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par

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Acknowledgements

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Lausanne, 12 Mars 2011 D. K.

Abstract

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Key words:

Résumé

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Mots clefs :

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1 Introduction

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Bibliography

2 Elastic gridshell

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Bibliography

Part I

Torsion

3 Geometry of smooth and discret curves

3.1 Introduction

Dans ce chapitre, après un bref rappel sur le cadre mathématique d'étude des courbes paramétrique de l'espace, on présente les notions de courbures et de torsion géométrique associées au repère de fraient. On montre ensuite le cas plus général d'un repère mobile quelconque attaché à une courbe gamma. On définit enfin la particularité d'un repère mobile adapté à un courbe, et on présente, en sus du repère de Frenet, une approche différente pour accrocher des repères le long d'une courbe (Bishop / RMF / Zéro-twisting frame)

3.2 Paramectric Curves

3.2.1 Definition

Let I be an interval [Bis75] of $\mathbb{R}$ and $F : t \mapsto F(t)$ be a map of $\mathcal{C}(I, \mathbb{R}^3)$. Then $\gamma = (I, F)$ is called a *parametric curve* and :

- The 2-uplet (I, F) is called a *parametrization* of γ
- $\gamma = F(I) = \{F(t), t \in I\}$ is called the *graph* or *trace* of γ
- γ is said to be $\mathcal{C}^k$ if $F \in \mathcal{C}^k(I, \mathbb{R}^3)$

Remark. Note that for a given graph in $\mathbb{R}^3$ they may be different possible parameterizations. From now, γ will simply refer to $F(I)$, its graph.

3.2.2 Regularity

Let $\gamma = (I, F)$ be a parametric [?] curve, and $t_0 \in I$ a parameter.

- A point of parameter t_0 is called *regular* if $F'(t_0) \neq 0$.
  The curve γ is called *regular* if γ is $\mathcal{C}^1$ and $F'(t) \neq 0, \forall t \in I$
- A point of parameter t_0 is called *biregular* if $F'(t_0)$ and $F''(t_0)$ are not collinear
  The curve γ is called *biregular* if γ is $\mathcal{C}^2$ and $F'(t) \cdot F''(t) \neq 0, \forall t \in I$

3.2.3 Reparametrization

Let $\gamma = (I, F)$ be a parametric curve of class $\mathcal{C}^k$, $J \in \mathbb{R}^3$ an interval, and $\varphi: I \mapsto J$ a $\mathcal{C}^k$ diffeomorphisme. Lets define $G = F \circ \varphi$. Then :

- $G \in \mathcal{C}^k(J, \mathbb{R}^3)$
- $G(J) = F(I)$
- φ is said to be an admissible *change of parameter* for γ
- (J, G) is said to be another *admissible parametrization* for γ

3.2.4 Natural parametrization

Let γ be a space curve of class $\mathcal{C}^1$. A parametrization (I, F) of γ is called *natural* if $\|F'(t)\| = 1, \forall t \in I$. Thus :

- The curve is necessarily regular
- F is strictly monotonic

3.2.5 Curve length

Let $\gamma = (I, F)$ be a parametric curve of class $\mathcal{C}^1$. The length of γ is define as :

$$L = \int_I \|F'(t)\| dt \tag{3.1}$$

Note that the length of γ is invariant under reparametrization.

3.2.6 Arc-length parametrization

Let $\gamma = (I, F)$ be a regular parametric curve of class $\mathcal{C}^1$. Let $t_0 \in I$ be a given parameter. The following map is said to be the *arc-length of origin* t_0 of γ :

$$s: t \mapsto \int_{t_0}^{t} \|F'(u)\| du \quad , \quad s \in I \times \mathbb{R} \tag{3.2}$$

The arc-length $s: I \mapsto s(I)$ is an admissible change of parameter for γ. Indeed, s is a $\mathcal{C}^1$ diffeomorphisme because it is bijective ($s' > 0$).

Lets define $G = F \circ s^{-1}$ and $J = s(I)$. Thus (J, G) is a natural reparametrization of γ and $\|G'(s)\| = 1, \forall s \in J$.

This parametrization is preferred because the natural parameter s traverses the image of γ at unit speed ($\|G'\| = 1$).

3.3 Frenet's Trihedron

En cinématique ou en géométrie différentielle, le repère de Frenet ou repère de Serret-Frenet est un outil d'étude du comportement local des courbes. Il s'agit d'un repère local associé à un point P, décrivant une courbe (C). Son mode de construction est différent selon que l'espace ambiant est de dimension 2 (courbe plane) ou 3 (courbe gauche) ; il est possible également de définir un repère de Frenet en toute dimension, pourvu que la courbe vérifie des conditions différentielles simples.

Le repère de Frenet, et les formules de Frenet donnant les dérivées des vecteurs de ce repère, permettent de mener de façon systématique des calculs de courbure, de torsion pour les courbes gauches et d'introduire des concepts géométriques associés aux courbes : cercle osculateur, plan osculateur (en), parallélisme des courbes

In this section we consider $\gamma = (J, G)$ to be a regular ($\|\gamma\| = 1$) parametric curve of class $\mathcal{C}^2$, parametrized by its arc-length (denoted s). For the sake of simplicity we will refer to $G(s)$ as $\gamma(s)$.

3.3.1 Tangent vector

The first vector of Frenet's trihedron is called the *unit tangent vector* ($\mathbf{t}$). At any given parameter $s_0 \in J$, it is defined as :

$$\mathbf{t}(s_0) = \frac{\gamma'(s_0)}{\|\gamma'(s_0)\|} = \gamma'(s_0) \quad , \quad \|\mathbf{t}(s_0)\| = 1 \tag{3.3}$$

3.3.2 Normal vector

The second vector of Frenet's trihedron is called the *unit normal vector* ($\mathbf{n}$). It is constructed from $\mathbf{t}'$ which is orthogonal to $\mathbf{t}$: $\|\mathbf{t}\| = 1 \Rightarrow \mathbf{t}' \cdot \mathbf{t} = 0 \Leftrightarrow \mathbf{t}' \perp \mathbf{t}$. Thus, at any given parameter $s_0 \in J$, it is define as :

$$\mathbf{n}(s_0) = \frac{\mathbf{t}'(s_0)}{\|\mathbf{t}'(s_0)\|} = \frac{\gamma''(s_0)}{\|\gamma''(s_0)\|} \quad , \quad \|\mathbf{n}(s_0)\| = 1 \tag{3.4}$$

3.3.3 Binormal vector

The third vector of Frenet's trihedron is called the *unit binormal vector* ($\mathbf{b}$). It is constructed from $\mathbf{t}$ and $\mathbf{n}$ to form an orthonormal direct basis of $\mathbb{R}^3$. Thus, at any given parameter $s_0 \in J$, it is define as :

$$\mathbf{b}(s_0) = \mathbf{t}(s_0) \times \mathbf{n}(s_0) \quad , \quad \|\mathbf{b}(s_0)\| = 1 \tag{3.5}$$

3.4 Curvature

Note that from a geometric point of view, $\frac{1}{\kappa(s_0)}$ represents the radius of the osculating circle of γ at the point of parameter s_0.

$\kappa(s_0) = \|\mathbf{t}'(s_0)\| = \|\gamma''(s_0)\|$

3.4.1 Osculating circle

Défini de façon directe, le cercle de courbure est le cercle le plus proche de la courbe en P, c'est l'unique cercle osculateur à la courbe en ce point. Ceci signifie qu'il constitue une très bonne approximation de la courbe, meilleure qu'un cercle tangent quelconque. En effet, il donne non seulement une idée de la direction dans laquelle la courbe avance (direction de la tangente), mais aussi de sa tendance à tourner de part ou d'autre de la tangente.

3.4.2 Curvature binormal vector

Finally, we define the *curvature binormal vector* at any given parameter $s_0 \in J$ as :

$$\kappa\mathbf{b}(s_0) = \mathbf{t}(s_0) \times \mathbf{t}'(s_0) = \kappa(s_0) \cdot \mathbf{b}(s_0) \quad , \quad \|\kappa\mathbf{b}(s_0)\| = \kappa(s_0) \tag{3.6}$$

3.5 Torsion

En géométrie différentielle, la torsion d'une courbe tracée dans l'espace mesure la manière dont la courbe se tord pour sortir de son plan osculateur (plan contenant le cercle osculateur). Ainsi, par exemple, une courbe plane a une torsion nulle et une hélice circulaire est de torsion constante. Prises ensemble, la courbure et la torsion d'une courbe de l'espace en définissent la forme comme le fait la courbure pour une courbe plane. La torsion apparait comme coefficient dans les équations différentielles du repère de Frenet.

The *torsion* measures the deviance of γ from being a planar curve and is defined at any given parameter $s_0 \in J$ as :

$$\tau_f(s_0) = \mathbf{n}'(s_0) \cdot \mathbf{b}(s_0) \tag{3.7}$$

3.6 Curve Framing

3.6.1 Moving frame

Soit $\gamma : s \to \gamma(s)$ une courbe bi-régulière de l'espace, paramétrée par son abscisse curviligne. On appelle *repère mobile* attaché à γ le trièdre orthonormé direct $\{\mathbf{d_3}(s), \mathbf{d_1}(s), \mathbf{d_2}(s)\}$.

Par construction, le repère mobile attaché à γ vérifie :

$$\begin{cases} \|\mathbf{d_i}(s)\| = 1 \\ \mathbf{d_i}(s) \cdot \mathbf{d_j}(s) = 0 \end{cases} \tag{3.8}$$

Governing equations

Par dérivation des relations précédentes on obtient les équations différentielles suivantes :

$$\begin{cases} \mathbf{d_i}'(s) \cdot \mathbf{d_i}(s) = 0 \\ \mathbf{d_i}'(s) \cdot \mathbf{d_j}(s) = -\mathbf{d_i}(s) \cdot \mathbf{d_j}'(s) \end{cases} \tag{3.9}$$

Il existe donc 3 fonctions scalaires $\tau(s)$, $\kappa_1(s)$, $\kappa_2(s)$ telles que :

$$\begin{cases} \mathbf{d_3}'(s) = \kappa_2(s)\mathbf{d_1}(s) - \kappa_1(s)\mathbf{d_2}(s) \\ \mathbf{d_1}'(s) = -\kappa_2(s)\mathbf{d_3}(s) + \tau(s)\mathbf{d_2}(s) \\ \mathbf{d_2}'(s) = \kappa_1(s)\mathbf{d_3}(s) - \tau(s)\mathbf{d_1}(s) \end{cases} \tag{3.10}$$

Ce système se réécrit sous forme matricielle de la façon suivante :

$$\begin{bmatrix} \mathbf{d_3}'(s) \\ \mathbf{d_1}'(s) \\ \mathbf{d_2}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_2(s) & -\kappa_1(s) \\ -\kappa_2(s) & 0 & \tau(s) \\ \kappa_1(s) & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{d_3}(s) \\ \mathbf{d_1}(s) \\ \mathbf{d_2}(s) \end{bmatrix} \tag{3.11}$$

On remarquera qu'ainsi définie, l'évolution des repères mobiles le long de la courbe γ est gouvernée par une équation différentielle d'ordre 1. Dès lors, un unique triplet $\{\tau$, κ_1, $\kappa_2\}$ engendre une famille de repères mobiles définis à une constante d'intégration prêt. Généralement, un repère mobile sera donc entièrement définit par la donnée de τ, κ_1, κ_2 et de $\{\mathbf{d_3}(s=0), \mathbf{d_1}(s=0), \mathbf{d_2}(s=0)\}$.

Darboux vector

Il est pertinent de considérer l'évolution d'un repère mobile le long de γ en introduisant son vecteur de Darboux ($\mathbf{\Omega}$), qui correspond au taux de rotation du trièdre $\{\mathbf{d_3}(s), \mathbf{d_1}(s), \mathbf{d_2}(s)\}$

selon l'abscisse curviligne. Les équations d'évolution du repère mobile s'écrivent alors :

$$\mathbf{d_i}'(s) = \mathbf{\Omega}(s) \times \mathbf{d_i}(s) \quad avec \quad \mathbf{\Omega}(s) = \begin{bmatrix} \tau(s) \\ \kappa_1(s) \\ \kappa_2(s) \end{bmatrix} \tag{3.12}$$

Géométriquement, les fonctions scalaires $\tau(s)$, $\kappa_1(s)$, $\kappa_2(s)$ correspondent respectivement aux taux de rotations du trièdre autour des axes dirigés par $\mathbf{d_3}(s), \mathbf{d_1}(s), \mathbf{d_2}(s)$:

$$\frac{d\theta_3}{dt}(s) = \tau(s) \quad , \quad \frac{d\theta_1}{dt}(s) = \kappa_1(s) \quad , \quad \frac{d\theta_2}{dt}(s) = \kappa_2(s) \tag{3.13}$$

3.6.2 Adapted frame

De plus, on dira qu'il est *adapté* à γ si en tout point $\gamma(s)$, $\mathbf{d_3}(s)$ est tangent à γ :

$$\mathbf{d_3}(s) = \mathbf{t}(s) = \frac{\gamma'(s)}{\|\gamma(s)\|} \tag{3.14}$$

Dans ce cas, la courbure κ de la courbe γ vaut : $\kappa \equiv \|\gamma''\| = \|\mathbf{t}'\| = \sqrt{\kappa_1^2 + \kappa_2^2}$

La courbure est une quantité géométrique intrinsèque, indépendante du choix du repère mobile attaché à la courbe. C'est donc un invariant. Et donc quelque soit le choix du repère mobile adapté $|\mathbf{t}'\| = \sqrt{\kappa_1^2 + \kappa_2^2}$ est un invariant (la courbure).

