Macroeconomics III - Problem Set 2

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October 2024

1 Literature Review: Garber (1990)

Garber, Peter M. 1990. "Famous First Bubbles." Journal of Economic Perspectives, 4 (2): 35–54.

Can we rationalize a speculative events? That is the question Garber try to answer looking at three historical episodes: the Dutch Tulipmania (1634-1637), the Mississippi Bubble (1719-20) and the South Sea Bubble (1720). If the "sunspot" literature often categorized them as bubbles caused by "outbursts of irrationality", this paper argues there are "reasonable" explanations behind these speculative events. By proposing several market fundamentals to explain the movements in asset prices, Garber offers valuable insights on why none of these events qualify as bubbles.

We focus on the case of the Dutch Tulipmania, characterized by an extraordinary rise in the prices of tulip bulbs from 1634 to their collapse in February 1637. Recently introduced in the Dutch market, tulips quickly became the object of conspicuous consumption as luxury goods. By 1636, the rapid price increase had attracted speculators, driven by the "perception of an increased probability of large returns", while new financial tools such as claims on future bulb harvests eased transactions. Table 9.1 displays the prices of certain bulb varieties at their peak, on February 5 1637. On that day, a coupon for a bulb of Witte Croonen was traded for 1668 Guilders (which is equivalent to 172,00 USD), when it would only be worth 37.5 Guilders five years later. However, Garber notices that the pricing pattern does not differ from the standard (and even modern) markets for new varieties of flowers. If the annual depreciation rate for the AvdE was 36%, it is not greatly different from the 18th century depreciation rate. That is why the author argues that the crash only could have accounted for no more than a 16% price decline. Garber the violence of the crash has to do with the one-month price surge that occurred before the crisis, due to the introduction common bulbs that started to be traded among middle-class. Overall, the movements in asset prices only followed the standard pattern.

We have selected this paper because we believe it casts light the rational mechanisms behind the phenomenon of speculative crisis, departing from the standard approach to propose a new interpretation of the so called "first speculative bubble in History". Interestingly enough, the Tulipmania did not seem to create any economic distress in the Dutch Republic; an argument often put forward was the relatively small size of the tulip market.

Table 9.1
Post-Collapse Bulb Prices in Guilders

Bulb	Jan. 1637	Feb. 5, 1637	1642 or 1643	Annual % Depreciation[1]
1. Witte Croonen (one-half pound)	64.	1668. (avg.)	37.5	76
2. English Admiral (bulb)		700. (25 aas bulb)	210.	24
3. Admirael van der Eyck (bulb)		1345. (wtd. avg.)	220.*	36
4. General Rotgans (Rotgansen)		805. (1000 azen)	138.	35

*Adjusted downward fl. 5 to account for the English Admiral outgrowth.
[1] From February 1637 peak.

Figure 1: Garber (1990), page 64

2 Exercise 1: Asset pricing in complete and incomplete markets

Part A: Preliminaries

1. Coefficient of relative risk aversion of the agent

   From lecture, we know that the coefficient of relative risk aversion of the agent equals to:

   $$A = -\mu \frac{u''(\mu)}{u'(\mu)}$$

   In this case the utility function has a logarithmic form:

   $$A = -\mu \frac{\frac{-1}{\mu^2}}{\frac{1}{\mu}} = 1$$

   Thus, the agent has a constant relative risk aversion (CRRA).

2. Aggregate income/consumption in a bad state of the economy

   We are in a complete market, which means lucky agents in the bad aggregate sate commit to share their income with unlucky agents such that all agents consume the same amount in the bad aggregate state. BY the law of large number, we consider that $(1-\lambda)$ corresponds to the proportion of lucky agents in the bad state of the economy and λ the proportion of unlucky ones. The aggregate income/consumption in a bad state of the economy takes the form of the expected income/consumption in a bad state of the economy for the representative agent:

   $$E(Y_1) = E(C_1) = (1-\lambda)\mu + \lambda(1 - \frac{\phi}{\lambda})\mu$$
   $$E(Y_1) = E(C_1) = \mu(1-\phi)$$

3. Budget constraints in t=0 and t=1

   One way to write the budget constraints is:

   In t=0:

   $$c_0 = \mu - qb - p_1 a_1 - p_2 a_2$$

   In t=1: (with one budget constraint by state)

   $$c_1^{Good} = \mu + b + (1+\pi_1)a_1 - a_2$$

   $$c_1^{Bad} = \bar{\mu} + b - a_1 - (1+\pi_2)a_2$$

With $\bar{\mu} = \mu(1-\phi)$, the expected income/consumption in a bad state with a complete market.

