Waveform Optimization with Quantum Circuits

Application to RADAR high resolution imagery

Les @tomes de Savoie

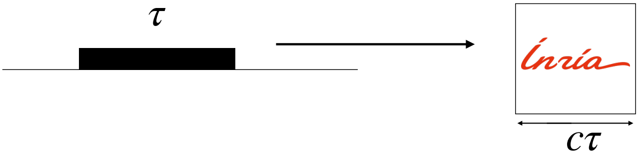
October, 03rd 2021

QuantX

THALES

Cambridge Quantum Computing

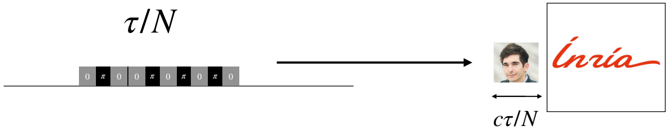
Send electromagnetic pulse to measure distance of object with echo delay



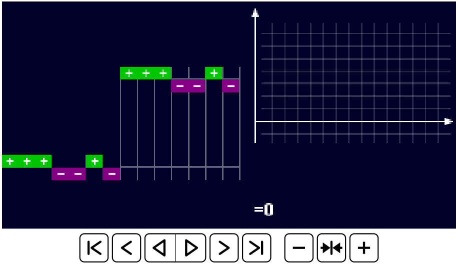
Civil and military applications: Sonar, Lidar, autonomous vehicles, etc.

Radar with phase modulation

To increase precision on distance, modulate the phase of the signal



In practice, measure the autocorrelation function of the signal and identify the peak



Our goal: optimize the height and the width of the central peak to increase accuracy.

Our signal is encoded by the values of the phase $(x_1, ..., x_N)$.

Define the auto-correlation function for a time delay k

$$\rho(k) = \sum_{i=1}^{N-k} x_i x_{i+k}. \tag{1}$$

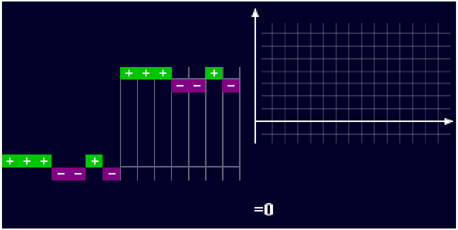
Minimize the integrated side lobes (our cost-function)

$$\mathcal{H}(x) = \sum_{k=1}^{N-1} \rho^2(k). \tag{2}$$

Naive solution: exhaustive search $\rightarrow 2^N$ evaluations of the cost function.

Classical algorithms: genetic algorithm, algorithm based on Fast-Fourier Transforms, etc.

A few optimal solutions known for $N \leq 13$. For example $N = 7$



Express the cost-function as the expectation value of a quantum Hamiltonian

$$\mathcal{H}(x) = \langle x| \sum_{k=1}^{N-1} \left(\sum_{i=1}^{N-k} \hat{\sigma}_z^i \hat{\sigma}_z^{i+k} \right)^2 |x\rangle = \langle x|\hat{H}|x\rangle \tag{3}$$

with the states $|x\rangle \equiv |x_1 \dots x_N\rangle$, and single-particle operators

$$\hat{\sigma}_z^i |x_1 \dots x_N\rangle = x_i |x_1 \dots x_N\rangle. \tag{4}$$

Hamiltonian of a spin-chain with 4-spin interaction terms, which is diagonal in the computational basis.

The initial classical problem boils down to the determination of the fundamental state of the spin chain.

Quantum Approximate Optimization Algorithm (QAOA)

Recent method designed to find *the ground state energy of a Hamiltonian* $\hat{H}$, inspired from the quantum annealing protocol

- Initialization in a superposition initial state $|\psi_0\rangle$

Recent method designed to find *the ground state energy of a Hamiltonian $\hat{H}$*, inspired from the quantum annealing protocol

- Initialization in a superposition initial state $|\psi_0\rangle$
- Trotterized evolution according to $\hat{H}$ with angles $\vec{\gamma}$ and mixing steps according to $\hat{H}_M$ with angles $\vec{\beta}$

$$\left|\psi(\vec{\beta},\vec{\gamma})\right\rangle = e^{-i\gamma_p \hat{H}} e^{-i\beta_p \hat{H}_M} \cdots e^{-i\gamma_1 \hat{H}} e^{-i\beta_1 \hat{H}_M} |\psi_0\rangle \tag{5}$$

with mixing Hamiltonian

$$\hat{H}_m = \sum_{i=1}^{N} \sigma_x^i. \tag{6}$$

Recent method designed to find *the ground state energy of a Hamiltonian $\hat{H}$*, inspired from the quantum annealing protocol