3.6.3 Frenet frame

Definition

The Frenet frame is a well-known particular adapted moving frame (§3.3). At any given regular point $\gamma(s_0)$ it is define as $\{\mathbf{t}(s_0), \mathbf{n}(s_0), \mathbf{b}(s_0)\}$ where :

$$\mathbf{t}(s_0) = \frac{\gamma'(s_0)}{\|\gamma'(s_0)\|} \quad , \quad \mathbf{n}(s_0) = \frac{\mathbf{t}'(s_0)}{\kappa(s_0)} \quad , \quad \mathbf{b}(s_0) = \mathbf{t}(s_0) \times \mathbf{n}(s_0) \tag{3.15}$$

Governing equations

The Frenet frame satisfies the *Frenet-Serret* formulas, which govern the evolution of the frame along the curve γ :

$$\begin{bmatrix} \mathbf{t}'(s) \\ \mathbf{n}'(s) \\ \mathbf{b}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau_f(s) \\ 0 & -\tau_f(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{bmatrix} \tag{3.16}$$

One can remember the generic differential equations of an adapted moving frame attached to a curve, where :

$$\mathbf{d_3}(s) = \mathbf{t}(s) = \frac{\gamma'(s)}{\|\gamma(s)\|} \quad , \quad \kappa_1(s) = 0 \quad , \quad \kappa_2(s) = \kappa(s) \quad , \quad \tau(s) = \tau_f(s) \tag{3.17}$$

Darboux vector

Consequently, the Darboux vector ($\mathbf{\Omega_f}$) of the Frenet frame is given by :

$$\mathbf{\Omega_f}(s) = \begin{bmatrix} \tau_f(s) \\ 0 \\ \kappa(s) \end{bmatrix} \tag{3.18}$$

Specific points

undefined when curvature vanishes : montrer des examples

not related to mechanical torsion

une perturbation de la courbe dans le sens le sens de la courbure engendre une variation de longueur de la courbe proportionnelle à l'inverse de la courbure (au premier ordre) + schéma

une perturbation de la courbe dans le sens de la binormale (en tout point) préserve la longueur de la courbe au 1er ordre : c'est un déplacement qui conserve l'hypothèse d'inextensibilité au premier ordre

Examiner la question de la fermeture sur une boucle fermée. Schéma.

3.6.4 Bishop frame

Definition

Different ways to frame a curve. The usual one is Frenet. But, it could not be as relevant as we want in our field of interest.

The Bishop frame is defined as a a well-known particular adapted moving frame (§3.3). At any given regular point $\gamma(s_0)$ it is define as $\{\mathbf{t}(s_0), \mathbf{n}(s_0), \mathbf{b}(s_0)\}$ where :

$$\mathbf{t}(s_0) = \frac{\gamma'(s_0)}{\|\gamma'(s_0)\|} \quad , \quad \mathbf{n}(s_0) = \frac{\mathbf{t}'(s_0)}{\kappa(s_0)} \quad , \quad \mathbf{b}(s_0) = \mathbf{t}(s_0) \times \mathbf{n}(s_0) \tag{3.19}$$

Governing equations

The Bishop frame evolution is governed by the following differential equations :

$$\begin{bmatrix} \mathbf{t}'(s) \\ \mathbf{u}'(s) \\ \mathbf{v}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_2(s) & -\kappa_1(s) \\ -\kappa_2(s) & 0 & 0 \\ \kappa_1(s) & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}(s) \\ \mathbf{u}(s) \\ \mathbf{v}(s) \end{bmatrix} \tag{3.20}$$

One can remember the generic differential equations of an adapted moving frame attached to a curve, where :

$$\mathbf{d_3}(s) = \mathbf{t}(s) = \frac{\gamma'(s)}{\|\gamma(s)\|} \quad , \quad \kappa_1(s) = 0 \quad , \quad \kappa_2(s) = \kappa(s) \quad , \quad \tau(s) = \tau_f(s) \tag{3.21}$$

Darboux vector

Consequently, the Darboux vector ($\mathbf{\Omega_b}$) of the Bishop frame is given by :

$$\mathbf{\Omega_b}(s) = \begin{bmatrix} 0 \\ \kappa_1(s) \\ \kappa_2(s) \end{bmatrix} \tag{3.22}$$

Specific points

well defined when curvature vanishes

related to mechanical torsion

expliquer la relation entre bishop et frenet : bishop est obtenu par rotation d'un angle $\alpha = \int \tau_f$ par rapport à frenet.

expliquer la notion de parallèle comme l'a formulé Laurent Hauswirth : la projection de u' et v' dans le plan normal à la tangente t est nulle, cad que d'un plan à un autre la projection de u et v est conservée + faire schéma.

Laurent Hauswirth : la complexité d'un problème est en général proportionnelle à la codimension de l'objet étudié et donc, de ce fait les courbes ($codim = 3 - 1 = 2$) sont des objets plus compliqués que les surfaces ($codim = 3 - 2 = 1$) ds $\mathbb{R}^3$.

Expliquer le défaut de fermeture sur une boucle fermée. Calcul du writhe. Quelle différence avec Frenet ?

Bibliography

[Bis75] Richard L. Bishop. There is more than one way to frame a curve. *Mathematical Association of America*, 1975.

4 Elastic rod : variational approch

4.1 Introduction

In this section a novel element with 4 degrees of freedom accounting for torsion and bending behaviours is presented. The beam is considered in Kirchhoff's theory framework, so that it is supposed to be inextensible and its sections are supposed to remain orthogonal to the centreline during deformation. The reduction from the classic 6-DoF model to this 4-DoF model is achieved by an appropriate curve-angle representation based on a relevant curve framing. Energies are then formulated and leads to internal forces and moments acting on the beam. The static equilibrium is deduced from a damped fictitious dynamic with an adapted dynamic relaxation algorithm.

Basile [BAV+10]

Basile [BWR+08] Je Basile [?]

Sina [Nab14]

[Ful78], [dV05], [Vau00], [Ber09]

4.2 Kirchhoff rod

The geometric configuration of the rod is described by its centerline $\boldsymbol{x}(s)$ and its cross sections. The centerline is parameterized by its arc-length. Cross sections orientations are followed along the centerline by their material frame $\{\boldsymbol{d}_3(s), \boldsymbol{d}_1(s), \boldsymbol{d}_2(s)\}$ which is an adapted orthonormal moving frame aligned to section's principal axes of inertia. Here, "adapted" means $\boldsymbol{d}_3(s) = \boldsymbol{x}'(s) = \boldsymbol{t}(s)$ is aligned to the centerline's tangent. In the literature, this description is also known as a *Cosserat Curve* [ref].

4.2.1 Inextensibility

Note the previous description is only valid for inextensible rods in order to follow material points by their arc-length indifferently in their rest or deformed configuration. As explained in [AAP10], this hypothesis is usually relevant for slender beams. Indeed, in practice, if a slender member faces substantial axial strain the bending behaviour would become negligible due to the important difference between axial and bending stiffness. The length of the rod will be denoted L and the arc-length s will vary (with no loss of generality) in $[0, L]$.

4.2.2 Euler-Bernouilli

Strains are supposed to remain small so that material frame remains orthogonal to the centerline in the deformed configuration. Thus, differentiating the conditions of orthonormality leads to the following differential equations governing the evolution of $\{\boldsymbol{d}_3(s), \boldsymbol{d}_1(s), \boldsymbol{d}_2(s)\}$ along the centerline :

$$\begin{bmatrix} \mathbf{d_3'}(s) \\ \mathbf{d_1'}(s) \\ \mathbf{d_2'}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_2(s) & -\kappa_1(s) \\ -\kappa_2(s) & 0 & \tau(s) \\ \kappa_1(s) & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{d_3}(s) \\ \mathbf{d_1}(s) \\ \mathbf{d_2}(s) \end{bmatrix} \tag{4.1}$$

La théorie des poutres est une application de la théorie de l'élasticité isotrope. Pour mener les calculs de résistance des matériaux, on considère les hypothèses suivantes :

(1) hypothèse de Bernoulli : au cours de la déformation, les sections droites restent perpendiculaires à la courbe moyenne ;

(2) les sections droites restent planes selon Navier-Bernoulli (pas de gauchissement).

L'hypothèse de Bernoulli permet de négliger le cisaillement dans le cas de la flexion : le risque de rupture est alors dû à l'extension des fibres situées à l'extérieur de la flexion, et la flèche est due au moment fléchissant. Cette hypothèse n'est pas valable pour les poutres courtes car ces dernières sont hors des limites de validité du modèle de poutre, à savoir que la dimension des sections doit être petite devant la longueur de la courbe moyenne. Le cisaillement est pris en compte dans le modèle de Timoshenko et Mindlin.

4.2.3 Darboux vector

Those equations can be formulated with the *Darboux vector* of the chosen material frame, which represents the rotational velocity of the frame along $\boldsymbol{x}(s)$:

$$\boldsymbol{d}_i'(s) = \boldsymbol{\Omega}_m(s) \times \boldsymbol{d}_i(s) \quad , \quad \boldsymbol{\Omega}_m(s) = \begin{bmatrix} \tau(s) \\ \kappa_1(s) \\ \kappa_2(s) \end{bmatrix} \tag{4.2}$$

Where $\kappa_1(s)$, $\kappa_2(s)$ and $\tau(s)$ represent respectively the rate of rotation of the material frame around the axis $\boldsymbol{d}_1(s)$, $\boldsymbol{d}_2(s)$ and $\boldsymbol{d}_3(s)$.

4.2.4 Curvatures and twist

The material curvatures are denoted $\kappa_1(s)$ and $\kappa_2(s)$ and represent the rod's flexion in the principal planes respectively normal to $\boldsymbol{d}_1(s)$ and $\boldsymbol{d}_2(s)$ .The material twist is denoted $\tau(s)$ and represents the section's rate of rotation around $\boldsymbol{d}_3(s)$. Those scalar functions measure directly the strain as defined in Kirchhoff's theory (Figure 4). Recall that the Frenet frame $\{\boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$ defines the osculating plane and the total curvature (κ) of a spatial curve :

$$\boldsymbol{t}'(s) = \kappa(s)\boldsymbol{n}(s) \quad , \quad \kappa(s) = \|\boldsymbol{t}'(s)\| \quad , \quad \boldsymbol{b}(s) = \boldsymbol{t}(s) \times \boldsymbol{n}(s) = \frac{\boldsymbol{t}(s) \times \boldsymbol{t}'(s)}{\kappa(s)} \tag{4.3}$$

To describe the osculating plane in which lies the bending part of the deformation, let's introduce the *curvature binormal* $\kappa\boldsymbol{b}(s) = \boldsymbol{t}(s) \times \boldsymbol{t}'(s)$, the vector of direction $\boldsymbol{b}(s)$ and norm $\kappa(s)$. At each point of arc-length s the osculating plane is normal to $\kappa\boldsymbol{b}(s)$.

4.2.5 Elastic energy

Kirrchhoff's theory assigns an elastic energy to beams according to their strain [AAP10]. In this theory, a beam is supposed to be inextensible. Thus the elastic energy ($\mathcal{E}_p$) only accounts for torsion and bending behaviors and is given by :

$$\mathcal{E}_p = \frac{1}{2}\int_0^L EI_1(\kappa_1 - \overline{\kappa_1})^2 + EI_2(\kappa_2 - \overline{\kappa_2})^2 ds + \frac{1}{2}\int_0^L \beta(\tau - \overline{\tau})^2 ds \tag{4.4}$$

Herer, $\overline{\kappa_1}$, $\overline{\kappa_2}$ and $\overline{\tau}$ denote the natural curvature and twist of the rod in the rest position (no stress).

4.3 Curve-angle representation

The previous paragraph has shown how the elastic potential energy of a rod can be computed following both its centerline and its cross sections orientations, which represents a model with 6-DoF : 3 for centerline positions and 3 for cross section orientations.

Following [BWR+08], let's introduce a reduced coordinate formulation of the rod that account for only 4-DoF. This reduction of DoF relies on the concept of zero-twisting frame which gives a reference frame with zero twist along a given centerline. Thus, cross section orientations $\{\boldsymbol{d}_3(s), \boldsymbol{d}_1(s), \boldsymbol{d}_2(s)\}$ can be tracked only by the measure of an angle θ from this reference frame denoted $\{\boldsymbol{d}_3(s), \boldsymbol{u}(s), \boldsymbol{v}(s)\}$ (Figure 5).

