-free asset, R_M is the return on the market portfolio, $\mu_M = E(R_M)$, σ_M is the standard deviation of the return on the market portfolio. The risk premium of R is $\mu_R - \mu_f$ and the risk premium of the market portfolio is $\mu_M - \mu_f$ (Ruppert, 2004).

3.1.2 Security Market Line

The security market line (SML) is a equation that derived by using CAPM, it can be shown as:

$$\mu_j - \mu_f = \beta_j(\mu_M - \mu_f). \tag{3.2}$$

where β_j is a variable in the linear equation, not the slope; more precisely, μ_j is a linear function of β_j with slope $\mu_M - \mu_f$. In other words, β_j is a slope in one context but is the independent variable in the SML (Ruppert, 2004).

SML can be applied to any asset. Generally, β_j measures how aggressive asset j is. By definition, $\beta_M = 1$ and

- if $\beta_j > 1$ or $\beta_j < -1$, it means it's a aggressive asset;
- if $0 < \beta_j < 1$ or $-1 < \beta_j < 0$, it means it's a less aggressive asset;
- if $\beta_j = 1$ or $\beta_j = -1$, it means it's an average risk asset (Marcel, 2017);

Regression Model

The SML is the regression model (Marcel, 2017).

$$R_{jt} = \mu_{ft} + \beta_j(R_{mt} - \mu_{ft}) + \epsilon_{jt} \tag{3.3}$$

where $\epsilon_{jt} \sim$ WhiteNoise$(0, \sigma^2_{\epsilon j})$, R_{jt} is the j th asset return at time t, R_{mt} is market portfolio, μ_{ft} is the risk free rate at time t. Also, we make a common assumption

$$Cov(\epsilon_{it}, \epsilon_{jt}) = 0, \;\; if i \neq j.$$

Let $\mu_{jt} = E(R_{jt}), \mu_{Mt} = E(R_{Mt})$, if we take expectation of equation (3.3), we have

$$\mu_{jt} = \mu_{ft} + \beta_j(\mu_{Mt} - \mu_{ft}).$$

Also, from equation (3.3), we get

$$Var(R_{jt}) = \sigma^2_j = \beta^2_j\sigma^2_M + \sigma^2_{\epsilon j}.$$

where $\beta^2_j\sigma^2_M$ is market/systemic component, and $\sigma^2_{\epsilon j}$ is unique/non-systemic component.

Moreover, we have

$$\begin{aligned}\sigma_{ij} &= Cov(R) = Cov(\mu_{ft} + \beta_i(R_{MT} - \mu_{ft}) + \epsilon_{it}, \mu_{ft} + \beta_j(R_{MT} - \mu_{ft}) + \epsilon_{jt}) \\ &= Cov(\beta_i R_{MT}, \beta_j R_{MT}) \\ &= \beta_i\beta_j\sigma^2_M\end{aligned}$$

$$\begin{aligned}Cov(R_{jt}, R_{MT}) &= Cov(\mu_{ft} + \beta_i(R_{MT} - \mu_{ft}) + \epsilon_{it}, R_{MT}) \\ &= \beta_j\sigma^2_M.\end{aligned}$$

If we let $R^*_{jt} = R_{jt} - \mu_{ft}$, which means excess return for asset j, then equation (3.3) becomes

$$R^*_{jt} = \beta_j R^*_{Mt} + \epsilon_{jt},$$

where $R^*_{Mt} = R_{Mt} - \mu_{ft}$

We can expand this to

$$R^*_{jt} = \alpha_j + \beta_j R^*_{Mt}.$$

It can test for mispricing of CAPM by testing a null hypothesis that $\alpha_j = 0$. If there is a time series for R^*_{jt}, R^*_{Mt}, we regress R^*_{jt} onto R^*_{MT} test $\hat{\alpha_j}$, if $\hat{\alpha_j} > 0$ in a statistically significant way, then we have found meaningful excee return that's not predicted by CAPM (Marcel, 2017).

