Form and Structure Factors: Modeling and Interactions

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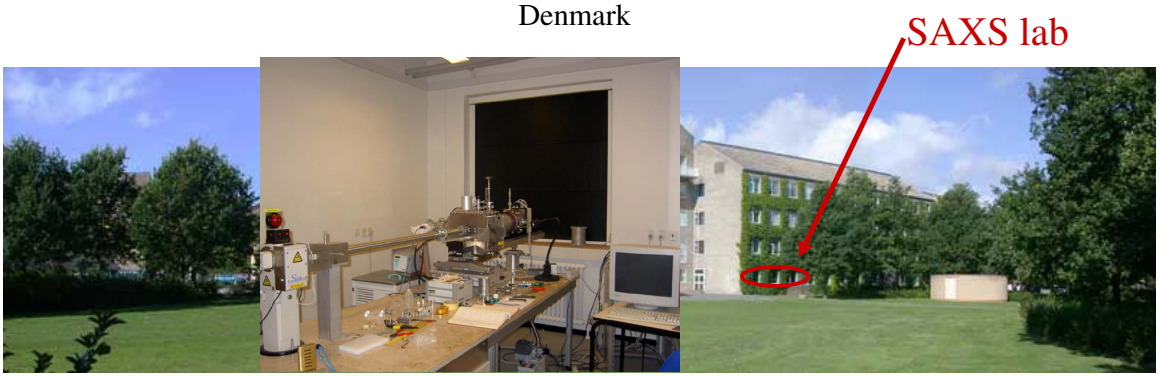


Outline

- Model fitting and least-squares methods
- Available form factors
  ex: sphere, ellipsoid, cylinder, spherical subunits…
  ex: polymer chain
- Monte Carlo integration for
  form factors of complex structures
- Monte Carlo simulations for
  form factors of polymer models
- Concentration effects and structure factors
  Zimm approach
  Spherical particles
  Elongated particles (approximations)
  Polymers

Motivation

- ***not*** to replace shape reconstruction and crystal-structure based modeling – we use the methods extensively

- alternative approaches to reduce the number of degrees of freedom in SAS data structural analysis
  (might make you aware of the limited information content of your data !!!)

- provide polymer-theory based modeling of flexible chains

- describe and correct for concentration effects

Literature

Jan Skov Pedersen, ***Analysis of Small-Angle Scattering Data from Colloids and Polymer Solutions: Modeling and Least-squares Fitting***
(1997). *Adv. Colloid Interface Sci.*, **70**, 171-210.

Jan Skov Pedersen
Monte Carlo Simulation Techniques Applied in the Analysis of Small-Angle Scattering Data from Colloids and Polymer Systems
in *Neutrons, X-Rays and Light*
P. Lindner and Th. Zemb (Editors) 2002 Elsevier Science B.V.
p. 381

Jan Skov Pedersen
Modelling of Small-Angle Scattering Data from Colloids and Polymer Systems
in *Neutrons, X-Rays and Light*
P. Lindner and Th. Zemb (Editors) 2002 Elsevier Science B.V.
p. 391

Rudolf Klein
Interacting Colloidal Suspensions
in *Neutrons, X-Rays and Light*
P. Lindner and Th. Zemb (Editors) 2002 Elsevier Science B.V.
p. 351

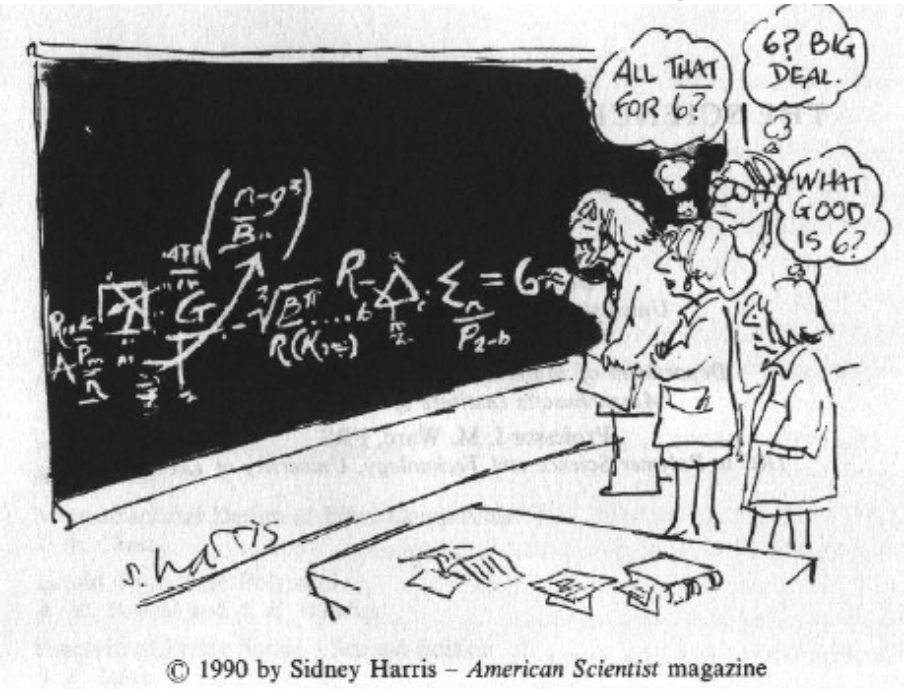
Form factors and structure factors

Warning 1:
Scattering theory – lots of equations!
= mathematics, Fourier transformations

Warning 2:
Structure factors:
Particle interactions = statistical mechanics

Not all details given
- but hope to give you an impression!

I will outline some calculations to show that it is not black magic !



Input data: Azimuthally averaged data

$$q_i, I(q_i), \sigma[I(q_i)] \quad i = 1,2,3,\ldots N$$

q_i calibrated

$I(q_i)$ calibrated, i.e. on absolute scale
- noisy, (smeared), truncated

$\sigma[I(q_i)]$ Statistical standard errors: Calculated from counting statistics by error propagation
- do not contain information on systematic error !!!!

Least-squared methods

Measured data: $I^{\exp}(q_i)$, $i = 1,\ldots,N$

Model: $I^{\mathrm{mod}}(q_i)$ a_i, $i = 1,\ldots,M$.

Chi-square: $$\chi^2 \equiv \sum_{i=1}^{N} \left(\frac{I^{\exp}(q_i) - I^{\mathrm{mod}}(q_i)}{\sigma_i} \right)^2$$

Reduced Chi-squared: $\chi_r^2 = \dfrac{\chi^2}{N - M}$ = goodness of fit (GoF)

Note that $\chi_r^2 = 1$ for $N >> M$ corresponds to $|I^{\exp}(q_i) - I^{\mathrm{mod}}(q_i)| = \sigma_i$
i.e. statistical agreement between model and data

Cross section

$d\sigma(q)/d\Omega$: number of scattered neutrons or photons per unit time, relative to the incident flux of neutron or photons, per unit solid angle at q per unit volume of the sample.

For system of monodisperse particles

$$\frac{d\sigma(q)}{d\Omega} = I(q) = n\,\Delta\rho^2 V^2 P(q) S(q) = c\,M\,\Delta\rho_m^2 P(q) S(q)$$

n is the number density of particles,
$\Delta\rho$ is the excess scattering length density, given by electron density differences
V is the volume of the particles,
$P(q)$ is the particle *form factor*, $P(q=0)=1$
$S(q)$ is the particle *structure factor*, $S(q=\infty)=1$

- $V \propto M$
- $n = c/M$
- $\Delta\rho$ can be calculated from partial specific density, composition

Form factors of geometrical objects

Form factors I

Homogenous rigid particles

1. Homogeneous sphere
2. Spherical shell:
3. Spherical concentric shells:
4. Particles consisting of spherical subunits:
5. Ellipsoid of revolution:
6. Tri-axial ellipsoid:
7. Cube and rectangular parallelepipedons:
8. Truncated octahedra:
9. Faceted Sphere:

9x Lens

10. Cube with terraces:
11. Cylinder:
12. Cylinder with elliptical cross section:
13. Cylinder with hemi-spherical end-caps:

13x Cylinder with 'half lens' end caps

14. Toroid:
15. Infinitely thin rod:
16. Infinitely thin circular disk:
17. Fractal aggregates:

Form factors II

'Polymer models'

18. Flexible polymers with Gaussian statistics:
19. Polydisperse flexible polymers with Gaussian statistics:
20. Flexible ring polymers with Gaussian statistics:
21. Flexible self-avoiding polymers:
22. Polydisperse flexible self-avoiding polymers:
23. Semi-flexible polymers without self-avoidance:
24. Semi-flexible polymers with self-avoidance:

24x Polyelectrolyte Semi-flexible polymers with self-avoidance:

25. Star polymer with Gaussian statistics:
26. Polydisperse star polymer with Gaussian statistics:
27. Regular star-burst polymer (dendrimer) with Gaussian statistics:
28. Polycondensates of A_f monomers:
29. Polycondensates of AB_f monomers:
30. Polycondensates of ABC monomers:
31. Regular comb polymer with Gaussian statistics:
32. Arbitrarily branched polymers with Gaussian statistics:
33. Arbitrarily branched semi-flexible polymers:
34. Arbitrarily branched self-avoiding polymers: (Block copolymer micelle)
35. Sphere with Gaussian chains attached:
36. Ellipsoid with Gaussian chains attached:
37. Cylinder with Gaussian chains attached:
38. Polydisperse thin cylinder with polydisperse Gaussian chains attached to the ends:
39. Sphere with corona of semi-flexible interacting self-avoiding chains of a corona chain.