Note that an alternative solution could be to parameterize the global rotations of local material frame and to compute the rotation needed to align two successive frames along the curve's tangent.

Ici, expliquer la succession des dépendances : les vecteurs matériaux dépendent du repère de bishop par la seule variable theta. Le repère de bishop quand à lui est entièrement déterminé (au choix d'une constante de départ près) par la donnée de la centerline x.

Faire un schéma explicatif.

quid du transport parallèle en temps et non en espace ?

4.3.1 Zero-twisting frame

Zero-twisting frame, also known as Bishop frame, was introduced by Bishop in 1964. Bishop remarked that there was more than one way to frame a curve [Bis75]. Indeed, for a given curve, any orthonormal moving frame would satisfy the following differential equations, where $k_1(s)$, $k_2(s)$ and $\tau(s)$ are scalar functions that define completely the moving frame :

$$\begin{bmatrix} \mathbf{e_3}'(s) \\ \mathbf{e_1}'(s) \\ \mathbf{e_2}'(s) \end{bmatrix} = \begin{bmatrix} 0 & k_2(s) & -\kappa_1(s) \\ -k_2(s) & 0 & \tau(s) \\ k_1(s) & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e_3}(s) \\ \mathbf{e_1}(s) \\ \mathbf{e_2}(s) \end{bmatrix} \tag{4.5}$$

For instance, a Frenet frame $\{\boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$ is a frame which satisfies $k_1(s) = 0$. Note that this frame suffers from major disadvantages : it is undefined where the curvature vanishes and it flips at inflexion points. A Bishop frame $\{\boldsymbol{t}(s), \boldsymbol{u}(s), \boldsymbol{v}(s)\}$ is a frame which satisfies $\tau(s) = 0$. By construction, this frame has no angular velocity (i.e. no twist) around the curve's tangent ($\boldsymbol{u} \cdot \boldsymbol{v}' = \boldsymbol{u}' \cdot \boldsymbol{v} = 0$). Its evolution along the curve is described by the corresponding Darboux vector : $\boldsymbol{\Omega}_b(s) = \kappa \boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{t}'$. Remark that $\boldsymbol{\Omega}_b(s)$ only depends on the centerline and is well defined even when the curvature vanishes.

Thus, by the help of $\boldsymbol{\Omega}_b(s)$, it's possible to transport a given vector $\boldsymbol{e}$ along the centerline with no twist : $\boldsymbol{e}' = \kappa \boldsymbol{b} \times \boldsymbol{e}$. This is called *parallel transport*.

4.3.2 Bending strains

Let's compute the bending strains κ_1 and κ_2 regarding the geometric configuration of the rod. Remark that :

$$\begin{cases} \kappa \boldsymbol{b} \cdot \boldsymbol{d}_1 = (\boldsymbol{d}_3 \times \boldsymbol{d}_3') \cdot \boldsymbol{d}_1 = (\boldsymbol{d}_1 \times \boldsymbol{d}_3) \cdot \boldsymbol{d}_3' = -\boldsymbol{d}_2 \cdot \boldsymbol{d}_3' = \kappa_1 \\ \kappa \boldsymbol{b} \cdot \boldsymbol{d}_2 = (\boldsymbol{d}_3 \times \boldsymbol{d}_3') \cdot \boldsymbol{d}_2 = (\boldsymbol{d}_2 \times \boldsymbol{d}_3) \cdot \boldsymbol{d}_3' = \boldsymbol{d}_1 \cdot \boldsymbol{d}_3' = \kappa_2 \end{cases} \tag{4.6}$$

That is to say $\kappa \boldsymbol{b}$ is orthogonal to $\boldsymbol{d}_3$:

$$\kappa \boldsymbol{b} = \kappa_1 \boldsymbol{d}_1 + \kappa_2 \boldsymbol{d}_2 \tag{4.7}$$

Thus, the vector of material curvatures ($\boldsymbol{\omega}$) expressed on material frame axes $\{\boldsymbol{d}_1(s), \boldsymbol{d}_2(s)\}$ is defined as :

$$\boldsymbol{\omega} = \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix} = \begin{bmatrix} \kappa \boldsymbol{b} \cdot \boldsymbol{d}_1 \\ \kappa \boldsymbol{b} \cdot \boldsymbol{d}_2 \end{bmatrix} = \begin{bmatrix} -\boldsymbol{x}'' \cdot \boldsymbol{d}_2 \\ \boldsymbol{x}'' \cdot \boldsymbol{d}_1 \end{bmatrix} \tag{4.8}$$

4.3.3 Torsion strain

Let's compute the twist or torsion strain τ regarding the geometric configuration of the rod. Decomposing the material frame on the bishop frame gives :

$$\begin{bmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} \tag{4.9}$$

Thus, the twist can be identified directly as the variation of θ along the curve :

$$\tau = \boldsymbol{d}_1' \cdot \boldsymbol{d}_2 = (\theta' \boldsymbol{d}_2 + \kappa \boldsymbol{b} \times \boldsymbol{d}_1) \cdot \boldsymbol{d}_2 = \theta' + \boldsymbol{d}_3 \cdot \kappa \boldsymbol{b} = \theta' \tag{4.10}$$

Note that the Frenet frame does not lead to a correct evaluation of the twist.

4.4 Elastic energy

Introducing $\boldsymbol{\omega}$ and θ, the elastic energy can be rewritten as follow :

$$\mathcal{E}_p = \mathcal{E}_b + \mathcal{E}_t = \frac{1}{2} \int_0^L (\boldsymbol{\omega} - \overline{\boldsymbol{\omega}})^T B (\boldsymbol{\omega} - \overline{\boldsymbol{\omega}}) ds + \frac{1}{2} \int_0^L \beta (\theta' - \overline{\theta}')^2 ds \tag{4.11}$$

Where B is the bending stiffness matrix along the principal axes of inertia and β is the torsional stiffness :

$$B = \begin{bmatrix} EI_1 & 0 \\ 0 & EI_2 \end{bmatrix} \quad , \quad \beta = GJ \tag{4.12}$$

Ici on peut remarquer que l'énergie de torsion est indépendante de $\boldsymbol{x}$ et ne dépend donc que de θ.

L'énergie de flexion quant à elle dépend à la fois de $\boldsymbol{x}$ et de θ par l'intermédiaire des vecteurs matériels (κ_1 et κ_2 sont les projections de $\kappa \boldsymbol{b}$ sur $\boldsymbol{d}_1$ et $\boldsymbol{d}_2$). Lorsque $EI_1 = EI_2 = EI$, l'énergie de flexion est proportionnelle à la courbure au carré ($EI_1\kappa_1^2 + EI_2\kappa_2^2 = EI\kappa^2$) et par conséquent de dépend plus de θ. En effet, la courbure ($\kappa = ||\kappa \boldsymbol{b}|| = ||\boldsymbol{x}' \times \boldsymbol{x}''||$) ne dépend que de la géométrie de la centerline et pas de l'orientation des sections. Il n'y a plus de couplage entre la flexion et la torsion et l'énergie totale peut-être minimisée indépendamment en trouvant le minimum de l'énergie de flexion par rapport à la centerline (c'est à dire par rapport à $\boldsymbol{x}$ et le minimum de l'énergie de torsion par rapport à l'orientation des sections (c'est à dire par rapport à θ).

4.5 Inextensibility

Recall that the rod is supposed to be inextensible in Kirchhoff's theory. Thus, there is no stretching energy associated with an axial strain. However, this constraint will be enforced via a penalty energy, which in practice is somehow very similar as considering an axial stiffness in the beam...

4.6 Time-scale assumption

Following [BWR⁺08], it is relevant to assume that the propagation of twist waves is instantaneous compared to the one of bending waves. Thus, internal forces $\boldsymbol{f}^{int}$ and moment $\boldsymbol{m}^{int}$ acts on two different timescales in the rod dynamic. Thus on the timescale of action of the force $\boldsymbol{f}^{int}$ on the center line, driving the bending waves, the twist waves propagate instantaneously, so that $\forall s \in [0, L]$, $\delta\mathcal{E}_p/\delta\theta = 0$ for the computation of $\boldsymbol{f}^{int}$. This assumption may not be enforced, as in [Nab14], but leads to simpler and faster computations.

4.7 Energy gradient with respect to θ : torsional moment

Internal torsional moments and forces acting on the rod are classically obtained by differentiating the potential energy of the system with respect to θ and $\boldsymbol{x}$. Here, the calculus is a bit tricky as far as the differentiation takes place in function spaces. After a brief reminder on functional derivative, the main results of the calculations of the energy derivatives are given.

4.7.1 Derivative of material directors with respect to θ

Recalling that θ and $\boldsymbol{x}$ are independant variables and that Bishop frame $\{\boldsymbol{u}, \boldsymbol{v}\}$ only depends on $\boldsymbol{x}$, the decomposition of material frame directors $\{\boldsymbol{d}_1, \boldsymbol{d}_2\}$ on Bishop frame leads directly to the following expression for the derivative of the material directors :

$$\begin{cases} \boldsymbol{D}_\theta \boldsymbol{d}_1(s) \cdot h_\theta = \dfrac{d}{d\lambda}\boldsymbol{d}_1[\theta + \lambda h_\theta]\Big|_{\lambda=0} = (-\sin\theta\, \boldsymbol{u} + \cos\theta\, \boldsymbol{v}) \cdot h_\theta = \boldsymbol{d}_2 \cdot h_\theta \\ \boldsymbol{D}_\theta \boldsymbol{d}_2(s) \cdot h_\theta = \dfrac{d}{d\lambda}\boldsymbol{d}_2[\theta + \lambda h_\theta]\Big|_{\lambda=0} = (-\cos\theta\, \boldsymbol{u} - \sin\theta\, \boldsymbol{v}) \cdot h_\theta = -\boldsymbol{d}_1 \cdot h_\theta \end{cases} \tag{4.13}$$

4.7.2 Derivative of the material curvatures vector with respect to θ

Regarding the definition of the material curvatures vector and the derivative of material directors with respect to θ, it follows immediately that :

$$\boldsymbol{D}_\theta \boldsymbol{\omega}(s) \cdot h_\theta = \frac{d}{d\lambda}\boldsymbol{\omega}[\theta + \lambda h_\theta]\Big|_{\lambda=0} = \begin{bmatrix} \kappa\boldsymbol{b} \cdot \boldsymbol{d}_2 \\ -\kappa\boldsymbol{b} \cdot \boldsymbol{d}_1 \end{bmatrix} \cdot h_\theta = -\boldsymbol{J}\boldsymbol{\omega} \cdot h_\theta \tag{4.14}$$

Where $\boldsymbol{J}$ is the matrix that acts on two dimensional vectors by counter-clockwise rotation of angle $\frac{\pi}{2}$:

$$\boldsymbol{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{4.15}$$

4.7.3 Computation of the moment of torsion

The moment of torsion is given by the functional derivative of the potential elastic energy with respect to θ which can be decomposed according to the chaine rule :

$$\begin{aligned} \langle -m(s)\,; h_\theta\rangle = \boldsymbol{D}_\theta\mathcal{E}_p(s)\cdot h_\theta &= \boldsymbol{D}_\theta\mathcal{E}_b(s)\cdot h_\theta + \boldsymbol{D}_\theta\mathcal{E}_t(s)\cdot h_\theta \\ &= \boldsymbol{D}_\theta\mathcal{E}_b[\boldsymbol{\omega}[\theta]](s)\cdot h_\theta + \boldsymbol{D}_\theta\mathcal{E}_t[\theta](s)\cdot h_\theta \end{aligned} \tag{4.16}$$

Derivative of the torsion energy with respect to θ

Decomposing the previous calculus gives:

$$\begin{aligned} \boldsymbol{D}_\theta\mathcal{E}_t[\theta](s)\cdot h_\theta &= \frac{d}{d\lambda}\mathcal{E}_t[\theta+\lambda h_\theta]\Big|_{\lambda=0} \\ &= \frac{d}{d\lambda}\left(\frac{1}{2}\int_0^L \beta\left((\theta+\lambda h_\theta)'-\overline{\theta'}\right)^2\,dt\right)\Big|_{\lambda=0} \\ &= \int_0^L \beta(\theta'-\overline{\theta'})\cdot h'_\theta\,dt \\ &= \left[\beta(\theta'-\overline{\theta'})\cdot h_\theta\right]_0^L - \int_0^L \left(\beta(\theta'-\overline{\theta'})\right)'\cdot h_\theta\,dt \\ &= \int_0^L \left(\beta(\theta'-\overline{\theta'})(\delta_L-\delta_0) - \left(\beta(\theta'-\overline{\theta'})\right)'\right)\cdot h_\theta\,dt \end{aligned} \tag{4.17}$$

