Correction Problem Set 2: Stochastic environment

Sciences Po - Macroeconomics 3 - Fall 2024
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Exercise 1: Asset pricing in complete and incomplete markets

Environment

It is an endowment economy, where agents face some risks at period 2, but not at period 1. At period 1 agents consume $y_0 = \mu$ (for sure). The endowment in $t = 1$ depends on the state of the economy, either "good" and "bad", as follows:

- probability of 1/2 that we are in a **good state** where: $y_1 = \mu$
- probability of 1/2 that we are in a **bad state** where:
  - probability of $(1 - \lambda)$ not to be affected $y_1 = \mu$
  - probability of λ to be affected $y_1 = (1 - \frac{\phi}{\lambda})\mu$

Assets

The economy is composed of three assets:

- one **safe asset**, denoted b at price q, which returns 1 whatever the state of the economy,
- a **risky asset 1**, denoted a_1 at price p_1, which delivers $1 + \pi_1$ in good states and -1 in bad states,
- a **risky asset 2**, denoted a_2 at price p_2, which delivers -1 in good states and $1 + \pi_2$ in bad states.

Moreover, we assume that the assets are not yet introduced in the economy (e.g. zero net supply). In other words, we price this asset considering exogenous consumption profile and equal to the income in both period 0 and 1.

Preferences

We assume that the consumer favors his inter temporal period utility. π_1 and π_2 are exogenous and we want to determine the prices of the assets p_1, p_2 and q.

$$\max_{b,a_1,a_2} ln(c_0) + \mathbb{E}ln(c_1)$$

Part A: Preliminaries

1. **What is the coefficient of relative risk aversion of the agent?**

$$R = \frac{-cu''(c)}{u'(c)} = 1$$

2. **Compute the aggregate income/consumption in a bad state of the economy.**

$$y_1^B = (1 - \lambda)\mu + \lambda(1 - \frac{\phi}{\lambda})\mu = (1 - \phi)\mu$$

3. **Write the budget constraints in $t = 0$ and $t = 1$**

   The budget in $t = 0$ writes:

$$\mu = c_0 + qb + p_1 a_1 + p_2 a_2$$

   The budget in $t = 1$ in good state writes:

$$\mu + b + (1 + \pi_1)a_1 - a_2 = c_1$$

   The budget in $t = 1$ in bad state in complete market writes:

$$(1 - \phi)\mu + b - a_1 + (1 + \pi_2)a_2 = c_1$$

   The budget in $t = 1$ in bad state in incomplete market for an unaffected agent writes:

$$\mu + b - a_1 + (1 + \pi_2)a_2 = c_1$$

   The budget in $t = 1$ in bad state in incomplete market for an affected agent writes:

$$(1 - \frac{\phi}{\lambda})\mu + b - a_1 + (1 + \pi_2)a_2 = c_1$$

Part B: Asset pricing in complete market

1. **Write down the program.**

   After replacing c through the associated budget constraints in Q3, the program of the agent in complete market writes:

$$\begin{aligned}\max_{b,a_1,a_2} \quad & ln(\mu - qb - p_1 a_1 - p_2 a_2) + \frac{1}{2} ln(\mu + b + (1 + \pi_1)a_1 - a_2) \\ & + \frac{1}{2} ln((1 - \phi)\mu + b - a_1 + (1 + \pi_2)a_2)\end{aligned}$$

2. **Derive the pricing equation of the three assets: risk free, risky 1 and risky 2.**

   We take the FOC of the three assets / choice variables.