However, note that we here assume zero net supply (consumption profile to be exogenous), the assets are not yet introduced in the economy, which implies:

$$-qb - p_1 a_1 - p_2 a_2 = 0$$

$$b + (1+\pi_1)a_1 - a_2 = 0$$

$$b - a_1 - (1+\pi_2)a_2 = 0$$

Thus the budget constraint in a complete market actually looks like:

$$c_0 = \mu - qb - p_1 a_1 - p_2 a_2$$

$$c_1^{Good} = \mu$$

$$c_1^{Bad} = \bar{\mu}$$

Part B: Asset pricing in complete market

1. Maximization program

   We assume the assets are not introduced yet in the economy (zero net supply) and we want to know the price of these assets if introduced in a very small amount such that it doesn't affect allocations.

   The maximization program looks like:

$$\max_{b,a_1,a_2} \ln c_0 + \mathbb{E} \ln c_1$$

   s.t.

$$c_0 = \mu - qb - p_1 a_1 - p_2 a_2$$

$$c_1^{Good} = \mu + b + (1+\pi_1)a_1 - a_2$$

$$c_1^{Bad} = \bar{\mu} + b - a_1 - (1+\pi_2)a_2$$

   And we can rewrite the program as follows:

$$\max_{b,a_1,a_2} \ln(\mu - qb - p_1 a_1 - p_2 a_2) + \frac{1}{2}\ln(\mu + b + (1+\pi_1)a_1 - a_2) + \frac{1}{2}\ln(\bar{\mu} + b - a_1 - (1+\pi_2)a_2)$$

2. Pricing equation of the three assets: b, a_1 and a_2

   We look for q:

$$\frac{\partial V}{\partial b} = 0 \Leftrightarrow qu'(\mu) = \frac{1}{2}u'(\mu) + \frac{1}{2}u'(\bar{\mu})$$

$$\Leftrightarrow q = \frac{1}{2} + \frac{1}{2}\frac{u'(\bar{\mu})}{u'(\mu)}$$

$$\Leftrightarrow q = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\bar{\mu}}$$

$$\Leftrightarrow q = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\mu(1-\phi)}$$

$$\Leftrightarrow q = \frac{(1-\phi)}{2(1-\phi)} + \frac{1}{2(1-\phi)}$$

$$\Leftrightarrow q = \frac{2-\phi}{2(1-\phi)}$$

We look for p_1 :

$$\frac{\partial V}{\partial a_1} = 0 \Leftrightarrow p_1 u'(\mu) = \frac{(1+\pi_1)}{2} u'\left(\mu\right) - \frac{1}{2} u'(\bar{\mu})$$

$$\Leftrightarrow p_1 = \frac{(1+\pi_1) - u'\left(\frac{\bar{\mu}}{\mu}\right)}{2}$$

$$\Leftrightarrow p_1 = \frac{(1+\pi_1) - \frac{\mu}{\bar{\mu}}}{2}$$

$$\Leftrightarrow p_1 = \frac{(1+\pi_1)}{2} - \frac{\mu}{2\bar{\mu}}$$

$$\Leftrightarrow p_1 = \frac{(1+\pi_1)}{2} - \frac{\mu}{2\mu(1-\phi)}$$

$$\Leftrightarrow p_1 = \frac{(1+\pi_1)(1-\phi) - 1}{2(1-\phi)}$$

$$\Leftrightarrow p_1 = \frac{\pi_1 - \phi - \pi_1\phi}{2(1-\phi)}$$

We look for p_2 :