Derivative of the bending energy with respect to θ

The derivative of $\mathcal{E}_b$ is obtained with the chaine rule :

$$\begin{aligned} \boldsymbol{D}_\omega\mathcal{E}_b[\boldsymbol{\omega}](s)\cdot\boldsymbol{h}_\omega &= \frac{d}{d\lambda}\mathcal{E}_b[\boldsymbol{\omega}+\lambda\boldsymbol{h}_\omega]\Big|_{\lambda=0} \\ &= \frac{d}{d\lambda}\left(\frac{1}{2}\int_0^L ((\boldsymbol{\omega}+\lambda\boldsymbol{h}_\omega)-\overline{\boldsymbol{\omega}})^T\boldsymbol{B}\,((\boldsymbol{\omega}+\lambda\boldsymbol{h}_\omega)-\overline{\boldsymbol{\omega}})\,dt\right)\Big|_{\lambda=0} \\ &= \int_0^L (\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{B}\cdot\boldsymbol{h}_\omega\,dt \end{aligned} \tag{4.18}$$

Finally, reminding eq 4.14 :

$$\begin{aligned} \boldsymbol{D}_\theta\mathcal{E}_b[\boldsymbol{\omega}[\theta]](s)\cdot h_\theta &= \boldsymbol{D}_\omega\mathcal{E}_b[\boldsymbol{\omega}](s)\cdot(\boldsymbol{D}_\theta\boldsymbol{\omega}[\theta](s)\cdot h_\theta) \\ &= -\int_0^L (\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{B}\boldsymbol{J}\boldsymbol{\omega}\cdot h_\theta\,dt \end{aligned} \tag{4.19}$$

Moment of torsion

Thus, the

$$\begin{aligned}\langle -m(s)\,;h_\theta\rangle &= \boldsymbol{D}_\theta\mathcal{E}_b[\boldsymbol{\omega}[\theta]](s)\cdot h_\theta + \boldsymbol{D}_\theta\mathcal{E}_t[\theta](s)\cdot h_\theta \\ &= \int_0^L \left(\left(\beta(\theta'-\overline{\theta'})(\delta_L-\delta_0) - \left(\beta(\theta'-\overline{\theta'})\right)'\right) - (\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{BJ\omega}\right)\cdot h_\theta\, dt\end{aligned} \tag{4.20}$$

Finally, we can conclude on the expression of the internal moment of torsion :

$$m(s) = -\left(\beta(\theta'-\overline{\theta'})(\delta_L-\delta_0) - \left(\beta(\theta'-\overline{\theta'})\right)'\right) + (\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{BJ\omega} \tag{4.21}$$

Quasistatic hypothesis

$$\left(\beta(\theta'-\overline{\theta'})\right)' + (\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{BJ\omega} = 0 \tag{4.22}$$

4.8 Energy gradient with respect to x : forces

Internal torsional moments and forces acting on the rod are classically obtained by differentiating the potential energy of the system with respect to θ and $\boldsymbol{x}$. Here, the calculus is a bit tricky as far as the differentiation takes place in function spaces. After a brief reminder on functional derivative, the main results of the calculations of the energy derivatives are given.

paragraphe entièrment à revoir. Expliquer le cheminement. x fixe bishop et theta fixe d1,d2 par rapport à bishop. x est indépendant de theta. Seul des CL peuvent créer des couplages entre x et theta

Donc les vrais degrés de liberté du problème sont en fait les vecteurs matériels et les positions x. Se reporter à une modélisation du pb dans SO3 comme Spillmann par exemple.

Le calcul des gradients se résume donc à calculer les gradients des vecteurs matériels par rapport à des perturbations infinitésimales en x et theta. Pour theta, c'est facile. Pour kb, c'est facile. Reste la variation par rapport à x, qui est en fait la variation de bishop qu'on explique avec le writhe (défaut de fermeture de bishop sur une boucle fermée) et le transport parallèle. Le calcul se fait aisément en écrivant la double rotation et en effectuant le DL au premier ordre.

Le reste est quasiment immédiat. Reste la question des CL et des termes aux bords.

Il faut aussi se positionner par rapport à l'article de Basile. Regarder la question applied displacement vs settlement pour imposer une CL.

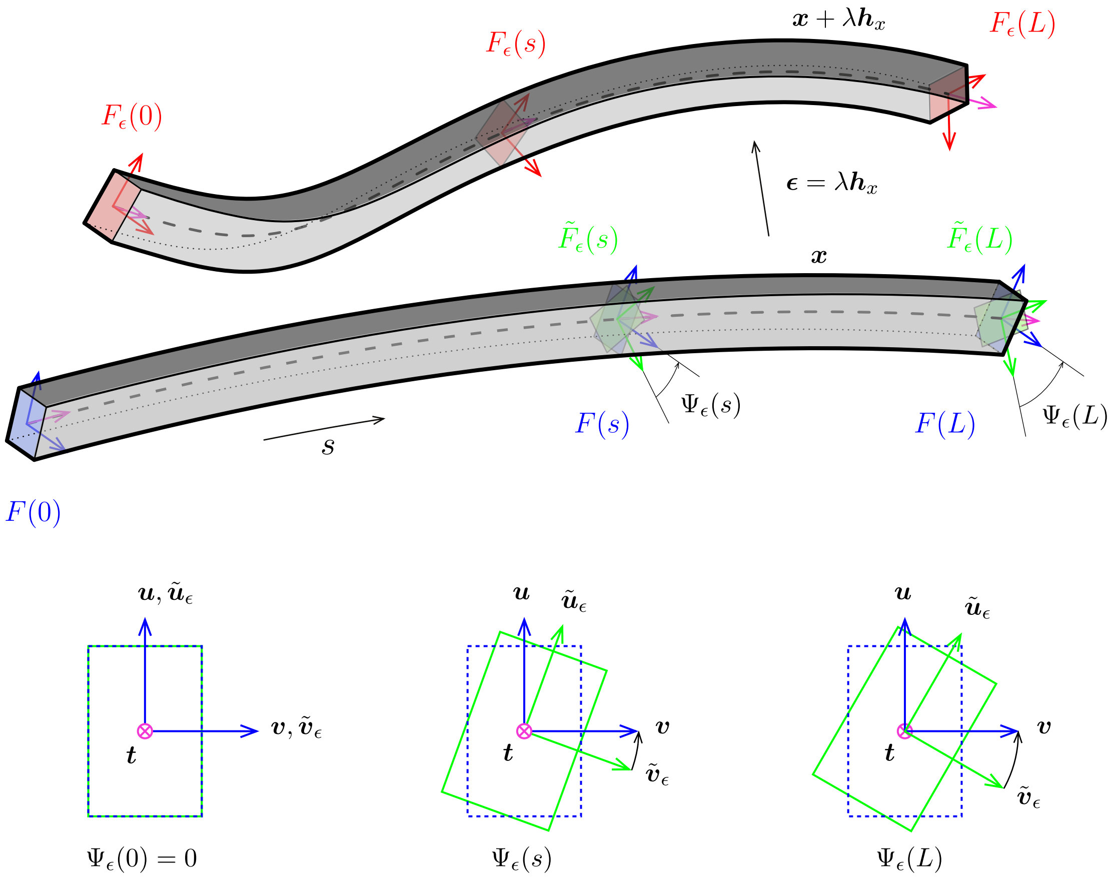

Figure 4.1 – Repères de Frenet attachés à γ.

4.8.1 Derivative of material directors with respect to $\boldsymbol{x}$

A variation of the centerline $\boldsymbol{x}$ by $\boldsymbol{\epsilon} = \lambda \boldsymbol{h}_x$ would cause a variation in the Bishop frame because parallel transport depends on the centerline itself. As far as $\boldsymbol{x}$ and θ are independent variables, this leads necessarily to a variation of the material frame.

$F_\epsilon = \{\boldsymbol{t}_\epsilon, \boldsymbol{u}_\epsilon, \boldsymbol{v}_\epsilon\}$ $\tilde{F}_\epsilon = \{\boldsymbol{t}, \tilde{\boldsymbol{u}}_\epsilon, \tilde{\boldsymbol{v}}_\epsilon\}$ $F = \{\boldsymbol{t}, \boldsymbol{u}, \boldsymbol{v}\}$

What we want to achive is to write at arc-length s the Bishop frame in the deformed configuration $\{\boldsymbol{t}_\epsilon, \boldsymbol{u}_\epsilon, \boldsymbol{v}_\epsilon\}$ on the Bishop frame in the reference configuration $\{\boldsymbol{t}, \boldsymbol{u}, \boldsymbol{v}\}$.

We first write $\{\boldsymbol{t}_\epsilon, \boldsymbol{u}_\epsilon, \boldsymbol{v}_\epsilon\}$ on the the basis $\{\boldsymbol{t}, \tilde{\boldsymbol{u}}_\epsilon, \tilde{\boldsymbol{v}}_\epsilon\}$. Recall that $\tilde{F}_\epsilon$ is obtained by parallel tranpsporting F_ϵ from $\boldsymbol{t}_\epsilon$ to $\boldsymbol{t}$. Denoting :

Calculation of Ψ_ϵ

This variation is closely related to the writhe of closed curves. As explained in [Ful78] when parallel transporting an adapted frame around a closed curve it might not realigned

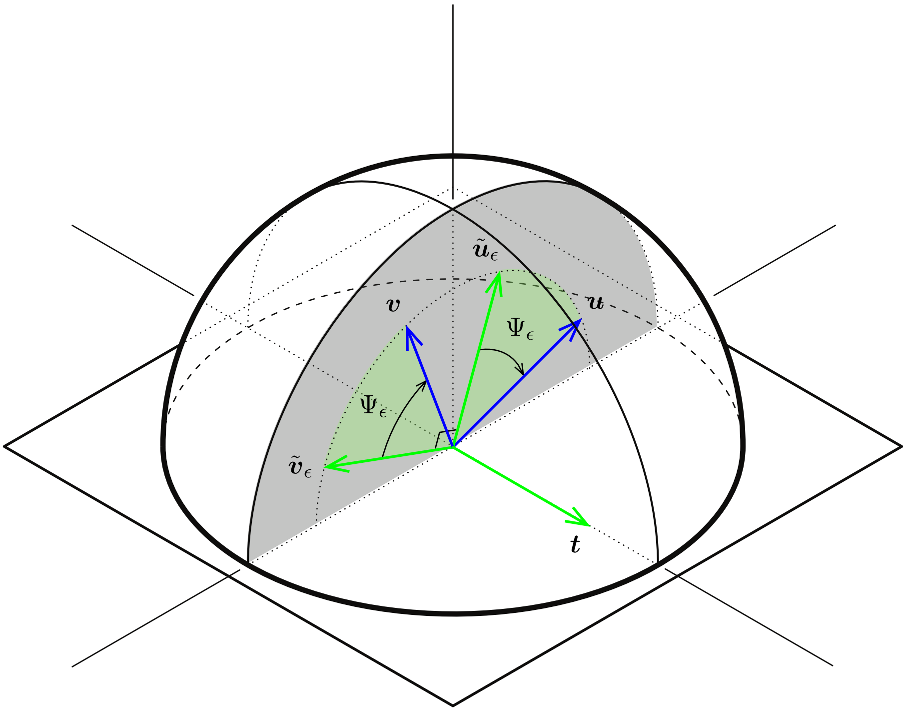

Figure 4.2 – $\tilde{F}$ is obtained by rotating $\tilde{F}_\epsilon$ around $\boldsymbol{t}$ of an angle Ψ_ϵ.

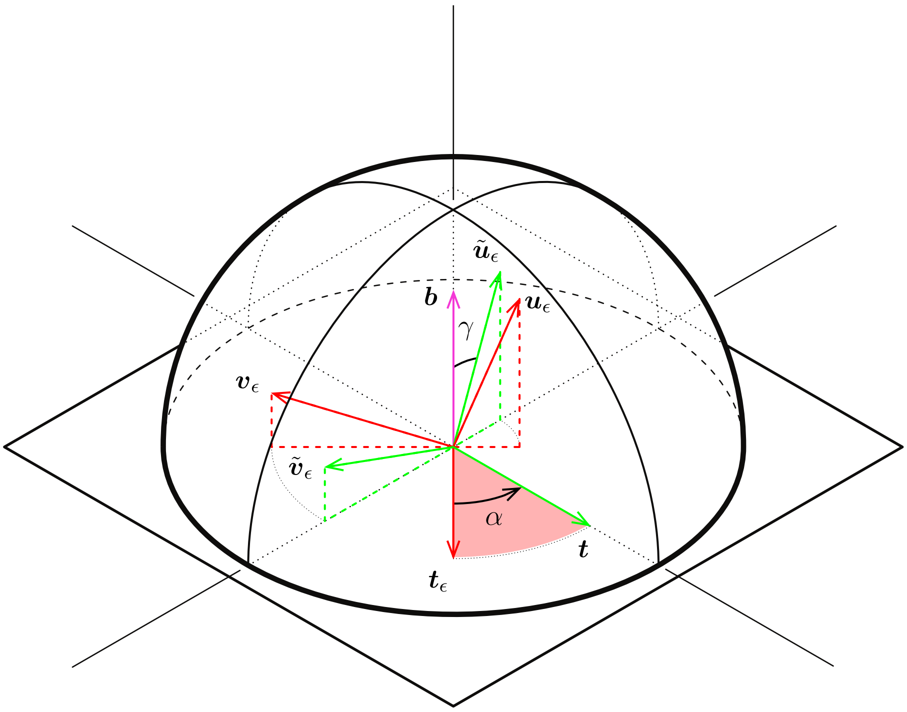

Figure 4.3 – $\tilde{F}_\epsilon$ is obtained by parallel tranpsporting F_ϵ from $\boldsymbol{t}_\epsilon$ to $\boldsymbol{t}$. This operation could be seen as a rotation around $\boldsymbol{t}_\epsilon \times \boldsymbol{t}$ of an angle α_ϵ.

with itself after one complete loop. This “lack of alignement” is directly measured by the change of writhe which can be computed with Fuller’s Formula [Ful78].