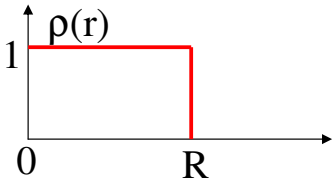
Form factors III

$P(q) = P_{cross\text{-}section}(q)\, P_{large}(q)$

40. Very anisotropic particles with local planar geometry:
Cross section:
(a) Homogeneous cross section
(b) Two infinitely thin planes
(c) A layered centro symmetric cross-section
(d) Gaussian chains attached to the surface
Overall shape:
(a) Infinitely thin spherical shell
(b) Elliptical shell
(c) Cylindrical shell
(d) Infinitely thin disk

41. Very anisotropic particles with local cylindrical geometry:
Cross section:
(a) Homogeneous circular cross-section
(b) Concentric circular shells
(c) Elliptical Homogeneous cross section.
(d) Elliptical concentric shells
(e) Gaussian chains attached to the surface
Overall shape:
(a) Infinitely thin rod
(b) Semi-flexible polymer chain with or without excluded volume

From factor of a solid sphere



$$A(q) = 4\pi \int_0^\infty \rho(r) \frac{\sin(qr)}{qr} r^2 dr = 4\pi \int_0^R \frac{\sin(qr)}{qr} r^2 dr$$

r

$$= \frac{4\pi}{q} \int_0^R \sin(qr)\, r dr =$$

$$\left(\int f' g\, dx = [fg] - \int fg'\, dx \right)$$

(partial integration)…

$$= \frac{4\pi}{q} \left(-\frac{R\cos qR}{q} + \left[\frac{\sin qr}{q} \right]_0^R \right)$$

$$= \frac{4\pi}{q} \left(-\frac{R\cos qR}{q} + \frac{\sin qr}{q^2} \right)$$

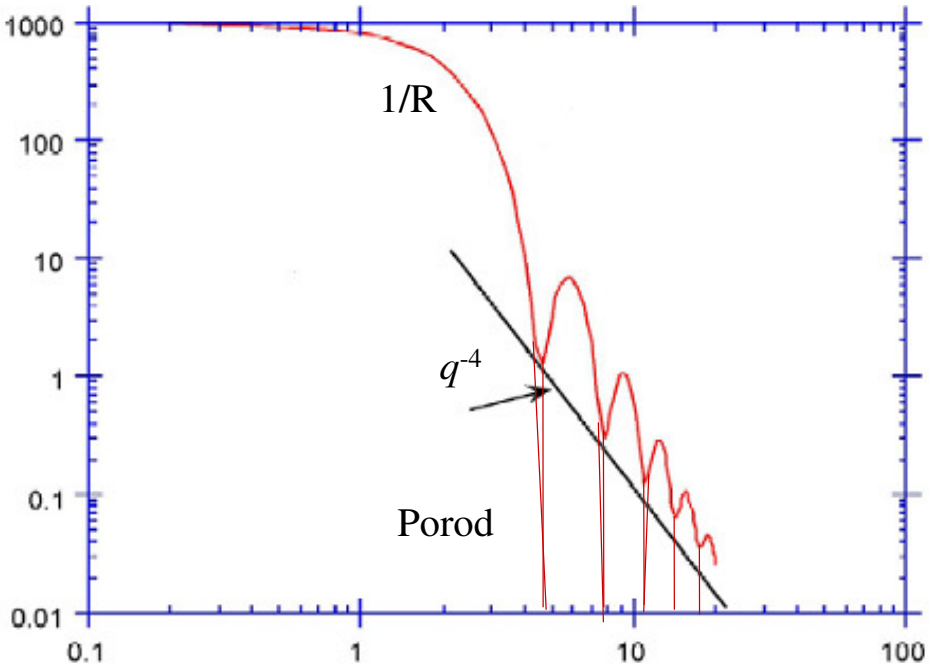
$$= \frac{4\pi}{q^3} (\sin qR - qR\cos qR)$$

$$= \frac{4}{3}\pi R^3 \frac{3[\sin(qR) - qR\cos(qR)]}{(qR)^3}$$

spherical Bessel function

Form factor of sphere

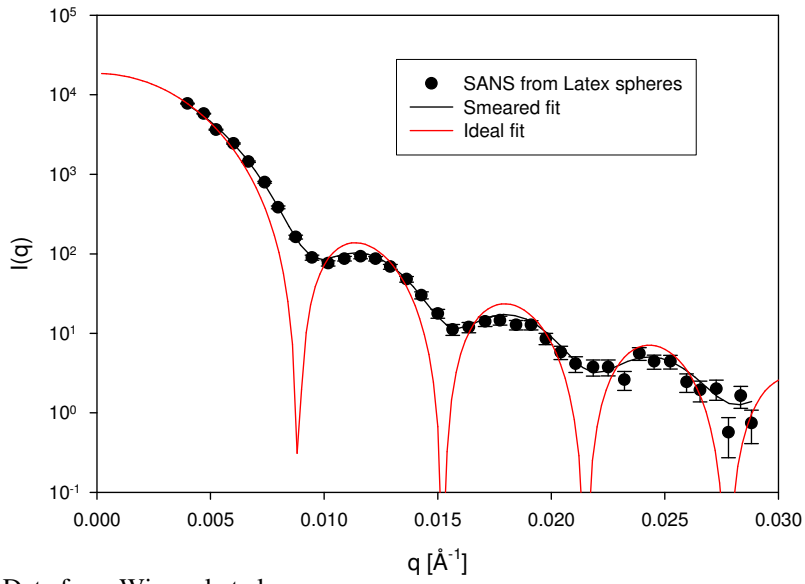
$P(q) = A(q)^2/V^2$



C. Glinka

Measured data from solid sphere (SANS)

$$I(\langle q \rangle) = \int R(\langle q \rangle, q) \frac{\mathrm{d}\sigma(q)}{\mathrm{d}\Omega}\, \mathrm{d}q \qquad \frac{d\sigma}{d\Omega}(q) = \Delta\rho^2 V^2 \left[\frac{3[\sin(qR) - qR\cos(qR)]}{(qR)^3}\right]^2$$



Instrumental smearing is routinely included in SANS data analysis

Data from Wiggnal et al.

Ellipsoid

5. Ellipsoid of revolution: This expression was determined by Guinier (1939). The averaging over orientations has to be done numerically. For the semi-axes $R, R, \epsilon R$:

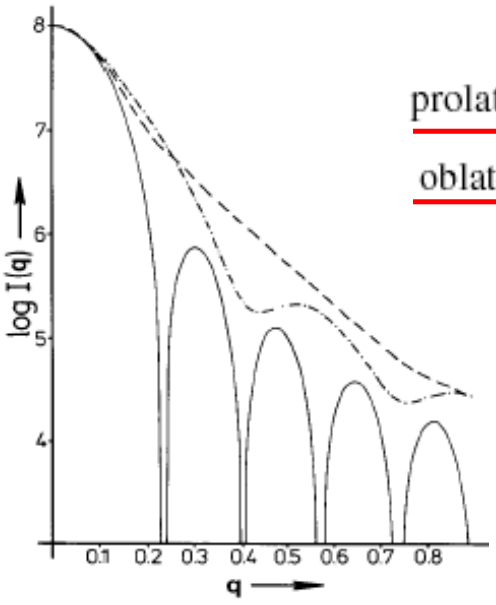
$$P_5(q,R,\epsilon) = \int_0^{\pi/2} F_1\left[q, r(R,\epsilon,\alpha)\right]^2 \sin\alpha \, d\alpha, \qquad (27)$$

where $r(R,\epsilon,\alpha) = R(\sin^2\alpha + \epsilon^2\cos^2\alpha)^{1/2}$.

And:

$$F_1(q,R) = \frac{3[\sin(qR) - qR\cos(qR)]}{(qR)^3}.$$

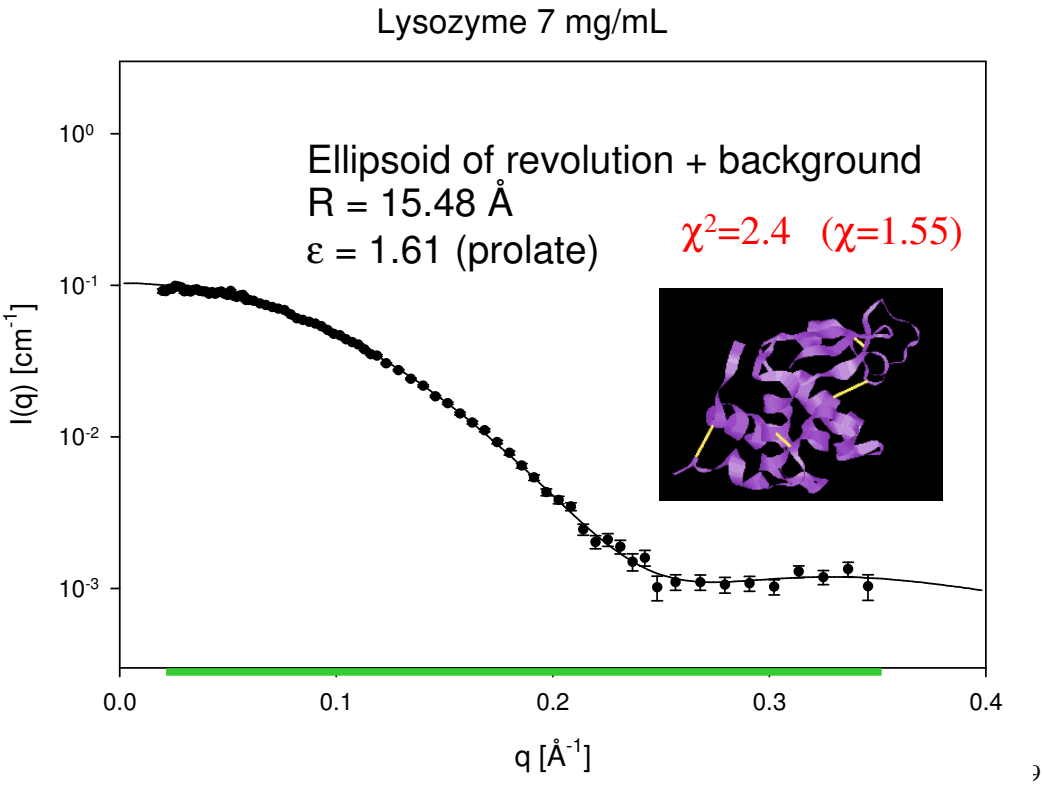
P(q): Ellipsoid of revolution



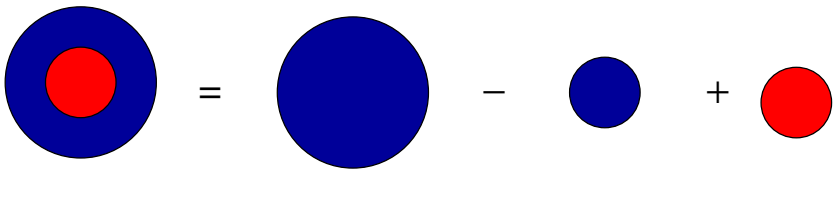
prolate ellipsoid of revolution 1 : 1 : 3 (– · – · –)

oblate ellipsoid of revolution 1 : 1 : 0.2 (– – – –)

Glatter



Core-shell particles:



$$A_{core-shell}(q) = \Delta\rho_{core}\left[\Delta\rho_{shell}V_{out}\Phi(qR_{out}) - (\Delta\rho_{shell} - \Delta\rho_{core})V_{in}\Phi(qR_{in})\right]$$

where

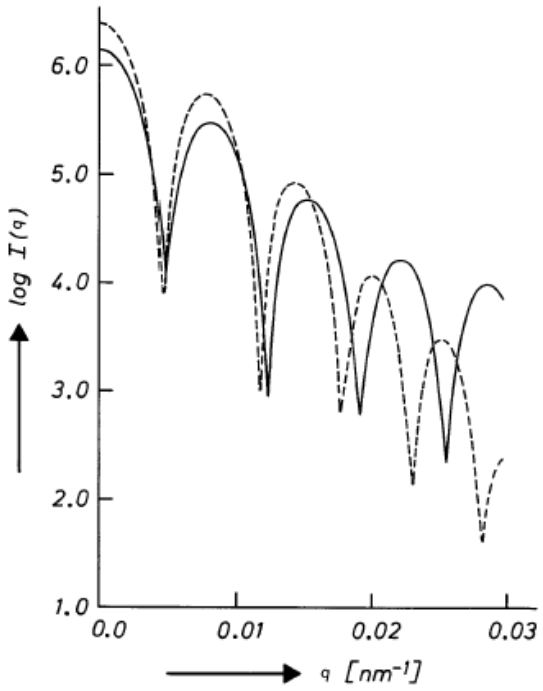
$V_{out} = 4\pi R_{out}^3/3$ and $V_{in} = 4\pi R_{in}^3/3$.