$$
\begin{aligned}
\frac{\partial V}{\partial a_2} = 0 \Leftrightarrow p_2 u'(\mu) &= -\frac{1}{2}u'(\mu) + \frac{(1+\pi_2)}{2}u'\left(\bar{\mu}\right) \\
\Leftrightarrow p_2 &= \frac{(1+\pi_2)\, u'\left(\frac{\bar{\mu}}{\mu}\right) - 1}{2} \\
\Leftrightarrow p_2 &= \frac{(1+\pi_2)\, \frac{\mu}{\bar{\mu}} - 1}{2} \\
\Leftrightarrow p_c &= \frac{(1+\pi_2)\, \mu}{2\bar{\mu}} - \frac{1}{2} \\
\Leftrightarrow p_2 &= \frac{(1+\pi_2)\, \mu - \bar{\mu}}{2\bar{\mu}} \\
\Leftrightarrow p_2 &= \frac{(1+\pi_2)\, \mu - \mu(1-\phi)}{2\mu(1-\phi)} \\
\Leftrightarrow p_2 &= \frac{1+\pi_2 - 1 + \phi}{2(1-\phi)} \\
\Leftrightarrow p_2 &= \frac{\pi_2 + \phi}{2(1-\phi)}
\end{aligned}
$$

3. How do the price ratios $\frac{p_1}{q}$ and $\frac{p_2}{q}$ vary with λ? With ϕ ? Compare and interpret these results.

   We start by computing the price ratios:

$$
\frac{p_1}{q} = \frac{\frac{\pi_1 - \phi - \pi_1\phi}{2(1-\phi)}}{\frac{2-\phi}{2(1-\phi)}} = \frac{\pi_1 - p\phi - \pi_1\phi}{2-\phi}
$$

$$
\frac{p_2}{q} = \frac{\frac{\pi_2 + \phi}{2(1-\phi)}}{\frac{2-\phi}{2(1-\phi)}} = \frac{\pi_2 + \phi}{2-\phi}
$$

   We notice that the price ratios do not depend on λ, this is because the market is complete. Agents get the same income in the bad state, whether they are lucky or unlucky.

   We take the derivative of $\frac{p_2}{q}$ wrt ϕ:

$$
\begin{aligned}
\frac{\partial \frac{p_1}{q}}{\partial \phi} &= \frac{(-1-\pi_1)(2-\phi) - (-1)(\pi_1 + \phi + \pi_1\phi)}{(2-\phi)^2} \\
\frac{\partial \frac{p_1}{q}}{\partial \phi} &= \frac{-2 + \phi - 2\pi_1 + \pi_1 - \phi - \phi\pi_1}{(2-\phi)^2} \\
\frac{\partial \frac{p_1}{q}}{\partial \phi} &= \frac{\pi_1 - 2\pi_1 - 2}{(2+\phi)^2} \\
\frac{\partial \frac{p_1}{q}}{\partial \phi} &= \frac{-\pi_1 - 2}{(2-\phi)^2} < 0
\end{aligned}
$$

We take the derivative of $\frac{p_2}{q}$ wrt ϕ:

$$\frac{\partial \frac{p_2}{q}}{\partial \phi} = \frac{2-\phi-(-1)(\pi+\phi)}{(2-\phi)^2}$$
$$\frac{\partial \frac{p_2}{q}}{\partial \phi} = \frac{2-\phi+\pi+\phi}{(2-\phi)^2}$$
$$\frac{\partial \frac{p_2}{q}}{\partial \phi} = \frac{2+\pi}{(2-\phi)^2} > 0$$

Thus we have $\frac{p_1}{q}$ decreasing with ϕ, while $\frac{p_2}{q}$ increasing with ϕ. This makes sense intuitively given that aggregate income/consumption in the bad state decreases with ϕ. In other words, agents will prefer getting more of a_2 to smooth their consumption and compensate for this loss of income in the bad state of the economy. On the other hand, a_1 gives negative returns in the bad state, where consumption is already lower, which is why its price needs to decrease for agents to be willing to buy some.

