Modeling of bending-torsion couplings in active-bending structures. Application to the design of elastic gridshell.

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4 Elastic rod : variational approach

This chapter should be understood as an extension of the work initiated by Tayeb 2015 [1] and published by du Peloux et al. 2015 [2] and Lefevre et al. 2017 [3].

4.1 Introduction

Elastic gridshells are lightweight structures made of interconnected slender beams (see **??**). Modeling the deformation process of such structures is complex as it involves to take account for the geometric non linearities induced by the large displacements and rotations of the grid. More over, the large number of connexions and the coupling between flexion and torsion highly increase the complexity of the analysis.

To facilitate the design process of elastic gridshells, architects and engineers need a dedicated numerical tool that provide a good level of interactivity – which means that numerical computations must converge within a reasonable time, if not in real time – and gives deep insights on the geometry and on the mechanical behavior of the grid. This tool must be able to model complex connexions and various types of support conditions to enhance the user experience during the formfinding stage and to improve his ability to explore the space of constructible shapes.

In this chapter, our goal is to develop a beam model suitable for the modeling of grids of interconnected slender beams in order to study the mechanics of elastic gridshells. For such beams, Kirchhoff's theory of rods is considered to be appropriate. We follow recent advances in the field of computer graphics about hair modeling [4] to build a reduced degrees of freedom rod model thanks to an appropriate curve-angle representation. This representation is based on a relevant curve framing, namely the Bishop frame presented in **??**. The rod will be considered inextensible. Moreover, it will be assumed that cross-sections remain normal to the rod centerline. The internal forces and moments acting on the rod will be deduced from the differentiation of the elastic energy of the beam with respect to

the degrees of freedom of the system.

This chapter is devoted to the development of the beam model. The formulation of a discrete element and its implementation in a numerical solver are treated in a dedicated chapter (see **??**).

4.1.1   Overview

We begin this chapter by establishing a representation for any geometric configuration of a Kirchhoff rod (see §4.2) based on a curve-angle description. This representation is

4.1.2   Contributions

- We consolidate the mathematical development of the beam model by introducing the Fréchet and Gateaux derivatives in function spaces.

- We clarify the independence of the the rotational degree of freedom (θ) with respect to translational degrees of freedom ($\boldsymbol{x}$).

- We factorize the expressions of the internal forces and moments by reusing the quasi-static assumption.

- We identify the contributions of axial and shear forces, bending and torsion moments in the expressions of internal forces and moments.

- We prove the equivalence with the shear force obtained from the dynamical equations of Kirchhoff.

- We suggest that the dynamical Kirchhoff equations should be a more straight forward starting point to build up similar theories.

4.1.3   Related work

Bergou et al. 2008 [4] present a discrete treatment of adapted framed curves, parallel transport, and holonomy. Based on this framework they propose a curve-angle representation of the geometric configuration of slender rods. In this representation, the orientation of the material frame is established with respect to the Bishop frame by a single scalar angle. Upon this representation they build a mechanical model for slender elastic rods with anisotropic cross-section and arbitrary rest configuration. In the dynamic the centerline is treated explicitly and material frames are treated as quasistatic. Bergou et al. 2010 [5] improve there previous model for the modeling of viscous threads.

Nabaei 2014 [6] implements the model developed by [4] in IPOPT, an interior point optimizer, to solve the static equilibrium of simply connected systems of twisted elastica.

Tayeb 2015 [1], du Peloux et al. 2015 [2] and Lefevre et al. 2017 [3] follow [4] to model grids of interconnected slender beams. They implement their model in a dynamic relaxation

solver to formfind elastic gridshells. They formulate a special connexion.

[7]. [8], [9], [10], [11] Mireille Gregoire Bertail Spilmann Hose Pour intro : [12]

4.2 Kirchhoff rod

Kirchhoff's theory of rods is presented thoroughly in the next chapter (see **??**). In this chapter, although the assumptions are not exactly the ones made by Kirchhoff in his theory, we will stick to this denomination as introduced by [4]. In the present theory we will assume that :

- the rod is inextensible,

- cross-sections remain plane,

- cross-sections remain perpendicular to the centerline,

- the material deforms elastically.

4.2.1 Description of the motion

We introduce a curvilinear coordinate system to describe the motion of our Kirchhoff rod, compatible with the model assumptions. It is composed of a parametric space curve, called the *centerline*, equipped with a moving frame, called the *material frame*.

Deformed configuration

The actual geometric configuration of the rod is described by its centerline $\boldsymbol{x}(s)$ and its cross-sections. The centerline is a space curve parameterized by its arc length, denoted s. Cross-sections orientation are followed along the centerline by a material frame $\{\boldsymbol{d}_3(s), \boldsymbol{d}_1(s), \boldsymbol{d}_2(s)\}$ which is an adapted orthonormal moving frame aligned with the principal axes of inertia of the cross-section.

Recall from **??** that *adapted* means that the material frame is aligned with the tangent vector of the centerline :

$$\boldsymbol{d}_3(s) = \boldsymbol{x}'(s) = \boldsymbol{t}(s) \tag{4.1}$$

Here, the prime symbol denotes the derivation with respect to the arc length parameter s. Recall also from **??** that $\|\boldsymbol{x}'(s)\| = 1$ because $\boldsymbol{x}$ is parametrized by arc length.

Stress-free configuration

Among all the possible geometric configurations of the rod we identify the *stress-free configuration* or *rest configuration*, that is the configuration in which the rod is stress-free under no external forces and moments applied top it (loads, supports, . . .). This

configuration is crucial as the elastic energy of a rod in a given configuration relies on both its actual and rest configuration.

Hereinafter, the symbols referring to this configuration will be denoted with an overbar (e.g. $\bar{\boldsymbol{x}}(s)$).

4.2.2 Inextensibility assumption

As explained by Audoly et al. 2010 [13], based on a scaling argument, two cases arise for slender beams : either the centerline stretches and bending and twisting forces become negligible compared to axial forces ; either the centerline remains inextensible. As we are not interested in the first case – in which the beam would behave like a cable, mainly in tension – we will assume that the rod is effectively inextensible.[1]

Remark that the previous description (see §4.2.1) is only valid for inextensible rods. Indeed, for an inextensible rod the centerline does not stretches and the arc length parameter for the rest configuration is also a valid arc length parameter for any other configuration.[2]

The inextensibility hypothesis also implies that any admissible perturbation $(\lambda\boldsymbol{h}_x)$ of the rod's centerline $(\boldsymbol{x})$ is locally orthogonal to the centerline itself. Indeed, at each arc length s an inextensible rod must satisfies :

$$\|\boldsymbol{x}'\| = \|(\boldsymbol{x} + \lambda\boldsymbol{h}_x)'\| = 1 \Rightarrow \boldsymbol{d}_3 \cdot \boldsymbol{h}_x' = -\tfrac{\lambda^2}{2}\|\boldsymbol{h}_x'\|^2 = o(\lambda) \simeq 0 \tag{4.2}$$

It is worth to mention here that this property $(\boldsymbol{d}_3 \cdot \boldsymbol{h}_x' = 0)$ will be used several times in the following sections.

Hereinafter, the length of the rod will be denoted L and the arc length s will vary in $[0, L]$, with no loss of generality.

4.2.3 Euler-Bernouilli assumption

Strains are supposed to remain small so that the cross-sections remain plane and the material frame remains orthonormal and adapted to the centerline during the motion of the rod. In other words the cross-sections undergo rigid body motions.[3]

4.2.4 Motion of the material frame

As a consequence of the Euler-Bernoulli assumption, we can differentiate the conditions of orthonormality of the material frame (see **??**). This leads to the following system of differential equations governing the evolution of the *material directors* $\{\boldsymbol{d}_3(s), \boldsymbol{d}_1(s), \boldsymbol{d}_2(s)\}$

---

[1]For a complete treatment of the question of inextensibility refer to **??** in **??**.

[2]For a complete treatment of the question of reparametrization refer to **??** in **??**.

[3]For a complete treatment of this point in Kirchhoff's theory refer to **??** in **??**.

along the centerline :

$$
\begin{bmatrix} \boldsymbol{d}_3'(s) \\ \boldsymbol{d}_1'(s) \\ \boldsymbol{d}_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_2(s) & -\kappa_1(s) \\ -\kappa_2(s) & 0 & \tau(s) \\ \kappa_1(s) & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_3(s) \\ \boldsymbol{d}_1(s) \\ \boldsymbol{d}_2(s) \end{bmatrix}
\tag{4.3}
$$

where $\tau(s)$, $\kappa_1(s)$ and $\kappa_2(s)$ are the rates of rotation of the material frame with respect to the arc length parameter s. These equations can be formulated with the *Darboux vector* of the chosen material frame, which represents the angular velocity vector of the frame along $\boldsymbol{x}(s)$:

$$
\boldsymbol{d}_i'(s) = \boldsymbol{\Omega}_m(s) \times \boldsymbol{d}_i(s)
\tag{4.4a}
$$

$$
\boldsymbol{\Omega}_m(s) = \begin{bmatrix} \tau(s) \\ \kappa_1(s) \\ \kappa_2(s) \end{bmatrix}
\tag{4.4b}
$$

That means that $\tau(s)$, $\kappa_1(s)$ and $\kappa_2(s)$ are the components of the angular velocity of the material frame around its axes $\boldsymbol{d}_3(s)$, $\boldsymbol{d}_1(s)$ and $\boldsymbol{d}_2(s)$ when it travels along the centerline at unit speed.[4]

Recall from **??** how the (geometric) curvature (κ) of a spatial curve parametrized by arc length is related to the Frenet frame $\{\boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$ by :

$$
\boldsymbol{t}'(s) = \boldsymbol{x}''(s) = \kappa(s) \cdot \boldsymbol{n}(s)
\tag{4.5a}
$$

$$
\kappa(s) = \|\boldsymbol{t}'(s)\| = \|\boldsymbol{x}''(s)\|
\tag{4.5b}
$$

$$
\boldsymbol{b}(s) = \boldsymbol{t}(s) \times \boldsymbol{n}(s)
\tag{4.5c}
$$

To describe the osculating plane in which lies the bending part of the deformation we rely on the *curvature binormal* introduced previously in **??**. We recall from **????** that :

$$
\boldsymbol{\kappa b}(s) = \kappa(s) \cdot \boldsymbol{b}(s) = \boldsymbol{t}(s) \times \boldsymbol{t}'(s) = \kappa_1(s)\boldsymbol{d}_1(s) + \kappa_2(s)\boldsymbol{d}_2(s)
\tag{4.6}
$$

The curvature binormal is the vector of direction $\boldsymbol{b}(s)$ and norm $\kappa(s)$, and at each point of arc length s the osculating plane is normal to $\boldsymbol{\kappa b}(s)$. Finally, recall from **??** that, as the material frame is adapted to the centerline, the following equation holds :

$$
\boldsymbol{\Omega}_m(s) = \boldsymbol{\kappa b}(s) + \tau(s)\boldsymbol{t}(s)
\tag{4.7}
$$

4.2.5 Material curvatures and twist

Kirchhoff's theory assigns a physical meaning to the components of $\boldsymbol{\Omega}_m(s)$:

- $\kappa_1(s)$ and $\kappa_2(s)$ are called the *material curvatures* and represent the rod's flexion in the principal planes, respectively normal to $\boldsymbol{d}_1(s)$ and $\boldsymbol{d}_2(s)$ (see fig. 4.1a) ;

---

[4]See **??** in **??** for a geometric interpretation of these rates of rotation.

