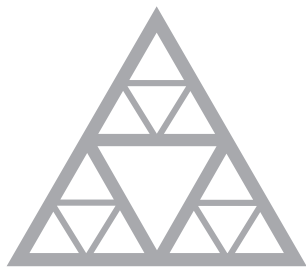


Modeling of bending-torsion couplings in active-bending structures. Application to the design of elastic gridshell.



École des Ponts
ParisTech

Thèse n. xxxxx
présenté le 01 décembre 2017
à l'Ecole des Ponts ParisTech
laboratoire Navier
Université Paris-Est

pour l'obtention du grade de Docteur ès Sciences
par

Lionel du Peloux

acceptée sur proposition du jury:

Prof Name Surname, président du jury
Prof Name Surname, directeur de thèse
Prof Name Surname, rapporteur
Prof Name Surname, rapporteur
Prof Name Surname, rapporteur

Paris, Ecole des Ponts ParisTech, 2016

Contents

Acknowledgements	v
Preface	vii
Abstract (English/Français/Deutsch)	ix
Contents	xiii
List of Figures	xix
List of Tables	xxiii
Index of notation	xxv
I Elastic gridshells	1
1 Elastic gridshells	3
1.1 Building free-forms	3
1.1.1 Non-standard forms	3
1.1.2 Importance of free-forms in modern architecture	3
1.1.3 Canonical approaches to build free-forms	3
1.1.4 Main challenges	3
1.2 Elastic gridshell	3
1.2.1 Structural Typology	4
1.2.2 Material Flexibility for Structural Rigidity	4
1.2.3 Erection Process	4
1.2.4 The benefits of composite materials	5
1.3 Built elastic gridshells : a review	6
1.3.1 The beginnings : from the first prototype to the German Pavilion	6
1.3.2 Mannheim Multihalle : the completion of a decade of research	10
1.3.3 The dry period : 25 years from Mannheim to Hannover	11
1.3.4 The signs of a renewal : Dorset and Doncaster	12
1.3.5 The renewal : Hannover, Downland and Savill	13

1.3.6	Gridshell in composite materials : a new perspective	19
1.3.7	Flourishing timber gridshell pavilions	21
1.3.8	Latest experiments	24
1.4	Research works in the field of elastic gridshells : a review	26
1.4.1	Mechanics	26
1.4.2	Geometry	27
1.4.3	Material	28
1.4.4	Technology	29
1.5	Conclusion	30
1.6	References	31
2	Ephemeral cathedral : the first GFRP gridshell building	35
2.1	Introduction	35
2.1.1	Overview	35
2.1.2	Goals and contributions	36
2.1.3	Related work	36
2.2	Project overview	36
2.2.1	Context and challenges	36
2.2.2	Architectural considerations on the form	38
2.2.3	Placing of the building on the site	38
2.2.4	Entrance	39
2.2.5	Daylight	39
2.2.6	Technical description	39
2.3	Construction process	44
2.3.1	Assembly of the grid	44
2.3.2	Deformation of the grid	44
2.3.3	Bracing of the grid	44
2.3.4	Covering of the gridshell	46
2.4	Structural design	46
2.4.1	Overall design process	46
2.4.2	3D modelling of the intended shape	47
2.4.3	Meshing the surface	48
2.4.4	Formfinding and bending prestress	49
2.4.5	As-built geometry	53
2.4.6	Structural analysis	53
2.5	Designing with GFRP materials	55
2.5.1	Properties of the tubes	55
2.5.2	Codes for composite materials	55
2.5.3	Flexural strength of the tubes	57
2.5.4	Partial safety factors	57
2.6	Construction details	59
2.6.1	The swivel coupler	59
2.6.2	The sleeve system	65
2.6.3	Foundations	70
2.6.4	The membrane	70

2.7	Hygrothermal behavior	72
2.7.1	Temperature	72
2.7.2	Moisture	73
2.8	Cost analysis	76
2.8.1	Overall cost for the client	76
2.8.2	Cost details for the building	76
2.8.3	Strengths and weaknesses	77
2.9	Conclusions	86
2.10	Acknowledgements	86
2.11	References	87

II Rich Kirchhoff beam model 89

3 Geometry of smooth and discrete space curves 91

3.1	Introduction	91
3.1.1	Goals and contributions	91
3.1.2	Related work	92
3.1.3	Overview	92
3.2	Parametric curves	92
3.2.1	Definition	92
3.2.2	Regularity	92
3.2.3	Reparametrization	93
3.2.4	Natural parametrization	93
3.2.5	Curve length	93
3.2.6	Arc length parametrization	93
3.3	Frenet trihedron	94
3.3.1	Tangent vector	94
3.3.2	Normal vector	95
3.3.3	Binormal vector	96
3.3.4	Osculating plane	96
3.4	Curves of double curvature	96
3.4.1	First invariant : the curvature	97
3.4.2	Second invariant : the torsion	99
3.4.3	Fundamental theorem of space curves	100
3.4.4	Serret-Frenet formulas	100
3.5	Curve framing	101
3.5.1	Moving frame	102
3.5.2	Adapted moving frame	104
3.5.3	Rotation-minimizing frame	105
3.5.4	Parallel transport	105
3.5.5	Frenet frame	106
3.5.6	Bishop frame	108
3.5.7	Comparison between Frenet and Bishop frames	110
3.6	Discrete curves	111