Risk Reduction by Diversification

Assume there are N assets with returns $R_{1t}, R_{2t}, \cdots, R_{Nt}$ holding period t with weight $\omega_1, \omega_2, \cdots, \omega_N$.

Then the portfolio return is

$$R_{Pt} = \sum_{i=1}^{N} = \omega_i R_{it},$$

where R_{Mt} is market portfolio return.

Combine with equation (3.3), we have

$$\begin{aligned} R_{Pt} &= \sum_{j=1}^{N} \omega_j [\mu_{ft} + \beta_j (R_{Mt} - \mu_{ft}) + \epsilon_{jt}] \\ &= \sum_{j=1}^{N} \omega_j \mu_{ft} + (R_{Mt} - \mu_{ft})(\sum_{j=1}^{N} \omega_j \beta_j) + \sum_{j=1}^{N} \omega_j \epsilon_{jt} \\ &= \mu_{ft} + (\sum_{j=1}^{N} \omega_j \beta_j)(R_{Mt} - \mu_{ft}) + \sum_{j=1}^{N} \omega_j \epsilon_{jt}. \end{aligned}$$

If the ϵ_{jt} are uncorrelated then

$$\sigma_P^2 = Var(R_{Pt}) = (\sum_{j=1}^{N} \omega_j \beta_j)^2 \sigma_M^2 + \sum_{j=1}^{N} \omega_j^2 \sigma_{\epsilon j}^2,$$

where $(\sum_{j=1}^{N} \omega_j \beta_j)^2$ is systemic risk, and $\sum_{j=1}^{N} \omega_j^2 \sigma_{\epsilon j}^2$ is unique/non-systemic risk.

3.2 Robust Regression

Def 1. *A robust regression estimate* $\begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix}$ *is defined as*

$$\begin{bmatrix} \hat{\alpha_i} \\ \hat{\beta_i} \end{bmatrix} = \underset{\alpha_i \beta_i}{argmin} \sum_{t=1}^{n} L(\frac{R_{it} - \alpha_i - \beta_i R_{Mt}}{S}),$$

where S is an estimate of $\sigma_{\epsilon i}$, L is a symmetric loss, real value function.

In this project, we introduce 4 different methods to estimate $\begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix}$.

3.2.1 Least Median of Squares

This method lessens the influence of outliers (Marcel, 2017).

$$\begin{bmatrix} \hat{\alpha}_{i,Med} \\ \hat{\beta}_{j,Med} \end{bmatrix} = \underset{\alpha_i \beta_i}{argmin} \underset{t=1,2,\cdots,n}{Median} \left(\frac{R_{it} - \alpha_i - \beta_i R_{Mt}}{S} \right).$$

3.2.2 Least Trimmed Squares

We set values smaller than the $\alpha-$quantile to that of the $\alpha-$quantile in left tail. We also set values larger than the $(1-\alpha)-$quantile to that of the $(1-\alpha)-$quantile in the right tail [Blais Note].
Def 2.

$$L(\epsilon) = \begin{cases} q_\alpha^2 & \textit{if } \epsilon \leq q_\alpha \\ \epsilon \leq q_\alpha & \textit{if } q_\alpha < \epsilon < q_{1-\alpha} \\ q_{1-\alpha}^2 & \textit{if } \epsilon \geq q_{1-\alpha} \end{cases}$$

$$\begin{bmatrix} \hat{\alpha}_{i,LTS} \\ \hat{\beta}_{j,LTS} \end{bmatrix} = \underset{\alpha_i \beta_i}{argmin} \sum_{t=1}^{n} L(R_{it} - \alpha_i - \beta_i R_{Mt}).$$

3.2.3 Constant beta Shrinkage Estimation

It can be applied for many asserts and few observations, it can exploit information across $\beta' s$ (Marcel, 2017).