   - Safe asset
$$\frac{\partial U}{\partial b} = 0 \iff q\frac{1}{\mu} = \frac{1}{2}\frac{1}{\mu} + \frac{1}{2}\frac{1}{(1 - \phi)\mu}$$
$$\iff q = \frac{2 - \phi}{2(1 - \phi)}$$
   - Risky asset 1
$$\frac{\partial U}{\partial a_1} = 0 \iff p_1\frac{1}{\mu} = \frac{1}{2}(1 + \pi_1)\frac{1}{\mu} - \frac{1}{2}\frac{1}{(1 - \phi)\mu}$$
$$\iff p_1 = \frac{\pi_1 - \pi_1\phi - \phi}{2(1 - \phi)}$$
   - Risky asset 2
$$\frac{\partial U}{\partial a_2} = 0 \iff p_2\frac{1}{\mu} = -\frac{1}{2}\frac{1}{\mu} + (1 + \pi_2)\frac{1}{2}\frac{1}{(1 - \phi)\mu}$$
$$\iff p_2 = \frac{\pi_2 + \phi}{2(1 - \phi)}$$

3. **How do the price ratios $\frac{p_1}{q}$ and $\frac{p_2}{q}$ vary with λ? With ϕ? Compare and interpret the results.**

$$\frac{p_1}{q} = \frac{\pi_1 - \phi - \phi pi_1}{2-\phi}$$

$$\frac{p_2}{q} = \frac{\pi_2 + \phi}{2-\phi}$$

We have:

$$\frac{\partial \frac{p_1}{q}}{\partial \lambda} = \frac{\partial \frac{p_2}{q}}{\partial \lambda} = 0$$

Because the idiosyncratic risk is not priced, as it does not affect the agents decisions.

$$\frac{\partial \frac{p_1}{q}}{\partial \phi} = \frac{-2-\pi_1}{(2-\phi)^2} < 0$$

$$\frac{\partial \frac{p_2}{q}}{\partial \phi} = \frac{2+\pi_2}{(2-\phi)^2} > 0$$

Part C: Asset pricing in incomplete market

1. **Write down the program.**

After replacing c through the associated budget constraints in Q3, the program of the agent in complete market writes:

$$\begin{aligned}\max_{b,a_1,a_2} \quad & ln(\mu - qb - p_1a_1 - p_2a_2) + \frac{1}{2}ln(\mu + b + (1+\pi_1)a_1 - a_2) \\ & +\frac{1}{2}\lambda ln((1-\frac{\phi}{\lambda})\mu + b - a_1 + (1+\pi_2)a_2) \\ & +\frac{1}{2}(1-\lambda)ln(\mu + b - a_1 + (1+\pi_2)a_2)\end{aligned}$$

2. **Derive the pricing equation of the three assets: risk free, risky 1 and risky 2.**

We take the FOC of the three assets / choice variables.

- Safe asset

$$\frac{\partial U}{\partial b} = 0 \iff q\frac{1}{\mu} = \frac{1}{2}\frac{1}{\mu} + \frac{1}{2}\lambda\frac{1}{(1-\frac{\phi}{\lambda})\mu} + \frac{1}{2}(1-\lambda)\frac{1}{\mu}$$

$$\iff q = \frac{2(\lambda-\phi)+\lambda\phi}{2(\lambda-\phi)}$$

- Risky asset 1

$$\frac{\partial U}{\partial a_1} = 0 \iff p_1\frac{1}{\mu} = \frac{1}{2}(1+\pi_1)\frac{1}{\mu} - \frac{1}{2}\lambda\frac{1}{(1-\frac{\phi}{\lambda})\mu} - \frac{1}{2}(1-\lambda)\frac{1}{\mu}$$

$$\iff p_1 = \frac{\pi_1\lambda - \pi_1\phi - \lambda\phi}{2(\lambda-\phi)}$$

- Risky asset 2

$$\frac{\partial U}{\partial a_2} = 0 \iff p_2\frac{1}{\mu} = -\frac{1}{2}\frac{1}{\mu} + (1+\pi_2)\frac{1}{2}\lambda\frac{1}{\mu} + (1+\pi_2)\frac{1}{2}(1-\lambda)\frac{1}{(1-\frac{\phi}{\lambda})\mu}$$

$$\iff p_2 = \frac{\lambda\phi + \lambda\pi_2 - \pi_2\phi + \lambda\phi\pi_2}{2(\lambda-\phi)}$$