3.6.1	Definition	111
3.6.2	Regularity	113
3.6.3	Parametrization	113
3.7	Discrete curvature	114
3.7.1	Definition from osculating circles	114
3.7.2	Benchmarking : sensitivity to non uniform discretization	117
3.7.3	Benchmarking : accuracy in bending energy representation	119
3.8	Discrete tangent vector	122
3.8.1	Circumscribed case	123
3.8.2	Inscribed case	125
3.9	Discrete parallel transport	128
3.9.1	The rotation method	129
3.9.2	The double reflexion method	129
3.10	Conclusion	131
3.11	References	132
4	Elastic rod : variational approach	135
4.1	Introduction	135
4.1.1	Goals and contribution	135
4.1.2	Related work	135
4.1.3	Overview	135
4.2	Kirchhoff rod	136
4.2.1	Inextensibility	136
4.2.2	Euler-Bernoulli	136
4.2.3	Darboux vector	137
4.2.4	Curvatures and twist	137
4.2.5	Elastic energy	137
4.3	Curve-angle representation	138
4.3.1	Zero-twisting frame	138
4.4	Strains	139
4.4.1	Axial strain	139
4.4.2	Bending strain	139
4.4.3	Torsional strain	139
4.5	Elastic energy	140
4.6	Quasistatic assumption	140
4.7	Energy gradient with respect to θ : moment of torsion	141
4.7.1	Derivative of material directors with respect to θ	141
4.7.2	Derivative of the material curvatures vector with respect to θ	141
4.7.3	Computation of the moment of torsion	141
4.8	Energy gradient with respect to x : internal forces	143
4.8.1	Derivative of material directors with respect to x	143
4.8.2	Derivative of the material curvatures vector with respect to x	149
4.8.3	Computation of the forces acting on the centerline	149
4.9	Conclusion	152
4.10	References	154

5	Elastic rod : equilibrium approach	155
5.1	Introduction	155
5.1.1	Goals and contribution	155
5.1.2	Related work	155
5.1.3	Overview	156
5.2	Introduction to the special Cosserat theory of rods	158
5.2.1	Description of the motion	158
5.2.2	Time evolution	162
5.2.3	Strains	163
5.2.4	Parametrization of the centerline	163
5.2.5	To go further	164
5.3	Kirchhoff theory of rods	166
5.3.1	Description of the motion	168
5.3.2	Reparametrization	170
5.3.3	Strains	171
5.3.4	Balance of momentum	172
5.3.5	Equations of motion	178
5.3.6	Hookean elasticity	178
5.3.7	Deformation of cross-sections	179
5.3.8	Strain tensor	181
5.3.9	Stress tensor	181
5.3.10	Constitutive equations for internal forces and moments	181
5.3.11	Summary of the theory	183
5.3.12	Comments	185
5.4	Geometric interpretation of Kirchhoff's equations	187
5.5	Conclusion	194
5.6	References	195
6	Numerical Model	199
6.1	Introduction	199
6.1.1	Main hypothesis	199
6.1.2	Discret beam model	200
6.1.3	Modeling discontinuities	203
6.1.4	Matrix notation	203
6.1.5	Discret extension and axial force	204
6.1.6	Discret curvature and bending moment	205
6.1.7	Discret rate of twist and twisting moment	208
6.1.8	Discret shear force	209
6.1.9	Interpolation	210
6.2	Conclusion	210
6.3	References	212

III	Appendix	213
A	Review of built elastic gridshells	215
A.1	References	218
B	Calculus of variations	219
B.1	Introduction	219
B.2	Spaces	219
B.2.1	Normed space	219
B.2.2	Inner product space	219
B.2.3	Euclidean space	220
B.2.4	Banach space	220
B.2.5	Hilbert space	220
B.3	Derivative	221
B.3.1	Fréchet derivative	221
B.3.2	Gâteaux derivative	222
B.3.3	Useful properties	223
B.3.4	Partial derivative	223
B.4	Gradient vector	224
B.5	Jacobian matrix	224
B.6	Hessian	225
B.7	Functional	225
B.8	References	226
C	Parabolic interpolation	227
C.1	Introduction	227
C.2	Lagrange interpolating polynomial	227
C.3	Reparametrization	228
C.4	Characteristic values	228
C.5	Extremum value	229
	Bibliography	231