Part C: Asset pricing in incomplete market

1. Maximization program

   In incomplete markets, the maximization program becomes:

$$\max_{b,a1,a2} \ln\left(\mu - qb - p_1 a_1 - p_2 a_1\right) + \frac{1}{2}\left(\mu + b + (1+\pi_1)\, a_1 - a_2\right)$$
$$+ \frac{1}{2}\left((1-\lambda)(\mu + b - a_1 + (1-\pi)_1 a_2)) + \lambda\left((1-\frac{\phi}{\lambda})\mu + b - a_1 + (1+\pi_2)a_2\right)\right)$$

2. Pricing equation of the three assets: b, a_1 and a_2

   We look for q:

$$
\begin{aligned}
\frac{\partial V}{\partial b} = 0 \Leftrightarrow q u'(\mu) &= \frac{1}{2}u'(\mu) + \frac{1}{2}\left((1-\lambda)u'(\mu) + \lambda u'((1-\frac{\phi}{\lambda})\mu)\right) \\
\Leftrightarrow q u'(\mu) &= \frac{1}{2}u'(\mu) + \frac{(1-\lambda)}{2}u'(\mu) + \frac{\lambda}{2}u'\left(\left((1-\frac{\phi}{\lambda})\mu\right)\right. \\
\Leftrightarrow q &= \frac{1}{2} + \frac{(1-\lambda)}{2} + \frac{\lambda}{2}\frac{u'\left((1-\frac{\phi}{\lambda}\right)\mu\Big)}{u'(\mu)} \\
\Leftrightarrow q &= \frac{1+(1-\lambda)}{2} + \frac{\lambda}{2}\frac{\mu}{\left(1-\frac{\phi}{\lambda}\right)\mu} \\
\Leftrightarrow q &= \frac{2-\lambda}{2} + \frac{\lambda}{2}\cdot\frac{1}{1-\frac{\phi}{\lambda}} \\
\Leftrightarrow q &= \frac{(2-\lambda)\left(1-\frac{\phi}{\lambda}\right)+\lambda}{2-\frac{2\phi}{\lambda}} \\
\Leftrightarrow q &= \frac{2-\frac{2\phi}{\lambda}-\lambda+\phi+\lambda}{\frac{2\lambda-2\phi}{\lambda}} \\
\Leftrightarrow q &= \frac{\left(2-\frac{2\phi}{\lambda}+\phi\right)\lambda}{2(\lambda-\phi)} \\
\Leftrightarrow q &= \frac{2\lambda-2\phi+\phi\lambda}{2(\lambda-\phi)} \\
\Leftrightarrow q &= \frac{2(\lambda-\phi)+\phi\lambda}{2(\lambda-\phi)}
\end{aligned}
$$

We look for p_1:

$$\frac{\partial V}{\partial a_1}=0 \Leftrightarrow p_1 u'(\mu)=\frac{1}{2}\left(1+\pi_1\right)u'(\mu)+\frac{1}{2}\left[(1-\lambda)(-1)u'(\mu)+\lambda(-1)u'\left(\left(1-\frac{\phi}{\lambda}\right)\mu\right)\right]$$

$$\Leftrightarrow pu_1'(\mu)=\frac{(1+\pi_1)}{2}u'(\mu)-\frac{(1-\lambda)}{2}u'(\mu)-\frac{\lambda}{2}u'\left(\left(1-\frac{\phi}{\lambda}\right)\mu\right)$$

$$c\Rightarrow p_1=\frac{(1+\pi_1)}{2}-\frac{(1-\lambda)}{2}-\frac{\lambda}{2}\frac{u'\left(\left(1-\frac{\phi}{\lambda}\right)\mu\right)}{u'(\mu)}$$

$$\Leftrightarrow p_1=\frac{\pi_1+\lambda}{2}-\frac{\lambda}{2}\cdot\frac{\mu}{\left(1-\frac{\phi}{\lambda}\right)\mu}$$