$\Delta\rho_{core}$ is the excess scattering length density of the core,

$\Delta\rho_{shell}$ is the excess scattering length density of the shell and:

$$\Phi(x) = \frac{3[\sin x - x\cos x]}{x^3}$$

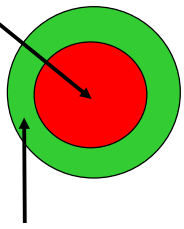
P(q): Core-shell



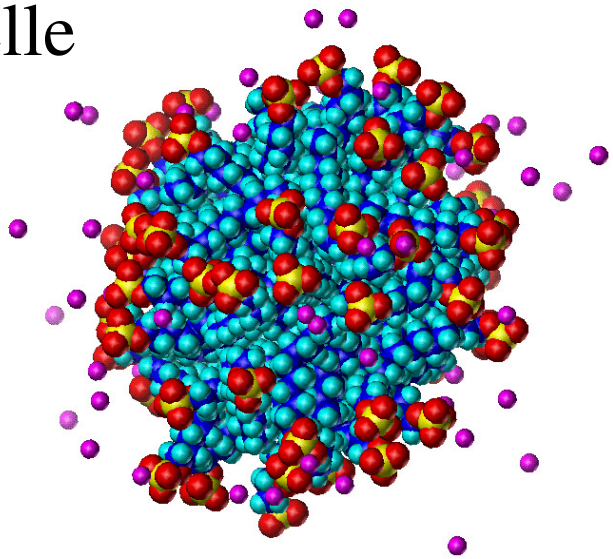
Glatter

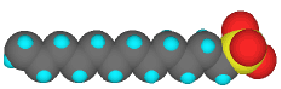
SDS micelle

Hydrocarbon core



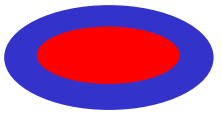
Headgroup/counterions





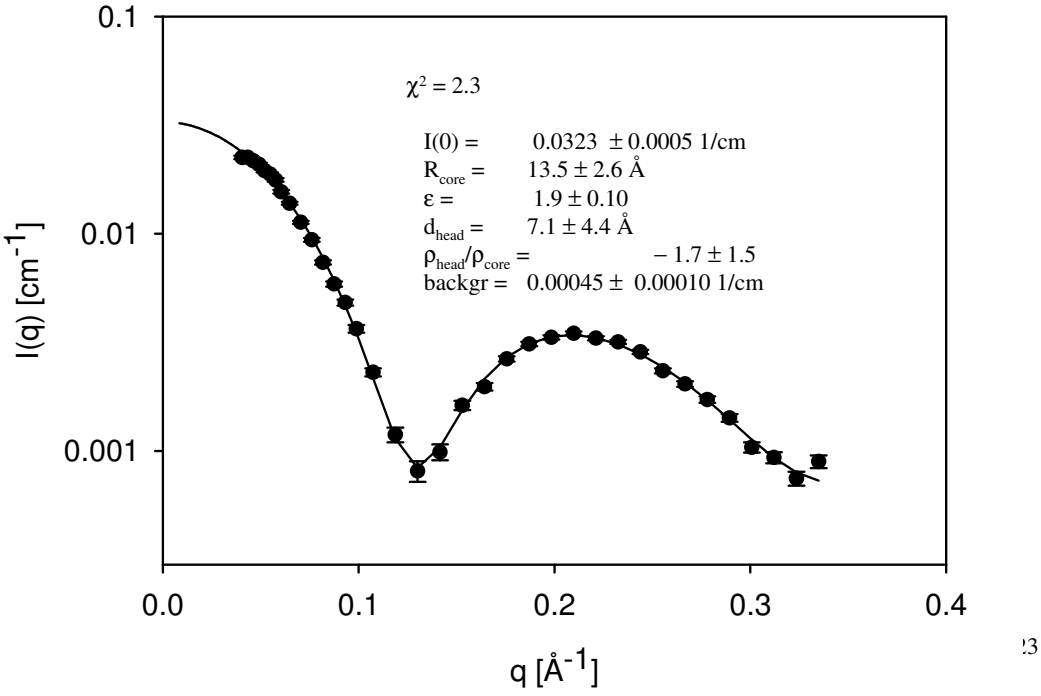
20 Å

www.psc.edu/.../charmm/tutorial/ mackerell/membrane.html
mrsec.wisc.edu/edetc/cineplex/ micelle.html



SDS micelles:

prolate ellispoid with shell of constant thickness



23

Molecular constraints:

$$\Delta\rho^{e}_{tail} = \frac{Z^{e}_{tail}}{V_{tail}} - \frac{Z^{e}_{H2O}}{V_{H2O}}$$

$$V_{core} = N_{agg} V_{tail}$$

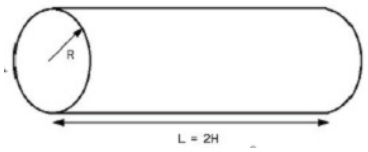
$$\Delta\rho^{e}_{head} = \frac{Z^{e}_{head} + nZ_{H2O}}{V_{head} + nV_{H2O}} - \frac{Z^{e}_{H2O}}{V_{H2O}}$$

n water molecules in headgroup shell

$$V_{shell} = N_{agg}(V_{head} + nV_{H2O})$$

$$n_{micelles} = \frac{c}{N_{agg} M_{\text{surfactant}}}$$

Cylinder



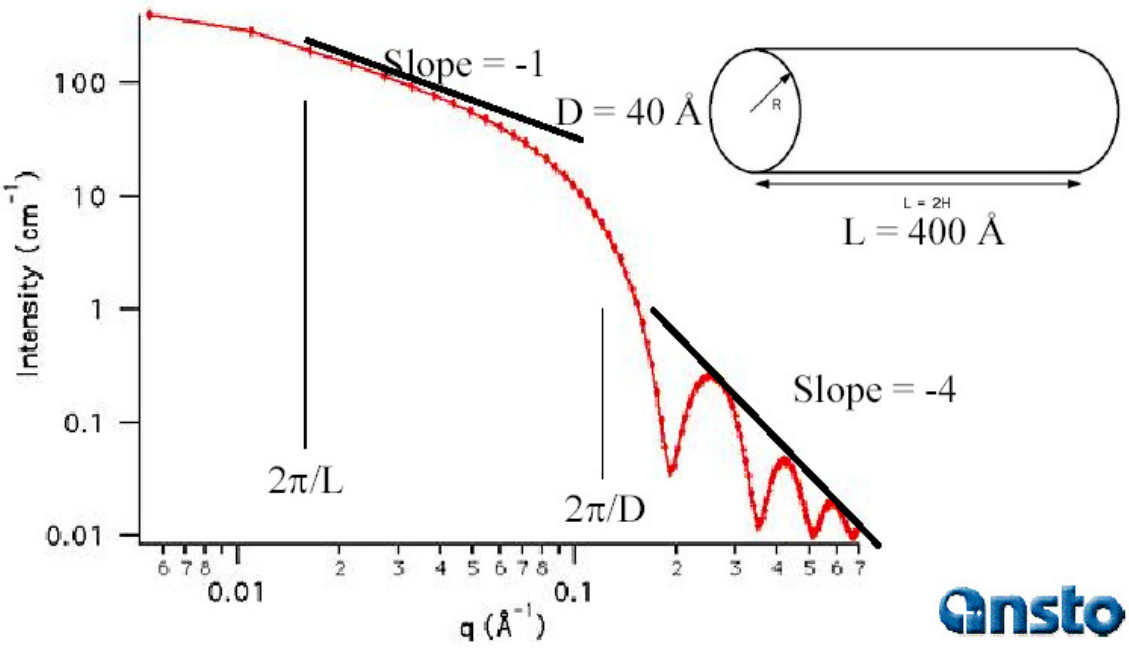
11. Cylinder: The expression for a cylinder with radius R and length L was given by Fournet (1949):

$$P_{11}(q) = \int_0^{\pi/2} \left[\frac{2J_1(qR\sin\alpha)}{qR\sin\alpha} \frac{\sin(qL\cos\alpha/2)}{qL\cos\alpha/2} \right]^2 \sin\alpha \, d\alpha, \qquad (30)$$

where $J_1(x)$ is the first order Bessel function.

P(q) cylinder

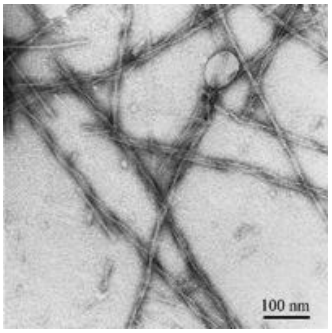
Dilute randomly oriented monodisperse rods

