Problem Set 3

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Exercise 3

1) Let

$$\mathbb{E}(Y \mid X = x_k) = \sum_y y\, P(Y = y_i, X = x_k)$$

$$\mathbb{E}(Y \mathbb{1}(X = x_k)) = \sum_y y\, P(Y = y_i, X = x_k)$$

As for any x_j for $j \neq k$, the indicator is 0.

Now, write:

$$P(Y = y_i, X = x_k) = P(Y = y_i, X = x_k)\, P(X = x_k)$$

Plugging in:

$$\mathbb{E}(Y \mathbb{1}(X = x_k)) - \sum_y y\, P(Y = y_i, X = x_k)\, P(X = x_k)$$

Then,

$$\frac{\mathbb{E}(Y \mathbb{1}(X = x_k))}{P(X = x_k)} = \sum_y y\, P(Y = y_i, X = x_k)$$

$$= \mathbb{E}(Y \mid X = x_k)$$

QED.

2) If $X \perp\!\!\!\perp Y$, $P(X|Y) = P(X)$. Thus:

$$
\begin{aligned}
\mathbb{E}(Y|X) &= \sum_y y\, P(Y|X) \\
&= \sum_y y\, \frac{P(X|Y)P(Y)}{P(X)} \qquad \text{(Bayes' Rule)} \\
&= \sum_y y\, P(Y)
\end{aligned}
$$

QED.

Exercise 4

1) $$\frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i^2 - 2 Y_i \bar{Y} + \bar{Y}^2)$$

$$= \frac{1}{n-1} \left(\sum_{i=1}^{n} Y_i^2 - 2 \sum_{i=1}^{n} Y_i \bar{Y} + n \bar{Y}^2 \right)$$

$$= \frac{1}{n-1} \left(\sum_{i=1}^{n} Y_i^2 - 2n Y_i^2 + n \bar{Y}^2 \right)$$

$$= \frac{1}{n-1} \sum_{i=1}^{n} Y_i^2 - \frac{n}{n-1} (\bar{Y})^2$$

2) $$\mathbb{E}\left(\frac{1}{n-1} \sum_{i=1}^{n} (Y_i)^2 \right) = \frac{1}{n-1} \sum_{i=1}^{n} \mathbb{E}(Y_i)^2$$

Recall that $V(Y_i) = E(Y_i^2) - \mathbb{E}(Y_i)^2$

$$\longrightarrow \mathbb{E}(Y_i^2) = V(Y_i) + \mathbb{E}(Y_i)^2$$

Plugging in:

$$= \frac{1}{n-1} \sum_{i=1}^{n} \left(V(Y_i) - \mathbb{E}(Y_i)^2 \right)$$

$$= \frac{1}{n-1} \sum_{i=1}^{n} V(Y_i) - \frac{1}{n-1} \sum_{i=1}^{n} \mathbb{E}(Y_i)^2$$

Observe that we can write both components as $\frac{1}{n-1} \frac{1}{n} n$. Then, factorize to obtain:

$$\mathbb{E}\left(\frac{1}{n-1} \sum_{i=1}^{n} (Y_i)^2 \right) = \frac{n}{n-1} \left(\frac{1}{n} \sum_{i=1}^{n} V(Y_i) - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(Y_i)^2 \right)$$

QED.

3) $\mathbb{E}(\overline{Y}^2) = \mathbb{E}((\frac{1}{n}\sum_{i=1}^{n} Y_i)^2$

$= \mathbb{E}(\frac{1}{n^2}\sum_{i=1}^{n} Y_i^2)$

$= \frac{1}{n^2}\sum_{i=1}^{n} \mathbb{E}(Y_i^2)$

which, using the previous result, amounts to :

$= \frac{1}{n^2}\sum_{i=1}^{n} (V(Y_i) + \mathbb{E}(Y_i)^2)$

$= \frac{1}{n^2}\sum_{i=1}^{n} V(Y_i) + \frac{1}{n^2}\sum_{i=1}^{n} \mathbb{E}(Y_i)^2$

Note: $\frac{1}{n^2}\sum_{i=1}^{n} \mathbb{E}(Y_i)^2 = (\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(Y_i))^2$

Finally,

$\mathbb{E}(\overline{Y}^2) = \frac{1}{n^2}\sum_{i=1}^{n} V(Y_i) + (\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(Y_i))^2$

QED.