- Initialization in a superposition initial state $|\psi_0\rangle$
- Trotterized evolution according to $\hat{H}$ with angles $\vec{\gamma}$ and mixing steps according to $\hat{H}_M$ with angles $\vec{\beta}$

$$\left|\psi(\vec{\beta},\vec{\gamma})\right\rangle = e^{-i\gamma_p \hat{H}} e^{-i\beta_p \hat{H}_M} \cdots e^{-i\gamma_1 \hat{H}} e^{-i\beta_1 \hat{H}_M} |\psi_0\rangle \tag{5}$$

  with mixing Hamiltonian

$$\hat{H}_m = \sum_{i=1}^{N} \sigma_x^i. \tag{6}$$

- Quantum measurement of the energy

$$\left\langle \psi(\vec{\beta},\vec{\gamma}) \right| \hat{H} \left| \psi(\vec{\beta},\vec{\gamma}) \right\rangle \tag{7}$$

Recent method designed to find *the ground state energy of a Hamiltonian $\hat{H}$*, inspired from the quantum annealing protocol

- Initialization in a superposition initial state $|\psi_0\rangle$
- Trotterized evolution according to $\hat{H}$ with angles $\vec{\gamma}$ and mixing steps according to $\hat{H}_M$ with angles $\vec{\beta}$

$$\left|\psi(\vec{\beta},\vec{\gamma})\right\rangle = e^{-i\gamma_p \hat{H}} e^{-i\beta_p \hat{H}_M} \dots e^{-i\gamma_1 \hat{H}} e^{-i\beta_1 \hat{H}_M} |\psi_0\rangle \tag{5}$$

with mixing Hamiltonian

$$\hat{H}_m = \sum_{i=1}^{N} \sigma_x^i. \tag{6}$$

- Quantum measurement of the energy

$$\left\langle \psi(\vec{\beta},\vec{\gamma}) \right| \hat{H} \left| \psi(\vec{\beta},\vec{\gamma}) \right\rangle \tag{7}$$

- Classical optimization on $\vec{\beta}$ and $\vec{\gamma}$:
Iteration of the procedure in order to find the minimal energy.

With Qiskit:

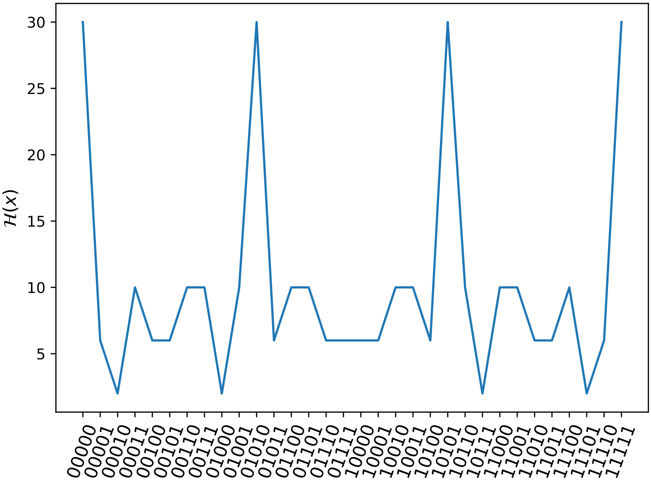
- Which gates to implement $e^{-i\gamma_p \hat{H}}$ and $e^{-i\beta_p \hat{H}_M}$?
- This reduces to three gates encoding $e^{-i\beta_p X}$, $e^{-i\gamma_p Z \otimes Z}$, and $e^{-i\gamma_p Z \otimes Z \otimes Z \otimes Z}$: R_X, R_{ZZ}, R_{ZZZZ}

$$R_{ZZ}(\theta) = exp(-i\frac{\theta}{2} Z \otimes Z) = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 & 0 & 0 \\ 0 & e^{i\frac{\theta}{2}} & 0 & 0 \\ 0 & 0 & e^{i\frac{\theta}{2}} & 0 \\ 0 & 0 & 0 & e^{-i\frac{\theta}{2}} \end{pmatrix} \tag{8}$$

With t|ket⟩:

- Simplification of the circuit especially for $e^{-i\gamma_p Z \otimes Z \otimes Z \otimes Z}$
- Use of symbolic parameters for the optimization