3. **How do the price ratios $\frac{p_1}{q}$ and $\frac{p_2}{q}$ vary with λ? With ϕ? Compare and interpret the results.**

$$\frac{p_1}{q} = \frac{\pi_1(\lambda - \phi) - \lambda\phi}{2(\lambda - \phi) + \lambda\phi}$$

$$\frac{p_2}{q} = \frac{\lambda\phi + \lambda\phi\pi_2 + \lambda\pi_2 - \phi\pi_2}{2(\lambda - \phi) + \lambda\phi}$$

$$\frac{\partial \frac{p_1}{q}}{\partial \phi} = \frac{-(\pi_1 + \lambda)(2(\lambda - \phi) + \lambda\phi) - (\pi_1(\lambda - \phi) - \lambda\phi)(-2 + \lambda)}{(2(\lambda - \phi) + \lambda\phi)^2}$$

$$\iff \frac{\partial \frac{p_1}{q}}{\partial \phi} = \frac{\lambda^2(-2 - \pi_1)}{(2(\lambda - \phi) + \lambda\phi)^2} < 0$$

$$\frac{\partial \frac{p_1}{q}}{\partial \lambda} = \frac{(\pi_1 - \phi)(2(\lambda - \phi) + \lambda\phi) - (\pi_1(\lambda - \phi) - \lambda\phi)(2 + \phi)}{(2(\lambda - \phi) + \lambda\phi)^2}$$

$$\iff \frac{\partial \frac{p_1}{q}}{\partial \lambda} = \frac{\phi^2(2 + \pi_1)}{(2(\lambda - \phi) + \lambda\phi)^2} > 0$$

$$\frac{\partial \frac{p_2}{q}}{\partial \phi} = \frac{(\lambda + \lambda\pi_2 - \pi_2)(2(\lambda - \phi) + \lambda\phi) - (\lambda\phi + \lambda\phi\pi_2 + \lambda\pi_2 - \phi\pi_2)(-2 + \lambda)}{(2(\lambda - \phi) + \lambda\phi)^2}$$

$$\iff \frac{\partial \frac{p_2}{q}}{\partial \phi} = \frac{\lambda^2(2 + \pi_2)}{(2(\lambda - \phi) + \lambda\phi)^2} > 0$$

$$\frac{\partial \frac{p_2}{q}}{\partial \lambda} = \frac{(\phi + \phi\pi_2 + \pi_2)(2(\lambda - \phi) + \lambda\phi) - (\lambda\phi + \lambda\phi\pi_2 + \lambda\pi_2 - \phi\pi_2)(2 + \phi)}{(2(\lambda - \phi) + \lambda\phi)^2}$$

$$\iff \frac{\partial \frac{p_2}{q}}{\partial \lambda} = \frac{-\phi^2(2 + \pi_2)}{(2(\lambda - \phi) + \lambda\phi)^2} < 0$$

Exercise 2: Asset pricing with internal habits

Assume that agents have the period utility function:

$$u(c_t - hc_{t-1}, l_t), \text{ with } h < 1$$

where l_t is labor and c_t denotes consumption. They are assumed to have a discount factor β and they understand that consumption affects their habit. The budget constraint is (w is constant) :

$$c_t + a_{t+1} = (1 + r_t)a_t + wl_t$$

1. **Write the Bellman equation of this problem.**

   Recall: there are as many FOCs as choice variables and as many envelope conditions as state variables.

$$V(a, c_{-1}) = \max_{c,l,a'} u(c - hc_{-1}, l) + \beta \mathbb{E} V(a', c)$$

   subject to

$$c + a' = (1 + r)a + wl$$

   Thus

$$V(a, c_{-1}) = \max_{l,a'} u((1 + r)a + wl - a' - hc_{-1}, l) + \beta \mathbb{E} V(a', (1 + r)a + wl - a')$$