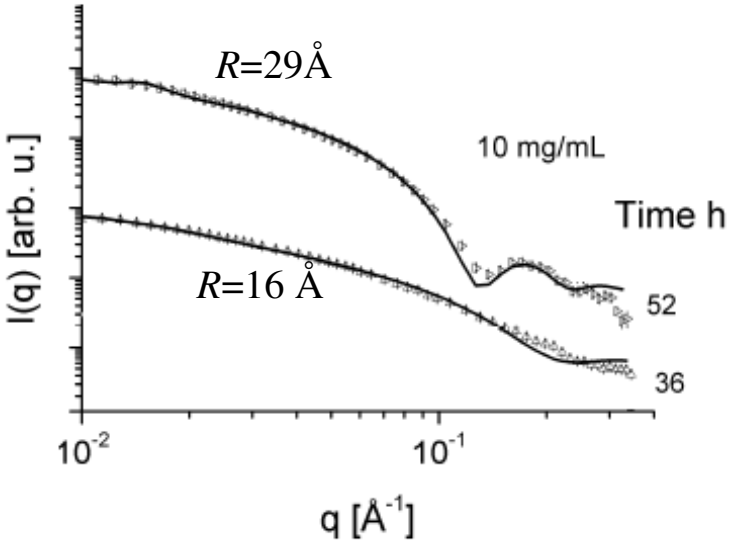


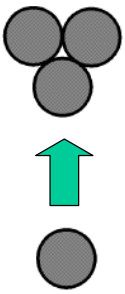
ansto

E. Gilbert

Glucagon Fibrils

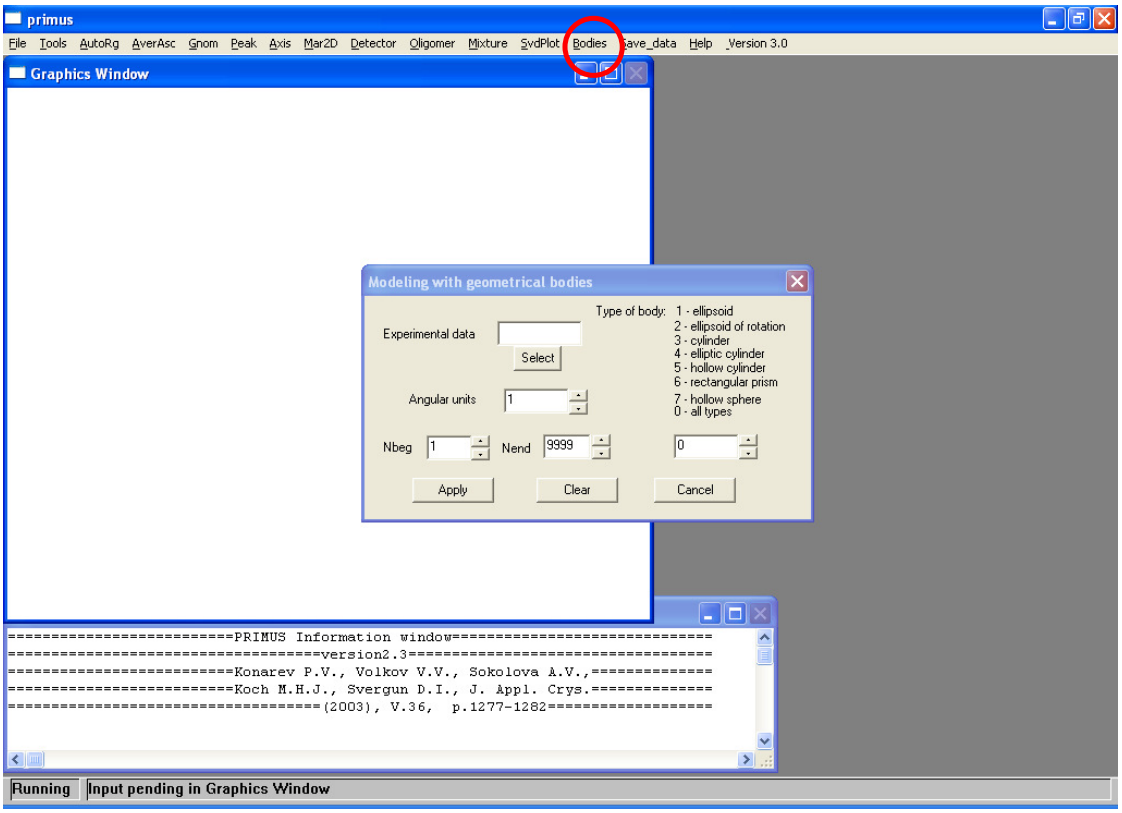




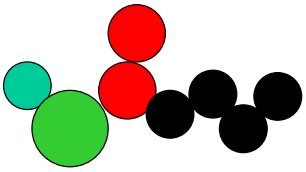


Cristiano Luis Pinto Oliveira, Manja A. Behrens, Jesper Søndergaard Pedersen, Kurt Erlacher, Daniel Otzen and Jan Skov Pedersen J. Mol. Biol. (2009) 387, 147–161

Primus



Collection of particles with spherical symmetry



center of sphere

$\vec{r} = r' + \vec{s}$

$$A(\vec{q}) = \sum b_j \exp(-i\vec{q}\cdot\vec{r}_j) = \sum b_{ij} \exp(-i\vec{q}\cdot(\vec{r}_i' + \vec{s}_j))$$

$$I(q) = \sum b_{ij} b_{kl} \exp(-i\vec{q}\cdot(\vec{r}_i' + \vec{s}_j - \vec{r}_k' - \vec{s}_l))$$

$$= \sum b_{ij} \exp(-i\vec{q}\cdot\vec{r}_i')\, b_{kl} \exp(i\vec{q}\cdot\vec{r}_k') \exp(-i\vec{q}\cdot(\vec{s}_j' - \vec{s}_k'))$$

$$= \sum V_j \Delta\rho_j \Phi(qR_j) V_k \Delta\rho_k \Phi(qR_k) \frac{\sin qd_{jk}}{qd_{jk}}$$

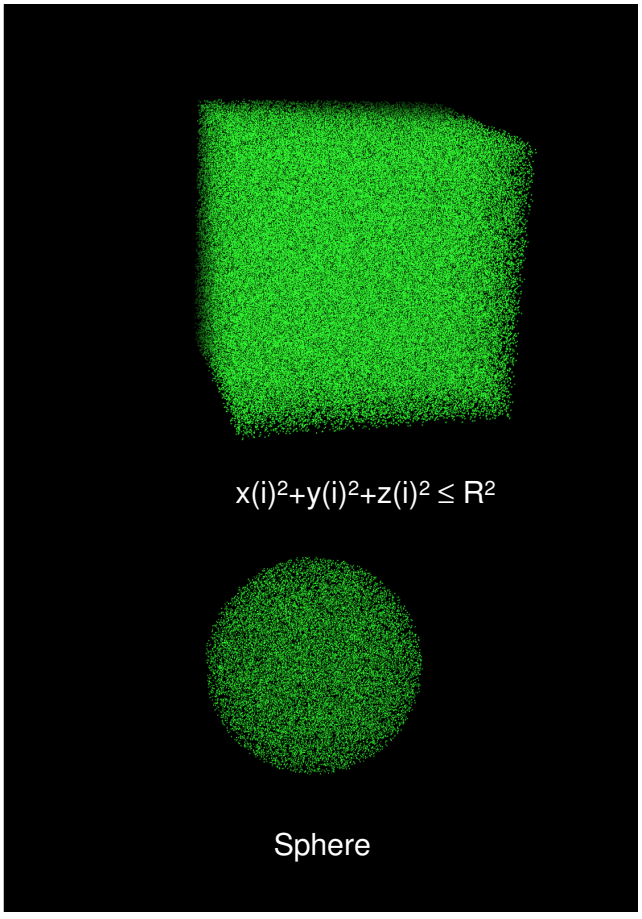
$$= \sum V_j^2 \Delta\rho_j^2 P(qR_j) + \sum_{j\neq k} V_j \Delta\rho_j \Phi(qR_j) V_k \Delta\rho_k \Phi(qR_k) \frac{\sin qd_{jk}}{qd_{jk}}$$

self-term interference term phase factor

Debye, 1915

Monte Carlo integration in calculation of form factors for complex structures

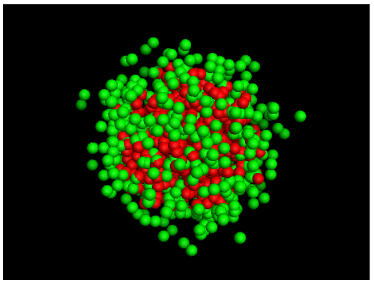
- Generate points in space by Monte Carlo simulations
- Select subsets by geometric constraints
- Caclulate histograms p(r) functions
- (Include polydispersity)
- Fourier transform



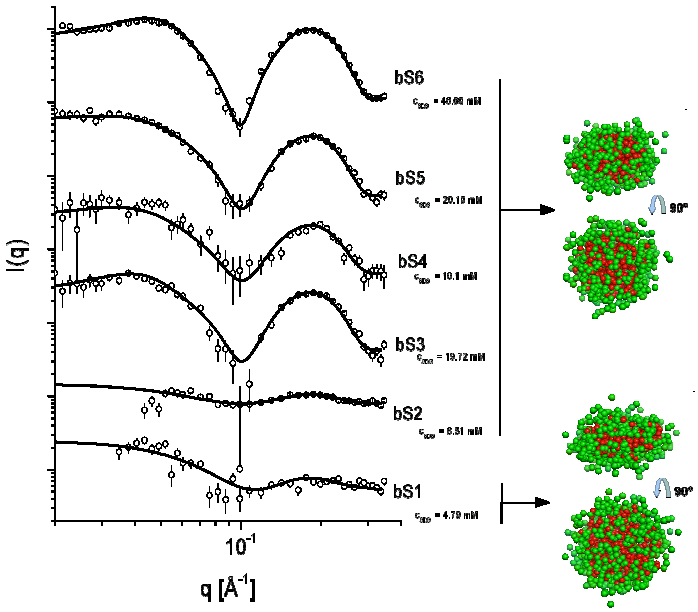
The Role of Decorated SDS Micelles in Sub-CMC Protein Denaturation and Association

Kell K. Andersen[1,2], Cristiano L. Oliveira[3], Kim L. Larsen[2], Flemming M. Poulsen[4], Thomas H. Callisen[5], Peter Westh[6], Jan S. Pedersen[6] and Daniel Otzen[1*]

J. Mol. Biol. (2009) **391**, 207–226



Pure SDS micelles in buffer:



C > CMC (~ 5 mM)

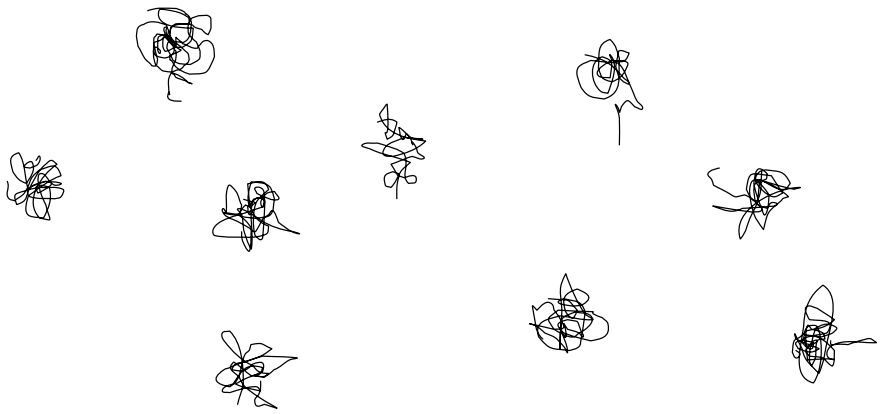
N_{agg}= 66±1
oblate ellipsoids
R=20.3±0.3 Å
ε=0.663±0.005

C ≤ CMC

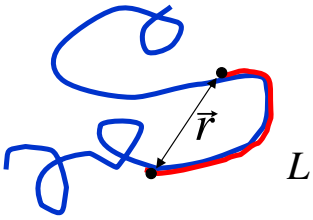
Polymer chains in solution

Gigantic ensemble of 3D random flights

- all with different configurations



Gaussian polymer chain



Look at two points: Contour separation: L
Spatial separation: r

Contribution to scattering: $I_2(q) = \frac{\sin(qr)}{qr}$

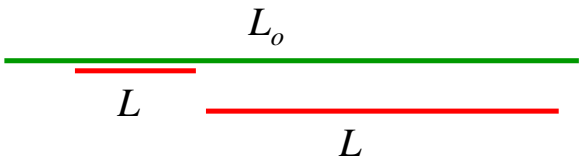
$$\exp\left[-q^2 Lb/6\right]$$

For an ensemble of polymers, points with
L has $\langle r^2 \rangle = Lb$
and r has a Gaussian distribution:

$$D(r) \propto \exp\left[-\frac{3r^2}{2\langle r^2 \rangle}\right]$$

Add scattering from all pair of points

'Density of points': $(L_o - L)$



Gaussian chains: The calculation

$$P(q) = \frac{1}{L_o^2}\int_0^\infty dr \int_0^{L_o} dL_1 \int_0^{L_o} dL_2 D(r, |L_2 - L_1|)\frac{\sin(qr)}{qr} r$$