Bibliography

- [1] B. Burkhardt, J. Henniecke, and E. Schauer, eds. *IL10 Grid Shells*. Institut für leichte Flächentragwerke (IL). Stuttgart, 1974.
- [2] E. Happold and I. Liddell. “Timber lattice roof for the Mannheim Bundesgartenschau”. In: *The structural engineer* 53.3 (1975), pp. 99–135.
- [3] C. Douthe, J.-F. Caron, and O. Baverel. “Gridshell structures in glass fibre reinforced polymers”. In: *Construction and Building Materials* 24.9 (2010), pp. 1580–1589.
- [4] B. Burkhardt, M. Chaitos, J. Langner, W. Langner, and G. Lubberger, eds. *IL13 Multihalle Mannheim*. Institut für leichte Flächentragwerke (IL). Stuttgart, 1978.
- [5] J. Chilton and G. Tang. *Timber Gridshells: Architecture, structure and craft*. Routledge, 2017.
- [6] M. Collins and T. Cosgrove. “A Review of the State of the Art of Timber Gridshell Design and Construction”. In: *Civil Engineering Research in Ireland*. 2016.
- [7] G. Quinn and C. Gengnagel. “A review of elastic grid shells, their erection methods and the potential use of pneumatic formwork”. In: *WIT Transactions on the Built Environment* 136 (2014), pp. 129–144.
- [8] P. Cuvilliers, C. Douthe, L. du Peloux, and R. Le Roy. “Hybrid structural skin : prototype of an elastic gridshell braced by a fibre-reinforced concrete envelope”. In: *Journal of The International Association for Shell and Spatial Structures* (2017), pp. 1–12.
- [9] L. Kim-Lan Vaultot. “Form-Finding of Elastic Gridshells”. PhD thesis. Massachusetts Institute of Technology, 2016.
- [10] C. Douthe. “Etude de structures élancées précontraintes en matériaux composites : application à la conception des gridshells”. PhD thesis. Université Paris Est, 2007, p. 274.
- [11] L. Bouhaya. “Optimisation structurelle des gridshells”. PhD thesis. Université Paris-Est, 2010, p. 154.
- [12] F. Tayeb. “Simulation numérique du comportement mécanique non linéaire de gridshells composés de poutres élancées en matériaux composites et de sections quelconques”. PhD thesis. 2015.

Bibliography

- [13] E. Lafuente Hernández. “Design and Optimisation of Elastic Gridshells”. PhD thesis. Universität der Künste Berlin, 2015.
- [14] B. Addis. “‘Toys that save millions’ - A history of using physical models in structural design”. In: *The Structural Engineer* 91.4 (2013), pp. 12–27.
- [15] I. Liddell. “Frei Otto and the development of gridshells”. In: *Case Studies in Structural Engineering* 4 (2015), pp. 39–49.
- [16] R. Harris, J. Romer, O. Kelly, and S. Johnson. “Design and construction of the Downland Gridshell”. eng. In: *Building research and information* 31.6 (2003), pp. 427–454.
- [17] R. Burton, M. Dickson, and R. Harris. “The use of roundwood thinnings in buildings : a case study”. In: *Building Research & Information* 26.2 (1998), pp. 76–93.
- [18] M. McQuaid, F. Otto, and S. Ban. “Building the Japan pavilion”. In: *Shigeru Ban*. Phaidon Press, 2006, pp. 8–11.
- [19] R. Harris and O. Kelly. “The structural engineering of the Downland gridshell”. In: *Space Structures* 5 1.April 2002 (2002), pp. 161–172.
- [20] O. Lowenstein. “Lothian Gridshell”. In: *Building for a Future Winter* (2002), p. 29.
- [21] Bdonline.co.uk. *The other gridshell*. 2002.
- [22] O. Lowenstein. “Gridshells in England. Ideas from the U.K. make stunning use of small timber”. In: *Wood Design & Building* (2004).
- [23] Fourthdoor.org. *Growing and making Flimwell’s chestnut gridshell*. 2003.
- [24] R. Harris, S. Haskins, and J. Roynon. “The Savill Garden gridshell : Design and construction”. In: *The Structural Engineer* 86.17 (2008), pp. 27–34.
- [25] Trada. “The Savill Building. A visitor centre with a timber gridshell roof Gridshell structures”. In: *TRADA* (2006), pp. 1–7.
- [26] C. Douthe, O. Baverel, and J.-F. Caron. “Formfinding of a grid shell in composite materials”. In: *Journal of the IASS* 47 (2006), pp. 53–62.
- [27] O. Baverel, J.-F. Caron, F. Tayeb, and L. du Peloux. “Gridshells in composite materials : construction of a 300 m² forum for the solidays’ festival in Paris”. In: *Structural Engineering International* 22.3 (2012), pp. 408–414.
- [28] L. du Peloux, F. Tayeb, O. Baverel, and J.-F. Caron. “Construction of a large composite gridshell structure: a lightweight structure made with pultruded glass fibre reinforced polymer tubes”. In: *Structural Engineering International* 26.2 (2016), pp. 160–167.
- [29] B. D’Amico, A. Kermani, and H. Zhang. “Form finding and structural analysis of actively bent timber grid shells”. In: *Engineering Structures* 81 (2014), pp. 195–207.
- [30] B. D’Amico, A. Kermani, H. Zhang, A. Pugnale, S. Colabella, and S. Pone. “Timber gridshells: Numerical simulation, design and construction of a full scale structure”. In: *Structures* 3.May 2016 (2015), pp. 227–235.
- [31] D. Naicu, R. Harris, and C. Williams. “Timber Gridshells: Design methods and their application to a temporary pavilion”. In: *World Conference on Timber Engineering* (2014), pp. 1–2.