- $\tau(s)$ is called the *material twist* and represents the cross-sections rate of rotation around $\boldsymbol{d}_3(s)$ (see fig. 4.1b).

These scalar functions are directly related to the components of the strain tensor as defined in Kirchhoff's theory.[5]

Note that In the literature these quantities are sometimes called *strain rates* (Antman 2005 [15]) or moment strains (Reissner 1973 [16]). However, this depart from the most common definition in which the strain is a dimensionless quantity : "The theory of elasticity deals with solids that, in the presence of mechanical stress, depart from their 'natural' configuration [...]. The strain provides a geometrical characterization of deformation : it measures by how much the solid departs from its natural configuration." [13, p. 19].

4.2.6 Material constitutive equations

The twisting moment (Q) and the components of the bending moment (M_1, M_2) are determined through the usual material constitutive equations :

$$Q = GJ(\tau_1 - \overline{\tau_1}) \tag{4.8a}$$

$$M_1 = EI_1(\kappa_1 - \overline{\kappa_1}) \tag{4.8b}$$

$$M_2 = EI_2(\kappa_2 - \overline{\kappa_2}) \tag{4.8c}$$

We also introduce the vector of internal moments as :

$$\boldsymbol{M} = Q\boldsymbol{d}_3 + M_1\boldsymbol{d}_1 + M_2\boldsymbol{d}_2 \tag{4.9}$$

4.2.7 Elastic energy

Kirchhoff's theory assigns an elastic energy to beams according to their strain [13]. In this theory, a beam is supposed to be inextensible. Thus the elastic energy $(\mathcal{E})$ only accounts for bending and torsion and behaviors and is given by :

$$\mathcal{E} = \tfrac{1}{2} \int_0^L EI_1(\kappa_1 - \overline{\kappa}_1)^2 + EI_2(\kappa_2 - \overline{\kappa}_2)^2 ds + \tfrac{1}{2} \int_0^L GJ(\tau - \overline{\tau})^2 ds \tag{4.10}$$

Here, $\overline{\kappa}_1$, $\overline{\kappa}_2$ and $\overline{\tau}$ denote the *natural* material curvatures and material twist of the rod in its rest configuration. E and G are the elastic and shear modulus of the material. I_1 and I_2 are the moments of inertia of the cross-section. J is the torsion constant of the cross-section.

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[5]For a complete treatment of the definition of strain in Kirchhoff's theory refer to **??** in **??** or Dill 1992 [14]

Initially straight isotropic rod : a special case

From this energy formulation an interesting an well-known result on elastic rods can be retrieved.[6]. This results stipulates that torsion is uniform in a rod with isotropic cross-section and naturally straight.

Indeed, by definition for a rod with isotropic cross-section : $EI_1 = EI_2 = EI$. If in addition the rod is straight in its rest configuration : $\bar{\kappa}_1 = \bar{\kappa}_2 = \bar{\tau} = 0$. The bending energy becomes : $\mathcal{E}_b = EI_1\,\kappa_1^2 + EI_2\,\kappa_2^2 = EI(\kappa_1^2 + \kappa_2^2) = EI\kappa^2$. Consequently, the elastic energy of the rod is composed of two independent terms : $\mathcal{E}_b[\boldsymbol{x}]$ and $\mathcal{E}_r[\theta]$. The coupling between bending and twisting disappears and the global minimum of elastic energy is reached when minimizing separately bending an twisting energies. This meansthat the geometry of the rod at static equilibrium ($\boldsymbol{x}$) is the one that minimized $\mathcal{E}_b[\boldsymbol{x}]$. And the minimum of $\mathcal{E}_t[\theta]$ is obviously achieved for a uniform twist ($\tau = cst$) along the centerline, only prescribed by the boundary conditions at the extremities of the rod (least-squares minimization).

In that particular case, the geometry of the centerline is not influenced by the amount of uniform twist (hence torsion) present in the rod.

4.3   Curve-angle representation

In the previous paragraph we have shown how the elastic energy of a rod can be computed with respect to the position of its centerline and the orientation of its cross-sections. This representation can be naturally described with six degrees of freedom (6-DOFs) : [7]

- 3-DOFs for the position of the centerline,

- and 3-DOFs for the orientation of the cross-sections.

Following Bergou et al. 2008 [4] we introduce a reduced coordinate formulation of the rod that accounts for only 4-DOFs :

- 3-DOFs for the position of the centerline,

- and only 1-DOF for the orientation of the cross-sections.

4.3.1   Definition of the representation

This reduction of the number of DOFs relies on the choice of a suitable reference frame, namely a Bishop (or zero-twisting) frame denoted $\{\boldsymbol{t}(s), \boldsymbol{u}(s), \boldsymbol{v}(s)\}$. Recall from **??** in **??** that this reference frame is adapted to the centerline and exhibits a null angular velocity

---

[6]For instance, this result is cited by Adriaenssens et al. 1999 [17].

[7]This is the usual choice in which the orientation of the material frame is parametrized by a rotation matrix, or equivalently a quaternion.

(a) bending strains

(b) angular deviation
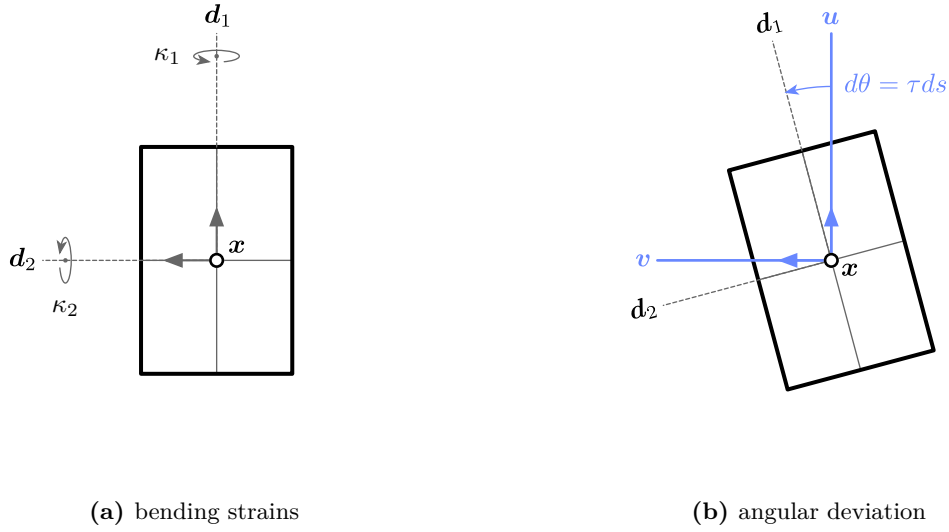
Figure 4.1 − Curve-angle representation of the rod. The orientation of the material frame is established from the reference frame by the deviation angle $\theta(s)$.

around the centerline's tangent vector (see **??**), which means :

$$\boldsymbol{u}(s) \cdot \boldsymbol{v}'(s) = \boldsymbol{u}'(s) \cdot \boldsymbol{v}(s) = 0 \tag{4.11}$$

The Bishop frame only depends on $\boldsymbol{x}$, the geometry of the centerline, and the choice of an initial condition. This reference frame is obtained all along the curve by propagating a given initial frame – usually chosen at $s = 0$ – with the parallel transport operator (see **??**). By construction, this reference frame evolves along the curve with the following angular velocity :

$$\boldsymbol{\Omega}_b(s) = \boldsymbol{\kappa b} = \boldsymbol{t} \times \boldsymbol{t}' = \boldsymbol{x}' \times \boldsymbol{x}'' \tag{4.12a}$$
$$\boldsymbol{t}'(s) \ = \boldsymbol{\kappa b} \times \boldsymbol{t} \tag{4.12b}$$
$$\boldsymbol{u}'(s) \ = \boldsymbol{\kappa b} \times \boldsymbol{u} \tag{4.12c}$$
$$\boldsymbol{v}'(s) \ = \boldsymbol{\kappa b} \times \boldsymbol{v} \tag{4.12d}$$

Remark that $\boldsymbol{\Omega}_b(s)$ only depends on the centerline and is well defined even when the curvature vanishes, unlike the Frenet frame. In that case the parallel transport operator becomes the rotation of null angle, which is still a valid transformation.

Hence, the orientation of the cross-sections, that is the material frame $\{\boldsymbol{d}_3(s), \boldsymbol{d}_1(s), \boldsymbol{d}_2(s)\}$, can be tracked only by the measure of a single angle $\theta(s)$ from this reference frame denoted $\{\boldsymbol{d}_3(s), \boldsymbol{u}(s), \boldsymbol{v}(s)\}$:

$$\begin{bmatrix} \boldsymbol{d}_1(s) \\ \boldsymbol{d}_2(s) \end{bmatrix} = \begin{bmatrix} \cos\theta(s) & \sin\theta(s) \\ -\sin\theta(s) & \cos\theta(s) \end{bmatrix} \begin{bmatrix} \boldsymbol{u}(s) \\ \boldsymbol{v}(s) \end{bmatrix} \tag{4.13}$$

Note that the choice of an initial condition for the definition of the Bishop frame is not a

matter of concern as only the derivative of θ appears in the elastic energy (see eq. (4.10)). Thus, we are free to choose this condition in the most convenient way.

4.3.2 Measurement of the material twist

With this 4-DOFs representation, the material twist is directly expressed in terms of the derivative of θ. Indeed, from eq. (4.12) and (4.13) we obtain :

$$
\begin{aligned}
\boldsymbol{d}_1' &= (\cos\theta\boldsymbol{u} + \sin\theta\boldsymbol{v})' \\
&= (-\sin\theta\boldsymbol{u} + \cos\theta\boldsymbol{v}) \cdot \theta' + \cos\theta\boldsymbol{u}' + \sin\theta\boldsymbol{v}' \\
&= \theta'\boldsymbol{d}_2 + \cos\theta\,(\boldsymbol{\kappa b} \times \boldsymbol{u}) + \sin\theta\,(\boldsymbol{\kappa b} \times \boldsymbol{v}) \\
&= \theta'\boldsymbol{d}_2 + \boldsymbol{\kappa b} \times \boldsymbol{d}_1
\end{aligned}
\tag{4.14}
$$

Finally, using the definition of τ from eq. (4.3) and the fact that $\boldsymbol{\kappa b}$ is perpendicular to $\boldsymbol{d}_3$ we can deduce that $\tau = \theta'$:

$$
\tau = \boldsymbol{d}_1' \cdot \boldsymbol{d}_2 = (\theta'\boldsymbol{d}_2 + \boldsymbol{\kappa b} \times \boldsymbol{d}_1) \cdot \boldsymbol{d}_2 = \theta' + \boldsymbol{d}_3 \cdot \boldsymbol{\kappa b} = \theta'
\tag{4.15}
$$

Here, the benefits of the curve-angle representation are revealed as the material twist is now simply given by the rate of θ with respect to the arc length parameter s. Everything happens as if the Bishop would enable a directe measurement of the mechanical torsion, getting ride of the intrinsic geometric torsion of the centerline itself (aka the torsion of Frenet).

