Correction Problem Set 1: Bellman equations

Sciences Po - Macroeconomics 3 - Fall 2024
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Exercise 2: Housing problem

Houses are durable goods from which households derive some utility. To model the demand for houses, a simple shortcut consists in introducing houses in the utility function. The goal of this exercise is to be able to use data on house prices and interest rates to derive properties of the demand for houses.

Households thus derive utility from consumption and from having houses. The instantaneous utility function is $u(c_t, H_t)$ where H_t is the amount of housing. Households also have access to financial savings denoted b_t at period t, remunerated at a real rate r_t between period t and $t+1$. Houses depreciate at rate δ. The program of the households is

$$\max_{\{c_t, H_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, H_t)$$

$$c_t + b_t + P_t H_t = y_t + (1 + r_{t-1}) b_{t-1} + P_t (1-\delta) H_{t-1}$$

1. **What are the underlying assumptions about the transactions in the housing market?**

2. **State the transversality conditions.**

   Solution:

   The transversality condition for housing is:

   $$\lim_{t \to \infty} H_t (1-\delta)^t = 0$$

   This condition implies that there is no possibility of becoming infinitely rich by starting with an initial stock H_0 of housing and simply keeping it forever.

   The transversality condition for financial wealth is:

   $$\lim_{t \to \infty} \frac{b_t}{\prod_{k=0}^{t} (1 + r_k)} = 0$$

   This is a no-Ponzi condition: if an individual plans to borrow indefinitely without repaying, no lender will lend to them.

3. **Write the Bellman equation of the problem, with the value function denoted as $V(b, H)$ (be careful about the timing notation). What are the control variable(s)? What are the state variable(s)?**

   Solution:

   The Bellman equation is:

   $$V(b, H) = \max_{c, H', b'} \left[u(c, H') + \beta V(b', H') \right]$$

   subject to the budget constraint:

$$c + b' + PH' = y + (1 + r)b + P(1 - \delta)H.$$

We could also write:

$$V(b_{-1}, H_{-1}) = \max_{c,H,b} [u(c, H) + \beta V(b, H)]$$

subject to the budget constraint:

$$c + b + PH = y + (1 + r)b_{-1} + P(1 - \delta)H_{-1}.$$

4. **Find the two Euler equations (step by step).**

Solution:

There are as many first-order conditions (FOCs) as choice variables. However, we have the budget constraint that provides a linear combination of our 3 choice variables, implying that one of the variables is predetermined. Therefore, only two FOCs are necessary. The problem can be rewritten as:

$$V(b, H) = \max_{H',b'} [u (y + (1 + r)b + P(1 - \delta)H - b' - PH', H') + \beta V(b', H')]$$

The FOC with respect to b' is:

$$\frac{\partial u(c, H')}{\partial c} = \beta \frac{\partial V(b', H')}{\partial b'}$$

The FOC with respect to H' is:

$$\frac{\partial u(c, H')}{\partial H'} + \beta \frac{\partial V(b', H')}{\partial H'} = P \frac{\partial u(c, H')}{\partial c}$$

The envelope conditions are as follows:

$$\frac{\partial V(b, H)}{\partial b} = (1 + r) \frac{\partial u(c, H')}{\partial c}$$

$$\frac{\partial V(b, H)}{\partial H} = P(1 - \delta) \frac{\partial u(c, H')}{\partial c}$$

By iterating forward the envelope conditions and plugging them into the FOC, we obtain the two Euler equations:

$$\frac{\partial V(b', H')}{\partial b'} = (1 + r') \frac{\partial u(c', H'')}{\partial c'}$$

$$\frac{\partial V(b', H')}{\partial H'} = P'(1 - \delta) \frac{\partial u(c', H'')}{\partial c'}$$

The first equation is the common Euler equation. The second equation shows the trade-off between housing and private consumption.

5. **Assume the utility function has the following form:**

$$u(c_t, H_t) = (c_t^\rho + H_t^\rho)^{\frac{1}{\rho}}$$

Explain what is the economic meaning of the ρ coefficient. Using the two Euler equations, express $\frac{c_t}{H_t}$ as a function of P_t, P_{t+1} and r_t. How can we get ρ from the data?

Solution:

This utility function is a CES (constant elasticity of substitution) form, where $\frac{1}{1-\rho}$ represents the elasticity of substitution between consumption and housing goods.