-
- [32] J. Haddal Mork, S. Dyvik Hillersøy, B. Manum, A. Rønnquist, and N. Labonnote. “Introducing the Segment Lath - a Simplified Modular Timber Gridshell Built in Trondheim Norway”. In: *World Conference on Timber Engineering*. 3. Vienna, Austria, 2016.
 - [33] N. Labonnote, J. H. Mork, S. H. Dyvik, A. Rønnquist, and B. Manum. “Experimental and Numerical Study of the Structural Performance of a Timber Gridshell”. In: *World Conference on Timber Engineering*. 2. Vienna, Austria, 2016, pp. 3–10.
 - [34] M. Toussaint. “A design tool for timber gridshells”. PhD thesis. Technische Universiteit Delft, 2007.
 - [35] S. Adriaenssens. “Stressed spline structures”. PhD thesis. University of Bath, 2000.
 - [36] S. Adriaenssens, M. Barnes, and C. Williams. “A new analytic and numerical basis for the form-finding and analysis of spline and gridshell structures”. In: *Computing Developments in Civil and Structural Engineering*. Ed. by B. Kumar and B. H. V. Topping. Edinburgh: Civil-Comp Press, 1999, pp. 83–91.
 - [37] M. Barnes. “Form finding and analysis of tension structures by dynamic relaxation”. In: *International Journal of Space Structures* 14.2 (1999), pp. 89–104.
 - [38] S. Adriaenssens and M. Barnes. “Tensegrity spline beam and grid shell structures”. In: *Engineering Structures* 23.1 (2001), pp. 29–36.
 - [39] M. Barnes, S. Adriaenssens, and M. Krupka. “A novel torsion/bending element for dynamic relaxation modeling”. In: *Computers & Structures* 119 (Apr. 2013), pp. 60–67.
 - [40] E. Poulsen. “Structural design and analysis of elastically bent gridshells”. PhD thesis. Chalmers University of Technology, Gothenburg Sweden, 2015.
 - [41] L. du Peloux, F. Tayeb, B. Lefevre, O. Baverel, and J.-F. Caron. “Formulation of a 4-DoF torsion / bending element for the formfinding of elastic gridshells”. In: *Proceedings of the International Association for Shell and Spatial Structures*. August. Amsterdam, 2015, pp. 1–14.
 - [42] B. D’Amico, H. Zhang, and A. Kermani. “A finite-difference formulation of elastic rod for the design of actively bent structures”. In: *Engineering Structures* 117 (2016), pp. 518–527.
 - [43] T. Bulenda and J. Knippers. “Stability of grid shells”. In: *Computers and Structures* 79.12 (2001), pp. 1161–1174.
 - [44] R. Mesnil, J. Ochsendorf, and C. Douthe. “Stability of Elastic Grid Shells”. In: *International Journal of Space Structures* 30.1 (2015), pp. 27–36.
 - [45] B. Lefevre, C. Douthe, and O. Baverel. “Buckling of elastic gridshells”. In: *Journal of the IASS* September (2015).
 - [46] S. R. Malek. “The effect of geometry and topology on the mechanics of grid shells”. PhD thesis. MIT, 2012.
 - [47] T. Jensen, O. Baverel, and C. Douthe. “Morphological and mechanical investigation of interconnected elastic gridshells”. In: *International Journal of Space Structures* 28.3-4 (2013), pp. 175–186.