Note that the derivative of θ with respect to $\boldsymbol{x}$ can be evaluated by the change of writhe in the curve as suggested in [dV05]. This approach is completly equivalent.

One can also see this lack of alignement in terms of rotation. Parallel transport being a propagation of frame from $s = 0$, the cumulated rotation of Bishop frame from the deformed configuration around the initial configuration at arc-length s is the cumulated angle of rotation of $\boldsymbol{u}[\boldsymbol{x} + \lambda \boldsymbol{h}_x]$ around $\boldsymbol{d}_3[\boldsymbol{x}]$. Recalling the rotation rate of $\boldsymbol{u}[\boldsymbol{x} + \lambda \boldsymbol{h}_x]$ is $\kappa \boldsymbol{b}[\boldsymbol{x} + \lambda \boldsymbol{h}_x]$ by definition of zero-twisting frame, one can write :

$$\Psi_\epsilon[\boldsymbol{x}](s) = -\int_0^s \kappa \boldsymbol{b}[\boldsymbol{x} + \lambda \boldsymbol{h}_x] \cdot \boldsymbol{d}_3[\boldsymbol{x}]\, dt \tag{4.23}$$

The calculation of $\kappa \boldsymbol{b}[\boldsymbol{x} + \lambda \boldsymbol{h}_x]$ is straight forward from the curvature binormal definition :

$$\begin{aligned}
\kappa \boldsymbol{b}[\boldsymbol{x} + \lambda \boldsymbol{h}_x] &= (\boldsymbol{x} + \lambda \boldsymbol{h}_x)' \times (\boldsymbol{x} + \lambda \boldsymbol{h}_x)'' \\
&= \kappa \boldsymbol{b}[\boldsymbol{x}] + \lambda(\boldsymbol{x}' \times \boldsymbol{h}_x'' + \boldsymbol{h}_x' \times \boldsymbol{x}'') + \lambda^2(\boldsymbol{h}_x' \times \boldsymbol{h}_x'') \\
&= \kappa \boldsymbol{b}[\boldsymbol{x}] + \lambda(\boldsymbol{x}' \times \boldsymbol{h}_x'' + \boldsymbol{h}_x' \times \boldsymbol{x}'') + o(\lambda)
\end{aligned} \tag{4.24}$$

Thus, reminding that $\boldsymbol{d}_3[\boldsymbol{x}] = \boldsymbol{x}'$ and $\kappa \boldsymbol{b}[\boldsymbol{x}] \cdot \boldsymbol{d}_3[\boldsymbol{x}] = 0$, and using the invariance of circular product by cyclic permutation, one can express :

$$\begin{aligned}
\Psi_\epsilon[\boldsymbol{x}](s) &= -\int_0^s \kappa \boldsymbol{b}[\boldsymbol{x} + \lambda \boldsymbol{h}_x] \cdot \boldsymbol{d}_3[\boldsymbol{x}]\, dt \\
&= -\lambda \int_0^s (\boldsymbol{x}' \times \boldsymbol{h}_x'' + \boldsymbol{h}_x' \times \boldsymbol{x}'') \cdot \boldsymbol{x}'\, dt + o(\lambda) \\
&= -\lambda \int_0^s \kappa \boldsymbol{b}[\boldsymbol{x}] \cdot \boldsymbol{h}_x'\, dt + o(\lambda)
\end{aligned} \tag{4.25}$$

By integration by parts, dropping the implicit reference to $\boldsymbol{x}$ in the notation, and denoting by δ_s and H_s the Dirac function and the Heaviside step function centered at s, $\Psi_\epsilon(s)$ could be rewritten as :

$$\begin{aligned}
\Psi_\epsilon(s) &= -\lambda \int_0^s \kappa \boldsymbol{b} \cdot \boldsymbol{h}_x'\, dt + o(\lambda) \\
&= -\lambda \left(\Big[\kappa \boldsymbol{b} \cdot \boldsymbol{h}_x \Big]_0^s - \int_0^s \kappa \boldsymbol{b}' \cdot \boldsymbol{h}_x\, dt \right) + o(\lambda) \\
&= -\lambda \left(\int_0^s \big((\delta_s - \delta_0)\, \kappa \boldsymbol{b} - \kappa \boldsymbol{b}' \big) \cdot \boldsymbol{h}_x\, dt \right) + o(\lambda) \\
&= -\lambda \left(\int_0^L \big((\delta_s - \delta_0)\, \kappa \boldsymbol{b} - (1 - H_s) \kappa \boldsymbol{b}' \big) \cdot \boldsymbol{h}_x\, dt \right) + o(\lambda)
\end{aligned} \tag{4.26}$$

Note that, as expected, $\Psi_\epsilon(s)$ is in first order of λ and thus gets negligible when λ tends to zero, that is to say when the perturbation of $\boldsymbol{x}$ is infinitesimal :

$$\lim_{\lambda \to 0} \Psi_\epsilon(s) = 0 \tag{4.27}$$

$H_s : t \mapsto \begin{cases} 0, & t < s \\ 1, & t \geqslant s \end{cases}$ est la fonction de Heaviside.

$\delta_s : t \mapsto \delta(t-s)$ est la distribution de dirac centrée en s.

Calculation of α_ϵ

Recall that $\tilde{F}_\epsilon$ is obtained by parallel tranpsporting F_ϵ from $\boldsymbol{t}_\epsilon$ to $\boldsymbol{t}$. $\tilde{F}_\epsilon$ results from the rotation of F_ϵ around $\boldsymbol{b} = \boldsymbol{t}_\epsilon \times \boldsymbol{t}$ by an angle α_ϵ.

Because the rod is supposed to be inextensible, the following properties hold :

$$\|\boldsymbol{t}\| = \|\boldsymbol{x}'\| = 1 \quad , \quad \|\boldsymbol{t}_\epsilon\| = \|(\boldsymbol{x}+\boldsymbol{\epsilon})'\| = 1 \tag{4.28}$$

Thus, when the perturbation $\boldsymbol{\epsilon}$ of the centerline gets infinitesimal $\boldsymbol{t}_\epsilon$ stays collinear to $\boldsymbol{t}$. Indeed :

$$\|\boldsymbol{t}_\epsilon\| = 1 \Rightarrow (\boldsymbol{x}+\boldsymbol{\epsilon})' \cdot (\boldsymbol{x}+\boldsymbol{\epsilon})' = 1 \Leftrightarrow \boldsymbol{x}' \cdot \boldsymbol{\epsilon}' = -\tfrac{\lambda^2}{2}\|\boldsymbol{h}_x\|^2 \tag{4.29}$$

Which leeds to :

$$\cos\alpha_\epsilon(s) = \boldsymbol{t}\cdot\boldsymbol{t}_\epsilon = \boldsymbol{x}'\cdot(\boldsymbol{x}+\boldsymbol{\epsilon})' = 1 + \boldsymbol{x}'\cdot\boldsymbol{\epsilon}' = 1 - \tfrac{\lambda^2}{2}\|\boldsymbol{h}_x\|^2 \tag{4.30}$$

Finally, it's possible to conclude that $\alpha_\epsilon(s)$ is in first order of λ and thus gets negligible when λ tends to zero :

$$\lim_{\lambda\to 0} \alpha_\epsilon(s) = 0 \tag{4.31}$$

Aligning $\tilde{F}_\epsilon$ towards F_ϵ

$$\begin{cases} \tilde{\boldsymbol{u}}_\epsilon = \cos\Psi_\epsilon\, \boldsymbol{u} + \sin\Psi_\epsilon\, \boldsymbol{v} \\ \tilde{\boldsymbol{v}}_\epsilon = -\sin\Psi_\epsilon\, \boldsymbol{u} + \cos\Psi_\epsilon\, \boldsymbol{v} \end{cases} \tag{4.32}$$

Aligning F_ϵ towards $\boldsymbol{t}$

Recall that $\tilde{F}_\epsilon$ is obtained by parallel tranpsporting F_ϵ from $\boldsymbol{t}_\epsilon$ to $\boldsymbol{t}$. This operation could be seen as a rotation around $\boldsymbol{t}_\epsilon \times \boldsymbol{t}$ of an angle α_ϵ. Where :

$$\boldsymbol{b} = \boldsymbol{t}_\epsilon \times \boldsymbol{t} = \cos\gamma\, \tilde{\boldsymbol{u}}_\epsilon + \sin\gamma\, \tilde{\boldsymbol{v}}_\epsilon = \cos\gamma\, \boldsymbol{u}_\epsilon + \sin\gamma\, \boldsymbol{v}_\epsilon \tag{4.33}$$

Expressing F_ϵ on the basis $\tilde{F}_\epsilon$ gives :

$$\begin{aligned} \boldsymbol{u}_\epsilon &= \sin\gamma\, \boldsymbol{b} + \cos\gamma\, \Big(\sin\alpha_\epsilon\, \tilde{\boldsymbol{t}} + \cos\alpha_\epsilon\, (\cos\gamma\, \tilde{\boldsymbol{u}}_{\boldsymbol{\epsilon}} - \sin\gamma\, \tilde{\boldsymbol{v}}_{\boldsymbol{\epsilon}})\Big) \\ &= \cos\gamma \sin\alpha_\epsilon\, \boldsymbol{t} + (\cos\alpha_\epsilon \cos\gamma^2 + \cos\gamma^2)\tilde{\boldsymbol{u}}_{\boldsymbol{\epsilon}} + \sin\gamma\cos\gamma\, (1-\cos\alpha_\epsilon)\tilde{\boldsymbol{v}}_{\boldsymbol{\epsilon}} \end{aligned} \tag{4.34}$$

And :

$$\begin{aligned} \boldsymbol{v}_\epsilon &= \cos\gamma\, \boldsymbol{b} + \sin\gamma \Big(-\sin\alpha_\epsilon\, \tilde{\boldsymbol{t}} + \cos\alpha_\epsilon\, (\sin\gamma\, \tilde{\boldsymbol{u}}_{\boldsymbol{\epsilon}} - \cos\gamma\, \tilde{\boldsymbol{v}}_{\boldsymbol{\epsilon}}) \Big) \\ &= -\sin\gamma \sin\alpha_\epsilon\, \boldsymbol{t} + \cos\gamma \sin\gamma\, (1 - \cos\alpha_\epsilon) \tilde{\boldsymbol{u}}_{\boldsymbol{\epsilon}} + (\cos\gamma^2 + \cos\alpha_\epsilon \sin\gamma^2) \tilde{\boldsymbol{v}}_{\boldsymbol{\epsilon}} \end{aligned} \tag{4.35}$$

Thus, one can finally express F_ϵ on the basis F as the composition of two rotations :

$$\boldsymbol{u}_\epsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Psi_\epsilon & -\sin\Psi_\epsilon \\ 0 & \sin\Psi_\epsilon & \cos\Psi_\epsilon \end{bmatrix} \begin{bmatrix} \cos\gamma \sin\alpha_\epsilon \\ 1 - 2\cos\gamma^2 \sin\alpha_\epsilon/2^2 \\ 2\sin\gamma\cos\gamma\sin\alpha_\epsilon/2^2 \end{bmatrix} = \begin{bmatrix} \alpha_\epsilon \cos\gamma \\ 1 \\ \Psi_\epsilon \end{bmatrix} + o(\lambda) \tag{4.36}$$

And :

$$\boldsymbol{v}_\epsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Psi_\epsilon & -\sin\Psi_\epsilon \\ 0 & \sin\Psi_\epsilon & \cos\Psi_\epsilon \end{bmatrix} \begin{bmatrix} -\sin\gamma \sin\alpha_\epsilon \\ 2\sin\gamma\cos\gamma\sin\alpha_\epsilon/2^2 \\ 1 - 2\sin\gamma^2 \sin\alpha_\epsilon/2^2 \end{bmatrix} = \begin{bmatrix} -\alpha_\epsilon \sin\gamma \\ -\Psi_\epsilon \\ 1 \end{bmatrix} + o(\lambda) \tag{4.37}$$

Here, the expressions avec been developed in first order of λ as far as α_ϵ and Ψ_ϵ it's been proofed in eq [] that those quantities tends two zero when the perturbation of the centerline is infinitesimal.