4.3.3 Vector of material curvatures

We introduce the vector of material curvatures ($\boldsymbol{\omega}$) as the 2-dimensional vector of components κ_1 and κ_2 :

$$
\boldsymbol{\omega} = \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\kappa b} \cdot \boldsymbol{d}_1 \\ \boldsymbol{\kappa b} \cdot \boldsymbol{d}_2 \end{bmatrix} = \begin{bmatrix} -\boldsymbol{x}'' \cdot \boldsymbol{d}_2 \\ \boldsymbol{x}'' \cdot \boldsymbol{d}_1 \end{bmatrix}
\tag{4.16}
$$

4.4 Definition of the variational problem

We now have all the ingredients to build the variational problem that will lead us to the determination of the internal forces ($\boldsymbol{f}$) and the internal twisting moment ($\boldsymbol{m}$) acting on the centerline. Indeed, the quasi-static out-of-balance internal forces and twisting moment acting on the beam are calculated by differentiating the elastic energy of the system with respect to the 4 DOFs of the system, namely $\boldsymbol{x}$ and θ.

4.4.1 Calculus of variations

Differentiating $\mathcal{E}$ with respect to $\boldsymbol{x}$ will yield the linear resultant of the internal forces ($\boldsymbol{f}$, see §4.5), while differentiating $\mathcal{E}$ with respect to θ will yield the linear resultant of the

internal twisting moment ($\boldsymbol{m}$, see §4.6). Main results for the calculus of variations are reminded in **??**.

These forces and moments will be used later in a damped explicit dynamic procedure to solve the equilibrium of the system. However, they are nothing but the gradient of the elastic energy and any other variational method can be employed to find a geometric configuration that minimize the elastic energy, that is a configuration in which the rod is at static equilibrium (our ultime goal).

Introducing $\boldsymbol{\omega}$ and θ, the elastic energy defined in eq. (4.10) can be rewritten in the form :

$$\mathcal{E} = \mathcal{E}_b + \mathcal{E}_t = \tfrac{1}{2}\int_0^L (\boldsymbol{\omega} - \overline{\boldsymbol{\omega}})^T B(\boldsymbol{\omega} - \overline{\boldsymbol{\omega}})ds + \tfrac{1}{2}\int_0^L \beta(\theta' - \overline{\theta}')^2 ds \qquad (4.17)$$

where B is the bending stiffness matrix given in material frame coordinate system and β is the torsional stiffness given by :

$$B(s) = \begin{bmatrix} EI_1(s) & 0 \\ 0 & EI_2(s) \end{bmatrix} \quad , \quad \beta(s) = (GJ)(s) \qquad (4.18)$$

The matrix notation introduced in eq. (4.17) will enable more compact forms for equations. It will be used through out this chapter. Remark who the scalar product of vectors is treated as matrix multiplication with a row and column vector.

The internal shear forces and the internal twisting moment are then given by :

$$\boldsymbol{f} \;\; = -\frac{\partial \mathcal{E}}{\partial \boldsymbol{x}}[\boldsymbol{x}, \theta] \qquad (4.19\text{a})$$

$$\boldsymbol{m} = -\frac{\partial \mathcal{E}}{\partial \theta}[\boldsymbol{x}, \theta] \cdot \boldsymbol{d}_3 \qquad (4.19\text{b})$$

It is important to notice that as the rod is supposed to be inextensible, the elastic energy does not include any term to characterized the axial strain. This assumption will have to be treated as a constraint when solving the variational problem. For instance, Bergou et al. 2008 [4] choose to enforce inextensibility at each time step with a reprojection algorithm while Lefevre et al. 2017 [3] choose to enforce this constraint thanks to a penalty energy. Hence, $\boldsymbol{f}$ will give only the internal shear forces acting on the centerline.[8]

4.4.2 Prerequisite for the computation of energy gradients

To compute the energy gradients with respect to the degrees of freedom of the rod it is primordial to understand how these DOFs are chained (see fig. 4.2) :

- $\boldsymbol{x}$ leads to the determination of the curvature binormal ($\boldsymbol{\kappa}\boldsymbol{b}$) and to the determination of the reference Bishop frame $\{\boldsymbol{t}, \boldsymbol{u}, \boldsymbol{v}\}$.

---

[8]This is somehow equivalent to what is remarked by Dill : "The resultant shears F_1 and F_2 are not determined by the constitutive equations. They are reactive parameters in the equations of balance of momentum as they are in the elementary beam theory." [14].

$$s \xrightarrow{\;\;x\;\;} \{t, u, v\} \xrightarrow{\;\;\theta\;\;} \{d_3, d_1, d_2\} \xrightarrow{\quad\quad} \tau, \kappa_1, \kappa_2$$

Figure 4.2 – Succession of the degrees of freedom.

- θ leads to the determination of the material frame $\{t, d_1, d_1\}$ from the reference frame $\{t, u, v\}$.

- κ_1 and κ_2 are the projections of κb on d_1 and d_2. This means that ω depends on x through the determination of κb and the determination of $\{t, u, v\}$; and depends on θ through the determination of $\{d_1, d_2\}$ from $\{u, v\}$.

- τ only depends on θ and is thus independent of x.

4.4.3   Coupling between bending and torsion

Remark that the twisting energy ($\mathcal{E}_t$) only depends on θ and is independent regarding x while the bending energy ($\mathcal{E}_b$) depends on both θ and x variables (remind that κ_1 and κ_2 are the projections of κb over d_1 over d_2). Thus, a coupling between bending and twisting appears and the minimum of the whole elastic energy is not necessarily reached for concomitant minimums of bending and twisting energies.

4.4.4   Quasistatic assumption

Following Bergou et al. 2008 [4], it is relevant to assume that the propagation of twist waves is instantaneous compared to the propagation of bending waves because for slender rods the axial stiffness is usually an order of magnitude higher than the bending stiffness. This means that the distributed internal forces (f) and the distributed internal moment of torsion (m) act on two different timescales in the rod dynamic.

Thus, on the timescale of action of the internal forces on the center line, driving the bending waves, the twist waves propagate instantaneously, so that :

$$\frac{\partial \mathcal{E}}{\partial \theta}[x, \theta] = 0 \tag{4.20}$$

for the computation of f.

This assumption is not mandatory – for instance it is not made by Nabaei 2014 [6] – but was found to lead to simpler and faster computations.

4.5   Energy gradient with respect to θ : twisting moment

For the calculus of variations, the reader is invited to refer to **??** where the notations employed through out this section are defined and where main results are reminded.

4.5.1 Derivative of material directors with respect to θ

Recalling that θ and $\boldsymbol{x}$ are independant variables and that the Bishop frame $\{\boldsymbol{u}, \boldsymbol{v}\}$ only depends on $\boldsymbol{x}$, the decomposition of the material frame directors $\{\boldsymbol{d}_1, \boldsymbol{d}_2\}$ on the Bishop frame leads directly to the following expression for the derivative of the material directors :

$$
\begin{aligned}
\boldsymbol{D}_\theta \boldsymbol{d}_1(s) \cdot h_\theta &= \frac{d}{d\lambda} \boldsymbol{d}_1[\theta + \lambda h_\theta]\Big|_{\lambda=0} \\
&= (-\sin\theta\, \boldsymbol{u} + \cos\theta\, \boldsymbol{v}) \cdot h_\theta \\
&= \boldsymbol{d}_2 \cdot h_\theta
\end{aligned}
\tag{4.21a}
$$

$$
\begin{aligned}
\boldsymbol{D}_\theta \boldsymbol{d}_2(s) \cdot h_\theta &= \frac{d}{d\lambda} \boldsymbol{d}_2[\theta + \lambda h_\theta]\Big|_{\lambda=0} \\
&= (-\cos\theta\, \boldsymbol{u} - \sin\theta\, \boldsymbol{v}) \cdot h_\theta \\
&= -\boldsymbol{d}_1 \cdot h_\theta
\end{aligned}
\tag{4.21b}
$$

where $h_\theta : s \longmapsto h_\theta(s)$ denotes a small perturbation of θ and $\boldsymbol{D}_\theta \boldsymbol{d}_i(s)$ denotes the derivative of $\boldsymbol{d}_i$ at s with respect to θ. When it is appropriate, brackets are employed to signal important functional dependencies, while parenthesis will denote the dependence with respect to the arc length parameter s.

4.5.2 Derivative of the material curvatures vector with respect to θ

Regarding the definition of the material curvatures vector and the derivative of material directors with respect to θ, it follows immediately that :

$$
\begin{aligned}
\boldsymbol{D}_\theta \boldsymbol{\omega}(s) \cdot h_\theta &= \frac{d}{d\lambda} \boldsymbol{\omega}[\theta + \lambda h_\theta]\Big|_{\lambda=0} \\
&= \begin{bmatrix} \boldsymbol{\kappa b} \cdot \boldsymbol{d}_2 \\ -\boldsymbol{\kappa b} \cdot \boldsymbol{d}_1 \end{bmatrix} \cdot h_\theta \\
&= -\boldsymbol{J}\boldsymbol{\omega} \cdot h_\theta
\end{aligned}
\tag{4.22}
$$

where $\boldsymbol{J}$ is the matrix that acts on two dimensional vectors by counter-clockwise rotation of angle $\frac{\pi}{2}$:

$$
\boldsymbol{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\tag{4.23}
$$

4.5.3 Computation of the twisting moment

The distributed moment of torsion is given by the functional derivative of the elastic energy with respect to θ, which can be decomposed into :

$$
\begin{aligned}
\langle -m(s)\,, h_\theta \rangle &= \boldsymbol{D}_\theta \mathcal{E}(s) \cdot h_\theta \\
&= \boldsymbol{D}_\theta \mathcal{E}_b(s) \cdot h_\theta + \boldsymbol{D}_\theta \mathcal{E}_t(s) \cdot h_\theta
\end{aligned}
\tag{4.24}
$$

Derivative of the torsion energy with respect to θ

We calculate the partial derivative of the twisting elastic energy with respect to θ as :

$$
\begin{aligned}
\boldsymbol{D}_\theta \mathcal{E}_t[\theta](s) \cdot h_\theta &= \frac{d}{d\lambda} \mathcal{E}_t[\theta + \lambda h_\theta]\Big|_{\lambda=0} \\
&= \frac{d}{d\lambda} \left(\tfrac{1}{2} \int_0^L \beta \left((\theta + \lambda h_\theta)' - \overline{\theta}' \right)^2 \, dt \right)\Big|_{\lambda=0} \\
&= \int_0^L \beta(\theta' - \overline{\theta}') \cdot h_\theta' \, dt \\
&= \left[\beta(\theta' - \overline{\theta}') \cdot h_\theta \right]_0^L - \int_0^L \left(\beta(\theta' - \overline{\theta}') \right)' \cdot h_\theta \, dt \\
&= \int_0^L \left(\beta(\theta' - \overline{\theta}')(\delta_L - \delta_0) - \left(\beta(\theta' - \overline{\theta}') \right)' \right) \cdot h_\theta \, dt
\end{aligned}
\tag{4.25}
$$

where δ_L and δ_0 are the Dirac distributions centered respectively on $s = L$ and $s = \check{r}$.