Reintegrating the subscript of time, the Euler equations becomes:

$$c_t^{\rho-1}(c_t^\rho + H_t^\rho)^{\frac{1-\rho}{\rho}} = \beta(1+r_t)c_{t+1}^{\rho-1}(c_{t+1}^\rho + H_{t+1}^\rho)^{\frac{1-\rho}{\rho}}$$

$$H_t^{\rho-1}(c_t^\rho + H_t^\rho)^{\frac{1-\rho}{\rho}} + \beta P_{t+1}(1-\delta)c_{t+1}^{\rho-1}(c_{t+1}^\rho + H_{t+1}^\rho)^{\frac{1-\rho}{\rho}} = P_t c_t^{\rho-1}(c_t^\rho + H_t^\rho)^{\frac{1-\rho}{\rho}}$$

By manipulating these equations, we express the ratio $\frac{c_t}{H_t}$ as:

$$\frac{H_t}{c_t} = \left(P_t - \frac{P_{t+1}(1-\delta)}{1+r_t}\right)^{\frac{1}{\rho-1}}$$

Thus, ρ can be obtained from the data by the equation:

$$\rho = 1 + \frac{\ln\left(P_t - \frac{P_{t+1}(1-\delta)}{1+r_t}\right)}{\ln\left(\frac{H_t}{c_t}\right)}$$

Exercise 3

Consider a single agent problem where each period, w total output is produced and can be divided into consumption of a perishable good, c_t, and investment in a durable good, d_{xt}. The durable depreciates like a capital good, but is not directly productive. The stock of durables at any date, d_t, produces a flow of services that enters the utility function. Thus, the problem faced by the household with initial stock d_0 is:

$$\max_{\{c_t, d_t, d_{xt}\}_{t=0}^\infty} \sum_{t=0}^{\infty} \beta^t [u_1(c_t) + u_2(d_t)]$$

subject to:

$$c_t + d_{xt} \leq w \quad \forall t$$

$$d_{t+1} \leq (1-\delta)d_t + d_{xt} \quad \forall t$$

$$c_t, d_t, d_{xt} \geq 0 \quad \forall t$$

$$d_0 \text{ given}$$

where both u_1 and u_2 are strictly increasing and continuous. Ignore the non-negativity constraints on d_{xt} while solving this problem.

1. **State a condition on either u_1 or u_2 (or both) such that you can write an equivalent problem in the following form:**

$$\max_{\{d_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(d_t, d_{t+1})$$
$$\text{s.t.} \quad d_{t+1} \in \Gamma(d_t) \quad \forall t$$

where $\Gamma(d_t) \in \mathbb{R}^+$. What is F? What is the correspondence Γ (i.e., this is the set of possible values where the variable d_{t+1} can be chosen)?

Solution:

Since u_1 and u_2 are both continuous and strictly increasing, we know the budget constraint and the law of motion for durable goods will always bind with equality. No further conditions are required to write the equivalent problem. Therefore,

$$F(d_t, d_{t+1}) = u_1(w + (1-\delta)d_t - d_{t+1}) + u_2(d_t)$$

and

$$\Gamma(d_t) = [0, w + (1-\delta)d_t]$$

.

For the remaining questions, assume that both u_1 and u_2 satisfy the Inada conditions and are continuously differentiable.

2. **Write the Bellman equation for this problem.**

Solution:

The Bellman equation is:

$$v(d) = \max_{d' \in \Gamma(d)} [F(d, d') + \beta v(d')]$$

$$v(d) = \max_{d' \in \Gamma(d)} [u_1(c) + u_2(d) + \beta v(d')]$$

with $c = w - d' + (1-\delta)d$

3. **State the envelope condition and the F.O.C.**

Solution:

The first-order condition is:

$$\frac{\partial v(d)}{\partial d'} = 0 \iff \frac{\partial c}{\partial d'}\frac{\partial u_1(c)}{\partial c} + \beta \frac{\partial v(d')}{\partial d'} = 0$$

which simplifies to:

$$\frac{\partial u_1(c)}{\partial c} = \beta \frac{\partial v(d')}{\partial d'}$$

The envelope condition is:

$$\frac{\partial v(d)}{\partial d} = \frac{\partial c}{\partial d}\frac{\partial u_1(c)}{\partial c} + \frac{\partial u_2(d)}{\partial d}$$

which simplifies to:

$$\frac{\partial v(d)}{\partial d} = (1-\delta)\frac{\partial u_1(c)}{\partial c} + \frac{\partial u_2(d)}{\partial d}$$

4. **Find the Euler equation of this problem.**

Solution:

The Euler equation is obtained by combining the first-order and envelope conditions:

$$\frac{\partial u_1(c)}{\partial c} = \beta \left[(1-\delta)\frac{\partial u_1(c')}{\partial c'} + \frac{\partial u_2(d')}{\partial d'}\right]$$

5. **Show that there is a unique steady state value of the stock, d^*, such that if $d_0 = d^*$, then $d_t = d^* \quad \forall t$. Show that $d^* > 0$.**

Solution:

In the steady state, $d = d' = d'' = d^*$. Then the budget constraint simplifies to:

$$c = w - \delta d^*$$

which simplifies the Euler equation to:

$$u_1'(w - \delta d^*) = \beta \left[(1-\delta) u_1'(w - \delta d^*) + u_2'(d^*)\right]$$

Solving this equation proves the existence and uniqueness of d^*, and since u_2 satisfies the Inada conditions, $d^* > 0$.