$$\Leftrightarrow p_1=\frac{\pi_1+\lambda}{2}-\frac{\lambda}{2\left(1-\frac{\phi}{\lambda}\right)}$$

$$\Leftrightarrow p_1=\frac{(\pi_1+\lambda)\left(1-\frac{\phi}{\lambda}\right)-\lambda}{2\left(1-\frac{\phi}{\lambda}\right)}$$

$$\Leftrightarrow p_1=\frac{\pi_1-\pi_1\frac{\phi}{\lambda}+\lambda-\varnothing-\lambda}{2\left(1-\frac{\varnothing}{\lambda}\right)}$$

$$\Leftrightarrow p_1=\frac{\pi_1\left(1-\frac{\phi}{\lambda}\right)-\varnothing}{2\left(1-\frac{\phi}{\lambda}\right)}$$

$$\Leftrightarrow p_1=\frac{(\pi_1\left(1-\frac{\phi}{\lambda}\right)-\phi)\lambda}{2(\lambda-\phi)}$$

$$\Leftrightarrow p_1=\frac{\lambda\pi_1\left(1-\frac{\phi}{\lambda}\right)-\lambda\phi}{2(\lambda-\phi)}$$

$$\Leftrightarrow p_1=\frac{\lambda\pi_1-\phi(\pi_1+\lambda)}{2(\lambda-\phi)}$$

We look for p_2:

$$\frac{\partial V}{\partial a_2}=0\Longleftrightarrow p_2u'(\mu)=\frac{1}{2}(1+\pi_2)((1-\lambda)u'(\mu)-\lambda u'((1-\phi/\lambda)\mu)-\frac{1}{2}u'(\mu)$$

$$\Longleftrightarrow p_2u'(\mu)=\frac{(1+\pi_2)(1-\lambda)}{2}u'(\mu)+\frac{(1+\pi_2)\lambda}{2}u'\left((1-\frac{\phi}{\lambda})\mu\right)-\frac{1}{2}u'(\mu)$$

$$\Longleftrightarrow p_2=\frac{(1+\pi_2)(1-\lambda)}{2}+\frac{(1+\pi_2)\lambda}{2}\frac{u'((1-\frac{\phi}{\lambda})}{u'(\mu)}-\frac{1}{2}$$

$$\Longleftrightarrow p_2=\frac{(1+\pi_2)(1-\lambda)-1}{2}+\frac{(1+\pi_2)\lambda}{2}\frac{\mu}{(1-\frac{\phi}{\lambda})\mu}$$

$$\iff p_2 = \frac{(\pi_2 - \lambda(1+\pi_2))}{2} + \frac{(1+\pi_2)\lambda}{2(1-\frac{\phi}{\lambda})}$$

$$\iff p_2 = \frac{(\pi_2 - \lambda(1+\pi_2))(1-\frac{\phi}{\lambda}) + (1+\pi_2)\lambda}{2(1-\frac{\phi}{\lambda})}$$

$$\iff p_2 = \frac{\pi_2(1-\frac{\phi}{\lambda}) + \phi(1+\pi_2)}{2(1-\frac{\phi}{\lambda})}$$

$$\iff p_2 = \frac{\pi_2\lambda(1-\frac{\phi}{\lambda}) + \phi\lambda(1+\pi_2)}{2(\lambda-\phi)}$$

$$\iff p_2 = \frac{\pi_2(\lambda-\phi) + \phi\lambda(1+\pi_2)}{2(\lambda-\phi)}$$

3. How do the price ratios $\frac{p_1}{q}$ and $\frac{p_2}{q}$ vary with λ? With ϕ ? Compare and interpret these results.

