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École Polytechnique Fédérale de Lausanne

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par

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Lausanne, EPFL, 2011

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Acknowledgements

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Lausanne, 12 Mars 2011 D. K.

Abstract

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Key words:

Résumé

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Mots clefs :

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Bibliography

2 Elastic gridshell

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Bibliography

Part I

Torsion

3 Geometry of smooth and discret curves

3.1 Introduction

Dans ce chapitre, après un bref rappel sur le cadre mathématique d'étude des courbes paramétrique de l'espace, on présente les notions de courbures et de torsion géométrique associées au repère de fraient. On montre ensuite le cas plus général d'un repère mobile quelconque attaché à une courbe gamma. On définit enfin la particularité d'un repère mobile adapté à un courbe, et on présente, en sus du repère de Frenet, une approche différente pour accrocher des repères le long d'une courbe (Bishop / RMF / Zéro-twisting frame)

3.2 Paramectric Curves

3.2.1 Definition

Let I be an interval [Bis75] of $\mathbb{R}$ and $F\colon t \mapsto F(t)$ be a map of $\mathcal{C}(I, \mathbb{R}^3)$. Then $\gamma = (I, F)$ is called a *parametric curve* and :

- The 2-uplet (I, F) is called a *parametrization* of γ
- $\gamma = F(I) = \{F(t), t \in I\}$ is called the *graph* or *trace* of γ
- γ is said to be $\mathcal{C}^k$ if $F \in \mathcal{C}^k(I, \mathbb{R}^3)$

Remark. Note that for a given graph in $\mathbb{R}^3$ they may be different possible parameterizations. From now, γ will simply refer to $F(I)$, its graph.

3.2.2 Regularity

Let $\gamma = (I, F)$ be a parametric [Blo] curve, and $t_0 \in I$ a parameter.

- A point of parameter t_0 is called *regular* if $F'(t_0) \neq 0$.
  The curve γ is called *regular* if γ is $\mathcal{C}^1$ and $F'(t) \neq 0, \forall t \in I$

- A point of parameter t_0 is called *biregular* if $F'(t_0)$ and $F''(t_0)$ are not collinear
  The curve γ is called *biregular* if γ is $\mathcal{C}^2$ and $F'(t) \cdot F''(t) \neq 0, \forall t \in I$

3.2.3 Reparametrization

Let $\gamma = (I, F)$ be a parametric curve of class $\mathcal{C}^k$, $J \in \mathbb{R}^3$ an interval, and $\varphi: I \mapsto J$ a $\mathcal{C}^k$ diffeomorphisme. Lets define $G = F \circ \varphi$. Then :

- $G \in \mathcal{C}^k(J, \mathbb{R}^3)$

- $G(J) = F(I)$

- φ is said to be an admissible *change of parameter* for γ

- (J, G) is said to be another *admissible parametrization* for γ

3.2.4 Natural parametrization

Let γ be a space curve of class $\mathcal{C}^1$. A parametrization (I, F) of γ is called *natural* if $\|F'(t)\| = 1, \forall t \in I$. Thus :

- The curve is necessarily regular

- F is strictly monotonic

3.2.5 Curve length

Let $\gamma = (I, F)$ be a parametric curve of class $\mathcal{C}^1$. The length of γ is define as :

$$L = \int_I \|F'(t)\| dt \tag{3.1}$$

Note that the length of γ is invariant under reparametrization.

3.2.6 Arc-length parametrization

Let $\gamma = (I, F)$ be a regular parametric curve of class $\mathcal{C}^1$. Let $t_0 \in I$ be a given parameter. The following map is said to be the *arc-length of origin* t_0 of γ :

$$s: t \mapsto \int_{t_0}^{t} \|F'(u)\| du \quad , \quad s \in I \times \mathbb{R} \tag{3.2}$$

The arc-length $s: I \mapsto s(I)$ is an admissible change of parameter for γ. Indeed, s is a $\mathcal{C}^1$ diffeomorphisme because it is bijective ($s' > 0$).

Lets define $G = F \circ s^{-1}$ and $J = s(I)$. Thus (J, G) is a natural reparametrization of γ and $\|G'(s)\| = 1, \forall s \in J$.

This parametrization is preferred because the natural parameter s traverses the image of γ at unit speed ($\|G'\| = 1$).

3.3 Frenet's Trihedron

En cinématique ou en géométrie différentielle, le repère de Frenet ou repère de Serret-Frenet est un outil d'étude du comportement local des courbes. Il s'agit d'un repère local associé à un point P, décrivant une courbe (C). Son mode de construction est différent selon que l'espace ambiant est de dimension 2 (courbe plane) ou 3 (courbe gauche) ; il est possible également de définir un repère de Frenet en toute dimension, pourvu que la courbe vérifie des conditions différentielles simples.