Variation of Bishop frame with respect to $\boldsymbol{x}$

Finally, one can express the variation of material directors with respect to an infinitesimal variation of rod's centerline :

$$\begin{cases} \boldsymbol{u}_\epsilon = \alpha_\epsilon \cos\gamma\, \boldsymbol{t} + \boldsymbol{u} + \Psi_\epsilon \boldsymbol{v} + o(\lambda) \\ \boldsymbol{v}_\epsilon = -\alpha_\epsilon \sin\gamma\, \boldsymbol{t} + \boldsymbol{v} - \Psi_\epsilon \boldsymbol{u} + o(\lambda) \end{cases} \tag{4.38}$$

Variation of material frame with respect to $\boldsymbol{x}$

Recalling the expression of the material frame expressed in the reference Bishop frame, it's now easy to deduce the variation of material frame with respect to a variation of the rod's centerline :

$$\begin{cases} \boldsymbol{d}_1[\boldsymbol{x} + \lambda \boldsymbol{h}_x] = \cos\theta\, \boldsymbol{u}_\epsilon + \sin\theta\, \boldsymbol{v}_\epsilon \\ \boldsymbol{d}_2[\boldsymbol{x} + \lambda \boldsymbol{h}_x] = -\sin\theta\, \boldsymbol{u}_\epsilon + \cos\theta\, \boldsymbol{v}_\epsilon \end{cases} \tag{4.39}$$

Which leads according to the previous équation to :

$$\begin{cases} \boldsymbol{d}_1[\boldsymbol{x} + \lambda \boldsymbol{h}_x] = \boldsymbol{d}_1[\boldsymbol{x}] + \Psi_\epsilon \boldsymbol{d}_2[\boldsymbol{x}] + \alpha_\epsilon \cos(\theta - \gamma)\, \boldsymbol{t}[\boldsymbol{x}] + o(\lambda) \\ \boldsymbol{d}_2[\boldsymbol{x} + \lambda \boldsymbol{h}_x] = \boldsymbol{d}_2[\boldsymbol{x}] - \Psi_\epsilon \boldsymbol{d}_1[\boldsymbol{x}] - \alpha_\epsilon \sin(\theta + \gamma)\, \boldsymbol{t}[\boldsymbol{x}] + o(\lambda) \end{cases} \tag{4.40}$$

4.8.2 Derivative of the material curvatures vector with respect to $\boldsymbol{x}$

It's know straightforward from the previous section to express the variation of the material curvatures with respect to a variation $\boldsymbol{\epsilon} = \lambda \boldsymbol{h}_x$ of $\boldsymbol{x}$ while θ remains unchanged.

$$\begin{cases} (\boldsymbol{x} + \lambda \boldsymbol{h}_x)'' \cdot \boldsymbol{d}_1[\boldsymbol{x} + \lambda \boldsymbol{h}_x] = (\boldsymbol{x}'' + \lambda \boldsymbol{h}_x'') \cdot \Big(\boldsymbol{d}_1 + \Psi_\epsilon \boldsymbol{d}_2 + \alpha_\epsilon \cos(\theta - \gamma)\, \boldsymbol{t} + o(\lambda) \Big) \\ (\boldsymbol{x} + \lambda \boldsymbol{h}_x)'' \cdot \boldsymbol{d}_2[\boldsymbol{x} + \lambda \boldsymbol{h}_x] = (\boldsymbol{x}'' + \lambda \boldsymbol{h}_x'') \cdot \Big(\boldsymbol{d}_2 - \Psi_\epsilon \boldsymbol{d}_1 - \alpha_\epsilon \sin(\theta + \gamma)\, \boldsymbol{t} + o(\lambda) \Big) \end{cases} \tag{4.41}$$

Thus, recalling that $\boldsymbol{x}'' \cdot \boldsymbol{d}_3 = 0$ and that α_ϵ and Ψ_ϵ are first order quantities in λ :

$$\begin{cases} (\boldsymbol{x} + \lambda \boldsymbol{h}_x)'' \cdot \boldsymbol{d}_1[\boldsymbol{x} + \lambda \boldsymbol{h}_x] = \boldsymbol{x}'' \cdot \boldsymbol{d}_1 + \Psi_\epsilon \boldsymbol{x}'' \cdot \boldsymbol{d}_2 + \lambda \boldsymbol{h}_x'' \cdot \boldsymbol{d}_1 + o(\lambda) \\ (\boldsymbol{x} + \lambda \boldsymbol{h}_x)'' \cdot \boldsymbol{d}_2[\boldsymbol{x} + \lambda \boldsymbol{h}_x] = \boldsymbol{x}'' \cdot \boldsymbol{d}_2 - \Psi_\epsilon \boldsymbol{x}'' \cdot \boldsymbol{d}_1 + \lambda \boldsymbol{h}_x'' \cdot \boldsymbol{d}_2 + o(\lambda) \end{cases} \tag{4.42}$$

Which finally leads to :

$$\boldsymbol{\omega}[\boldsymbol{x} + \lambda \boldsymbol{h}_x] = \boldsymbol{\omega}[\boldsymbol{x}] - \Psi_\epsilon \boldsymbol{J} \boldsymbol{\omega}[\boldsymbol{x}] + \lambda \begin{bmatrix} -\boldsymbol{h}_x'' \cdot \boldsymbol{d}_2 \\ \boldsymbol{h}_x'' \cdot \boldsymbol{d}_1 \end{bmatrix} + o(\lambda) \tag{4.43}$$

Reminding the expression of Ψ_ϵ computed in paragraphe[], one can express the derivative of the material curvatures vector with respect to $\boldsymbol{x}$:

$$\boldsymbol{D}_x \boldsymbol{\omega}(s) \cdot \boldsymbol{h}_x = \left(\int_0^L \big((\delta_s - \delta_0)\, \kappa \boldsymbol{b} - (1 - H_s) \kappa \boldsymbol{b}' \big) \cdot \boldsymbol{h}_x \, dt \right) \boldsymbol{J} \boldsymbol{\omega} + \begin{bmatrix} -\boldsymbol{d}_2{}^T \\ \boldsymbol{d}_1{}^T \end{bmatrix} \cdot \boldsymbol{h}_x'' \tag{4.44}$$

4.8.3 Computation of the forces acting on the centerline

The forces acting on the centerline are given by the functional derivative of the potential elastic energy with respect to x which can be decomposed according to the chaine rule :

$$\begin{aligned} \langle -f(s)\,; \boldsymbol{h}_x \rangle = \boldsymbol{D}_x \mathcal{E}_p(s) \cdot \boldsymbol{h}_x &= \boldsymbol{D}_x \mathcal{E}_b(s) \cdot \boldsymbol{h}_x + \boldsymbol{D}_x \mathcal{E}_t(s) \cdot \boldsymbol{h}_x \\ &= \boldsymbol{D}_x \mathcal{E}_b[\boldsymbol{\omega}[\boldsymbol{x}]](s) \cdot \boldsymbol{h}_x + \boldsymbol{D}_x \mathcal{E}_t[\boldsymbol{x}](s) \cdot \boldsymbol{h}_x \end{aligned} \tag{4.45}$$

Derivative of the torsion energy with respect to x

Recall that the torsion energy only depends on θ which is independent of x. Thus, $\mathcal{E}_t$ is independent of x :

$$\boldsymbol{D}_x \mathcal{E}_t[\boldsymbol{x}](s) \cdot \boldsymbol{h}_x = \frac{d}{d\lambda} \mathcal{E}_t[\boldsymbol{x} + \lambda \boldsymbol{h}_x] \Big|_{\lambda=0} = 0 \tag{4.46}$$

Derivative of the bending energy with respect to x

The derivative of $\mathcal{E}_b$ is obtained with the chaine rule :

$$\boldsymbol{D}_\omega \mathcal{E}_b[\boldsymbol{\omega}](s) \cdot \boldsymbol{h}_\omega = \frac{d}{d\lambda} \mathcal{E}_b[\boldsymbol{\omega} + \lambda \boldsymbol{h}_\omega] \Big|_{\lambda=0} = \int_0^L (\boldsymbol{\omega} - \overline{\boldsymbol{\omega}})^T \boldsymbol{B} \cdot \boldsymbol{h}_\omega \, dt \tag{4.47}$$

Finally, reminding eq 4.14 :

$$\boldsymbol{D}_x\mathcal{E}_b[\boldsymbol{\omega}[\boldsymbol{x}]](s)\cdot\boldsymbol{h}_x = \boldsymbol{D}_\omega\mathcal{E}_b[\boldsymbol{\omega}](s)\cdot(\boldsymbol{D}_x\boldsymbol{\omega}[\boldsymbol{x}](s)\cdot\boldsymbol{h}_x) = \mathcal{A}+\mathcal{B}+\mathcal{C} \tag{4.48}$$

Where :

$$\begin{aligned}
\mathcal{A} &= \int_0^L (\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{B}\begin{bmatrix}-\boldsymbol{d_2}^T\\ \boldsymbol{d_1}^T\end{bmatrix}\cdot\boldsymbol{h}_x''\,dt\\
\mathcal{B} &= \int_{t=0}^L (\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{BJ\omega}\left(\int_{u=0}^L(\delta_t-\delta_0)\,\kappa\boldsymbol{b}\cdot\boldsymbol{h}_x\,du\right)\,dt\\
\mathcal{C} &= \int_{t=0}^L -(\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{BJ\omega}\left(\int_{u=0}^L(1-H_t)\,\kappa\boldsymbol{b}'\cdot\boldsymbol{h}_x\,du\right)\,dt
\end{aligned} \tag{4.49}$$

Calculus of $\mathcal{A}$:

$$\begin{aligned}
\mathcal{A} &= \int_0^L (\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{B}\begin{bmatrix}-\boldsymbol{d_2}^T\\ \boldsymbol{d_1}^T\end{bmatrix}\cdot\boldsymbol{h}_x''\,dt\\
&= \left[(\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{B}\begin{bmatrix}-\boldsymbol{d_2}^T\\ \boldsymbol{d_1}^T\end{bmatrix}\cdot\boldsymbol{h}_x'\right]_0^L - \int_0^L\left((\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{B}\begin{bmatrix}-\boldsymbol{d_2}^T\\ \boldsymbol{d_1}^T\end{bmatrix}\right)'\cdot\boldsymbol{h}_x'\,dt\\
&= \left[(\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{B}\begin{bmatrix}-\boldsymbol{d_2}^T\\ \boldsymbol{d_1}^T\end{bmatrix}\cdot\boldsymbol{h}_x' - \left((\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{B}\begin{bmatrix}-\boldsymbol{d_2}^T\\ \boldsymbol{d_1}^T\end{bmatrix}\right)'\cdot\boldsymbol{h}_x\right]_0^L\\
&+ \int_0^L\left((\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{B}\begin{bmatrix}-\boldsymbol{d_2}^T\\ \boldsymbol{d_1}^T\end{bmatrix}\right)''\cdot\boldsymbol{h}_x\,dt
\end{aligned} \tag{4.50}$$

Calculus of $\mathcal{B}$:

$$\begin{aligned}
\mathcal{B} &= \int_{t=0}^L (\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{BJ\omega}\left(\int_{u=0}^L(\delta_t-\delta_0)\,\kappa\boldsymbol{b}\cdot\boldsymbol{h}_x\,du\right)\,dt\\
&= -\left(\kappa\boldsymbol{b}\cdot\boldsymbol{h}_x\right)(0)\int_{t=0}^L(\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{BJ\omega}\,dt + \int_{t=0}^L(\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{BJ\omega}\kappa\boldsymbol{b}\cdot\boldsymbol{h}_x\,dt
\end{aligned} \tag{4.51}$$

Calculus of $\mathcal{C}$:

$$\begin{aligned}
\mathcal{C} &= \int_{t=0}^L -(\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{BJ\omega}\left(\int_{u=0}^L(1-H_t)\,\kappa\boldsymbol{b}'\cdot\boldsymbol{h}_x\,du\right)\,dt\\
&= \int_{u=0}^L\int_{t=u}^L -\left((\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{BJ\omega}\right)(t)\left(\kappa\boldsymbol{b}'\cdot\boldsymbol{h}_x\right)(u)\,dt\,du\\
&= \int_{u=0}^L -\left(\int_{t=u}^L(\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{BJ\omega}\,dt\right)\left(\kappa\boldsymbol{b}'\cdot\boldsymbol{h}_x\right)\,du
\end{aligned} \tag{4.52}$$