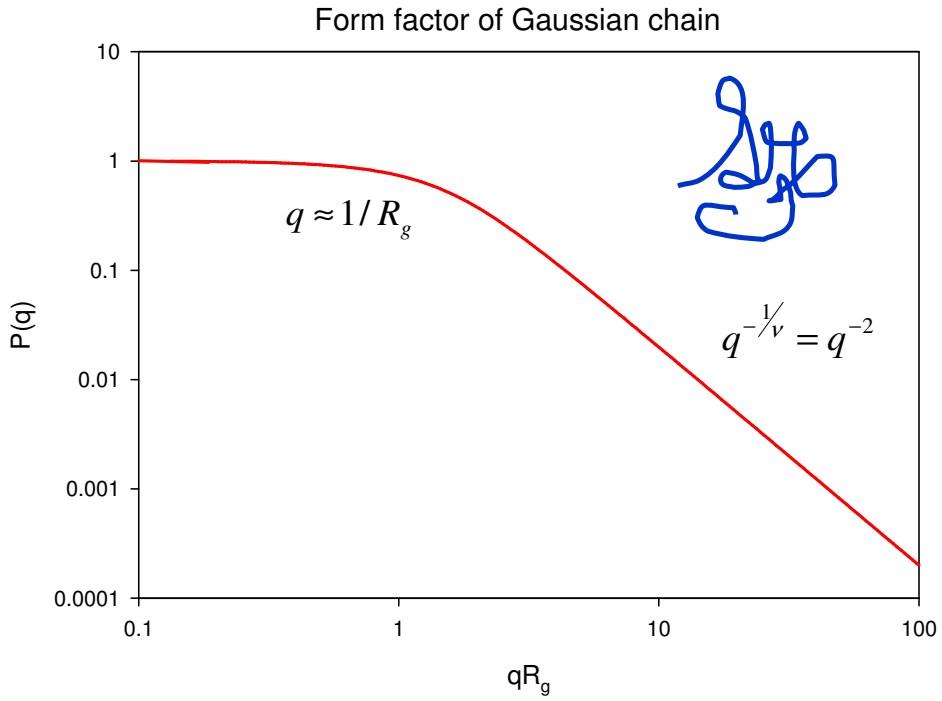
$$= \frac{1}{L_o}\int_0^\infty dr \int_0^{L_o} dL (L_o - L) D(r, L)\frac{\sin(qr)}{qr} r^2$$

$$= \frac{1}{L_o}\int_0^{L_o} dL (L_o - L)\exp\left[-q^2 Lb/6\right]$$

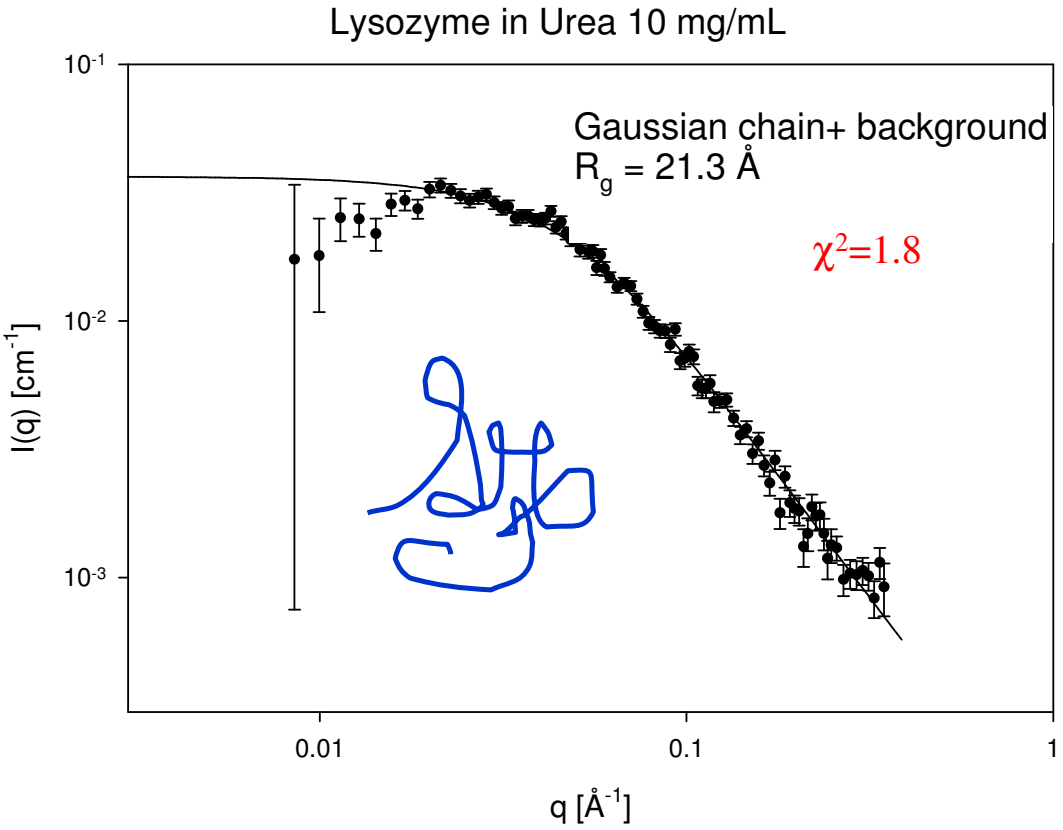
$$= \frac{2[\exp(-x) - 1 + x]}{x^2} \qquad x = R_g^2 q^2 \qquad R_g^2 = Lb/6$$

How does this function look?

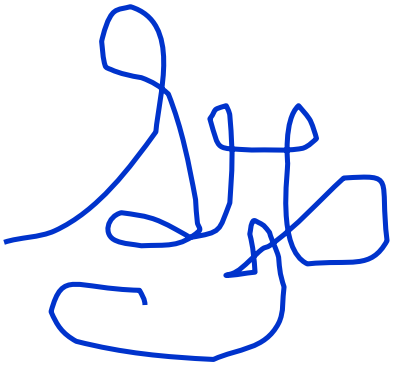
Polymer scattering



Lysozyme in urea

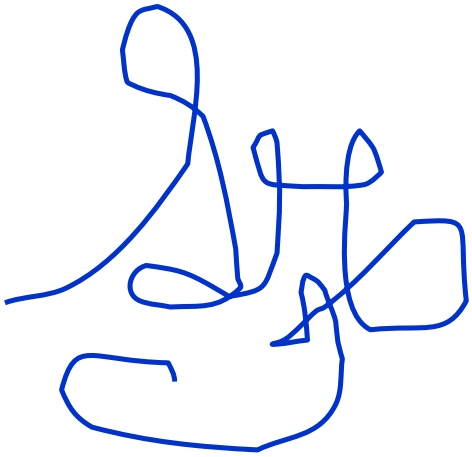


Self avoidence



No excluded volume

$$P(x = R_g^2 q^2) = \frac{2[\exp(-x) - 1 + x]}{x^2}$$



No excluded volume
=> expansion

no analytical solution!

Monte Carlo simulation approach

Pedersen and Schurtenberger 1996

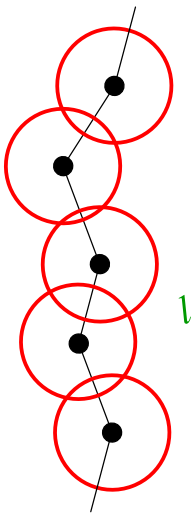
(1) Choose model

(2) Vary parameters in a broad range
Generate configs., sample P(q)

(3) Analyze P(q) using physical insight

(4) Parameterize P(q) using physical insight

(5) Fit experimental data using numerical expressions for P(q)



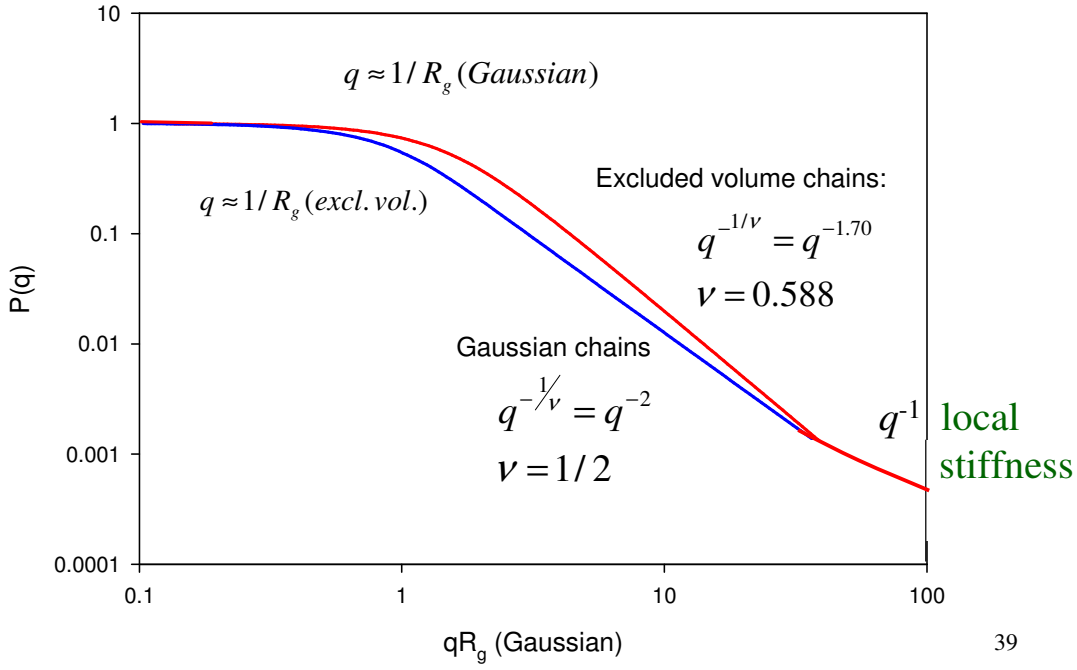
$P(q,L,b)$

L = contour length

b = Kuhn ('step') length

Expansion = 1.5

Form factor polymer chains



C16E6 micelles with ‘C

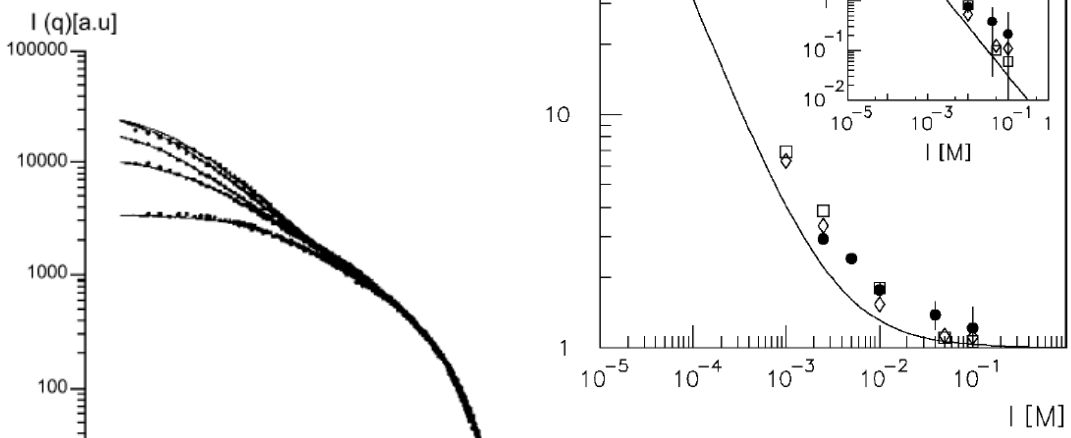


Fig. 3. Fit to the SANS data from doped C16E6 micelles (doping level of 6% of ionic surfactant) with different salt concentrations at low surfactant concentrations ($c = 0.4$ mg/ml) [32•]. From top to bottom: 0.1, 0.01, 0.005, 0.0025 and 0.001 M salt.