Assume there are N assets, for each asset run OLS regression for $\hat{\beta}_{i,OLS}$, we get

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\beta}_{1,OLS} \\ \vdots \\ \hat{\beta}_{N,OLS} \end{bmatrix}.$$

Shrinkage Technique: James-Stein Estimator

$$\hat{\beta}_{i,JS} = \bar{\beta} + \alpha_i(\hat{\beta}_{i,OLS} - \bar{\beta}) = (1-\alpha_i)\bar{\beta} + \alpha_i \hat{\beta}_{i,OLS},$$

where $\bar{\beta}$ is the mean of $\hat{\beta}_{i,OLS}$, $i = 1, 2, \cdots, N$.

In this project, we simply use $\alpha_i = \frac{2}{3}$.

3.2.4 Exponential Weighted Moving Average

Exponential Smoothing

Problems: β may be time varying (Marcel, 2017).

1. Questionable stationarity assumption.
2. Firms may change
    - Mergers & Acquisition

- Change leverage
- Old data points may not be as reliable as recent data points.

We use EWMA to fix these problems (Marcel, 2017).

- Covariance estimate:

$$\begin{aligned}
\sigma_{i,M,t} &= (1-\lambda)\sum_{j=1}^{\infty}\lambda^{j-1}R_{i,t-j}R_{M,t-j} \\
&= (1-\lambda)[R_{i,t-1}R_{M,t-1} + \sum_{j=2}^{\infty}\lambda^{j-1}R_{i,t-j}R_{M,t-j}] \\
&= (1-\lambda)[R_{i,t-1}R_{M,t-1} + \lambda\sum_{j=2}^{\infty}\lambda^{j-2}R_{i,t-j}R_{M,t-j}] \\
&= (1-\lambda)R_{i,t-1}R_{M,t-1} + (1-\lambda)\lambda\sum_{j=1}^{\infty}\lambda^{j-1}R_{i,t-j-1}R_{M,t-j-1} \\
&= (1-\lambda)R_{i,t}R_{M,t} + \lambda\sigma_{i,M,t-1}.
\end{aligned}$$

- Variance estimate:

$$\sigma_{M,t}^2 = (1-\lambda)\sum_{j=1}^{\infty}\lambda^{j-1}R_{M,t-j}^2 = (1-\lambda)R_{M,t}^2 + \lambda\sigma_{M,t-1}^2.$$

Standard value:

- Monthly data: $\lambda = 0.97$.
- Daily data: $\lambda = 0.94$.

Chapter 4

Implementation

4.1 Traditional Approach

We just use traditional approach that is using historical returns data to calculate $\beta' s$. Here we list our assets in increasing order of their β_j.