   We start by computing the price ratios:

$$\frac{p_1}{q} = \frac{\frac{\lambda\pi_1 - \phi(\pi_1+\lambda)}{2(\lambda-\phi)}}{\frac{2(\lambda-\phi)+\phi}{2(\lambda-\varnothing)}}$$

$$\frac{p_1}{q} = \frac{\lambda(\pi_1-\phi) - \pi_1\phi}{\lambda(2+\phi) - 2\phi} = \frac{\pi_1\lambda - \phi(\pi_1+\lambda)}{\phi(\lambda-2)+2\lambda}$$

   And

$$\frac{p_2}{q} = \frac{\frac{\pi_2(\lambda-\phi)+\phi\lambda(1+\pi_2)}{2(\lambda-\phi)}}{\frac{2(\lambda-\phi)+\phi}{2(\lambda-\varnothing)}}$$

$$\frac{p_2}{q} = \frac{\lambda(\pi_2+\phi+\phi\pi_2) - \pi_2\phi}{2(\lambda-\phi)+\phi\lambda} = \frac{\phi(-\pi_2+\lambda+\lambda\pi_2) + \pi_2\lambda}{2(\lambda-\phi)+\phi\lambda}$$

   We take the derivatives of $\frac{p_1}{q}$ wrt λ:

$$\frac{\partial\frac{p_1}{q}}{\partial\lambda} = \frac{(\pi_1-\phi)(\lambda(2+\phi)-2\phi) - (2+\phi)(\lambda(\pi_1-\phi)-\pi_1\phi)}{(\lambda(2+\phi)-2\phi)^2}$$

$$\frac{\partial\frac{p_1}{q}}{\partial\lambda} = \frac{(\pi_1-\phi)(2\lambda+\lambda\phi-2\phi) - (2+\phi)(\lambda\pi_1-\lambda\phi-\pi_1\phi)}{(2\lambda+\lambda\phi-2\phi)^2}$$

$$\frac{\partial\frac{p_1}{q}}{\partial\lambda} = \frac{2\phi^2+\pi_1\phi^2}{(2\lambda+\lambda\phi-\lambda\phi)^2} > 0$$

   And wrt ϕ:

$$\frac{\partial \frac{p_1}{q}}{\partial \phi} = \frac{(-\pi_1 - \lambda)\left(\phi(\lambda-2)+2\lambda\right) - (\lambda-2)\left(\pi_1\lambda - \phi\left(\pi_1+\lambda\right)\right)}{(\phi(\lambda-2)+2\lambda)^2}$$
$$\frac{\partial \frac{p_1}{q}}{\partial \phi} = \frac{(-\pi_1 - \lambda)\left(\phi\lambda - \phi 2+2\lambda\right) - (\lambda-2)\left(\pi_1\lambda - \phi\pi_1+\phi\lambda\right)}{(\phi(\lambda-2)+2\lambda)^2}$$
$$\frac{\partial \frac{p_1}{q}}{\partial \phi} = \frac{-\pi_1\lambda^2 - 2\lambda^2}{(\phi(\lambda-2)+2\lambda)^2}$$
$$\frac{\partial \frac{p_1}{q}}{\partial \phi} = \frac{-\lambda^2\left(\pi_1+2\right)}{(\phi(\lambda-2)+2\lambda)^2} \quad < 0$$

Now we take the derivatives of $\frac{p_2}{q}$ wrt λ:

$$\frac{\partial \frac{p_2}{q}}{\partial \lambda} = \frac{(\pi_2+\phi+\phi\pi)\left(2(\lambda-\phi)+\phi\lambda\right) - (2+\phi)\left(\lambda\left(\pi_2+\phi+\phi\pi_2\right)\pi_2\phi\right)}{(2(\lambda-\phi)+\phi\lambda)^2}$$
$$\frac{\partial \frac{p_2}{q}}{\partial \lambda} = \frac{(\pi_2+\phi+\phi\pi)\left(2\lambda-2\phi+\phi\lambda\right) - (2+\phi)\left(\lambda\pi_2+\lambda\phi+\pi_2\lambda\phi-\pi_2\phi\right)}{(2(\lambda-\phi)+\phi\lambda)^2}$$
$$\frac{\partial \frac{p_2}{q}}{\partial \lambda} = \frac{\pi_2\phi^2 - 2\phi^2 - 2\phi^2\pi}{(2(\lambda-\phi)+\phi\lambda)^2}$$
$$\frac{\partial \frac{p_2}{q}}{\partial \lambda} = \frac{-\phi^2(2+\pi_2)}{(2(\lambda-\phi)+\phi\lambda)^2} < 0$$