works also for
polyelectrolytes
like unfolded proteins

Models II

Polydispersity: Spherical particles as e.g. vesicles

No interaction effects: Size distribution $D(R)$

$$\frac{\mathrm{d}\sigma(q)}{\mathrm{d}\Omega} = \Delta\rho^2 \int_0^\infty D(R)\, V(R)^2 P(q,R)\, \mathrm{d}R.$$

Oligomeric mixture: Discrete particles

$$\frac{d\sigma(q)}{d\Omega} = c\,\Delta\rho^2 \sum_i M_i f_i P_i(q)$$

Application to insulin:

Pedersen, Hansen, Bauer (1994).
European Biophysics Journal 23, 379-389.
(Erratum). *ibid* 23, 227-229.

f_i = mass fraction $\quad \sum_i f_i = 1$

Used in PRIMUS 'Oligomers'

Concentration effects

Concentration effects in protein solutions

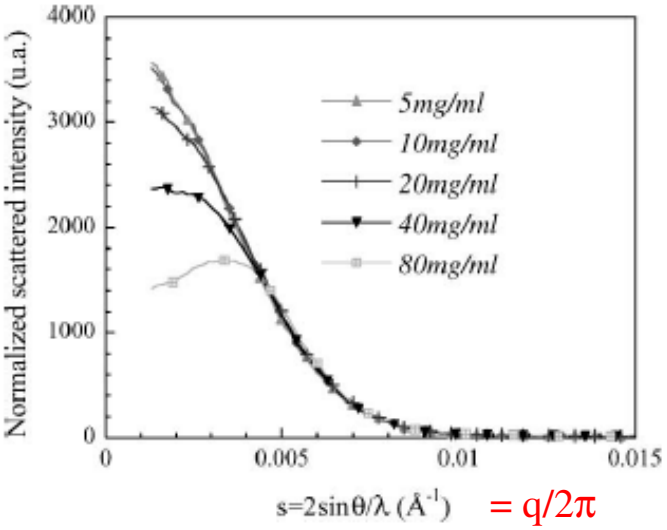


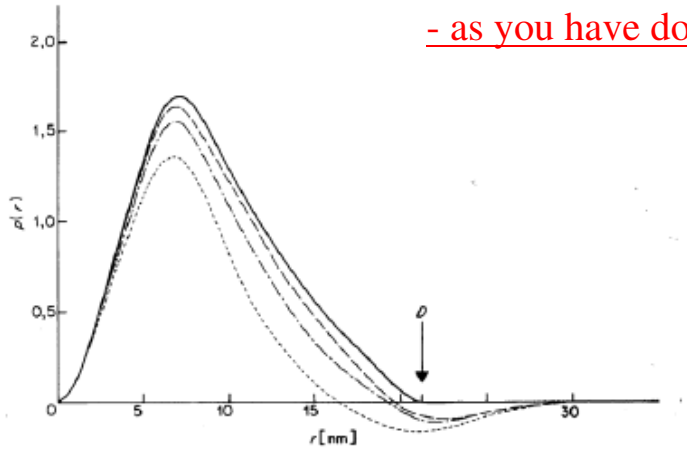
Fig. 1. Variation of normalized scattered intensity of α-crystallin solutions as a function of protein concentration, at 20°C, in 150 mM phosphate buffer at pH = 6.8.

S. Finet*,[1], A. Tardieu

α-crystallin
eye lens protein

p(r) by Indirect Fourier Transformation (IFT)

- as you have done by GNOM !



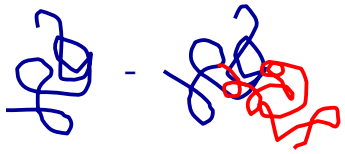
I. Pilz, 1982

At high concentration, the neighborhood is different from the average further away!

(1) Simple approach: Exclude low *q* data.
(2) Glatter: Use Generalized Indirect Fourier Transformation (GIFT)

Low concentraions

Zimm 1948 – originally for light scattering

$P'(q) =$  Subtract overlapping configuration

$$P'(q) = P(q) - \upsilon P(q)^2$$
$$= P(q)[1 - \upsilon P(q)]$$

$\upsilon \sim$ concentration

Higher order terms:

$$P'(q) = P(q)[1 - \upsilon P(q)\{1 - \upsilon P(q)\}]$$
$$= P(q)[1 - \upsilon P(q) + \upsilon^2 P(q)^2]$$

…

$$= P(q)[1 - \upsilon P(q) + \upsilon^2 P(q)^2 - \upsilon^3 P(q)^3 \ldots..]$$

$$= \frac{P(q)}{1 + \upsilon P(q)}$$

= Random-Phase Approximation (RPA)

Zimm approach

$$I(q) = K \frac{P(q)}{1 + \upsilon P(q)}$$

$$I(q)^{-1} = K^{-1} \frac{1 + \upsilon P(q)}{P(q)} = K^{-1}\left[\frac{1}{P(q)} + \upsilon\right]$$

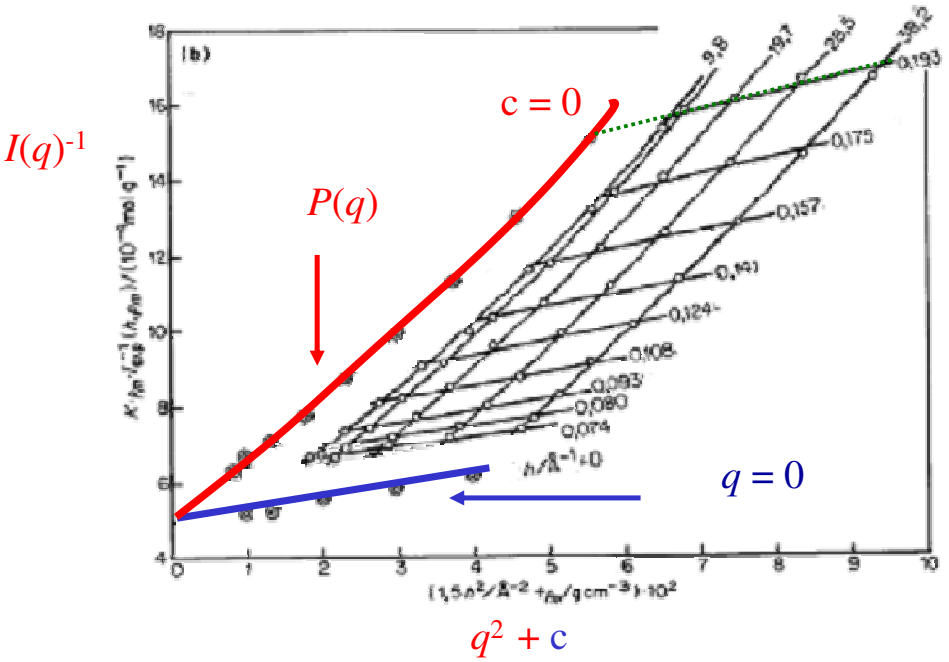
With $P(q) \approx \dfrac{1}{1 + q^2 R_g^2 / 3}$

$$I(q)^{-1} = K^{-1}\left[1 + q^2 R_g^2 / 3 + \upsilon\right]$$

Plot $I(q)^{-1}$ versus $q^2 + c$ and extrapolate to $q=0$ and $c=0$!

Zimm plot

Kirste and Wunderlich
PS in toluene



My suggestion:

(- which includes also information from what follows)

- Minimum 3 concentrations for same system.
- Fit data simultaneously all data sets

$$\frac{I(q_i)}{c} = \frac{P_i}{1 + c a_1 \exp(-q_i^2 a_2^2 / 3)}$$

With a_1, a_2, and P_i fit parameters

But now we look at the information content related to these effects…

Understand the fundamental processes and principles governing aggregation and crystallization

Why is the eye lens transparent despite a protein concentration of 30-40% ?

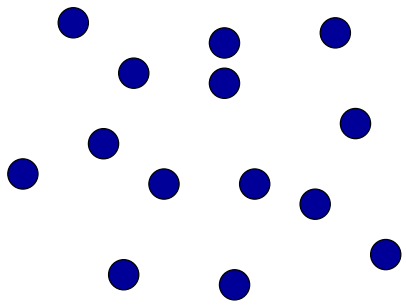
Structure factor

Spherical monodisperse particles

$$I(q) = V^2 \Delta\rho^2 P(q) \sum \frac{\sin qr_{jk}}{qr_{jk}}$$

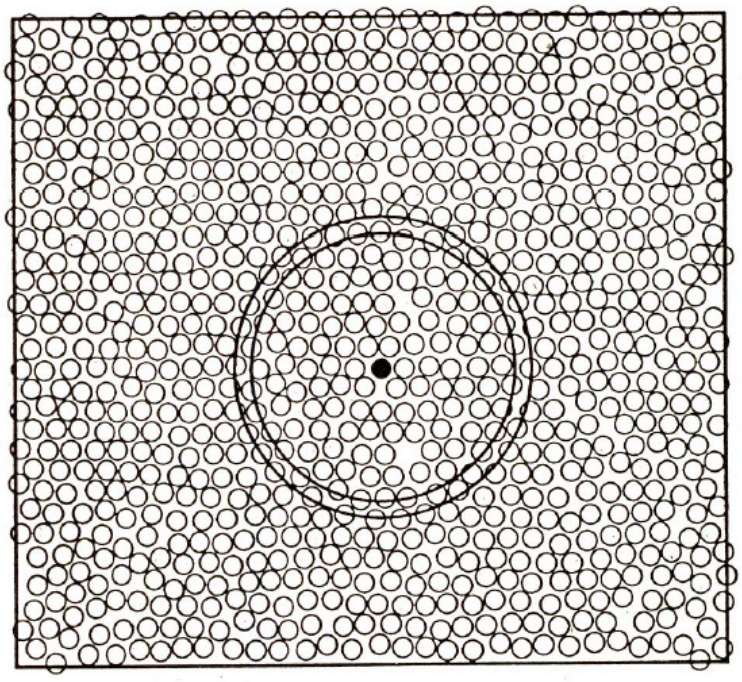
$$= V^2 \Delta\rho^2 P(q)\, S(q)$$

$$= V^2 \Delta\rho^2 P(q) \sum p(r_j) \frac{\sin qr_j}{qr_j}$$



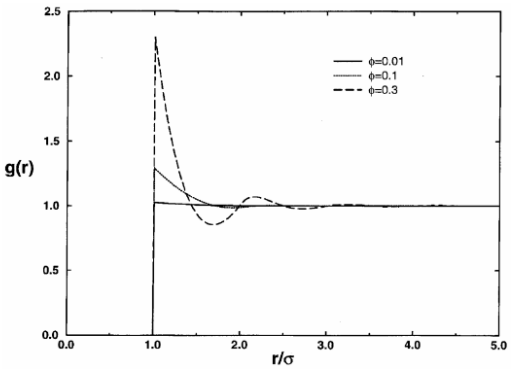
$S(q)$ is related to the probability distribution function of inter-particles distances, i.e. the pair correlation function $g(r)$