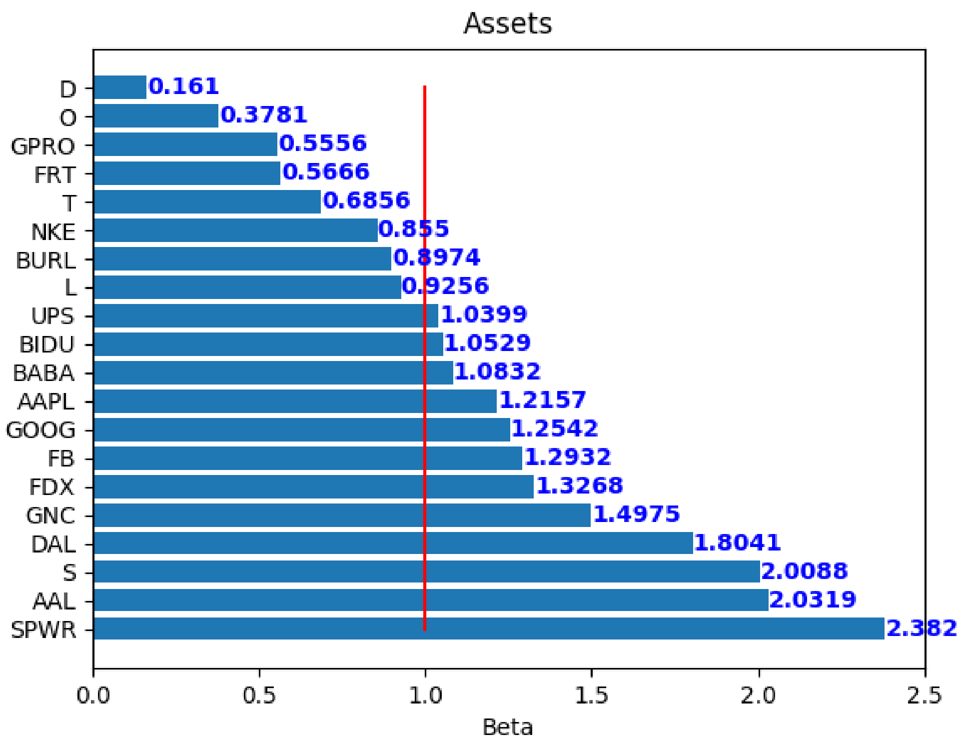


FIGURE 4.1: Increasing order of β_j using traditional approach.

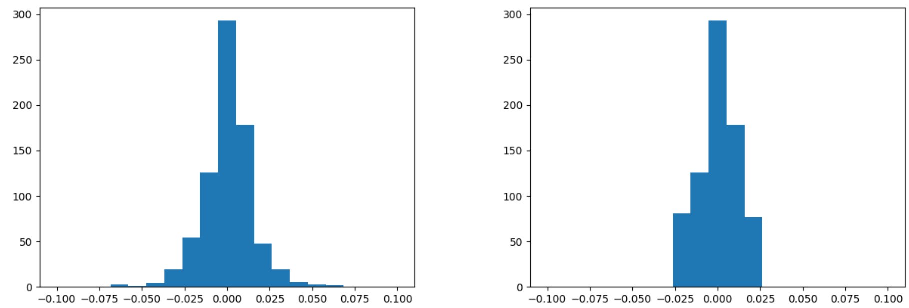
In this situation, our portfolio β is:

$$1.0263$$

which indicates the portfolio is an aggressive portfolio.

4.2 Least Trimmed Squares

We choose AAPL as our representative asset, and plot its returns distribution and the trimmed distribution.



(A) Return distribution. (B) Trimmed distribution.

FIGURE 4.2: Two distributions.

4.3 Shrinkage Estimation

We chose $\alpha_i = \frac{2}{3}$, and the result is:

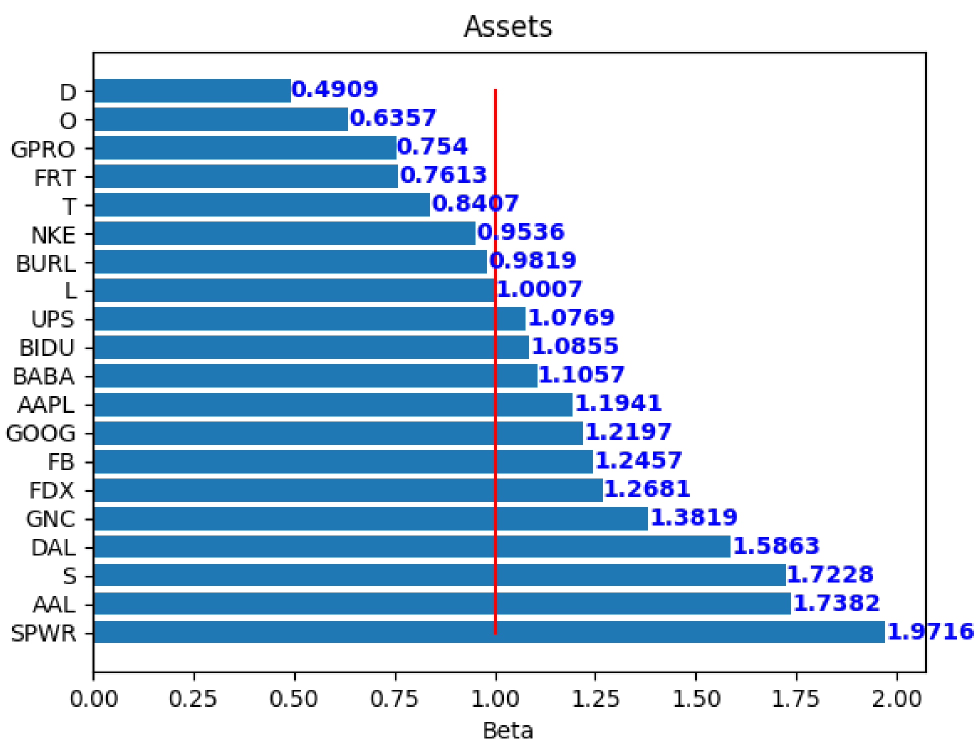


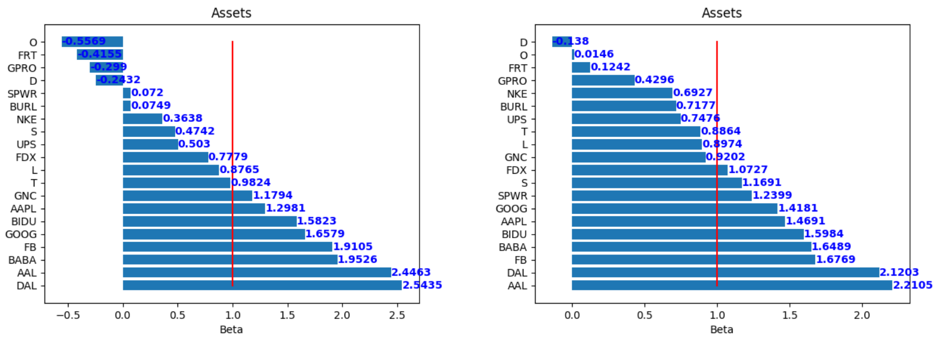
FIGURE 4.3: Increasing order of β_j using shrinkage approach.

In this situation, our portfolio β is:

$$1.0678$$

which indicates the portfolio is an aggressive portfolio.

4.4 Exponentially Weighted Moving Average



(A) $\lambda = 0.94$. (B) $\lambda = 0.97$.

FIGURE 4.4: Assets β_j using EWMA approach.