2. **Derive the pricing kernel.**

   FOCs:

$$\frac{\partial V(a,c_{-1})}{\partial a'} = 0 \iff \frac{\partial u(c-hc_{-1},l)}{\partial (c-hc_{-1})} = \beta\mathbb{E}\frac{\partial V(a',c)}{\partial a'} - \beta\mathbb{E}\frac{\partial V(a',c)}{\partial c}$$

$$\frac{\partial V(a,c_{-1})}{\partial l} = 0 \iff w\frac{\partial u(c-hc_{-1},l)}{\partial (c-hc_{-1})} + \frac{\partial u(c-hc_{-1},l)}{\partial l} = -\beta w\mathbb{E}\frac{\partial V(a',c)}{\partial c}$$

   Envelope conditions:

$$\frac{\partial V(a,c_{-1})}{\partial a} = (1+r)\frac{\partial u(c-hc_{-1},l)}{\partial (c-hc_{-1})} + \beta(1+r)\mathbb{E}\frac{\partial V(a',c)}{\partial c}$$

$$\frac{\partial V(a,c_{-1})}{\partial c_{-1}} = -h\frac{\partial u(c-hc_{-1},l)}{\partial (c-hc_{-1})}$$

   After iterating on the envelope conditions, we get:

$$\frac{\partial u(c-hc_{-1},l)}{\partial (c-hc_{-1})} = \beta(1+r')\mathbb{E}\frac{\partial u(c'-hc,l')}{\partial (c'-hc)} - \beta^2(1+r')h\mathbb{E}\frac{\partial u(c''-hc',l'')}{\partial (c''-hc')} + \beta h\mathbb{E}\frac{\partial u(c'-hc,l')}{\partial (c'-hc)}$$

$$\iff \frac{\partial u(c-hc_{-1},l)}{\partial (c-hc_{-1})} - \beta h\mathbb{E}\frac{\partial u(c'-hc,l')}{\partial (c'-hc)} = (1+r')\beta\left(\mathbb{E}\frac{\partial u(c'-hc,l')}{\partial (c'-hc)} - \beta h\mathbb{E}\frac{\partial u(c''-hc',l'')}{\partial (c''-hc')}\right)$$

   The pricing kernel satisfies (note: the pricing kernel is slightly different in the internal habit formation case, because of the double intertemporal role of the consumption level on the relative marginal utility between tomorrow and today):

$$M = \frac{1}{1+r'} = \beta\mathbb{E}\frac{\frac{\partial u(c'-hc,l')}{\partial (c'-hc)} - \beta h\frac{\partial u(c''-hc',l'')}{\partial (c''-hc')}}{\frac{\partial u(c-hc_{-1},l)}{\partial (c-hc_{-1})} - \beta h\frac{\partial u(c'-hc,l')}{\partial (c'-hc)}}$$

3. **What is the steady state interest rate?**

   In steady state:

$$\beta(1+r) = 1$$

   Thus, the steady-state interest rate is:

$$1+r = \frac{1}{\beta}$$

   Same as before! Habits change only the dynamics.

4. **Assume that $a_t = 0$ and that**

$$u(c,l) = \frac{c^{1-\sigma}-1}{1-\sigma} - \frac{l^{1+\varepsilon}}{1+\varepsilon}$$

   How does the steady-state labor supply change when agents care more about habits?

   The first-order condition (FOC) with respect to labor l_t is:

$$(1-\beta h)(1-h)^{-\sigma}C^{-\sigma} = wl^{\varepsilon}$$

The budget constraint, using $A_t = A_{t+1} = 0$, becomes:

$$c = wl$$

Therefore:

$$l^{\varepsilon+\sigma} = \left(\frac{(1-\beta h)}{(1-h)^{\sigma}}\right) w^{1-\sigma}$$

Taking the log and the derivative with respect to h, we get the sign S of this derivative:

$$S = \frac{\sigma}{1-h} - \frac{\beta}{1-\beta h}$$

When h is close to 0, the sign of this derivative is the sign of $\sigma - \beta$.