Macroeconomics III - Problem Set 3 (Part 1)

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There are three types of agents: entrepreneurs, rich agents, and poor agents, who alternate between being rich and poor. They all maximize:

$$\max_a \sum_{t=0}^{\infty} \beta^t u(c_{i,t})$$

Part A: Preliminaries

Question 1

Entrepreneur profits are:

$$\pi_{e,t} = f(k_t) - (1+r)k_t$$

The first-order condition (FOC) implies:

$$\frac{\delta \pi}{\delta k_t} = 0$$

$$f'(k_t) = 1 + r$$

Thus:

$$\alpha k^{\alpha-1} = 1 + r$$

The expression for $c_{e,t}$ implies:

$$c_{e,t} = k_t^{\alpha} - (1+r)k_t$$

Then, plugging in the expression for r:

$$c_{e,t} = k_t^{\alpha} - (1 + \alpha k_t^{\alpha-1} - 1)k_t$$

$$c_{e,t} = k_t^{\alpha} - \alpha k_t^{\alpha}$$

Yielding the desired result:

$$c_{e,t} = (1-\alpha)k_t^{\alpha}$$

Question 2

Writing the constraints of all agents:

$$c_{e,t} = f(k_t) - (1+r)k_t$$

$$c_A + a_{A,t+1} = e_A + a_{A,t}(1+r)$$

$$c_A + a_{B,t+1} = e_A + a_{B,t}(1+r)$$

Combining yields:

$$c_e + c_A + c_B + a_{A,t+1} + a_{B,t+1} = f(k_t) - (1+r)k_t + e_A + e_B + (1+r)(a_{A,t} + a_{B,t})$$

We know $a_{A,t+1} + a_{B,t+1} = k_{t+1}$,

$$c_e + c_A + c_B + k_{t+1} = f(k_t) - (1+r)k_t + e_A + e_B + (1+r)(a_{A,t} + a_{B,t})$$

Noting that total endowment $e_A + e_B = 1$ and $a_{A,t+1} + a_{B,t+1} = k_{t+1}$,

$$c_e + c_A + c_B + k_{t+1} = 1 + f(k_t) - (1+r)k_t + (1+r)k_t$$

$$c_e + c_A + c_B + k_{t+1} = 1 + f(k_t)$$

Since, by full depreciation, $k_{t+1} = k_t = k$, we get:

$$c_e + c_A + c_B + k = 1 + f(k_t)$$

The expression implies that total consumption must be equal to total endowment and total production, assuming that markets clear.

Part B: Unconstrained Economy

Question 1

The program of the rich agent is:

$$V^R(a) = \max_{a'} u(c^R) + \beta V^P(a')$$

The budget constraint is:

$$e = c^R - a^P(1+r) + a'^R$$

By definition of an unconstrained setting, there is no borrowing constraint. The state variable is

$$a$$

and the choice variables are

$$c, a'$$

which lead us to the following Bellman equation:

$$V^R(a) = u(e + a^R(1+r) - a^{R'}) + \beta V^P(a')$$

The usual steps yield the following FOC and envelope condition:

$$\frac{\delta V^R(a)}{\delta c} = 0$$

implies

$$\frac{\delta u(e - (1+r)a - a')}{\delta c} = \beta V'^P(a')$$

The Envelope condition is:

$$\frac{\delta V^R(a)}{\delta a} = (1+r)\frac{\delta u(e - (1+r)a - a')}{\delta c}$$

By the Envelope Theorem:

$$\frac{\delta V^R(a')}{\delta a'} = (1+r')\frac{\delta u(e' - (1+r)a' - a'')}{\delta c'}$$

Question 2

The program of the poor agent is:

$$V^P(a) = \max_a \sum_{t=0}^{\infty} \beta^t u(c^P) + \beta V^R(a')$$

The budget constraint is:

$$e = c^P + a^R(1+r) - a'^P$$

which leads us to the following Bellman equation:

$$V^P(a) = \max_a u(e - a^P(1+r) - a'^P) + \beta V^R(a')$$

The usual steps yield the following FOC and envelope condition:

$$\frac{\delta V^P(a)}{\delta a'} = 0$$

implies

$$\frac{\delta u(e-(1+r)a-a')}{\delta c} = \beta V'^R(a')$$

The Envelope condition is:

$$\frac{\delta V^P(a)}{\delta a} = (1+r)\frac{\delta u(e-(1+r)a-a')}{\delta c}$$

By the Envelope Theorem:

$$\frac{\delta V^P(a')}{\delta a'} = (1+r')\frac{\delta u(e-(1+r)a-a')}{\delta c'}$$

Question 3

Combining both yields the following Euler equations:

$$\frac{u'(c^R)}{u'(c'^P)} = \beta(1+r')$$

$$\frac{u'(c^P)}{u'(c'^R)} = \beta(1+r')$$

Question 4

In steady-state, consumption is constant by definition. Thus,

$$\frac{u'(c^P)}{u'(c^R)} = \frac{u'(c^R)}{u'(c^P)} = 1 = \beta(1+r)$$

$$\frac{1}{\beta} = 1 + r$$

Since

$$\frac{u'(c^P)}{u'(c^R)} = 1$$

it follows that $c^R = c^P = c$.