Derivative of the bending energy with respect to θ

To compute the partial derivative of $\mathcal{E}_b$ with respect to θ we first calculate the derivative of $\mathcal{E}_b$ with respect to $\boldsymbol{\omega}$:

$$
\begin{aligned}
\boldsymbol{D}_\omega \mathcal{E}_b[\boldsymbol{\omega}](s) \cdot \boldsymbol{h}_\omega &= \frac{d}{d\lambda} \mathcal{E}_b[\boldsymbol{\omega} + \lambda \boldsymbol{h}_\omega]\Big|_{\lambda=0} \\
&= \frac{d}{d\lambda} \left(\tfrac{1}{2} \int_0^L \left((\boldsymbol{\omega} + \lambda \boldsymbol{h}_\omega) - \overline{\boldsymbol{\omega}} \right)^T \boldsymbol{B} \left((\boldsymbol{\omega} + \lambda \boldsymbol{h}_\omega) - \overline{\boldsymbol{\omega}} \right) \, dt \right)\Big|_{\lambda=0} \\
&= \int_0^L (\boldsymbol{\omega} - \overline{\boldsymbol{\omega}})^T \boldsymbol{B} \cdot \boldsymbol{h}_\omega \, dt
\end{aligned}
\tag{4.26}
$$

where $\boldsymbol{h}_\omega : s \longmapsto \boldsymbol{h}_\omega(s)$ denotes a small perturbation of $\boldsymbol{\omega}$. Then, we calculate the derivative of $\mathcal{E}_b$ with respect to θ from the chain rule and with eq. (4.22) and (4.26) :

$$
\begin{aligned}
\boldsymbol{D}_\theta \mathcal{E}_b[\boldsymbol{\omega}[\theta]](s) \cdot h_\theta &= \boldsymbol{D}_\omega \mathcal{E}_b[\boldsymbol{\omega}](s) \cdot (\boldsymbol{D}_\theta \boldsymbol{\omega}[\theta](s) \cdot h_\theta) \\
&= - \int_0^L (\boldsymbol{\omega} - \overline{\boldsymbol{\omega}})^T \boldsymbol{B} \boldsymbol{J} \boldsymbol{\omega} \cdot h_\theta \, dt
\end{aligned}
\tag{4.27}
$$

Twisting moment

The gradient of the elastic energy with respect to θ is obtained from eq. (4.24) with eq. (4.25) and (4.27) :

$$
\begin{aligned}
\langle -m(s)\,, h_\theta \rangle &= \boldsymbol{D}_\theta \mathcal{E}_b[\boldsymbol{\omega}[\theta]](s) \cdot h_\theta + \boldsymbol{D}_\theta \mathcal{E}_t[\theta](s) \cdot h_\theta \\
&= \int_0^L \left(\left(\beta(\theta' - \overline{\theta}')(\delta_L - \delta_0) - (\beta(\theta' - \overline{\theta}'))' \right) - (\boldsymbol{\omega} - \overline{\boldsymbol{\omega}})^T \boldsymbol{B} \boldsymbol{J} \boldsymbol{\omega} \right) \cdot h_\theta \; dt
\end{aligned}
\tag{4.28}
$$

Finally, we can conclude on the expression of the distributed internal twisting moment :

$$
m(s) = -\left(\beta(\theta' - \overline{\theta}')(\delta_L - \delta_0) - (\beta\,(\theta' - \overline{\theta}'))' \right)(s) + \left((\boldsymbol{\omega} - \overline{\boldsymbol{\omega}})^T \boldsymbol{B} \boldsymbol{J} \boldsymbol{\omega} \right)(s)
\tag{4.29}
$$

Remark with eq. (4.8a) and (4.15), respectively with eq. (4.8b) and (4.8c), that :

$$
\beta\,(\theta' - \overline{\theta}') = Q
\tag{4.30a}
$$

$$
(\boldsymbol{\omega} - \overline{\boldsymbol{\omega}})^T \boldsymbol{B} \boldsymbol{J} \boldsymbol{\omega} = \kappa_1 M_2 - \kappa_2 M_1
\tag{4.30b}
$$

Hence, the quasi-static distributed internal twisting moment acting on the centerline is given for all s in $]0, L[$ by :

$$
m(s) = Q'(s) + \kappa_1(s) M_2(s) - \kappa_2(s) M_1(s)
\tag{4.31}
$$

Quasistatic hypothesis

The quasistatic assumption (see eq. (4.20)) stipulates that the gradient of the elastic energy with respect to θ can be considered null ($\boldsymbol{D}_\theta \mathcal{E}(s) = 0$) for the calculation of the internal forces, which implies that for all s in $[0, L]$:

$$
\left((\beta(\theta' - \overline{\theta}'))' + (\boldsymbol{\omega} - \overline{\boldsymbol{\omega}})^T \boldsymbol{B} \boldsymbol{J} \boldsymbol{\omega} \right)(s) = 0
\tag{4.32}
$$

or equivalently :

$$
Q'(s) + \kappa_1(s) M_2(s) - \kappa_2(s) M_1(s) = 0
\tag{4.33}
$$

4.6   Energy gradient with respect to x : internal forces

For the calculus of variations, the reader is invited to refer to **??** where the notations employed through out this section are defined and where main results are reminded.

4.6.1   Derivative of material directors with respect to x

A variation of the centerline $\boldsymbol{x}$ by $\boldsymbol{\epsilon} = \lambda \boldsymbol{h}_x$ would cause a variation of the Bishop frame because parallel transport depends on the centerline itself. As far as $\boldsymbol{x}$ and θ are independent variables, this leads necessarily to a variation of the material frame. Let us denote :

- $F = \{t, u, v\}$: the Bishop frame in the reference configuration ;

- $F_\epsilon = \{t_\epsilon, u_\epsilon, v_\epsilon\}$: the Bishop frame in the deformed configuration ;

- $\tilde{F}_\epsilon = \{t, \tilde{u}_\epsilon, \tilde{v}_\epsilon\}$: the frame obtained by parallel transporting F_ϵ back on F.

From fig. 4.3 we start with an (arbitrary) initial reference frame defined in the rest configuration and denoted $F(0)$, positioned at arc length parameter $s = 0$:

- Firstly, the intial frame $F(0)$ is parallel transported along the centerline of the rest configuration into the frame $F(s)$ at arc length parameter s (see fig. 4.3).

- Secondly the initial frame $F(0)$ is parallel transported at the starting point of the centerline of the deformed configuration into the frame $F_\epsilon(0)$. The frame $F_\epsilon(0)$ is then parallel transported along the centerline of the deformed configuration into the frame $F_\epsilon(s)$ at arc length parameter s. Finally, $F_\epsilon(s)$ is parallel transported back onto the frame $F(s)$ of the reference configuration. This frame is denoted $\tilde{F}_\epsilon(s)$ (see fig. 4.3).

The two frames $F(s)$ and $\tilde{F}_\epsilon(s)$ are not aligned as a variation of the centerline has caused a variation of the bishop frame. We call $\Psi_\epsilon(s)$ the amount of rotation around $t(s)$ needed to realign $\tilde{F}_\epsilon(s)$ onto $F(s)$ (see fig. 4.3 where $\Psi_\epsilon(s) < 0$). $\Psi_\epsilon(s)$ characterizes precisely the variation of the parallel transport operator with respect to a perturbation of the centerline.

The sequence of transformations described previously and illustrated in fig. 4.3 can be decomposed into only two rotations that contribute to $\Psi_\epsilon(s)$:

- $F_\epsilon \rightarrow \tilde{F}_\epsilon$: parallel transporting F_ϵ from t_ϵ to t. This is equivalent to a rotation around $b = t_\epsilon \times t$ by an angle α_ϵ. This rotation is described in fig. 4.4.

- $\tilde{F}_\epsilon \rightarrow F$: aligning $\tilde{F}_\epsilon$ over F. This is equivalent to a rotation around t by an angle Ψ_ϵ. This rotation is described in fig. 4.5.

Firstly, let's decompose $\{t_\epsilon, u_\epsilon, v_\epsilon\}$ on the basis $\{t, \tilde{u}_\epsilon, \tilde{v}_\epsilon\}$. Note that $\tilde{F}_\epsilon$ is expressed by rotating $\tilde{F}_\epsilon$ by an angle $\Psi_\epsilon[x](s)$ around t because $\tilde{F}_\epsilon$ is obtained by parallel transporting F_ϵ from t_ϵ to t.

Calculation of Ψ_ϵ

This variation is closely related to the writhe of closed curves. As explained by Fuller 1978 [8] when parallel transporting an adapted frame around a closed curve it might not realigned with itself after one complete loop. This "lack of alignement" is directly measured by the change of writhe which can be computed with Fuller's Formula [8].