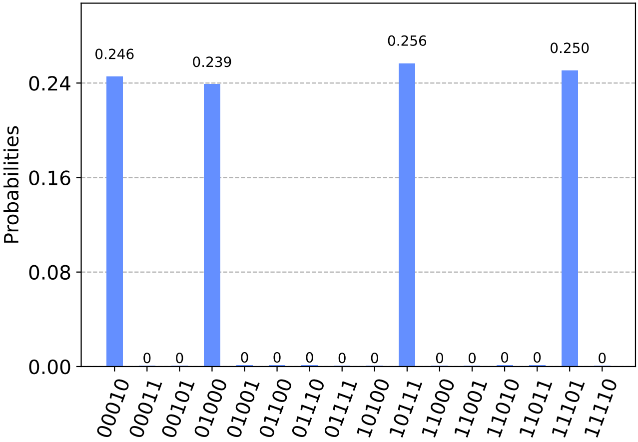
Results: simple case $N = 5$



We want to retrieve with our algorithm the x values minimizing $\mathcal{H}(x)$: 00010, 01000, 10111, 11101

Results: simple case $N = 5$

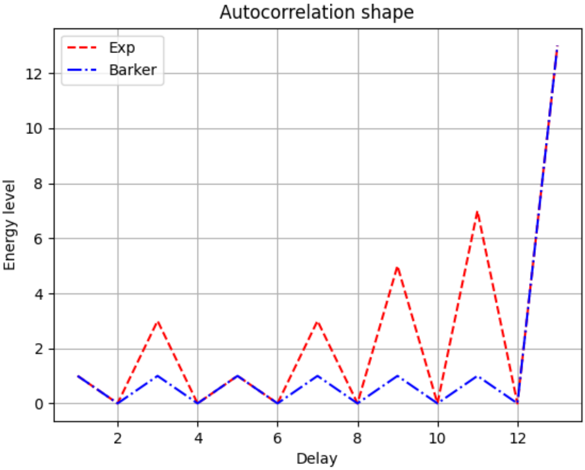
With $n_{\text{shots}} = 2048$ and $\dim\{\gamma_{\text{p}}, \beta_{\text{p}}\} = 40$



$\rightarrow$ Optimal configurations are retrieved!

Results: larger N values

For larger N, we cannot find the optimal configurations. Still, the results remain reasonable.



Results: the mismatched case

For x and y two sequences of length N:

$$\rho(k) = \sum_{i=1}^{N-k} x_i y_{i+k} \quad \text{for } k \geq 0 \tag{9}$$

$$= \sum_{i=1}^{N+k} y_i x_{i-k} \quad \text{for } k < 0 \tag{10}$$

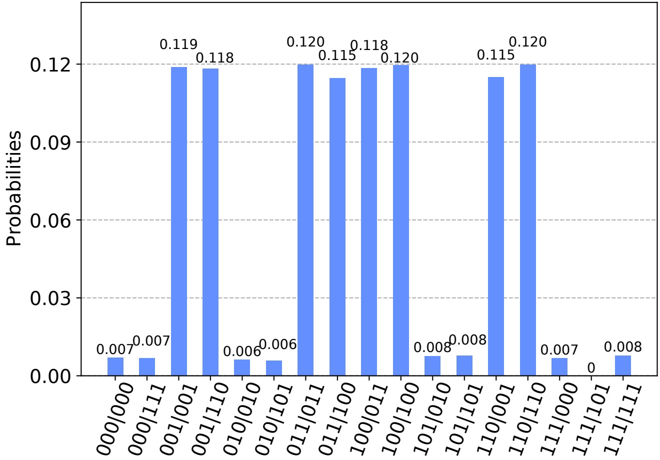
New cost function for mismatched case

$$\mathcal{H}(x, y) = \sum_{k=-N+1, k\neq 0}^{N-1} \rho^2(k) - \alpha\rho^2(0) \tag{11}$$

We want to minimize $\mathcal{H}(x, y) \rightarrow$ Double the number of qubits!

Results: the mismatched case

For $N = 3$, we retrieve the Barker codes.



- Multiphase case:

$$e^{2\pi iq/Q}$$

  with $Q > 2$.

- Other resolution methods: quantum annealing with D-Wave's simulator (need to be reformulated as a QUBO).

- Other formulations? Implementation of a Grover algorithm with an oracle built on the cost function.