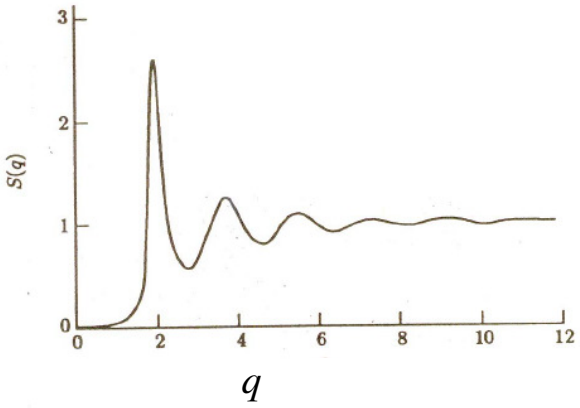
Correlation function $g(r)$



$g(r)$ =
<u>*Average*</u> of the
normalized
density of atoms
in a shell [r ; r+ dr]
from the center of
a particle

$g(r)$ and $S(q)$





$$S(q) = 1 + n\,4\pi \int (g(r) - 1) \frac{\sin(qr)}{qr} r^2 dr$$

GIFT

Glatter: Generalized Indirect Fourier Transformation (GIFT)

$$I(q) = 4\pi \int p(r) \frac{\sin(qr)}{qr} dr$$

With concentration effects

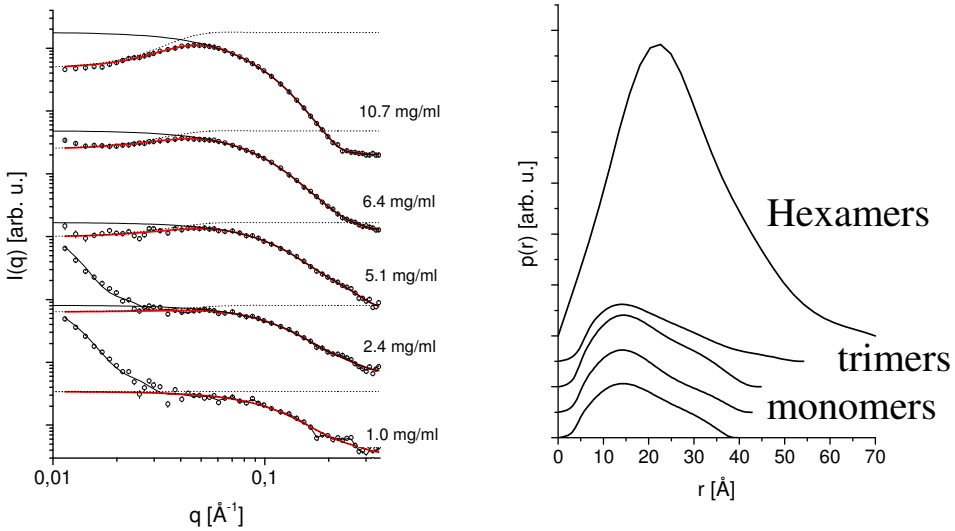
$$I(q) = S_{eff}(q, \eta, R, \sigma(R), Z, c_{salt}) 4\pi \int p(r) \frac{\sin(qr)}{qr} dr$$

Optimized by constrained non-linear least-squares method

- works well for globular models and provides $p(r)$

A SAXS study of the small hormone glucagon: equilibrium aggregation and fibrillation

Home-written software



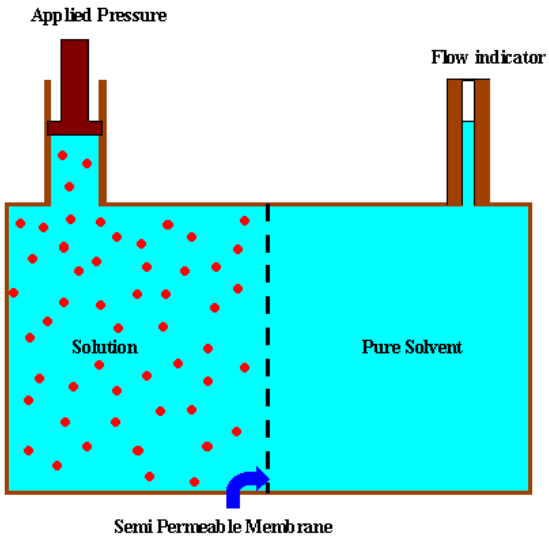
A SAXS Study of Glucagon Fibrillation

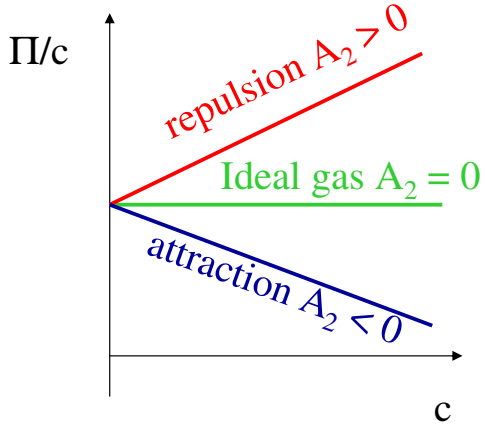
Cristiano Luis Pinto Oliveira[1]*, Manja Annette Behrens[1], Jesper Søndergaard Pedersen[2], Kurt Erlacher[1], Daniel Otzen[2,3] and Jan Skov Pedersen[1]

J. Mol. Biol. (2009) **387**, 147–161

Osmometry (Second virial coeff A_2)

$$\Pi = c\ (RT/M)\ (1 + A_2 c + A_3 c^2 + \ldots)$$





S(*q*), virial expansion and Zimm

From statistical mechanics…:

$$S(q=0) = RT\left(\frac{\partial \Pi}{\partial c}\right)^{-1} = \frac{1}{1 + 2cMA_2 + 3c^2 MA_3 + \ldots}$$

➔ In Zimm approach $\nu = 2cMA_2$

A_2 in lysozyme solutions

Isoelectric point

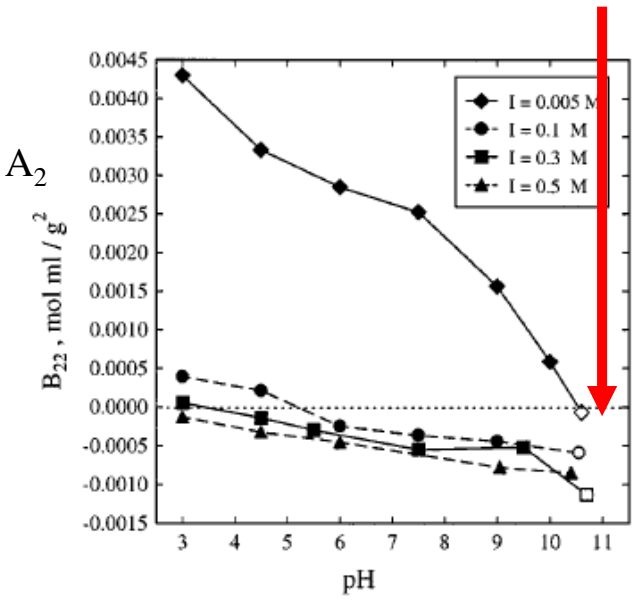


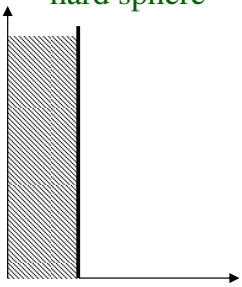
FIGURE 2 A summary of virial coefficients of lysozyme obtained from the SLS data at varying pH and four electrolyte concentrations (each point corresponds to one data set similar to those plotted in Fig. 1). The open symbols denote experimental conditions where the onset of aggregation was detected.

O. D. Velev, E. W. Kaler, and A. M. Lenhoff

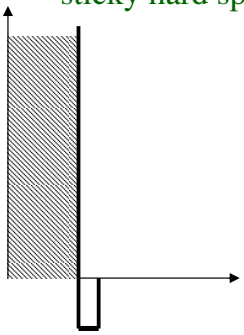
Colloidal interactions

- Excluded volume 'repulsive' interactions ('hard-sphere')
- Short range attractive van der Waals interaction ('stickiness')
- Short range attractive hydrophobic interactions
  (solvent mediated 'stickiness')
- Electrostatic repulsive interaction (or attractive for patchy charge distribution!)
  (effective Debye-Hückel potential)
- Attractive depletion interactions (co-solute (polymer) mediated)

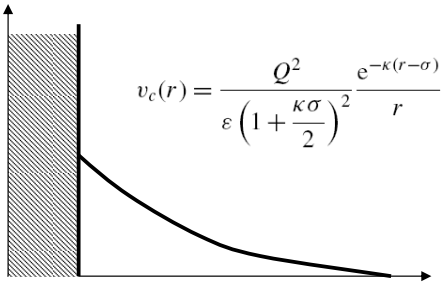
hard sphere



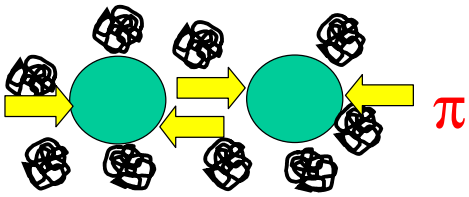
sticky hard sphere

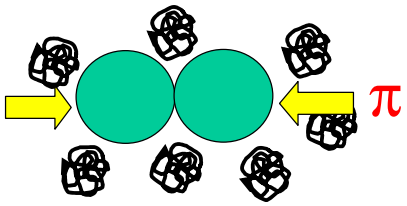


Debye-Hückel
screened Coulomb potential



Depletion interactions



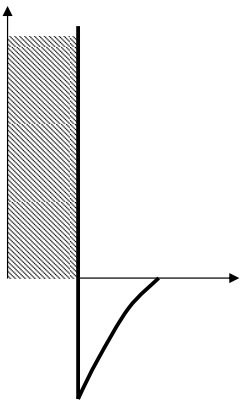


$$v_{\text{eff}}(r;\rho_1,\rho_2,T)=\begin{cases}\infty; & r<\sigma_1\\ v_{A0}(r); & \sigma_1<r<\sigma_1+\sigma_2\\ 0; & r>\sigma_1+\sigma_2\end{cases}$$

where

$$v_{A0}(r)=-k_BT\rho_2\frac{\pi\sigma_1^3(1+q)^3}{6}\left[1-\frac{3r}{2(1+q)\sigma_1}+\frac{r^3}{2(1+q)^3\sigma_1^3}\right]$$

with $q=\sigma_2/\sigma_1$ the ratio of diameters.