Le repère de Frenet, et les formules de Frenet donnant les dérivées des vecteurs de ce repère, permettent de mener de façon systématique des calculs de courbure, de torsion pour les courbes gauches et d'introduire des concepts géométriques associés aux courbes : cercle osculateur, plan osculateur (en), parallélisme des courbes

In this section we consider $\gamma = (J, G)$ to be a regular ($\|\gamma\| = 1$) parametric curve of class $\mathcal{C}^2$, parametrized by its arc-length (denoted s). For the sake of simplicity we will refer to $G(s)$ as $\gamma(s)$.

3.3.1 Tangent vector

The first vector of Frenet's trihedron is called the *unit tangent vector* ($\mathbf{t}$). At any given parameter $s_0 \in J$, it is defined as :

$$\mathbf{t}(s_0) = \frac{\gamma'(s_0)}{\|\gamma'(s_0)\|} = \gamma'(s_0) \quad , \quad \|\mathbf{t}(s_0)\| = 1 \tag{3.3}$$

3.3.2 Normal vector

The second vector of Frenet's trihedron is called the *unit normal vector* ($\mathbf{n}$). It is constructed from $\mathbf{t}'$ which is orthogonal to $\mathbf{t}$: $\|\mathbf{t}\| = 1 \Rightarrow \mathbf{t}' \cdot \mathbf{t} = 0 \Leftrightarrow \mathbf{t}' \perp \mathbf{t}$. Thus, at any given parameter $s_0 \in J$, it is define as :

$$\mathbf{n}(s_0) = \frac{\mathbf{t}'(s_0)}{\|\mathbf{t}'(s_0)\|} = \frac{\gamma''(s_0)}{\|\gamma''(s_0)\|} \quad , \quad \|\mathbf{n}(s_0)\| = 1 \tag{3.4}$$

3.3.3 Binormal vector

The third vector of Frenet's trihedron is called the *unit binormal vector* ($\mathbf{b}$). It is constructed from $\mathbf{t}$ and $\mathbf{n}$ to form an orthonormal direct basis of $\mathbb{R}^3$. Thus, at any given parameter

$s_0 \in J$, it is define as :

$$\mathbf{b}(s_0) = \mathbf{t}(s_0) \times \mathbf{n}(s_0) \quad , \quad \|\mathbf{b}(s_0)\| = 1 \tag{3.5}$$

3.4 Curvature

Note that from a geometric point of view, $\frac{1}{\kappa(s_0)}$ represents the radius of the osculating circle of γ at the point of parameter s_0.

$\kappa(s_0) = \|\mathbf{t}'(s_0)\| = \|\gamma''(s_0)\|$

3.4.1 Osculating circle

Défini de façon directe, le cercle de courbure est le cercle le plus proche de la courbe en P, c'est l'unique cercle osculateur à la courbe en ce point. Ceci signifie qu'il constitue une très bonne approximation de la courbe, meilleure qu'un cercle tangent quelconque. En effet, il donne non seulement une idée de la direction dans laquelle la courbe avance (direction de la tangente), mais aussi de sa tendance à tourner de part ou d'autre de la tangente.

3.4.2 Curvature binormal vector

Finally, we define the *curvature binormal vector* at any given parameter $s_0 \in J$ as :

$$\kappa\mathbf{b}(s_0) = \mathbf{t}(s_0) \times \mathbf{t}'(s_0) = \kappa(s_0) \cdot \mathbf{b}(s_0) \quad , \quad \|\kappa\mathbf{b}(s_0)\| = \kappa(s_0) \tag{3.6}$$

3.5 Torsion

En géométrie différentielle, la torsion d'une courbe tracée dans l'espace mesure la manière dont la courbe se tord pour sortir de son plan osculateur (plan contenant le cercle osculateur). Ainsi, par exemple, une courbe plane a une torsion nulle et une hélice circulaire est de torsion constante. Prises ensemble, la courbure et la torsion d'une courbe de l'espace en définissent la forme comme le fait la courbure pour une courbe plane. La torsion apparait comme coefficient dans les équations différentielles du repère de Frenet.

The *torsion* measures the deviance of γ from being a planar curve and is defined at any given parameter $s_0 \in J$ as :

$$\tau_f(s_0) = \mathbf{n}'(s_0) \cdot \mathbf{b}(s_0) \tag{3.7}$$

3.6 Curve Framing

3.6.1 Moving frame

Soit $\gamma : s \to \gamma(s)$ une courbe bi-régulière de l'espace, paramétrée par son abscisse curviligne. On appelle *repère mobile* attaché à γ le trièdre orthonormé direct $\{\mathbf{d_3}(s), \mathbf{d_1}(s), \mathbf{d_2}(s)\}$.