Question 5

Since both types of agents behave the same, we have the identical outcome of the typical RBC-model, since we can aggregate agents - there is no heterogeneity.

Question 6

Taxes should have no effect on the interest rate by Ricardian equivalence. Consumers will anticipate the tax increase and adjust consumption and saving accordingly. As they are not constrained, this does not change aggregate outcomes.

Part C: Constrained Economy

Question 1

The program of the poor agent becomes:

$$V^P(a) = \max_a \sum_{t=0}^{\infty} \beta^t u(c^P) + \lambda a' \beta V^R(a')$$

subject to

$$0 = c^P + a^P(1+r) - a'^P$$

which leads us to the following Bellman equation:

$$V^P(a) = \max_a u(0 - a^P(1+r) - a'^P) + \lambda a' + \beta V^R(a')$$

The usual steps yield the following FOC and envelope condition:

$$\frac{\delta V^P(a)}{\delta a'} = 0$$

implies

$$\frac{\delta u(e-(1+r)a-a')}{\delta c} = \lambda + \beta V'^R(a')$$

The Envelope condition is:

$$\frac{\delta V^P(a)}{\delta a} = (1+r)\frac{\delta u(e-(1+r)a-a')}{\delta c}$$

By the Envelope Theorem:

$$\frac{\delta V^P(a')}{\delta c} = (1+r')\frac{\delta u(e-(1+r)a-a')}{\delta c'}$$

We then obtain, in steady-state:

$$\frac{u'(c^P)}{u'(c^R)} = \lambda + \beta(1+r) > \beta(1+r)$$

For the rich agent, nothing changes, so we have from before:

$$\frac{u'(c^R)}{u'(c^P)} = \beta(1+r)$$

Question 2

From the previous part, we know that the constraint of the rich agent is:

$$c^R + a^R = e - (1+r)a^P$$

Since $e = 1$ and agents are now constrained in "poor periods", the expression simplifies to

$$c^R + a^R = 1$$

Similarly,

$$c^P + a^P = e + (1+r)a^R$$

where $e = 0$ and $a^P = 0$. Thus we are left with

$$c^P = (1+r)a^P$$

Finally, since the rich are unconstrained and the poor cannot borrow, then the rich will save by lending to the entrepreneur. Thereby $a^R = k$ follows.

Question 3

We know that

$$\frac{u'(c^R)}{u'(c^P)} = \beta(1+r)$$

Using $u(c) = log(c)$ and $\frac{\delta u(c)}{\delta c} = \frac{1}{c}$, we can write:

$$\frac{\frac{1}{c^R}}{\frac{1}{c^P}} = \frac{c^P}{c^R} = \beta(1+r)$$

Replacing the respective constraints:

$$\frac{c^P}{c^R} = \frac{a^R(1+r)}{1-a^R} = \beta(1+r)$$

$$a^R = (1-a^R)\beta$$

$$a^R(1-\beta) = \beta$$

$$\frac{\beta}{1-\beta} = a^R = k$$

which completes the proof.

Question 4

We know that

$$1 + r = \alpha k^{\alpha - 1}$$

Plugging in k yields:

$$1 + r = \alpha(\frac{\beta}{1 - \beta})^{\alpha - 1} > \frac{1}{\beta}$$

The equilibrium interest late is lower: agents wish to insure themselves against the periods in which they are poor, so they prefer to borrow when they are rich and repay their debt when they are poor. This drives the interest rate down.

In the unconstrained economy, we know that

$$1 + r = \frac{1}{\beta} = \alpha k^{1 - \alpha}$$

$$k = (\frac{1}{\alpha\beta})^{\frac{1}{1-\alpha}}$$

This implies an overaccumulation of capital under credit constraints. As the interest rate is lower, borrowing to accumulate capital becomes cheaper.

Questions 5 & 6

The previous result now no longer holds. Under credit constraints, taxes reduce consumption and/or savings further as agents are less able to insure themselves. This reduces available credit in the economy, so r must increase to attract capital from the perspective of the entrepreneur. From the perspective of the consumer, higher taxes mean that agents will pay a higher price to maintain their insurance against the bad state. This is different from the unconstrained case as agents could simply adjust their borrowing behaviour to account for the tax, eliminating any effect. This is no longer the case under credit constraints.