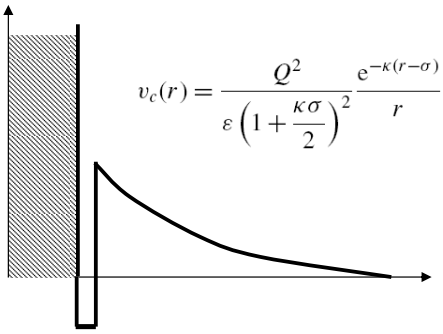


Asakura & Oosawa, 58

Theory for colloidal stability

DLVO theory: (Derjaguin-Landau-Vewey-Overbeek)

Debye-Hückel
screened Coulomb potential
+ attractive interaction



$$v_c(r)=\frac{Q^2}{\varepsilon\left(1+\frac{\kappa\sigma}{2}\right)^2}\frac{e^{-\kappa(r-\sigma)}}{r}$$

Integral equation theory

Relate $g(r)$ [or $S(q)$] to $V(r)$

At low concentration $g(r) = \exp(-V(r)/k_BT) \approx 1 - V(r)/k_BT$
Boltzmann approximation (weak interactions)

Make expansion around uniform state [Ornstein-Zernike eq.]

$g(r) = 1 - n\,c(r)$ –[3 particle] – [4 particle] - …
$= 1 - n\,c(r) - n^2\,c(r) * c(r) - n^3\,c(r) * c(r) * c(r) - \ldots$
[* = convolution] $c(r)$ = direct correlation function

$$S(q) = 1 - n\,c(q) - n^2\,c(q)^2 - n^3\,c(q)^3 - \ldots = \frac{1}{1 - nc(q)}$$

$\Rightarrow$ but we still need to relate $c(r)$ to $V(r)$!!!

Closure relations

Systematic density expansion….

Mean-spherical approximation MSA: $c(r) = -V(r)/k_BT$
(analytical solution for screened Coulomb potential
- but not accurate for low densities)

Percus-Yevick approximation PY: $c(r) = g(r)\,[\exp(V(r)/k_BT) - 1]$
(analytical solution for hard-sphere potential + sticky HS)

Hypernetted chain approximation HNC:
$c(r) = -V(r)/k_BT + g(r) - 1 - \ln(g(r))$
(Only numerical solution
- but quite accurate for Coulomb potential)

Rogers and Young closure RY:
Combines PY and HNC in a self-consistent way
(Only numerical solution
- but very accurate for Coulomb potential)

α-crystallin

S. Finet, A. Tardieu

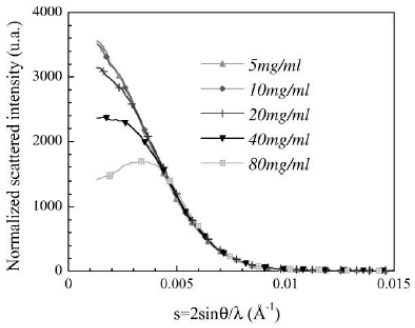
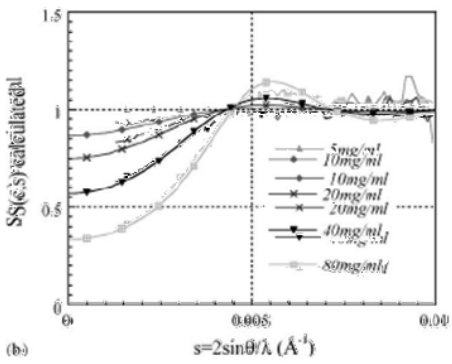


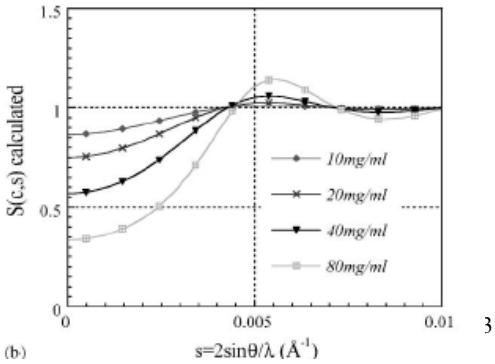
Fig. 1. Variation of normalized scattered intensity of α-crystallin solutions as a function of protein concentration, at 20°C, in 150 mM phosphate buffer at pH = 6.8.

DLVO potential

$$v(r) = \begin{cases} Z_p^2 e^2 / \{4\pi\varepsilon_0 \varepsilon k T r (1 + 0.5\sigma/\lambda_D)^2\} \\ \times \exp[-(r-\sigma)/\lambda_d] \\ -J\sigma/r \exp[-(r-\sigma)/d] & \text{for } r > \sigma, \\ +\infty & \text{for } r \leqslant \sigma. \end{cases}$$

HNC and numerical solution



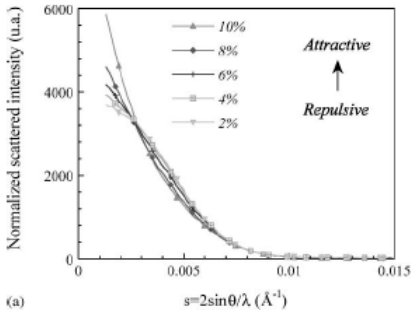


3

α-crystallin + PEG 8000

S. Finet, A. Tardieu

Depletion interactions:

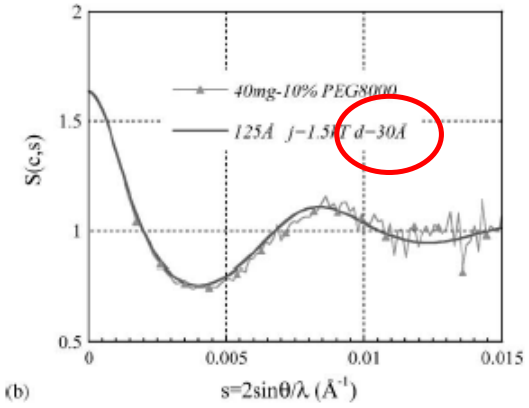


40 mg/ml α-crystallin solution pH 6.8, 150 mM ionic strength.

DLVO potential

$$v(r) = \begin{cases} Z_p^2 e^2 / \{4\pi\varepsilon_0 \varepsilon k T r (1 + 0.5\sigma/\lambda_D)^2\} \\ \times \exp[-(r-\sigma)/\lambda_d] \\ -J\sigma/r \exp[-(r-\sigma)/d] & \text{for } r > \sigma, \\ +\infty & \text{for } r \leqslant \sigma. \end{cases}$$

HNC and numerical solution



Anisotropy

$$\frac{d\sigma(q)}{d\Omega} = \Delta\rho^2 V^2 \left[\sum_i F_i(q,e_i)^2 + \frac{1}{N} \sum_{i,j} F_i(q,e_i) F_j(q,e_j) [S_{i,j}(q,e_i,e_j) - 1] \right]$$

Small: decoupling approximation (Kotlarchyk and Chen,1984) :

$$\frac{d\sigma(q)}{d\Omega} = \Delta\rho^2 V^2 P(q)[1 + \beta(q)(S(q) - 1)],$$

$$\beta(q) = \langle F(q) \rangle_o^2 / \langle F^2(q) \rangle_o,$$

Measured structure factor:

$$S_{meas}(q) \equiv \frac{\frac{\partial\sigma(q)}{\partial\Omega}}{n\Delta\rho^2 P(q)} = 1 + \beta(q)[S(q) - 1] \neq S(q) \quad !!!!!$$

Large anisotropy

Polymers, cylinders…

Anisotropy, large: Random-Phase Approximation (RPA):

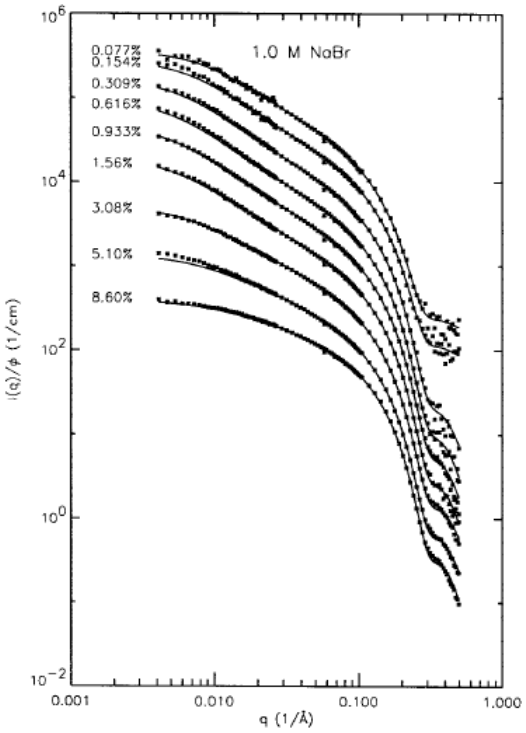
$$\frac{d\sigma}{d\Omega}(q) = n\Delta\rho^2 V^2 \frac{P(q)}{1 + \nu P(q)} \qquad \upsilon \sim \textit{concentration}$$

Anisotropy, large: Polymer Reference Interaction Site Model (PRISM)
Integral equation theory – equivalent site approximation

$$\frac{d\sigma}{d\Omega}(q) = n\Delta\rho^2 V^2 \frac{P(q)}{1 - nc(q)P(q)}$$

c(q) direct correlation function
related to FT of *V(r)*

Empirical PRISM expression



SDS micelle in 1 M NaBr

$$\frac{d\sigma}{d\Omega}(q) = n\Delta\rho^2 V^2 \frac{P(q)}{1 - nc(q)P(q)}$$

$c(q)$ = rod formfactor
- empirical from MC simulation

Arleth, Bergström and Pedersen

Overview: Available Structure factors

1) **Hard-sphere potential:** Percus-Yevick approximation
2) **Sticky hard-sphere potential**: Percus-Yevick approximation
3) **Screened Coulomb potential:**
   Mean-Spherical Approximation (MSA).
   Rescaled MSA (RMSA).
   Thermodynamically self-consistent approaches (Rogers and Young closure)
4) **Hard-sphere potential, polydisperse system:** Percus-Yevick approximation
5) **Sticky hard-sphere potential, polydisperse system:** Percus-Yevick approximation
6) **Screened Coulomb potential, polydisperse system:** MSA, RMSA,
7) **Cylinders, RPA**
8) **Cylinders, `PRISM':**
9) **Solutions of flexible polymers, RPA:**
10) **Solutions of semi-flexible polymers, `PRISM':**
11) **Solutions of polyelectrolyte chains 'PRISM':**

Summary

- Model fitting and least-squares methods
- Available form factors
  ex: sphere, ellipsoid, cylinder, spherical subunits…
  ex: polymer chain
- Monte Carlo integration for
  form factors of complex structures
- Monte Carlo simulations for
- form factors of polymer models

- Concentration effects and structure factors
  Zimm approach
  Spherical particles
  Elongated particles (approximations)
  Polymers

Literature

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