Linear Algebra Exercices

Lionel Cheng

2021

1 Reduction

1.1 Exercice 1

Solve in $\mathcal{M}_3(\mathbb{C})$ the following equation in X:

$$X^2 = A = \begin{pmatrix} 1 & 3 & -7 \\ 2 & 6 & -14 \\ 1 & 3 & -7 \end{pmatrix} \tag{1}$$

1.1.1 Solution

Computing first the characteristic polynomial of A

$$\chi(A) = \begin{vmatrix} 1-x & 3 & -7 \\ 2 & 6-x & -14 \\ 1 & 3 & -7-x \end{vmatrix} = \begin{vmatrix} -x & 0 & x \\ 2 & 6-x & -14 \\ 1 & 3 & -7-x \end{vmatrix} \tag{2}$$

$$= x \begin{vmatrix} -1 & 0 & 1 \\ 2 & 6-x & -14 \\ 1 & 3 & -7-x \end{vmatrix} = x \begin{vmatrix} -1 & 0 & 1 \\ 2 & 6-x & -14 \\ 1 & 3 & -7-x \end{vmatrix} = \tag{3}$$

$$= x \left[- \begin{vmatrix} 6-x & -14 \\ 3 & -7-x \end{vmatrix} + x \right] = -x^3 \tag{4}$$

Hence $S_p(A) = 0$. The matrice needs to be trigonalized. It is obvious that:

$$\dim(\ker(A)) = 2 \tag{5}$$

Moreover $A^2 = 0$ and hence $\mathrm{Im}(A) \subset \mathrm{Ker}(A)$. Let us first take a vector e_3 that is not in $\mathrm{Ker}(A)$ and $e_2 = Ae_3$. We complete e_2 with e_1 such that (e_1, e_2) is a basis of $\mathrm{Ker}(A)$. That way:

$$A = P \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} P^{-1} \tag{6}$$

Let us find a suitable form for P:

$$P = \begin{pmatrix} 3 & -7 & 0 \\ -1 & -14 & 0 \\ 0 & -7 & 1 \end{pmatrix} \tag{7}$$

and we have an equation for $\ker(A)$:

$$x + 3y - 7z = 0 \tag{8}$$

X now needs to be simplified. We search for the decomposition of X in the basis (e_1, e_2, e_3):

$$X' = P^{-1} X P \tag{9}$$

X and A commute, so that $\mathrm{Im}(A)$ and $\mathrm{Ker}(A)$ are stable by X:

$$X' = \begin{pmatrix} a & 0 & d \\ b & c & e \\ 0 & 0 & f \end{pmatrix} \tag{10}$$

And since X is also nilpotent, $\mathrm{Sp}(X) = \{0\}$ and $a = c = f = 0$. Finally:

$$X' = \begin{pmatrix} 0 & 0 & b \\ a & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \tag{11}$$

Solving the equation:

$$X'^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & ab \\ 0 & 0 & 0 \end{pmatrix} \tag{12}$$

Comparing with the reduced form of A:

$$ab = 1 \implies X' = \begin{pmatrix} 0 & 0 & 1/a \\ a & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \tag{13}$$