When $\lambda = 0.94$, our portfolio β is

$$1.8262.$$

When $\lambda = 0.97$, our portfolio β is

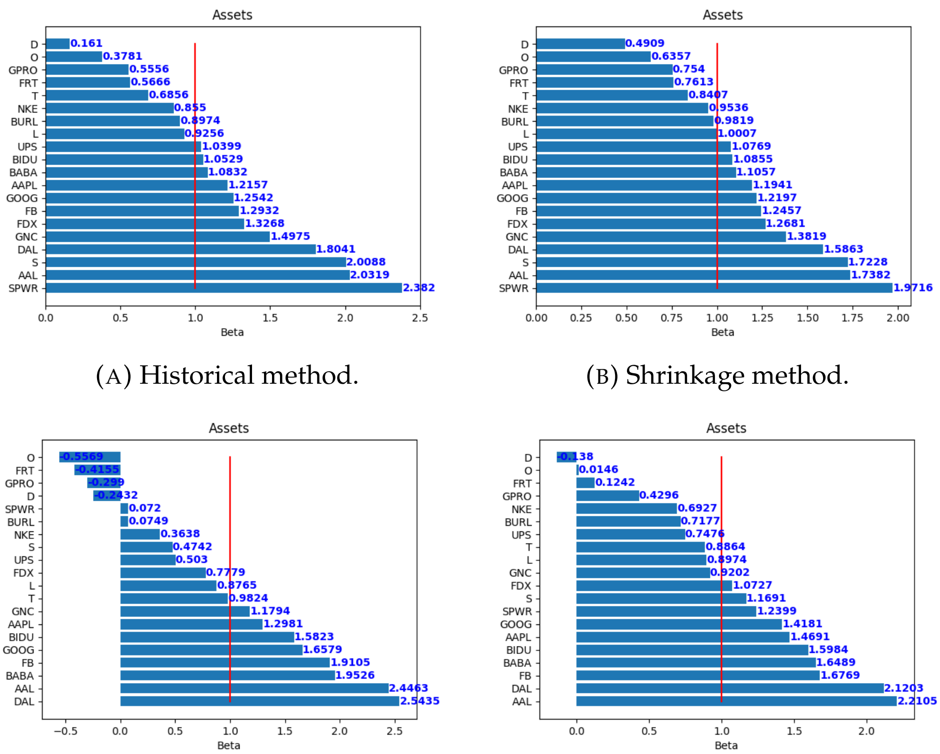
$$1.6076.$$

Which all indicate our portfolio is an aggressive portfolio.

4.5 Conclusion

We can see from figure 4.5, Historical method and Shrinkage method give us a same rank of our $\beta' s$ for each asset, and the pattern of these two methods all almost same. For EWMA method, the pattern changes a lot, and produces some negative β_j when comparing with Historical method and Shrinkage method. Moreover, the rank of $\beta' s$ in EWMA method is different with previous two methods, especially there are different rank between different λ value in EWMA method.

As a financial analysis we will more focus on the expected return of each asset. When given the market portfolio return and risk-free rate fixed, the bigger the beta is, the more we will get from this asset. At the same time, we looked at our data and found out the exponential smoothing method may be the most suitable one. Because it's suitable when dealing with firms that may change leverage over time and sometimes old data vary much from the recent one; compared to other method, Shrinkage method is more suitable when we have many assets (20 is not that much) and few observations which our observations are enough to do the estimations; and least-trimmed squares is just a way to help remove some outliers so it's not good enough. Then look into the result, the beta of the portfolio is the largest above all method, so it's more attractive to the investors and we will have some negative values which means if we have chance to redo the asset selection, they should be removed from our portfolio because their return will be smaller than the risk-free return. The only thing is we have to do more research to decide which λ is the best to choose for smoothing because we can see little change in λ can lead to changes in betas overall.



(A) Historical method. (B) Shrinkage method.

(C) $\lambda = 0.94$. (D) $\lambda = 0.97$.

FIGURE 4.5: Four different β list.

Values	EWMA(lambda = 0.94)	EWMA(lambda = 0.97)	Shrinkage	Traditional
AAL	2.4463	2.2105	1.7382	2.0319
AAPL	1.2981	1.4691	1.1941	1.2157
BABA	1.9526	1.6489	1.1057	1.0832
BIDU	1.5823	1.5984	1.0855	1.0529
BURL	0.0749	0.7177	0.9819	0.8974
D	-0.2432	-0.138	0.4909	0.161
DAL	2.5435	2.1203	1.5863	1.8041
FB	1.9105	1.6769	1.2457	1.2932
FDX	0.7779	1.0727	1.2681	1.3268
FRT	-0.4155	0.1242	0.7613	0.5666
GNC	1.1794	0.9202	1.3819	1.4975
GOOG	1.6579	1.4181	1.2197	1.2542
GPRO	-0.299	0.4296	0.754	0.5556
L	0.8765	0.8974	1.0007	0.9256
NKE	0.3638	0.6927	0.9536	0.855
O	-0.5569	0.0146	0.6357	0.3781
S	0.4742	1.1691	1.7228	2.0088
SPWR	0.072	1.2399	1.9716	2.382
T	0.9824	0.8864	0.8407	0.6856
UPS	0.503	0.7476	1.0769	1.0399

FIGURE 4.6: All methods $\beta' s$.

Values	EWMA(lambda = 0.94)	EWMA(lambda = 0.97)	Shrinkage	Traditional
Portfolio Beta	1.6076	1.8262	1.0678	1.0263

FIGURE 4.7: Portfolio β.

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