Bibliography

- [48] G. Filz and D. Naicu. “2 Landscapes – interaction of 2 gridshells based on a modified Stewart - Gough - principle”. In: *Proceedings of the IASS Working Groups 12 + 18*. Tokyo, Japan, 2015.
- [49] F. Tayeb, J.-F. Caron, O. Baverel, and L. du Peloux. “Stability and robustness of a 300m2 composite gridshell structure”. In: *Construction & Building Materials* (2013).
- [50] L. Bouhaya, O. Baverel, and J.-F. Caron. “Mapping two-way continuous elastic grid on an imposed surface : application to grid shells”. In: *IASS 2009*. Valencia, Spain, 2009, pp. 989–998.
- [51] L. Bouhaya, O. Baverel, and J. F. Caron. “Optimization of gridshell bar orientation using a simplified genetic approach”. In: *Structural and Multidisciplinary Optimization* 50.5 (2014), pp. 839–848.
- [52] E. Lafuente Hernández, S. Sechelmann, T. Rörig, and C. Gengnagl. “Topology optimisation of regular and irregular elastic gridshells by means of a non-linear variational method”. In: *Advances in Architectural Geometry*. 2012, pp. 147–160.
- [53] L. du Peloux, O. Baverel, J.-F. Caron, and F. Tayeb. “From shape to shell : a design tool to materialize freeform shapes using gridshell structures”. In: *Design Modelling Symposium*. Berlin, Deutschland, 2011.
- [54] Y. Masson and L. Monasse. “Existence of global Chebyshev nets on surfaces of absolute Gaussian curvature less than 2π ”. In: *Journal of Geometry* 108.1 (2017), pp. 25–32.
- [55] Y. Masson. “Existence et construction de réseaux de Chebyshev avec singularités et application aux gridshells”. PhD thesis. Université Paris-Est, 2017.
- [56] S. Pone, G. Mirra, E. Pignatelli, D. Lancia, and S. Colabella. “Specialised algorithms for different project stages in a post-formed timber gridshell design”. In: *Structures and Architecture* September (2016), pp. 259–266.
- [57] C. Douthe, R. Mesnil, H. Orts, and O. Baverel. “New shapes for elastic gridshells covered by planar facets”. In: *Proceedings of the IASS Annual Symposium*. October. Tokyo, Japan, 2016, pp. 1–9.
- [58] R. Mesnil. “Structural explorations of fabrication-aware design spaces for non-standard architecture”. PhD thesis. Paris-Est University, 2017, p. 231.
- [59] N. Kotelnikova-Weiler, C. Douthe, E. Lafuente Hernández, O. Baverel, C. Gengnagel, and J.-F. Caron. “Materials for actively-Bent structures”. In: *International Journal of Space Structures* 28.3-4 (2013), pp. 229–240.
- [60] N. Kotelnikova-Weiler. “Optimisation mécanique et énergétique d’enveloppes en matériaux composites pour les bâtiments”. PhD thesis. Université Paris-Est, 2012, p. 310.
- [61] G. Quinn, C. Gengnagel, and K.-U. Bletzinger. “Imulation of Pneumatic Inflation of a Strained Grid Shells via a Dynamic Relaxation”. In: *Proceedings of the IASS Annual Symposium*. October. Tokyo, Japan, 2016.

-
- [62] Liuti, A. Pugnale, and B. D’Amico. “Building timber gridshells with air : Numerical simulations and technique challenges”. In: *Structures and Architecture* (2016).
 - [63] E. Lafuente Hernández and C. Gengnagel. “A new hybrid: Elastic gridshells braced by membranes”. In: *WIT Transactions on the Built Environment* 136 (2014), pp. 157–170.
 - [64] B. D’Amico, A. Kermani, H. Zhang, P. Sheperd, and C. Williams. “Optimisation of cross-section of actively bent grid shells with strength and geometric compatibility constraints”. In: *Computers & Structures* 154 (2015), pp. 163–176.
 - [65] A. Menges, O. D. Krieg, and T. Schwinn. *Advancing Wood Architecture. A computational approach*. Routledge, July 2016, p. 256.
 - [66] Site Sécurité. *Règlement ERP type CTS (big tops & tents)*.
 - [67] J. L. Clarke. *Eurocomp design code and handbook : structural design of polymer composites*. 2003, p. 672.
 - [68] L. C. Bank. *Composites for construction : structural design with FRP materials*. 2006, p. 560.
 - [69] Fiberline Composites A/S. *Fiberline design manual*. May. 2003, pp. 1–326.
 - [70] J. Delcourt. “Analyse et géométrie, histoire des courbes gauches de Clairaut à Darboux”. In: *Archive for History of Exact Sciences* 65.3 (2011), pp. 229–293.
 - [71] J. L. Coolidge. *A history of geometrical methods*. Ed. by C. Corporation. Dover Books on Mathematics. Dover Publications, 2013, p. 480.
 - [72] R. Bishop. “There is more than one way to frame a curve”. In: *Mathematical Association of America* (1975).
 - [73] M. Bergou, M. Wardetzky, S. Robinson, B. Audoly, and E. Grinspun. “Discrete elastic rods”. In: *ACM SIGGRAPH* (2008), pp. 1–12.
 - [74] T. Hoffmann. “Discrete Differential Geometry of Curves and Surfaces”. In: 18 (2008).
 - [75] F. Frenet. “Sur les courbes à double courbure”. In: *Journal de Mathématiques Pures et Appliquées* 17 (1852), pp. 437–447.
 - [76] J. Delcourt. “Analyse et géométrie : les courbes gauches de Clairaut à Serret”. PhD thesis. Université de Paris VI, 2007, p. 302.
 - [77] R. T. Farouki, C. Giannelli, M. L. Sampoli, and A. Sestini. “Rotation-minimizing osculating frames”. In: *Computer Aided Geometric Design* 31.1 (2014), pp. 27–42.
 - [78] H. Guggenheimer. “Computing frames along a trajectory”. In: *Computer Aided Geometric Design* 6.1 (1989), pp. 77–78.
 - [79] F. Klok. “Two moving coordinate frames for sweeping along a 3D trajectory”. In: *Computer Aided Geometric Design* 3.3 (1986), pp. 217–229.
 - [80] J. Bernoulli. *Johannis Bernoulli... Opera omnia, tam antea sparsim edita, quam hactenus inedita*. IV. sumptibus Marci-Michaelis Bousquet, 1728, p. 584.
 - [81] H. Pitot. “Quadrature de la moitié d’une courbe des arcs appelée la compagne de la cycloïde”. In: *Mémoires de l’Académie Royale des Sciences (1724)* (1726), pp. 107–113.