By several integration by parts, using Fubini's theorem once and supposing that the terms vanishes at $s = 0$ and $s = L$:

$$\begin{aligned}
\mathcal{B}+\mathcal{C} &= \int_{t=0}^{L}\left((\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{BJ\omega}\kappa\boldsymbol{b}-\left(\int_{u=t}^{L}(\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{BJ\omega}\,du\right)\kappa\boldsymbol{b}'\right)\cdot\boldsymbol{h}_x\,dt \\
&= \int_{t=0}^{L}\left(-\left(\int_{u=t}^{L}(\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{BJ\omega}\,du\right)'\kappa\boldsymbol{b}-\left(\int_{u=t}^{L}(\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{BJ\omega}\,du\right)\kappa\boldsymbol{b}'\right)\cdot\boldsymbol{h}_x\,dt \\
&= \int_{t=0}^{L}\left(-\left(\int_{u=t}^{L}(\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{BJ\omega}\,du\right)\kappa\boldsymbol{b}\right)'\cdot\boldsymbol{h}_x\,dt
\end{aligned} \tag{4.53}$$

Which can be rewritted using the quasi-static hypothesis :

$$\begin{aligned}
\mathcal{B}+\mathcal{C} &= \int_{t=0}^{L}\left(-\left(\int_{u=t}^{L}(\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{BJ\omega}\,du\right)\kappa\boldsymbol{b}\right)'\cdot\boldsymbol{h}_x\,dt \\
&= \int_{t=0}^{L}\left(-\left(\int_{u=t}^{L}\beta(\theta'-\overline{\theta'})(\delta_L-\delta_0)-\left(\beta(\theta'-\overline{\theta'})\right)'\,du\right)\kappa\boldsymbol{b}\right)'\cdot\boldsymbol{h}_x\,dt \\
&= \int_{t=0}^{L}\left(-\left(\beta(\theta'-\overline{\theta'})(L)-\left[\beta(\theta'-\overline{\theta'})\right]_t^L\right)\kappa\boldsymbol{b}\right)'\cdot\boldsymbol{h}_x\,dt \\
&= \int_{t=0}^{L}-\left(\beta(\theta'-\overline{\theta'})\kappa\boldsymbol{b}\right)'\cdot\boldsymbol{h}_x\,dt
\end{aligned} \tag{4.54}$$

Finally :

$$\boldsymbol{D}_x\mathcal{E}_b[\boldsymbol{\omega}[\boldsymbol{x}]](s)\cdot\boldsymbol{h}_x = \int_0^L\left(\left((\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{B}\begin{bmatrix}-\boldsymbol{d}_2{}^T\\ \boldsymbol{d}_1{}^T\end{bmatrix}\right)''-\left(\beta(\theta'-\overline{\theta'})\kappa\boldsymbol{b}\right)'\right)\cdot\boldsymbol{h}_x\,dt \tag{4.55}$$

Internal forces

Thus, the

$$\begin{aligned}
\langle -f(s)\,;\boldsymbol{h}_x\rangle &= \boldsymbol{D}_x\mathcal{E}_p(s)\cdot\boldsymbol{h}_x \\
&= \int_0^L\left(\left((\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{B}\begin{bmatrix}-\boldsymbol{d}_2{}^T\\ \boldsymbol{d}_1{}^T\end{bmatrix}\right)''-\left(\beta(\theta'-\overline{\theta'})\kappa\boldsymbol{b}\right)'\right)\cdot\boldsymbol{h}_x\,dt
\end{aligned} \tag{4.56}$$

Finally, we can conclude on the expression of the internal moment of torsion :

$$f(s) = -\left((\boldsymbol{\omega}-\overline{\boldsymbol{\omega}})^T\boldsymbol{B}\begin{bmatrix}-\boldsymbol{d}_2{}^T\\ \boldsymbol{d}_1{}^T\end{bmatrix}\right)''+\left(\beta(\theta'-\overline{\theta'})\kappa\boldsymbol{b}\right)' \tag{4.57}$$

Remarquer ici que l'espression est purement locale. Elle ne dépend pas du sens de parcours de la poutre, contrairement au raisonnement suivi. Cette différence est notable avec la démarche de B. Audoly.

4.9 Numerical Model

4.10 Discretization

4.11 Connection

Internal forces

Note that :

$$\boldsymbol{M} = (\boldsymbol{\omega} - \overline{\boldsymbol{\omega}})^T \boldsymbol{B} = (\kappa_1 - \overline{\kappa_1}) EI_1 \boldsymbol{d}_1 + (\kappa_2 - \overline{\kappa_2}) EI_2 \boldsymbol{d}_2 \tag{4.58}$$

And

$$\boldsymbol{Q} = \left(\beta\left(\theta - \bar{\theta}\right)\right)' \boldsymbol{d}_3 \tag{4.59}$$

Bibliography

[AAP10] Basile Audoly, M Amar, and Yves Pomeau. *Elasticity and geometry*. 2010.

[BAV+10] Miklós Bergou, Basile Audoly, Etienne Vouga, Max Wardetzky, and Eitan Grinspun. Discrete viscous threads. *ACM Transactions on ...*, pages 1–10, 2010.

[Ber09] Mitchell Berger. Topological Quantities: Calculating Winding, Writhing, Linking, and Higher order Invariants. *Lecture Notes in Mathematics*, 1973:75–97, 2009.

[Bis75] Richard L. Bishop. There is more than one way to frame a curve. *Mathematical Association of America*, 1975.

[BWR+08] Miklós Bergou, Max Wardetzky, Stephen Robinson, Basile Audoly, and Eitan Grinspun. Discrete elastic rods. *ACM SIGGRAPH*, pages 1–12, 2008.

[dV05] Renko de Vries. Evaluating changes of writhe in computer simulations of supercoiled DNA. *The Journal of Chemical Physics*, 122(6), 2005.

[Ful78] F Brock Fuller. Decomposition of the linking number of a closed ribbon : A problem from molecular biology. 75(8):3557–3561, 1978.

[Nab14] Sina Nabei. *Mechanical form-finding of timber fabric structures*. PhD thesis, EPFL, 2014.

[Vau00] Rue Vauquelin. Writhing Geometry at Finite Temperature : Random Walks and Geometric phases for Stiff Polymers. (1), 2000.

5 Elastic rod : a novel element from kirchhoff equations

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5.1 Introduction

Dans ce chapitre, après un bref rappel sur le cadre mathématique d'étude des courbes paramétrique de l'espace, on présente les notions de courbures et de torsion géométrique associées au repère de fraient. On montre ensuite le cas plus général d'un repère mobile quelconque attaché à une courbe gamma. On définit enfin la particularité d'un repère mobile adapté à un courbe, et on présente, en sus du repère de Frenet, une approche différente pour accrocher des repères le long d'une courbe (Bishop / RMF / Zéro-twisting frame)

Connection Part II

6 Calculus of variations

6.1 Introduction

In this appendix we drawback essential mathematical concepts for the calculus of variations. Recall how the notion of energy, gradients are extended to function spaces.

[AMR02]

6.2 Spaces

6.2.1 Normed space

A *normed space* $V(\mathbb{K})$ is a vector space V over the scalar field $\mathbb{K}$ with a norm $\|.\|$.

A *norm* is a map $\|.\| : V \times V \longmapsto \mathbb{K}$ which satisfies :

$$\forall x \in V, \qquad \|x\| = 0_{\mathbb{K}} \Rightarrow x = 0_V \tag{6.1a}$$

$$\forall x \in V, \forall \lambda \in \mathbb{K}, \qquad \|\lambda x\| = |\lambda| \, \|x\| \tag{6.1b}$$

$$\forall (x, y) \in V^2, \qquad \|x + y\| \leqslant \|x\| + \|y\| \tag{6.1c}$$

6.2.2 Inner product space

A *inner product space* or *pre-hilbert space* $E(\mathbb{K})$ is a vector space E over the scalar field $\mathbb{K}$ with an inner product.

An *inner product* is a map $\langle\, ; \rangle : E \times E \longmapsto \mathbb{K}$ which is bilinear, symmetric, positive-definite

:

$$\forall (x,y,z) \in E^3, \forall (\lambda,\mu) \in \mathbb{K}^2, \qquad \langle \lambda x \ + \mu y \,;\, z\rangle = \lambda\langle x\,;\,z\rangle + \mu\langle y\,;\,z\rangle \tag{6.2a}$$

$$\langle x\,;\,\lambda y + \mu z\rangle = \lambda\langle x\,;\,y\rangle + \mu\langle x\,;\,z\rangle$$

$$\forall (x,y) \in E^2, \qquad \langle x\,;\,y\rangle = \langle y\,;\,x\rangle \tag{6.2b}$$

$$\forall x \in E, \qquad \langle x\,;\,x\rangle \geqslant 0_{\mathbb{K}} \tag{6.2c}$$

$$\forall x \in E, \qquad \langle x\,;\,x\rangle = 0_{\mathbb{K}} \Rightarrow x = 0_E \tag{6.2d}$$

Moreover, an inner product naturally induces a norm on E defined by :

$$\forall x \in E, \quad \|x\| = \sqrt{\langle x\,;\,x\rangle} \tag{6.3}$$

Thus, an inner product vector space is also naturally a normed vector space.

6.2.3 Euclidean space

An *Euclidean space* $\mathcal{E}(\mathbb{R})$ is a finite-dimensional real vector space with an inner product. Thus, distances and angles between vectors could be defined and measured regarding to the norm associated with the chosen inner product.

An Euclidean space is nothing but a finite-dimensional real pre-hilbert space.

6.2.4 Banach space

A *Banach space* $\mathcal{B}(\mathbb{K})$ is a complete normed vector space, which means that it is a normed vector space in which every Cauchy sequence of $\mathcal{B}$ converges in $\mathcal{B}$ for the given norm.

Thus, a Banach space is a vector space with a metric that allows the computation of vector length and distance between vectors and is complete in the sense that a Cauchy sequence of vectors always converges to a well defined limit in that space.

6.2.5 Hilbert space

A *Hilbert space* is an inner product vector space $\mathcal{H}(\mathbb{K})$ such that the natural norm induced by the inner product turns $\mathcal{H}$ into a complete metric space (i.e. every Cauchy sequence of $\mathcal{H}$ converges in $\mathcal{H}$).

The Hilbert space concept is a generalization of the Euclidean space concept. In physics it's common to encounter Hilbert spaces as infinite-dimensional function spaces.

Hilbert spaces are Banach spaces, but the converse does not hold generally.

For example, $\mathcal{L}^2([a,b])$ is an infinite-dimensional Hilbert space with the canonical inner product $\langle f\,;\,g\rangle = \int_a^b fg$.

Note that $\mathcal{L}^2$ is the only Hilbert space among the $\mathcal{L}^p$ spaces.

6.3 Derivative

The well known notion of function derivative in $\mathbb{R}^{\mathbb{R}}$ can be extended to maps between Banach spaces. This is useful in physics when formulating problems as variational problems, usually in terms of energy minimization. Indeed, energy is generally defined over a functional vector space and not simply over the real line.

In this case, the research of minimal values of a potential energy rests on the calculus of variations of the energy function compared to variations to other functions defining the problem (geometry, materials, boundary conditions, ...).

Mathematical concepts extended well-known notions of derivative, jacobian and hessian in Euclidean spaces (typically $\mathbb{R}^2$ or $\mathbb{R}^3$) for Banach functional spaces.

6.3.1 Fréchet derivative

Differentiability

Let $\mathcal{B}_V$ and $\mathcal{B}_W$ be two Banach spaces and $U \subset \mathcal{B}_V$ an open subset of $\mathcal{B}_V$. Let $f : u \longmapsto f(u)$ be a function of $U^{\mathcal{B}_W}$. f is said to be *Fréchet differentiable* at $u_0 \in U$ if there exists a continious linear operator $\boldsymbol{D}f(u_0) \in \mathcal{L}(\mathcal{B}_V, \mathcal{B}_W)$ such that :

$$\lim_{h \to 0} \frac{f(u_0 + h) - f(u_0) - \boldsymbol{D}f(u_0) \cdot h}{\|h\|} = 0 \tag{6.4a}$$

Or, equivalently :

$$f(u_0 + h) = f(u_0) + \boldsymbol{D}f(u_0) \cdot h + o(h) \quad , \quad \lim_{h \to 0} \frac{o(h)}{\|h\|} = 0 \tag{6.4b}$$

In the literature, it is common to found the following notations : $df = \boldsymbol{D}f(u_0) \cdot h = \boldsymbol{D}f_{u_0}(h) = \boldsymbol{D}f(u_0, h)$ for the differential of f, which means nothing but $\boldsymbol{D}f(u_0)$ is linear regarding h. The dot denotes the evaluation of $\boldsymbol{D}f(u_0)$ at h. This notation can be ambiguous as far as the linearity of $\boldsymbol{D}f(u_0)$ in h is denoted as a product which is not explicitly defined.