Note that the derivative of θ with respect to x can be evaluated by the change of writhe in the curve as suggested in [9]. This approach is completly equivalent.
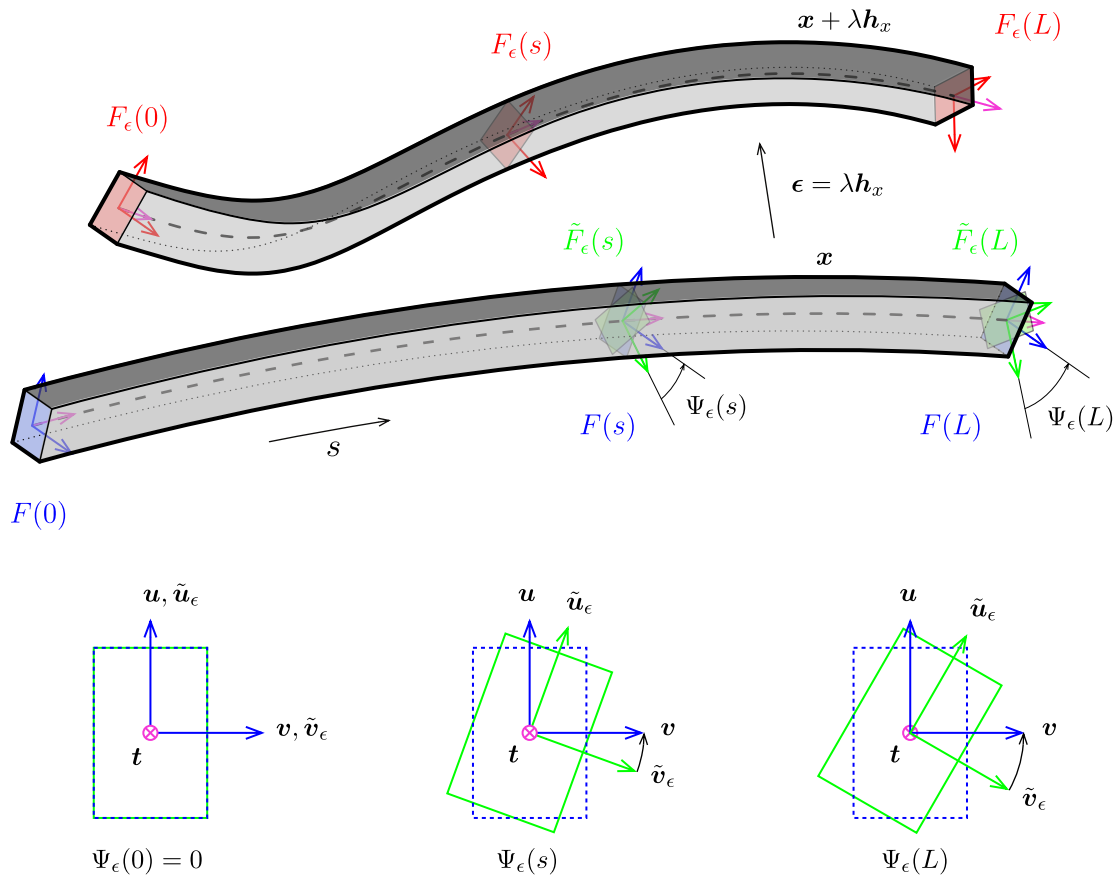
Figure 4.3 − Variation of the Bishop frame for a perturbation of the centerline.
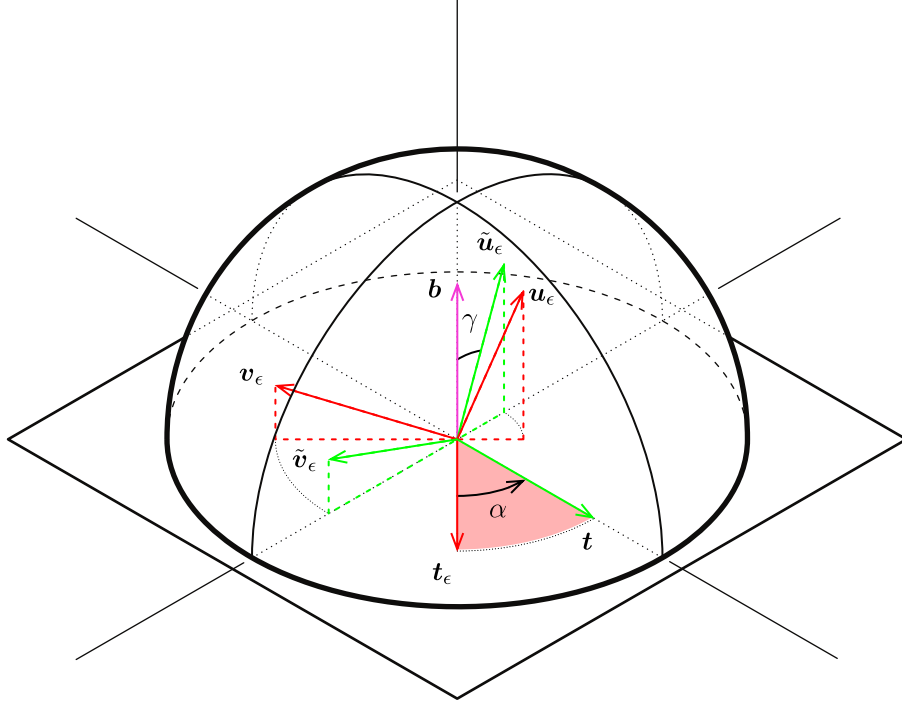
Figure 4.4 – Measuring the variation of parallel transport (α_ϵ). $\tilde{F}_\epsilon$ is obtained by parallel transpsorting F_ϵ from $\boldsymbol{t}_\epsilon$ to $\boldsymbol{t}$. This operation could be seen as a rotation around the axis $\boldsymbol{b} = \boldsymbol{t}_\epsilon \times \boldsymbol{t}$ by an angle α_ϵ.
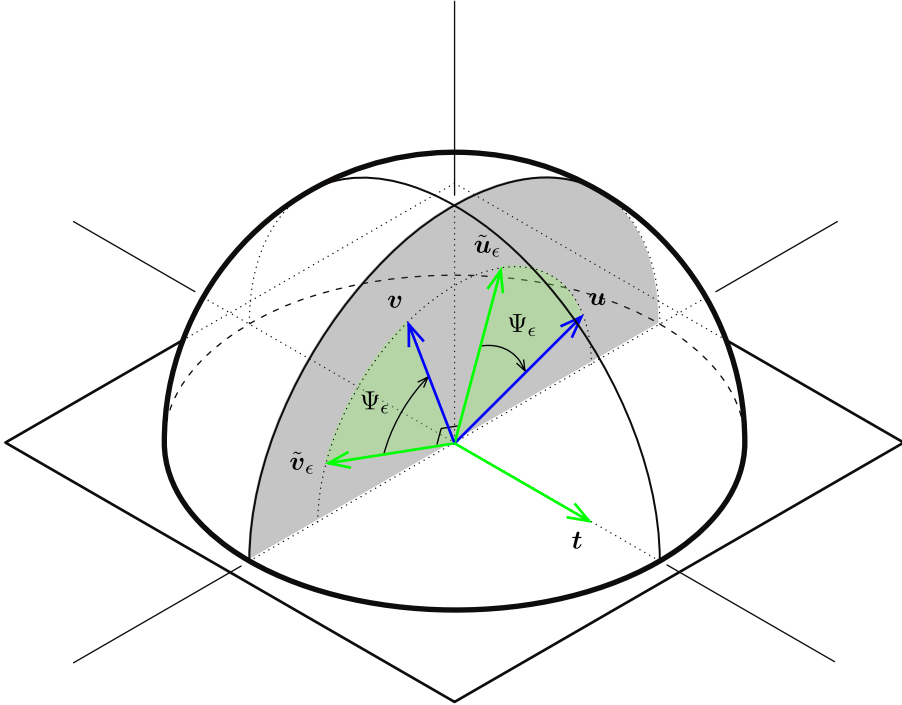
Figure 4.5 – Measuring the variation of parallel transport (Ψ_ϵ). F is obtained by rotating $\tilde{F}_\epsilon$ around $\boldsymbol{t}$ by an angle Ψ_ϵ.

One can also see this lack of alignement in terms of rotation. Parallel transport being a propagation of frame from $s = 0$, the cumulated rotation of the Bishop frame from the deformed configuration around the initial configuration at arc length s is the cumulated angle of rotation of $\boldsymbol{u}[\boldsymbol{x} + \lambda\boldsymbol{h}_x]$ around $\boldsymbol{d}_3[\boldsymbol{x}]$. Recalling that the rotation rate of $\boldsymbol{u}[\boldsymbol{x} + \lambda\boldsymbol{h}_x]$ is $\boldsymbol{\kappa b}[\boldsymbol{x} + \lambda\boldsymbol{h}_x]$ by definition of zero-twisting frame, one can write :

$$\Psi_\epsilon[\boldsymbol{x}](s) = -\int_0^s \boldsymbol{\kappa b}[\boldsymbol{x} + \lambda\boldsymbol{h}_x] \cdot \boldsymbol{d}_3[\boldsymbol{x}]\, dt \tag{4.34}$$

The calculation of $\boldsymbol{\kappa b}[\boldsymbol{x} + \lambda\boldsymbol{h}_x]$ is straight forward from the definition of the curvature binormal (see eq. (4.12a)) :

$$\begin{aligned}
\boldsymbol{\kappa b}[\boldsymbol{x} + \lambda\boldsymbol{h}_x] &= (\boldsymbol{x} + \lambda\boldsymbol{h}_x)' \times (\boldsymbol{x} + \lambda\boldsymbol{h}_x)'' \\
&= \boldsymbol{\kappa b}[\boldsymbol{x}] + \lambda(\boldsymbol{x}' \times \boldsymbol{h}_x'' + \boldsymbol{h}_x' \times \boldsymbol{x}'') + \lambda^2(\boldsymbol{h}_x' \times \boldsymbol{h}_x'') \\
&= \boldsymbol{\kappa b}[\boldsymbol{x}] + \lambda(\boldsymbol{x}' \times \boldsymbol{h}_x'' + \boldsymbol{h}_x' \times \boldsymbol{x}'') + o(\lambda)
\end{aligned} \tag{4.35}$$

Thus, reminding that $\boldsymbol{d}_3[\boldsymbol{x}] = \boldsymbol{x}'$ and $\boldsymbol{\kappa b}[\boldsymbol{x}] \cdot \boldsymbol{d}_3[\boldsymbol{x}] = 0$, and using the invariance of circular product by cyclic permutation, one can express :

$$\begin{aligned}
\Psi_\epsilon[\boldsymbol{x}](s) &= -\int_0^s \boldsymbol{\kappa b}[\boldsymbol{x} + \lambda\boldsymbol{h}_x] \cdot \boldsymbol{d}_3[\boldsymbol{x}]\, dt \\
&= -\lambda \int_0^s (\boldsymbol{x}' \times \boldsymbol{h}_x'' + \boldsymbol{h}_x' \times \boldsymbol{x}'') \cdot \boldsymbol{x}'\, dt + o(\lambda) \\
&= -\lambda \int_0^s \boldsymbol{\kappa b}[\boldsymbol{x}] \cdot \boldsymbol{h}_x'\, dt + o(\lambda)
\end{aligned} \tag{4.36}$$