- [82] G. Monge. *Application de l'analyse à la géométrie*. 1809, p. 475.
- [83] L. Euler. “Continens analysin, pro incuruatione fili in fingulis punctis inueniendia”. In: *Novi Commentarii academiae scientiarum Petropolitanae* 19 (1775), pp. 340–370.
- [84] E. Vouga. “Plane Curves”. In: *Lectures in discrete differential geometry*. 2014. Chap. 1, pp. 1–11.
- [85] A. Gray, E. Abbena, and S. Salamon. *Modern differential geometry of curves and surfaces with Mathematica*. Ed. by E. A. Salamon and Simon. Third. CRC Press, 2006, p. 1016.
- [86] W. Wang, B. Jüttler, D. Zheng, and Y. Liu. “Computation of rotation minimizing frames”. In: *ACM Transactions on Graphics* 27.1 (2008), pp. 1–18.
- [87] J. Bloomenthal. “Calculation of reference frames along a space curve”. In: *Graphics Gems*. Ed. by A. S. Glassner. Vol. 1. San Diego, CA, USA: Academic Press Professional, Inc., 1990, pp. 567–571.
- [88] T. Poston, S. Fang, and W. Lawton. “Computing and approximating sweeping surfaces based on rotation minimizing frames”. In: *Proceedings of the 4th International Conference on CAD/CG*. 1995, pp. 1–8.
- [89] T. Menninger. *Frenet curves and successor curves : generic parametrizations of the helix and slant helix*. Tech. rep. 2013, pp. 1–15. arXiv: [1302.3175](#).
- [90] A. J. Hanson and H. Ma. *Parallel Transport Approach to Curve Framing*. Tech. rep. 1995.
- [91] D. Carroll, E. Hankins, E. Kose, and I. Sterling. “A survey of the differential geometry of discrete curves”. In: *The Mathematical Intelligencer* 36.4 (2014), pp. 28–35. arXiv: [arXiv:1311.5862v1](#).
- [92] A. Bobenko. *Discrete differential geometry*. Second. 2015, p. 144.
- [93] P. Romon. *Introduction à la géométrie différentielle discrète*. Références sciences. Ellipses, 2013, p. 216.
- [94] M. Bergou, B. Audoly, E. Vouga, M. Wardetzky, and E. Grinspun. “Discrete viscous threads”. In: *ACM Transactions on ...* (2010), pp. 1–10.
- [95] B. Audoly, M. Amar, and Y. Pomeau. *Elasticity and geometry*. 2010, p. 600.
- [96] S. Nabaei. “Mechanical form-finding of timber fabric structures”. PhD thesis. EPFL, 2014.
- [97] F. B. Fuller. “Decomposition of the linking number of a closed ribbon : A problem from molecular biology”. In: 75.8 (1978), pp. 3557–3561.
- [98] R. de Vries. “Evaluating changes of writhe in computer simulations of supercoiled DNA”. In: *The Journal of Chemical Physics* 122.6 (2005).
- [99] R. Vauquelin. “Writhing Geometry at Finite Temperature : Random Walks and Geometric phases for Stiff Polymers”. In: 1 (2000).
- [100] M. Berger. “Topological Quantities: Calculating Winding, Writhing, Linking, and Higher order Invariants”. In: *Lecture Notes in Mathematics* 1973 (2009), pp. 75–97.