4) $$\mathbb{E}\left(\frac{1}{n-1}\sum_{i=1}^{n}(Y_i-\overline{Y})^2\right) = \frac{1}{n-1}\sum_{i=1}^{n}\mathbb{E}(Y_i-\overline{Y})^2$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}\mathbb{E}(Y_i^2 - 2Y_i\overline{Y} + \overline{Y}^2)$$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n}\mathbb{E}(Y_i^2) - 2n\,\mathbb{E}(\overline{Y}^2) + n\,\mathbb{E}(\overline{Y}^2)\right)$$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n}\mathbb{E}(Y_i^2) - n\,\mathbb{E}(\overline{Y}^2)\right)$$

$$= \frac{1}{n-1}\Bigg[\left(\sum_{i=1}^{n}\left(V(Y_i) - \mathbb{E}(Y_i)^2\right)\right) -$$

$$\left(n\frac{1}{n^2}\sum_{i=1}^{n}V(Y_i) + \left(\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(Y_i)\right)^2\right)\Bigg]$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}V(Y_i) - \frac{1}{n-1}\sum_{i=1}^{n}\mathbb{E}(Y_i)^2 - \frac{1}{(n-1)n}\sum_{i=1}^{n}V(Y_i)$$

$$- \frac{1}{(n-1)}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(Y_i)^2\right)$$

Note: $$\frac{1}{n-1} - \frac{1}{(n-1)n} = \frac{(n-1)n-(n-1)}{n(n-1)^2} = \frac{(n-1)^2}{n(n-1)^2} = \frac{1}{n}$$

Now:

$$= \frac{1}{n}\sum_{i=1}^{n}V(Y_i) - \frac{1}{n-1}\sum_{i=1}^{n}\mathbb{E}(Y_i)^2 - \left(\frac{1}{n^2}\sum_{i=1}^{n}\mathbb{E}(Y_i)^2\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}V(Y_i) + \frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(Y_i)^2 - \left(\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(Y_i)^2\right)\right)$$

We must now show that

$$\sum_{i=1}^{n}\left(\frac{1}{n}\mathbb{E}(Y_i)^2 - \left(\frac{1}{n}\mathbb{E}(Y_i)\right)^2\right) = \left(\mathbb{E}(Y_1) - \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(Y_i)\right)^2$$

Write:

$$\left(\sum_{i=1}^{n}\frac{1}{n}E(Y_i)-\frac{1}{n^2}\sum_{i=1}^{n}E(Y_i)\right)^2=$$

$$=\sum_{i=1}^{n}\left(\frac{1}{n^2}E(Y_i)^2-2\frac{1}{n}\sum_{i=1}^{n}E(Y_i)\frac{1}{n^2}\sum_{i=1}^{n}E(Y_i)+\frac{1}{n^2}\sum_{i=1}^{n}E(Y_i)\sum_{i=1}^{n}E(X_i)\right)$$

$$=\sum_{i=1}^{n}\left(E(Y_i)^2-\frac{1}{n}\sum_{i=1}^{n}E(Y_i)^2\right) \quad \text{by factorizing.}$$

Thus,

$$E(\bar{Y})^2=\frac{1}{n}\sum_{i=1}^{n}V(X_i)+\frac{n}{n-1}\frac{1}{n}\sum_{i=1}^{n}\left(E(Y_i)-\frac{1}{n}\sum_{i=1}^{n}E(X_i)\right)^2$$

5) We want to show that

$$\frac{n}{n-1}\frac{1}{n}\sum_{i=1}^{n}\left(E(Y_i)-\frac{1}{n}\sum_{i=1}^{n}E(Y_i)\right)^2\geq 0 \quad \Big| \times (n-1)n$$

$$\sum_{i=1}^{n}\left(E(Y_i)-\frac{1}{n}\sum_{i=1}^{n}E(Y_i)\right)^2\geq 0$$

$$\sum_{i=1}^{n}\left(E(Y_i)^2-2E(Y_i)\frac{1}{n}\sum_{i=1}^{n}E(X_i)+\frac{1}{n^2}E(Y_i)^2\right)\geq 0$$

$$\sum_{i=1}^{n}E(Y_i)^2-2\sum_{i=1}^{n}\left(E(Y_i)E(Y_i)\right)+\frac{1}{n}E(Y_i)^2)\geq 0$$

$$\sum_{i=1}^{n}E(Y_i)^2-\frac{1}{n}E(Y_i)^2\geq 0 \quad , \quad \text{using } \frac{1}{n}-2n=-\frac{1}{n}$$

Since $n>1$ and the expectation is squared, the result follows.

6) Then,

$$\sum_{i=1}^{n}\left(E(Y_i)-\frac{1}{n}\sum_{i=1}^{n}E(Y_i)\right)^2=0 \qquad \text{implies}$$

$E(Y_i)=\frac{1}{n}\sum_{i=1}^{n}E(Y_i)$, which holds when

Y_i is iid. !