By integration by parts and dropping the implicit reference to $\boldsymbol{x}$ in the notation, $\Psi_\epsilon(s)$ can be rewritten as :

$$\begin{aligned}
\Psi_\epsilon(s) &= -\lambda \int_0^s \boldsymbol{\kappa b} \cdot \boldsymbol{h}_x'\, dt + o(\lambda) \\
&= -\lambda \left(\left[\boldsymbol{\kappa b} \cdot \boldsymbol{h}_x \right]_0^s - \int_0^s \boldsymbol{\kappa b}' \cdot \boldsymbol{h}_x\, dt \right) + o(\lambda) \\
&= -\lambda \left(\int_0^s \left((\delta_s - \delta_0)\,\boldsymbol{\kappa b} - \boldsymbol{\kappa b}' \right) \cdot \boldsymbol{h}_x\, dt \right) + o(\lambda) \\
&= -\lambda \left(\int_0^L \left((\delta_s - \delta_0)\,\boldsymbol{\kappa b} - (1 - H_s)\boldsymbol{\kappa b}' \right) \cdot \boldsymbol{h}_x\, dt \right) + o(\lambda)
\end{aligned} \tag{4.37}$$

where δ_s and H_s are the Dirac function and the Heaviside step function centered at s :

$$H_s : t \mapsto \begin{cases} 0, & t < s \\ 1, & t \geqslant s \end{cases} \tag{4.38a}$$

$$\delta_s : t \mapsto \delta(t - s) \tag{4.38b}$$

Note that, as expected, $\Psi_\epsilon(s)$ is in first order of λ and thus gets negligible when λ tends to zero, that is to say when the perturbation of $\boldsymbol{x}$ is infinitesimal :

$$\lim_{\lambda \to 0} \Psi_\epsilon(s) = 0 \tag{4.39}$$

Calculation of α_ϵ

Recall that $\tilde{F}_\epsilon$ is obtained by parallel transporting F_ϵ from $\boldsymbol{t}_\epsilon$ to $\boldsymbol{t}$. $\tilde{F}_\epsilon$ results from the rotation of F_ϵ around $\boldsymbol{b} = \boldsymbol{t}_\epsilon \times \boldsymbol{t}$ by an angle α_ϵ.

Recall from eq. (4.2) that because the rod is supposed to be inextensible, $\boldsymbol{t}_\epsilon$ stays collinear to $\boldsymbol{t}$, at first order in λ, for an infinitesimal perturbation of the centerline :

$$\|\boldsymbol{t}\| = \|\boldsymbol{t}_\epsilon\| = 1 \Rightarrow (\boldsymbol{x} + \boldsymbol{\epsilon})' \cdot (\boldsymbol{x} + \boldsymbol{\epsilon})' = 1 \Leftrightarrow \boldsymbol{x}' \cdot \boldsymbol{\epsilon}' = -\tfrac{\lambda^2}{2}\|\boldsymbol{h}'_x\|^2 \tag{4.40}$$

Which yields :

$$\cos \alpha_\epsilon = \boldsymbol{t} \cdot \boldsymbol{t}_\epsilon = \boldsymbol{x}' \cdot (\boldsymbol{x} + \boldsymbol{\epsilon})' = 1 + \boldsymbol{x}' \cdot \boldsymbol{\epsilon}' = 1 - \tfrac{\lambda^2}{2}\|\boldsymbol{h}'_x\|^2 \tag{4.41}$$

Remark that the second order of the developpement is also accessible and can lead to the computation of the hessian of the system, which might be useful for improving the convergence of the minimization algorithm :

$$\cos \alpha_\epsilon = 1 - \tfrac{\lambda^2}{2}\|\boldsymbol{h}'_x\|^2 \tag{4.42a}$$
$$\sin \alpha_\epsilon = \sqrt{1 - \cos^2 \alpha_\epsilon} = \lambda\|\boldsymbol{h}'_x\| + o(\lambda^2) \tag{4.42b}$$
$$\sin^2 \alpha_\epsilon/2 = \tfrac{\lambda^2}{4}\|\boldsymbol{h}'_x\|^2 \tag{4.42c}$$

Finally, it's possible to conclude that $\alpha_\epsilon(s)$ is in first order of λ and thus gets negligible when λ tends to zero :

$$\lim_{\lambda \to 0} \alpha_\epsilon(s) = 0 \tag{4.43}$$

Aligning $\tilde{F}_\epsilon$ towards F_ϵ

Recall that aligning $\tilde{F}_\epsilon$ over F is nothing but a rotation around $\boldsymbol{t}$ by an angle Ψ_ϵ. This leads to :

$$\tilde{\boldsymbol{u}}_\epsilon = \cos \Psi_\epsilon\, \boldsymbol{u} + \sin \Psi_\epsilon\, \boldsymbol{v} \tag{4.44a}$$
$$\tilde{\boldsymbol{v}}_\epsilon = -\sin \Psi_\epsilon\, \boldsymbol{u} + \cos \Psi_\epsilon\, \boldsymbol{v} \tag{4.44b}$$

Aligning F_ϵ towards $\boldsymbol{t}$

Recall that $\tilde{F}_\epsilon$ is obtained by parallel transporting F_ϵ from $\boldsymbol{t}_\epsilon$ to $\boldsymbol{t}$. This operation could be seen as a rotation around $\boldsymbol{t}_\epsilon \times \boldsymbol{t}$ of an angle α_ϵ. Where :

$$\boldsymbol{b} = \boldsymbol{t}_\epsilon \times \boldsymbol{t} = \cos \gamma\, \tilde{\boldsymbol{u}}_\epsilon + \sin \gamma\, \tilde{\boldsymbol{v}}_\epsilon = \cos \gamma\, \boldsymbol{u}_\epsilon + \sin \gamma\, \boldsymbol{v}_\epsilon \tag{4.45}$$

Expressing F_ϵ on the basis $\tilde{F}_\epsilon$ gives for $\boldsymbol{u}_\epsilon$ and $\boldsymbol{v}_\epsilon$:

$$\boldsymbol{u}_\epsilon = \sin\gamma\, \boldsymbol{b} + \cos\gamma \left(\sin\alpha_\epsilon\, \tilde{\boldsymbol{t}} + \cos\alpha_\epsilon \left(\cos\gamma\, \tilde{\boldsymbol{u}}_\epsilon - \sin\gamma\, \tilde{\boldsymbol{v}}_\epsilon \right) \right) \tag{4.46a}$$

$$\boldsymbol{v}_\epsilon = \cos\gamma\, \boldsymbol{b} + \sin\gamma \left(-\sin\alpha_\epsilon\, \tilde{\boldsymbol{t}} + \cos\alpha_\epsilon \left(\sin\gamma\, \tilde{\boldsymbol{u}}_\epsilon - \cos\gamma\, \tilde{\boldsymbol{v}}_\epsilon \right) \right) \tag{4.46b}$$

Which can be rearranged in :

$$\boldsymbol{u}_\epsilon = \cos\gamma \sin\alpha_\epsilon\, \boldsymbol{t} + (\cos\alpha_\epsilon \cos^2\gamma + \cos^2\gamma)\tilde{\boldsymbol{u}}_\epsilon + \sin\gamma \cos\gamma\, (1-\cos\alpha_\epsilon)\tilde{\boldsymbol{v}}_\epsilon \tag{4.47a}$$

$$\boldsymbol{v}_\epsilon = -\sin\gamma \sin\alpha_\epsilon\, \boldsymbol{t} + \cos\gamma \sin\gamma\, (1-\cos\alpha_\epsilon)\tilde{\boldsymbol{u}}_\epsilon + (\cos^2\gamma + \cos\alpha_\epsilon \sin^2\gamma)\tilde{\boldsymbol{v}}_\epsilon \tag{4.47b}$$

Variation of the Bishop frame with respect to x

Finally, one can express F_ϵ on the basis F as the composition of two rotations :

$$\boldsymbol{u}_\epsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Psi_\epsilon & -\sin\Psi_\epsilon \\ 0 & \sin\Psi_\epsilon & \cos\Psi_\epsilon \end{bmatrix} \begin{bmatrix} \cos\gamma \sin\alpha_\epsilon \\ 1 - 2\cos^2\gamma \sin^2\alpha_\epsilon/2 \\ 2\sin\gamma \cos\gamma \sin^2\alpha_\epsilon/2 \end{bmatrix} = \begin{bmatrix} \alpha_\epsilon \cos\gamma \\ 1 \\ \Psi_\epsilon \end{bmatrix} + o(\lambda) \tag{4.48a}$$

$$\boldsymbol{v}_\epsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Psi_\epsilon & -\sin\Psi_\epsilon \\ 0 & \sin\Psi_\epsilon & \cos\Psi_\epsilon \end{bmatrix} \begin{bmatrix} -\sin\gamma \sin\alpha_\epsilon \\ 2\sin\gamma \cos\gamma \sin^2\alpha_\epsilon/2 \\ 1 - 2\sin\gamma^2 \sin^2\alpha_\epsilon/2 \end{bmatrix} = \begin{bmatrix} -\alpha_\epsilon \sin\gamma \\ -\Psi_\epsilon \\ 1 \end{bmatrix} + o(\lambda) \tag{4.48b}$$

Here, the expressions have been developed in first order of λ. It has been prooved in eq. (4.39) and (4.43) that α_ϵ and Ψ_ϵ tend toward zero when the perturbation of the centerline is infinitesimal.

Finally, one can express the variation of the material directors with respect to an infinitesimal variation of rod's centerline by :

$$\boldsymbol{u}_\epsilon = \alpha_\epsilon \cos\gamma\, \boldsymbol{t} + \boldsymbol{u} + \Psi_\epsilon \boldsymbol{v} + o(\lambda) \tag{4.49a}$$

$$\boldsymbol{v}_\epsilon = -\alpha_\epsilon \sin\gamma\, \boldsymbol{t} + \boldsymbol{v} - \Psi_\epsilon \boldsymbol{u} + o(\lambda) \tag{4.49b}$$

Variation of the material frame with respect to $\boldsymbol{x}$

Recalling the expression of the material frame expressed in the reference Bishop frame, it's now easy to deduce the variation of material frame with respect to a variation of the rod's centerline :

$$\boldsymbol{d}_1[\boldsymbol{x} + \lambda\boldsymbol{h}_x] = \cos\theta\, \boldsymbol{u}_\epsilon + \sin\theta\, \boldsymbol{v}_\epsilon \tag{4.50a}$$

$$\boldsymbol{d}_2[\boldsymbol{x} + \lambda\boldsymbol{h}_x] = -\sin\theta\, \boldsymbol{u}_\epsilon + \cos\theta\, \boldsymbol{v}_\epsilon \tag{4.50b}$$