And wrt ϕ:

$$\frac{\frac{\partial p_2}{q}}{\partial \phi} = \frac{(\lambda-\pi_2+\lambda\pi_2)\left(2(\lambda-\phi)+\phi\lambda\right) - (-2+\lambda)\left(\phi\left(\lambda-\pi_2+\lambda\pi_2\right)+\pi_2\lambda\right)}{(2(\lambda-\phi)+\phi\lambda)^2}$$
$$\frac{\frac{\partial p_2}{q}}{\partial \phi} = \frac{(\lambda-\pi_2+\lambda\pi_2)\left(2\lambda-2\phi+\phi\lambda\right) - (\lambda+\lambda)\left(\phi\lambda-\phi\pi_2+\lambda\phi\pi_2+\pi_2\lambda\right)}{(2(\lambda-\phi)+\phi\lambda)^2}$$
$$\frac{\frac{\partial p_2}{q}}{\partial \phi} = \frac{2\lambda^2+2\lambda^2\pi_2-\pi_2\lambda^2}{(2(\lambda-\phi)+\phi\lambda)^2}$$
$$\frac{\frac{\partial p_2}{q}}{\partial \phi} = \frac{\lambda^2\left(2+2\pi_2-\pi_2\right)}{(2(\lambda-\phi)+\phi\lambda)^2}$$
$$\frac{\frac{\partial p_2}{q}}{\partial \phi} = \frac{\lambda^2\left(2+\pi_2\right)}{(2(\lambda-\phi)+\phi\lambda)^2} > 0$$

Thus we have $\frac{p_1}{q}$ decreasing with ϕ, while $\frac{p_2}{q}$ increasing with ϕ, the intuition for complete markets still holds since the expected income of agents in the bad state decreases as ϕ increases. But now we also have $\frac{p_1}{q}$ increasing with λ, while $\frac{p_2}{q}$ decreasing with λ. The reason behind this is that, even though the agent is more likely to be unlucky in a bad state of the world, in this situation we notice that their income $(1-\frac{\phi}{\lambda})\mu$ actually increases with λ. In other words, even though their chance of being unlucky in the bad state increases, the penalty on income/consumption is lower. That is why agents will prefer buying more a_1 and less a_2, as they need to insure themselves less if they end up in that situation.

3 Problem 2: Asset Pricing with Internal Habits

We are given the utility function $u(c_t - hc_{t-1}, l)$ with the following budget constraint:

$$c_t + a_{t+1} = (1 + r_t)a_t + wl_t$$

3.1 Bellman Equation

We define the state variables as c_{t-1}, a_t and the control variables as c_t, a_{t+1}, l_t. Suppose monetary policy, i.e., the interest rate, is exogenous.

The Bellman equation is:

$$V(c_{-1}, a) = \max_{c,a'} [u(c - hc_{-1}, l) + \beta V(c, a')]$$

or

$$V(c_{-1}, a) = \max_{c,a'} [u((1 + r)a + wl - a' - hc_{-1}, l) + \beta V((1 + r)a + wl - a' - hc_{-1}, a')]$$

using the budget constraint.

Since consumption c is a function of a, we will derive only two first-order conditions (FOCs).

3.2 Pricing Kernel

Taking the first-order condition with respect to the choice variables, we obtain:

$$\frac{\partial V(c_{-1}, a)}{\partial a'} = 0$$

Thus

$$\frac{\partial u(c - hc_{-1}, l)}{\partial c} = -\beta \frac{\partial V}{\partial c} + \beta \frac{\partial V}{\partial a'} = \beta V_1'(c, a') - \beta V_2'(c, a')$$

Further

$$\frac{\partial V(c_{-1}, a)}{\partial l} = 0$$

Implies

$$w \frac{\partial u(c - hc_{-1}, l)}{\partial c} = \frac{\partial u(c - hc_{-1}, l)}{\partial l} + \beta V_2'(c, a')$$