-
- [101] P. Jung, S. Leyendecker, J. Linn, and M. Ortiz. “A discrete mechanics approach to the Cosserat rod theory—Part 1: static equilibria”. eng. In: *International journal for numerical methods in engineering* 85.1 (2010), pp. 31–60.
 - [102] W. Lewis. *Tension structures: form and behaviour*. Telford, Thomas, 2003, p. 201.
 - [103] J. Spillmann and M. Teschner. “CORDE : Cosserat Rod Elements for the Dynamic Simulation of One-Dimensional Elastic Objects”. In: *Eurographics/ ACM SIGGRAPH Symposium on Computer Animation* (2007), pp. 1–10.
 - [104] E. H. Dill. “Kirchhoff’s theory of rods”. In: *Archive for History of Exact Sciences* (1992), p. 23.
 - [105] S. Neukirch. “Enroulement, contact et vibrations de tiges élastiques”. PhD thesis. 2009, p. 97.
 - [106] A. Theetten. “Splines dynamiques géométriquement exactes : simulation haute performance et interaction”. PhD thesis. Université des Sciences et Technologies de Lille, 2007.
 - [107] P. C. J. Hoogenboom. “7 Vlasov torsion theory”. In: October (2006), pp. 1–12.
 - [108] H. Lang and J. Linn. “Lagrangian field theory in space-time for geometrically exact Cosserat rods”. In: 150 (2009), p. 21.
 - [109] J. Spillmann. “CORDE : Cosserat rod elements for the animation of interacting elastic rods”. PhD thesis. 2008.
 - [110] S. Antman. *Nonlinear problems of elasticity*. Applied mathematical sciences. New York: Springer, 2005, p. 838.
 - [111] G. Kirchhoff. *Über das gleichgewicht und die bewegung einer elastischen scheibe*. Berlin, 1850, p. 38.
 - [112] G. Kirchhoff. *Vorlesungen über mathematische, physik, mechanik*. Ed. by B. G. Teubner. Leipzig, 1876, p. 483.
 - [113] A. Clebsch. *Théorie de l’élasticité des corps solides*. Paris: Dunod, 1883, p. 900.
 - [114] A. Love. *A treatise on the mathematical theory of elasticity*. First. Cambridge University Press, 1892.
 - [115] S. Timoshenko. “On the correction for shear of the differential equation for transverse vibrations of prismatic bars”. In: *Philosophical Magazine Series 6* 41.245 (1921), pp. 744–746.
 - [116] S. Timoshenko. “On the transverse vibrations of bars of uniform cross-section”. In: *Philosophical Magazine Series 6* 43.253 (1922), pp. 125–131.
 - [117] S. Timoshenko and J. N. Goodier. *Theory of elasticity*. Second. New York: McGraw-Hill, 1951, p. 506.
 - [118] J. Langer and D. Singer. “Lagrangian Aspects of the Kirchhoff elastic rod”. In: *Society for Industrial and Applied Mathematics Review* 38.4 (1996), pp. 605–618.
 - [119] A. Day. “An Introduction to dynamic relaxation”. In: *The Engineer* (1965).
 - [120] D. Wakefield. “Dynamic relaxation analysis of pretensioned networks supported by compression arches”. PhD thesis. City University London, 1980, p. 281.