Which leads according to the previous equation to :

$$\boldsymbol{d}_1[\boldsymbol{x} + \lambda\boldsymbol{h}_x] = \boldsymbol{d}_1[\boldsymbol{x}] + \Psi_\epsilon \boldsymbol{d}_2[\boldsymbol{x}] + \alpha_\epsilon \cos(\theta - \gamma)\, \boldsymbol{t}[\boldsymbol{x}] + o(\lambda) \tag{4.51a}$$

$$\boldsymbol{d}_2[\boldsymbol{x} + \lambda\boldsymbol{h}_x] = \boldsymbol{d}_2[\boldsymbol{x}] - \Psi_\epsilon \boldsymbol{d}_1[\boldsymbol{x}] - \alpha_\epsilon \sin(\theta + \gamma)\, \boldsymbol{t}[\boldsymbol{x}] + o(\lambda) \tag{4.51b}$$

4.6.2 Derivative of the vector of material curvatures with respect to x

It is now straightforward from the previous section to express the variation of the material curvatures with respect to a variation $\boldsymbol{\epsilon} = \lambda \boldsymbol{h}_x$ of $\boldsymbol{x}$ while θ remains unchanged :

$$(\boldsymbol{x} + \lambda \boldsymbol{h}_x)'' \cdot \boldsymbol{d}_1[\boldsymbol{x} + \lambda \boldsymbol{h}_x] = (\boldsymbol{x}'' + \lambda \boldsymbol{h}_x'') \tag{4.52a}$$
$$\cdot \left(\boldsymbol{d}_1 + \Psi_\epsilon \boldsymbol{d}_2 + \alpha_\epsilon \cos(\theta - \gamma) \, \boldsymbol{t} + o(\lambda) \right)$$

$$(\boldsymbol{x} + \lambda \boldsymbol{h}_x)'' \cdot \boldsymbol{d}_2[\boldsymbol{x} + \lambda \boldsymbol{h}_x] = (\boldsymbol{x}'' + \lambda \boldsymbol{h}_x'') \tag{4.52b}$$
$$\cdot \left(\boldsymbol{d}_2 - \Psi_\epsilon \boldsymbol{d}_1 - \alpha_\epsilon \sin(\theta + \gamma) \, \boldsymbol{t} + o(\lambda) \right)$$

Thus, recalling that $\boldsymbol{x}'' \cdot \boldsymbol{d}_3 = 0$ and that α_ϵ and Ψ_ϵ are first order quantities in λ :

$$(\boldsymbol{x} + \lambda \boldsymbol{h}_x)'' \cdot \boldsymbol{d}_1[\boldsymbol{x} + \lambda \boldsymbol{h}_x] = \boldsymbol{x}'' \cdot \boldsymbol{d}_1 + \Psi_\epsilon \boldsymbol{x}'' \cdot \boldsymbol{d}_2 + \lambda \boldsymbol{h}_x'' \cdot \boldsymbol{d}_1 + o(\lambda) \tag{4.53a}$$
$$(\boldsymbol{x} + \lambda \boldsymbol{h}_x)'' \cdot \boldsymbol{d}_2[\boldsymbol{x} + \lambda \boldsymbol{h}_x] = \boldsymbol{x}'' \cdot \boldsymbol{d}_2 - \Psi_\epsilon \boldsymbol{x}'' \cdot \boldsymbol{d}_1 + \lambda \boldsymbol{h}_x'' \cdot \boldsymbol{d}_2 + o(\lambda) \tag{4.53b}$$

Which finally leads to :

$$\boldsymbol{\omega}[\boldsymbol{x} + \lambda \boldsymbol{h}_x] = \boldsymbol{\omega}[\boldsymbol{x}] - \Psi_\epsilon \boldsymbol{J} \boldsymbol{\omega}[\boldsymbol{x}] + \lambda \begin{bmatrix} -\boldsymbol{h}_x'' \cdot \boldsymbol{d}_2 \\ \boldsymbol{h}_x'' \cdot \boldsymbol{d}_1 \end{bmatrix} + o(\lambda) \tag{4.54}$$

Reminding the expression of Ψ_ϵ computed in eq. (4.37), one can express the derivative of the vector of material curvatures with respect to $\boldsymbol{x}$ as :

$$\boldsymbol{D}_x \boldsymbol{\omega}(s) \cdot \boldsymbol{h}_x = \left(\int_0^L \left((\delta_s - \delta_0) \, \boldsymbol{\kappa} \boldsymbol{b} - (1 - H_s) \boldsymbol{\kappa} \boldsymbol{b}' \right) \cdot \boldsymbol{h}_x \, dt \right) \boldsymbol{J} \boldsymbol{\omega} + \begin{bmatrix} -\boldsymbol{d}_2^T \\ \boldsymbol{d}_1^T \end{bmatrix} \cdot \boldsymbol{h}_x'' \tag{4.55}$$

Here, we have introduce a condensed matrix notation to write $\begin{bmatrix} -\boldsymbol{d}_2^T \\ \boldsymbol{d}_1^T \end{bmatrix}$ as a 2x3 matrix. The entries of the first line are $-\boldsymbol{d}_2^T = \begin{bmatrix} 0, & 0, & -1 \end{bmatrix}$ while the entries of the second line are $\boldsymbol{d}_1^T = \begin{bmatrix} 0, & 1, & 0 \end{bmatrix}$. Thus, the matrix entries in the material frame coordinate system are given by :

$$\begin{bmatrix} -\boldsymbol{d}_2^T \\ \boldsymbol{d}_1^T \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{4.56}$$

Remark also how the scalar product $\boldsymbol{h}_x'' \cdot \boldsymbol{d}_2$ in vector notation is treated as a product in matrix notation : $\boldsymbol{d}_2^T \cdot \boldsymbol{h}_x'' = (\boldsymbol{h}_x'')^T \cdot \boldsymbol{d}_2$.

4.6.3 Computation of the forces acting on the centerline

The distributed internal forces acting on the centerline are given by the functional derivative of the elastic energy with respect to $\boldsymbol{x}$, which can be decomposed into :

$$\langle -\boldsymbol{f}(s), \boldsymbol{h}_x \rangle = \boldsymbol{D}_x \mathcal{E}(s) \cdot \boldsymbol{h}_x$$
$$= \boldsymbol{D}_x \mathcal{E}_b(s) \cdot \boldsymbol{h}_x + \boldsymbol{D}_x \mathcal{E}_t(s) \cdot \boldsymbol{h}_x \tag{4.57}$$

Derivative of the torsion energy with respect to x

Recall that the torsion energy only depends on θ which is independent of x. Thus $\mathcal{E}_t$ is independent of x and :

$$\boldsymbol{D}_x \mathcal{E}_t[\boldsymbol{x}](s) \cdot \boldsymbol{h}_x = \frac{d}{d\lambda} \mathcal{E}_t[\boldsymbol{x} + \lambda \boldsymbol{h}_x] \bigg|_{\lambda=0} = 0 \tag{4.58}$$

Derivative of the bending energy with respect to x

To compute the partial derivative of $\mathcal{E}_b$ with respect to $\boldsymbol{x}$ we first calculate the derivative of $\mathcal{E}_b$ with respect to $\boldsymbol{\omega}$:

$$\boldsymbol{D}_\omega \mathcal{E}_b[\boldsymbol{\omega}](s) \cdot \boldsymbol{h}_\omega = \frac{d}{d\lambda} \mathcal{E}_b[\boldsymbol{\omega} + \lambda \boldsymbol{h}_\omega] \bigg|_{\lambda=0} = \int_0^L (\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^T \boldsymbol{B} \cdot \boldsymbol{h}_\omega \, dt \tag{4.59}$$

Then, we calculate the partial derivative of $\mathcal{E}_b$ with respect to $\boldsymbol{x}$ from the chain rule and with eq. (4.55) and (4.59) :

$$\boldsymbol{D}_x \mathcal{E}_b[\boldsymbol{\omega}[\boldsymbol{x}]](s) \cdot \boldsymbol{h}_x = \boldsymbol{D}_\omega \mathcal{E}_b[\boldsymbol{\omega}](s) \cdot (\boldsymbol{D}_x \boldsymbol{\omega}[\boldsymbol{x}](s) \cdot \boldsymbol{h}_x) = \mathcal{A} + \mathcal{B} + \mathcal{C} \tag{4.60}$$

where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are given by :

$$\mathcal{A} = \int_0^L (\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^T \boldsymbol{B} \begin{bmatrix} -\boldsymbol{d_2}^T \\ \boldsymbol{d_1}^T \end{bmatrix} \cdot \boldsymbol{h}_x'' \, dt \tag{4.61a}$$

$$\mathcal{B} = \int_{t=0}^L (\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^T \boldsymbol{B} \boldsymbol{J} \boldsymbol{\omega} \left(\int_{u=0}^L (\delta_t - \delta_0) \, \boldsymbol{\kappa} \boldsymbol{b} \cdot \boldsymbol{h}_x \, du \right) dt \tag{4.61b}$$

$$\mathcal{C} = \int_{t=0}^L -(\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^T \boldsymbol{B} \boldsymbol{J} \boldsymbol{\omega} \left(\int_{u=0}^L (1 - H_t) \, \boldsymbol{\kappa} \boldsymbol{b}' \cdot \boldsymbol{h}_x \, du \right) dt \tag{4.61c}$$

Calculus of $\mathcal{A}$:

$$\mathcal{A} = \int_0^L (\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^T \boldsymbol{B} \begin{bmatrix} -\boldsymbol{d_2}^T \\ \boldsymbol{d_1}^T \end{bmatrix} \cdot \boldsymbol{h}_x'' \, dt \tag{4.62}$$

One can remark that the (row) vector found in eq. (4.62) can be rewritten as : [9]

$$(\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^T \boldsymbol{B} \begin{bmatrix} -\boldsymbol{d_2}^T \\ \boldsymbol{d_1}^T \end{bmatrix} = M_2 \boldsymbol{d_1}^T - M_1 \boldsymbol{d_2}^T = -(\boldsymbol{d_3} \times \boldsymbol{M})^T \tag{4.63}$$

---

[9]Here, we mix up vector and matrix notations. The matrix form of a vector is given by the components of the vector in the material frame coordinate system. For instance, $(\boldsymbol{d_3} \times \boldsymbol{M})^T$ is a row vector that writes in its matrix form : $\begin{bmatrix} 0, & -M2, & M1 \end{bmatrix}$.