The Envelope condition is:

$$\frac{\partial V(c_{-1}, a)}{\partial a} = (1 + r) \frac{\partial u(c - hc_{-1})}{\partial c} + \beta(1 + r)V_2'(c, a')$$

$$\frac{\partial V(c_{-1},a)}{\partial c_{-1}} = -h\frac{\partial u(c-hc_{-1})}{\partial c}$$

By the Envelope Theorem, we write:

$$\frac{\partial V(c,a')}{\partial a'} = (1+r)\frac{\partial u(c'-hc,l')}{\partial c'} + \beta(1+r)V_2'(c',a'')$$

$$\frac{\partial V(c,a')}{\partial c} = -h\frac{\partial u(c'-hc,l')}{\partial c'}$$

The Euler equation is thus:

$$\frac{\partial u(c-hc_{-1},l)}{\partial c} = \beta\left[(1+r)\frac{\partial u(c'-hc,l')}{\partial c'} + \beta(1+r)V_2'(c',a'') - h\frac{\partial u(c'-hc,l')}{\partial c'}\right]$$

$$= \beta\left[(1+r)\frac{\partial u(c'-hc,l')}{\partial c'} - \beta(1+r)h\frac{\partial u(c''-hc',l'')}{\partial c'} + h\frac{\partial u(c'-hc,l')}{\partial c'}\right]$$

Which I dare to rewrite as:

$$\frac{\partial u(c-hc_{-1},l)}{\partial c} = \beta\left[(1+r)\frac{\partial u(c'-hc,l')}{\partial c'} - \beta(1+r)h\frac{\partial u(c''-hc',l'')}{\partial c'} + h\frac{\partial u(c'-hc,l')}{\partial c'}\right]$$

$$\frac{\partial u(c-hc_{-1},l)}{\partial c} - \beta h\frac{\partial u(c'-hc,l')}{\partial c'} = \beta(1+r)\left[\frac{\partial u(c'-hc,l')}{\partial c'} - \beta h\frac{\partial u(c''-hc',l'')}{\partial c'}\right]$$

By dividing the right-hand-side by the left-hand-side of the equation, it follows that the pricing kernel is:

$$\frac{1}{1+r_t} = \beta\frac{\left[\frac{\partial u(c'-hc,l')}{\partial c'} - \beta h\frac{\partial u(c''-hc',l'')}{\partial c'}\right]}{\left[\frac{\partial u(c-hc_{-1},l)}{\partial c} - \beta h\frac{\partial u(c'-hc,l')}{\partial c'}\right]}$$

Which looks somewhat like our usual expression for the pricing kernel:

$$m_{t+1} = \mathbb{E}\frac{1}{(1+r_t)} = \beta\frac{u'(c_{t+1})}{u'(c)}$$

3.3 Steady-State Interest Rate

By definition of the steady state, $c_t = c_{t+1} = c$. Rearranging the pricing kernel yields:

$$1 = \beta(1+r)$$

which gives the steady-state interest rate as:

$$r = \frac{1}{\beta} - 1$$

3.4 Labor Supply

From the first-order condition for labor, we have:

$$w\frac{\partial u(c-hc_{-1},l)}{\partial c} = \frac{\partial u(c-hc_{-1},l)}{\partial l} - \beta h\frac{\partial u(c'-hc,l')}{\partial c'}$$

Recall that in steady-state, consumption across all periods is constant. Since

$$u(c,l) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{l^{1+\epsilon}}{1+\epsilon}$$

$$wc^{-\sigma} = l^{\epsilon} - \beta hc^{-\sigma}$$

$$\frac{w+\beta h}{c^{\sigma}} = \frac{w+\beta h}{(c(1-h))^{\sigma}} = l^{\epsilon}$$

in steady-state. Thus, labor supply is determined by:

$$l = \left(\frac{w+\beta h}{c(1-h)^{\sigma}}\right)^{\frac{1}{\epsilon}}$$

This expression is decreasing in h (the habit parameter), since stronger habits reduce effective consumption and increase marginal utility.