Bibliography

- [121] Y. Duan, D. Li, and P. F. Pai. “Geometrically exact physics-based modeling and computer animation of highly flexible 1D mechanical systems”. In: *Graphical Models* 75.2 (2013), pp. 56–68.
- [122] C. Meier, A. Popp, and W. A. Wall. “An objective 3D large deformation finite element formulation for geometrically exact curved Kirchhoff rods”. In: *Computer Methods in Applied Mechanics and Engineering* 278.August (2014), pp. 445–478.
- [123] A. Ibrahimbegović. “On finite element implementation of geometrically nonlinear Reissner’s beam theory: three-dimensional curved beam elements”. In: *Computer Methods in Applied Mechanics and Engineering* 122.1-2 (1995), pp. 11–26.
- [124] C. Lázaro, S. Monleón, J. Bessini, and J. Casanova. “A review on geometrically exact models for very flexible rods”. In: *Proceedings of the IASS Annual Symposium 2016* (2016), pp. 1–10.
- [125] J. C. Simo and L. Vu-Quoc. “A Geometrically-exact rod model incorporating shear and torsion-warping deformation”. In: *International Journal of Solids and Structures* 27.3 (1991), pp. 371–393.
- [126] E. Reissner. “On one-dimensional large-displacement finite-strain beam theory.” In: *Studies in Applied Mathematics* 52.2 (1973), pp. 87–95.
- [127] E. Benvenuto. *An introduction to the history of structural mechanics : vaulted structures and elastic systems*. New York: Springer Verlag, 1991, pp. XXI, 255.
- [128] P. Villaggio. *Mathematical models for elastic structures*. Cambridge University Press, 1997.
- [129] M. A. Dias and B. Audoly. ““Wunderlich, meet kirchhoff” : a general and unified description of elastic ribbons and thin rods”. In: *Journal of Elasticity* 119.1 (2015), pp. 49–66. arXiv: [arXiv:1403.2094v2](https://arxiv.org/abs/1403.2094v2).
- [130] Y. Vetyukov. *Nonlinear mechanics of thin-walled structures : asymptotics, direct approach and numerical analysis*. 1st ed. 2014, pp. X, 272.
- [131] M. A. Dias and B. Audoly. “A non-linear rod model for folded elastic strips”. In: *Journal of the Mechanics and Physics of Solids* 62.1 (2014), pp. 57–80. arXiv: [1306.5035](https://arxiv.org/abs/1306.5035).
- [132] S. Antman. “Kirchhoff’s problem for nonlinearly elastic rods”. In: *Quarterly of Applied Mathematics* XXXIV.3 (1974), pp. 221–240.
- [133] Y. Shi, A. E. Borovik, and J. E. Hearst. “Elastic rod model incorporating shear and extension, generalized nonlinear Schrödinger equations, and novel closed-form solutions for supercoiled DNA”. In: *The Journal of Chemical Physics* 103.8 (1995), pp. 3166–3183.
- [134] D. Moulton, T. Lessinnes, and A. Goriely. “Morphoelastic rods. Part I : A single growing elastic rod”. In: *Journal of the Mechanics and Physics of Solids* 61.2 (2013), pp. 398–427.
- [135] B. Coleman, E. H. Dill, M. Lembo, L. Zheng, and I. Tobias. “On the dynamics of rods in the theory of Kirchhoff and Clebsch”. In: *Archive for Rational Mechanics and Analysis* 121.4 (1993), pp. 339–359.

-
- [136] J. Cisternas and P. Holmes. “Buckling of extensible thermoelastic rods”. In: *Mathematical and Computer Modelling* 36.3 (2002), pp. 233–243.
 - [137] S. Timoshenko. “Theory of bending, torsion and buckling of thin-walled members of open cross section : Part I.” In: *Journal of The Franklin Institute* 239.3 (1945), pp. 201–219.
 - [138] S. Timoshenko. “Theory of bending, torsion and buckling of thin-walled members of open cross section : Part II.” In: *Journal of The Franklin Institute* 249.4 (1945), pp. 249–268.
 - [139] S. Timoshenko. “Theory of bending, torsion and buckling of thin-walled members of open cross section : Part III.” In: *Journal of The Franklin Institute* 239.5 (1945), pp. 343–336.
 - [140] V. Z. Vlasov. *Thin-walled elastic beams*. Second. National Technical Information Service, 1961.
 - [141] E. Elter. “Two formulí of the shear center”. In: *Periodica Polytechnica Mechanical Engineering* 28.2-3 (1984), pp. 179–193.
 - [142] D. Manta and R. Gonçalves. “A geometrically exact Kirchhoff beam model including torsion warping”. In: *Computers and Structures* 177 (2016), pp. 192–203.
 - [143] H. Weiss. “Dynamics of geometrically nonlinear rods : Mechanical models and equations of motion”. In: *Nonlinear Dynamics* 30 (2002), pp. 357–381.
 - [144] E. Benvenuto. *An introduction to the history of structural mechanics : statics and resistance of solids*. New York: Springer Verlag, 1991, pp. XXI, 306.
 - [145] E. Reissner. “On the effect of shear center location on the values of axial and lateral cantilever buckling loads for singly symmetric cross-section beams”. In: *Journal of Applied Mathematics and Physics* 32.1 (1981), pp. 182–188.
 - [146] J. M. Alves. “Dynamic analysis of bridge girders subjected to moving loads : Numerical and analytical beam models considering warping effects”. PhD thesis. Tecnico Lisboa, 2014, p. 129.
 - [147] A. Campanile, M. Mandarino, V. Piscopo, and A. Pranzitelli. “On the exact solution of non-uniform torsion for beams with asymmetric cross-section”. In: *World Academy of Science, Engineering and Technology* 31 (2009), pp. 36–45.
 - [148] R. Casati and F. Bertails-descoubes. “Super space clothoids”. In: *SIGGRAPH*. 2013.
 - [149] L. Prandtl. “Zur torsion von prismatischen stäben”. In: *Physikalische Zeitschrift* 4 (1903), pp. 758–770.
 - [150] K. Koohestani. “Nonlinear force density method for the form-finding of minimal surface membrane structures”. In: *Communications in Nonlinear Science and Numerical Simulation* 19.6 (2014), pp. 2071–2087.
 - [151] R. Abraham, J. E. Marsde, and T. Ratiu. *Manifolds, Tensor Analysis, and Applications (Ralph Abraham, Jerrold E. Marsden and Tudor Ratiu)*. 2002, p. 617.