Thus, $\mathcal{A}$ could be rewritten in it's vectoriel form :

$$
\begin{aligned}
\mathcal{A} &= -\int_0^L (\boldsymbol{d}_3 \times \boldsymbol{M}) \cdot \boldsymbol{h}_x'' \, dt \\
&= -\left[(\boldsymbol{d}_3 \times \boldsymbol{M}) \cdot \boldsymbol{h}_x'\right]_0^L + \int_0^L (\boldsymbol{d}_3 \times \boldsymbol{M})' \cdot \boldsymbol{h}_x' \, dt \\
&= -\left[(\boldsymbol{d}_3 \times \boldsymbol{M}) \cdot \boldsymbol{h}_x'\right]_0^L + \int_0^L \left((\boldsymbol{d}_3 \times \boldsymbol{M}') \cdot \boldsymbol{h}_x' + (\boldsymbol{h}_x' \times \boldsymbol{d}_3') \cdot \boldsymbol{M}\right) \, dt
\end{aligned}
\tag{4.64}
$$

Recall that from eq. (4.2) that $\boldsymbol{h}_x' \cdot \boldsymbol{d}_3 = 0$ and from eq. (4.3) that $\boldsymbol{d}_3' \cdot \boldsymbol{d}_3 = 0$. Hence, $\boldsymbol{h}_x' \times \boldsymbol{d}_3'$ is colinear to $\boldsymbol{d}_3$. Or by definition $\boldsymbol{M}$ is orthogonal to $\boldsymbol{d}_3$. Thus, $(\boldsymbol{h}_x' \times \boldsymbol{d}_3') \cdot \boldsymbol{M} = 0$. Finally, after a second integration by parts :

$$
\begin{aligned}
\mathcal{A} &= -\left[(\boldsymbol{d}_3 \times \boldsymbol{M}) \cdot \boldsymbol{h}_x'\right]_0^L + \int_0^L (\boldsymbol{d}_3 \times \boldsymbol{M}') \cdot \boldsymbol{h}_x' \, dt \\
&= \left[(\boldsymbol{d}_3 \times \boldsymbol{M}') \cdot \boldsymbol{h}_x'' - (\boldsymbol{d}_3 \times \boldsymbol{M}) \cdot \boldsymbol{h}_x'\right]_0^L - \int_0^L (\boldsymbol{d}_3 \times \boldsymbol{M}')' \cdot \boldsymbol{h}_x \, dt
\end{aligned}
\tag{4.65}
$$

Calculus of $\mathcal{B}$:

$$
\begin{aligned}
\mathcal{B} &= \int_{t=0}^L (\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^T \boldsymbol{B} \boldsymbol{J} \boldsymbol{\omega} \left(\int_{u=0}^L (\delta_t - \delta_0) \, \boldsymbol{\kappa} \boldsymbol{b} \cdot \boldsymbol{h}_x \, du\right) \, dt \\
&= -(\boldsymbol{\kappa} \boldsymbol{b} \cdot \boldsymbol{h}_x)(0) \int_{t=0}^L (\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^T \boldsymbol{B} \boldsymbol{J} \boldsymbol{\omega} \, dt + \int_{t=0}^L (\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^T \boldsymbol{B} \boldsymbol{J} \boldsymbol{\omega} \boldsymbol{\kappa} \boldsymbol{b} \cdot \boldsymbol{h}_x \, dt
\end{aligned}
\tag{4.66}
$$

Calculus of $\mathcal{C}$:

$$
\begin{aligned}
\mathcal{C} &= \int_{t=0}^L -(\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^T \boldsymbol{B} \boldsymbol{J} \boldsymbol{\omega} \left(\int_{u=0}^L (1 - H_t) \, \boldsymbol{\kappa} \boldsymbol{b}' \cdot \boldsymbol{h}_x \, du\right) \, dt \\
&= \int_{u=0}^L \int_{t=u}^L -\left((\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^T \boldsymbol{B} \boldsymbol{J} \boldsymbol{\omega}\right)(t) \, (\boldsymbol{\kappa} \boldsymbol{b}' \cdot \boldsymbol{h}_x)(u) \, dt \, du \\
&= \int_{u=0}^L -\left(\int_{t=u}^L (\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^T \boldsymbol{B} \boldsymbol{J} \boldsymbol{\omega} \, dt\right) (\boldsymbol{\kappa} \boldsymbol{b}' \cdot \boldsymbol{h}_x) \, du
\end{aligned}
\tag{4.67}
$$

By several integration by parts, using Fubini's theorem once and supposing that the terms vanishes at $s = 0$ and $s = L$:

$$
\begin{aligned}
\mathcal{B} + \mathcal{C} &= \int_{t=0}^L \left((\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^T \boldsymbol{B} \boldsymbol{J} \boldsymbol{\omega} \boldsymbol{\kappa} \boldsymbol{b} - \left(\int_{u=t}^L (\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^T \boldsymbol{B} \boldsymbol{J} \boldsymbol{\omega} \, du\right) \boldsymbol{\kappa} \boldsymbol{b}'\right) \cdot \boldsymbol{h}_x \, dt \\
&= \int_{t=0}^L \left(-\left(\int_{u=t}^L (\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^T \boldsymbol{B} \boldsymbol{J} \boldsymbol{\omega} \, du\right)' \boldsymbol{\kappa} \boldsymbol{b} - \left(\int_{u=t}^L (\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^T \boldsymbol{B} \boldsymbol{J} \boldsymbol{\omega} \, du\right) \boldsymbol{\kappa} \boldsymbol{b}'\right) \cdot \boldsymbol{h}_x \, dt \\
&= \int_{t=0}^L \left(-\left(\int_{u=t}^L (\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^T \boldsymbol{B} \boldsymbol{J} \boldsymbol{\omega} \, du\right) \boldsymbol{\kappa} \boldsymbol{b}\right)' \cdot \boldsymbol{h}_x \, dt
\end{aligned}
\tag{4.68}
$$

Which can be rewritted using the quasi-static hypothesis eq. (4.33) :

$$
\begin{aligned}
\mathcal{B} + \mathcal{C} &= \int_{t=0}^{L} \left(-\left(\int_{u=t}^{L} (\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^T \boldsymbol{BJ\omega} \, du \right) \boldsymbol{\kappa b} \right)' \cdot \boldsymbol{h}_x \, dt \\
&= \int_{t=0}^{L} \left(-\left(\int_{u=t}^{L} \beta(\theta' - \overline{\theta}')(\delta_L - \delta_0) - (\beta(\theta' - \overline{\theta}'))' \, du \right) \boldsymbol{\kappa b} \right)' \cdot \boldsymbol{h}_x \, dt \\
&= \int_{t=0}^{L} \left(-\left(\beta(\theta' - \overline{\theta}')(L) - [\beta(\theta' - \overline{\theta}')]_t^L \right) \boldsymbol{\kappa b} \right)' \cdot \boldsymbol{h}_x \, dt \\
&= \int_{t=0}^{L} -\left(\beta(\theta' - \overline{\theta}')\boldsymbol{\kappa b} \right)' \cdot \boldsymbol{h}_x \, dt
\end{aligned}
\tag{4.69}
$$

Finally, combining eq. (4.65) and (4.69) into eq. (4.60) yields :

$$
\boldsymbol{D}_x \mathcal{E}_b[\boldsymbol{\omega}[\boldsymbol{x}]](s) \cdot \boldsymbol{h}_x = \int_0^L \left(-(\boldsymbol{d}_3 \times \boldsymbol{M}')' - (\beta(\theta' - \overline{\theta}')\boldsymbol{\kappa b})' \right) \cdot \boldsymbol{h}_x \, dt
\tag{4.70}
$$

Internal forces

The gradient of the elastic energy with respect to $\boldsymbol{x}$ is obtained from eq. (4.57) with eq. (4.58) and (4.70) :

$$
\langle -\boldsymbol{f}(s), \boldsymbol{h}_x \rangle = \boldsymbol{D}_x \mathcal{E}(s) \cdot \boldsymbol{h}_x = -\int_0^L \left((\boldsymbol{d}_3 \times \boldsymbol{M}')' + (\beta(\theta' - \overline{\theta}')\boldsymbol{\kappa b})' \right) \cdot \boldsymbol{h}_x \, dt
\tag{4.71}
$$

Finally, we can conclude on the expression of the distributed internal forces acting on the centermine :

$$
\boldsymbol{f}(s) = \left(\boldsymbol{d}_3 \times \boldsymbol{M}'(s) + \beta \left(\theta' - \overline{\theta}' \right) \boldsymbol{\kappa b} \right)'(s)
\tag{4.72}
$$

Remark that this expression can be rewritten thanks to eq. (4.30a) as :

$$
\boldsymbol{f}(s) = \left(\boldsymbol{d}_3 \times \boldsymbol{M}' + Q\boldsymbol{\kappa b} \right)'(s)
\tag{4.73}
$$

4.7 Internal forces and moments acting on the rod

We gather and recall from eq. (4.31) and (4.73) the expressions of the quasi-static internal forces and internal twisting moment acting on the rod :

$$
\boldsymbol{f}(s) = \left(\boldsymbol{d}_3 \times \boldsymbol{M}' + Q\boldsymbol{\kappa b} \right)'(s)
\tag{4.74a}
$$
$$
m(s) = Q'(s) + \kappa_1(s)M_2(s) - \kappa_2(s)M_1(s)
\tag{4.74b}
$$

Remark that we fall back on the static member of the dynamical equations of rods in Kirchhoff's theory (see **??**). Hence, we have proved the equivalence between the present approach (based on the calculus of variations and the formulation of an elastic energy) and a more direct approach from the well-established dynamical equations.

4.8 Conclusion

We have build upon Bergou et al. 2008 [4] a reduced coordinate beam theory for the modeling of slender rods with anisotropic cross-section and arbitrary natural geometry. This model assumes that the rod is inextensible and that cross-sections remain plane and perpendicular to the centerline. The material is supposed to work linearly. Unlike Bergou et al. 2008 [4] and Nabaei 2014 [6] :

- Our expressions for the internal forces and twisting moment acting on the rod have the advantage to be fully local, which leads to simpler and faster numerical evaluations.

- We have retrieved the physical meaning of the energy gradients in terms of shear, bending and twisting of the rod. This is a critical point for the post-analysis of the results given by the model as our goal is to understand and predict the behavior or real structures.

- Our model is developed in the smooth world and thus the choice of the discretization is left to the stage of the numerical implementation, which we believe gives more flexibility.

On retrouve les équations de kirchhoff dynamique où l'on a négligé les forces inertielles de rotation d'un élément de poutre autour de $\boldsymbol{d}_1$ et $\boldsymbol{d}_2$ pour ne garder que la dynamique de rotation de la section autour de la fibre neutre $\boldsymbol{d}_3$.

The shear force acting on the right side of a section is nothing but $\boldsymbol{T} = \int \boldsymbol{f}$:

$$\boldsymbol{T}(s) = \boldsymbol{d}_3 \times \boldsymbol{M}' + Q\boldsymbol{\kappa}\boldsymbol{b} \tag{4.75}$$

The linear momentum acting on the centerline is given by :

$$m(s) = Q' - \boldsymbol{\kappa}\boldsymbol{b} \cdot (\boldsymbol{d}_3 \times \boldsymbol{M}) \tag{4.76}$$

The main hypothesis are :

- Bernoulli : $\boldsymbol{d}_i \cdot \boldsymbol{d}_j = \delta_{ij}$

- Inextensibility : $\|\boldsymbol{d}_3'\| = 1$

- Quasistatic : at each timestep regarding bending motion $m(s) = 0$

4.9   References

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