Derivative

If f is Fréchet differentiable at $u_0 \in U$, the continous linear operator $\boldsymbol{D}f(u_0) \in \mathcal{L}(\mathcal{B}_V, \mathcal{B}_W)$ is called the *Fréchet derivative* of f at u_0 and is also denoted :

$$f'(u_0) = \boldsymbol{D}f(u_0) \tag{6.5}$$

f is said to be $\mathcal{C}^1$ in the sens of Fréchet if f is Fréchet differentiable for all $u \in U$ and the function $\boldsymbol{D}f : u \longmapsto f'(u)$ of $U^{\mathcal{L}(\mathcal{B}_V,\mathcal{B}_W)}$ is continuous.

Differential or total derivative

$df = \boldsymbol{D}f(u_0) \cdot h$ is sometimes called the *differential* or *total derivative* of f and represents the change in the function f for a perturbation h from u_0.

Higer derivatives

Because the differential of f is a linear map from $\mathcal{B}_V$ to $\mathcal{L}(\mathcal{B}_V,\mathcal{B}_W)$ it is possible to look for the differentiability of $\boldsymbol{D}f$. If it exists, it is denoted $\boldsymbol{D}^2 f$ and maps $\mathcal{B}_V$ to $\mathcal{L}(\mathcal{B}_V,\mathcal{L}(\mathcal{B}_V,\mathcal{B}_W))$.

6.3.2 Gâteaux derivative

Directional derivative

Let $\mathcal{B}_V$ and $\mathcal{B}_W$ be two Banach spaces and $U \subset \mathcal{B}_V$ an open subset of $\mathcal{B}_V$. Let $f : u \longmapsto f(u)$ be a function of $U^{\mathcal{B}_W}$. f is said to have a *derivative in the direction* $h \in \mathcal{B}_V$ at $u_0 \in U$ if :

$$\frac{d}{d\lambda} f(u_0 + \lambda h)\bigg|_{\lambda=0} = \lim_{\lambda\to 0} \frac{f(u_0+\lambda h) - f(u_0)}{\lambda} \tag{6.6}$$

exists. This element of $\mathcal{B}_W$ is called the *directional derivative* of f in the direction h at u_0.

Differentiability

Let $\mathcal{B}_V$ and $\mathcal{B}_W$ be two Banach spaces and $U \subset \mathcal{B}_V$ an open subset of $\mathcal{B}_V$. Let $f : u \longmapsto f(u)$ be a function of $U^{\mathcal{B}_W}$. f is said to be *Gâteaux differentiable* at $u_0 \in U$ if there exists a continious linear operator $\boldsymbol{D}f(u_0) \in \mathcal{L}(\mathcal{B}_V,\mathcal{B}_W)$ such that :

$$\forall h \in \mathcal{U}, \quad \lim_{\lambda\to 0} \frac{f(u_0+\lambda h) - f(u_0)}{\lambda} = \frac{d}{d\lambda} f(u_0 + \lambda h)\bigg|_{\lambda=0} = \boldsymbol{D}f(u_0) \cdot h \tag{6.7a}$$

Or, equivalently :

$$\forall h \in \mathcal{U}, \quad f(u+\lambda h) = f(u) + \lambda \boldsymbol{D}f(u_0) \cdot h + o(\lambda) \quad , \quad \lim_{\lambda\to 0} \frac{o(\lambda)}{\lambda} = 0 \tag{6.7b}$$

In other words, it means that all the directional derivatives of f exist at u_0.

Derivative

If f is Gâteaux differentiable at $u_0 \in U$, the continous linear operator $\boldsymbol{D}f(u_0) \in \mathcal{L}(\mathcal{B}_V, \mathcal{B}_W)$ is called the *Gâteaux derivative* of f at u_0 and is also denoted :

$$f'(u_0) = \boldsymbol{D}f(u_0) \tag{6.8}$$

f is said to be $\mathcal{C}^1$ in the sens of Gâteaux if f is Gâteaux differentiable for all $u \in U$ and the function $\boldsymbol{D}f : u \longmapsto f'(u)$ of $U^{\mathcal{L}(\mathcal{B}_V, \mathcal{B}_W)}$ is continuous.

The Gâteaux derivative is a weaker form of derivative than the Fréchet derivative. If f is Fréchet differentiable, then it is also Gâteaux differentiable and its Fréchet and Gâteaux derivatives agree, but the converse does not hold generally.

6.3.3 Useful properties

Let $\mathcal{B}_V$, $\mathcal{B}_W$ and $\mathcal{B}_Z$ be three Banach spaces. Let $f, g : \mathcal{B}_V \longmapsto \mathcal{B}_W$ and $h : \mathcal{B}_W \longmapsto \mathcal{B}_Z$ be three Gâteaux differentiable functions. Then, the following useful properties holds :

$$\boldsymbol{D}(f + g)(u) = \boldsymbol{D}f(u) + \boldsymbol{D}g(u) \tag{6.9}$$

$$\boldsymbol{D}(f \circ h)(u) = \boldsymbol{D}h(f(u)) \circ \boldsymbol{D}f(u) = \boldsymbol{D}h(f(u)) \cdot \boldsymbol{D}f(u) \tag{6.10}$$

Recall that the composition of $\boldsymbol{D}h(f(u))$ with $\boldsymbol{D}f(u)$ means "$\boldsymbol{D}h(f(u))$ applied to $\boldsymbol{D}f(u)$" and is also denoted by $\cdot$ as explained previously.

6.3.4 Partial derivative

Following [AMR02] the main results on partial derivatives of two-variables functions are presented here. They are generalizable to n-variables functions.

Definition

Let $\mathcal{B}_{V_1}$, $\mathcal{B}_{V_2}$ and $\mathcal{B}_W$ be three Banach spaces and $U \subset \mathcal{B}_{V_1} \oplus \mathcal{B}_{V_2}$ an open subset of $\mathcal{B}_{V_1} \oplus \mathcal{B}_{V_2}$. Let $f : u \longmapsto f(u)$ be a function of $U^{\mathcal{B}_W}$. Let $u_0 = (u_{01}, u_{02}) \in U$. If the derivatives of the following functions exist :

$$\begin{array}{rccc} f_1 : & \mathcal{B}_{V_1} & \longrightarrow & \mathcal{B}_W \\ & u_1 & \longmapsto & f(u_1, u_{02}) \end{array} \quad , \quad \begin{array}{rccc} f_2 : & \mathcal{B}_{V_2} & \longrightarrow & \mathcal{B}_W \\ & u_2 & \longmapsto & f(u_{01}, u_2) \end{array} \tag{6.11}$$

they are called *partial derivatives* of f at u_0 and are denoted $\boldsymbol{D}_1 f(u_0) \in \mathcal{L}(\mathcal{B}_{V_1}, \mathcal{B}_W)$ and $\boldsymbol{D}_2 f(u_0) \in \mathcal{L}(\mathcal{B}_{V_2}, \mathcal{B}_W)$.

Differentiability

Let $\mathcal{B}_{V_1}$, $\mathcal{B}_{V_2}$ and $\mathcal{B}_W$ be three Banach spaces and $U \subset \mathcal{B}_{V_1} \oplus \mathcal{B}_{V_2}$ an open subset of $\mathcal{B}_{V_1} \oplus \mathcal{B}_{V_2}$. Let $f : u \longmapsto f(u)$ be a function of $U^{\mathcal{B}_W}$. If f is differentiable, then the partial derivatives exist and satisfy for all $h = (h_1, h_2) \in \mathcal{B}_{V_1} \oplus \mathcal{B}_{V_2}$:

$$\boldsymbol{D}_1 f(u) \cdot h_1 = \boldsymbol{D} f(u) \cdot (h_1, 0) \tag{6.12}$$

$$\boldsymbol{D}_2 f(u) \cdot h_2 = \boldsymbol{D} f(u) \cdot (0, h_2) \tag{6.13}$$

$$\boldsymbol{D} f(u) \cdot (h_1, h_2) = \boldsymbol{D}_1 f(u) \cdot h_1 + \boldsymbol{D}_2 f(u) \cdot h_2 \tag{6.14}$$

6.4 Gradient vector

Let $\mathcal{H}$ be a Hilbert space with the inner product denoted $\langle\, ; \rangle$. Let $U \subset \mathcal{H}$ an open subset of $\mathcal{H}$. Let $F : u \longmapsto F(u)$ be a scalar function of $U^{\mathbb{R}}$. The *gradient* of F is the map $grad\, F : x \longmapsto (grad\, F)(x)$ of $U^{\mathcal{H}}$ such that :

$$\forall h \in \mathcal{H}, \quad \langle (grad\, F)(x)\, ; h \rangle = \boldsymbol{D} F(x) \cdot h \tag{6.15}$$

Note that the gradient vector depends on the chosen inner product. For $\mathcal{H} = \mathbb{R}^n$ with the canonical inner product, one can recall the usual definition of the gradient vector and the corresponding linear approximation of F :

$$\boldsymbol{F}_{x+h} = \boldsymbol{F}_x + (grad\, F)_x^T H + \boldsymbol{o}(H) \quad , \quad grad\, F_x = \begin{bmatrix} \frac{\partial F}{\partial x_1} \\ \vdots \\ \frac{\partial F}{\partial x_n} \end{bmatrix} \in \mathbb{R}^n \tag{6.16}$$

Recall that the canonical inner product on $\mathbb{R}^n$ is such that $\langle x\, ; y \rangle = X^T Y$ in a column vector representation. In this case it is common to denote $grad\, F = \nabla F$.

For function spaces the usual definition of the gradient can be extended. For instance if F is a scalar function on $\mathcal{L}^2$, the gradient of F is the unique function (if it exists) from $\mathcal{L}^2$ which satisfies :

$$\forall h \in \mathcal{L}^2, \quad \boldsymbol{D} F(x) \cdot h = \langle (grad\, F)(x)\, ; h \rangle = \int (grad\, F) h \tag{6.17}$$

In this case it is common to denote $grad\, F = \frac{\delta F}{\delta x}$. The gradient is also known as the *functional derivative*. The existence and unicity of $grad\, F$ is ensured by the *Riesz representation theorem*.

6.5 Jacobian matrix

Let f be a differentiable function from $\mathbb{R}^n$ to $\mathbb{R}^m$. The *differential* or *total derivative* of such a fonction is a linear application from $\mathbb{R}^n$ to $\mathbb{R}^m$ which could be represented with the following matrix called the *jacobian matrix* :

$$\boldsymbol{D}f(x) = \boldsymbol{J}_x = \frac{df}{dx} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \in \mathcal{M}_{m,n}(\mathbb{R}) \tag{6.18}$$

Thus, with the matrix notation, the Taylor expansion takes the following form :

$$\boldsymbol{F}_{x+h} = \boldsymbol{F}_x + \boldsymbol{J}_x H + \boldsymbol{o}(H) \tag{6.19}$$

In the cas $m = 1$, the jacobian matrix of the functional F is nothing but the gradient vector transpose itself :

$$\boldsymbol{D}F(x) = \boldsymbol{J}_x = \frac{dF}{dx} = \begin{bmatrix} \frac{\partial F}{\partial x_1} & \cdots & \frac{\partial F}{\partial x_n} \end{bmatrix} = \nabla F^T \tag{6.20}$$

6.6 Hessian

Let F be a differentiable scalar function from $\mathbb{R}^n$ to $\mathbb{R}$. The second order differential of such a fonction is a linear application from $\mathbb{R}^n$ to $\mathbb{R}^n$ which could be represented with the following matrix called the *hessian matrix* :

$$\boldsymbol{D}^2F(x) = \boldsymbol{H}_x = \frac{d^2F}{dx}(x) = \begin{bmatrix} \frac{\partial F_1^2}{\partial x_1^2} & \frac{\partial F_1^2}{\partial x_1 \partial x_2} & \cdots & \frac{\partial F_1^2}{\partial x_1 \partial x_n} \\ \frac{\partial F_1^2}{\partial x_2 \partial x_1} & \frac{\partial F_1^2}{\partial x_2^2} & \cdots & \frac{\partial F_1^2}{\partial x_2 \partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial F_p^2}{\partial x_n \partial x_1} & \frac{\partial F_p^2}{\partial x_n \partial x_2} & \cdots & \frac{\partial F_p^2}{\partial x_n^2} \end{bmatrix} \in \mathcal{M}_{n,n}(\mathbb{R}) \tag{6.21}$$

Thus, with the matrix notation, the Taylor expansion takes the following form :

$$\boldsymbol{F}_{x+h} = \boldsymbol{F}_x + \boldsymbol{J}_x H + \tfrac{1}{2} H^T \boldsymbol{H}_x H + \boldsymbol{o}(H) \tag{6.22}$$

6.7 Functional

A *functional* is a map from a vector space $E(\mathbb{K})$ into its underlying scalar field $\mathbb{K}$. Here $\mathcal{E}_p[\boldsymbol{x}, \theta]$ is a functional depending over $\boldsymbol{x}$ and θ.

Bibliography

[AMR02] Ralph Abraham, Jerrold E. Marsde, and Tudor Ratiu. *Manifolds, Tensor Analysis, and Applications (Ralph Abraham, Jerrold E. Marsden and Tudor Ratiu)*. 2002.