Par construction, le repère mobile attaché à γ vérifie :

$$\begin{cases} \|\mathbf{d_i}(s)\| = 1 \\ \mathbf{d_i}(s) \cdot \mathbf{d_j}(s) = 0 \end{cases} \tag{3.8}$$

Governing equations

Par dérivation des relations précédentes on obtient les équations différentielles suivantes :

$$\begin{cases} \mathbf{d}'_\mathbf{i}(s) \cdot \mathbf{d_i}(s) = 0 \\ \mathbf{d}'_\mathbf{i}(s) \cdot \mathbf{d_j}(s) = -\mathbf{d_i}(s) \cdot \mathbf{d}'_\mathbf{j}(s) \end{cases} \tag{3.9}$$

Il existe donc 3 fonctions scalaires $\tau(s)$, $\kappa_1(s)$, $\kappa_2(s)$ telles que :

$$\begin{cases} \mathbf{d}'_\mathbf{3}(s) = \kappa_2(s)\mathbf{d_1}(s) - \kappa_1(s)\mathbf{d_2}(s) \\ \mathbf{d}'_\mathbf{1}(s) = -\kappa_2(s)\mathbf{d_3}(s) + \tau(s)\mathbf{d_2}(s) \\ \mathbf{d}'_\mathbf{2}(s) = \kappa_1(s)\mathbf{d_3}(s) - \tau(s)\mathbf{d_1}(s) \end{cases} \tag{3.10}$$

Ce système se réécrit sous forme matricielle de la façon suivante :

$$\begin{bmatrix} \mathbf{d}'_\mathbf{3}(s) \\ \mathbf{d}'_\mathbf{1}(s) \\ \mathbf{d}'_\mathbf{2}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_2(s) & -\kappa_1(s) \\ -\kappa_2(s) & 0 & \tau(s) \\ \kappa_1(s) & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{d_3}(s) \\ \mathbf{d_1}(s) \\ \mathbf{d_2}(s) \end{bmatrix} \tag{3.11}$$

On remarquera qu'ainsi définie, l'évolution des repères mobiles le long de la courbe γ est gouvernée par une équation différentielle d'ordre 1. Dès lors, un unique triplet $\{\tau,\ \kappa_1,\ \kappa_2\}$ engendre une famille de repères mobiles définis à une constante d'intégration prêt. Généralement, un repère mobile sera donc entièrement définit par la donnée de τ, κ_1, κ_2 et de $\{\mathbf{d_3}(s=0), \mathbf{d_1}(s=0), \mathbf{d_2}(s=0)\}$.

Darboux vector

Il est pertinent de considérer l'évolution d'un repère mobile le long de γ en introduisant son vecteur de Darboux ($\mathbf{\Omega}$), qui correspond au taux de rotation du trièdre $\{\mathbf{d_3}(s), \mathbf{d_1}(s), \mathbf{d_2}(s)\}$ selon l'abscisse curviligne. Les équations d'évolution du repère mobile s'écrivent alors :

$$\mathbf{d}'_\mathbf{i}(s) = \mathbf{\Omega}(s) \times \mathbf{d_i}(s) \quad avec \quad \mathbf{\Omega}(s) = \begin{bmatrix} \tau(s) \\ \kappa_1(s) \\ \kappa_2(s) \end{bmatrix} \tag{3.12}$$

Géométriquement, les fonctions scalaires $\tau(s)$, $\kappa_1(s)$, $\kappa_2(s)$ correspondent respectivement aux taux de rotations du trièdre autour des axes dirigés par $\mathbf{d_3}(s), \mathbf{d_1}(s), \mathbf{d_2}(s)$:

$$\frac{d\theta_3}{dt}(s) = \tau(s) \quad , \quad \frac{d\theta_1}{dt}(s) = \kappa_1(s) \quad , \quad \frac{d\theta_2}{dt}(s) = \kappa_2(s) \tag{3.13}$$

3.6.2 Adapted frame

De plus, on dira qu'il est *adapté* à γ si en tout point $\gamma(s)$, $\mathbf{d_3}(s)$ est tangent à γ :

$$\mathbf{d_3}(s) = \mathbf{t}(s) = \frac{\gamma^{'}(s)}{\|\gamma(s)\|} \tag{3.14}$$

Dans ce cas, la courbure κ de la courbe γ vaut : $\kappa \equiv \|\gamma''\| = \|\mathbf{t'}\| = \sqrt{\kappa_1^2 + \kappa_2^2}$

La courbure est une quantité géométrique intrinsèque, indépendante du choix du repère mobile attaché à la courbe. C'est donc un invariant. Et donc quelque soit le choix du repère mobile adapté $|\mathbf{t'}\| = \sqrt{\kappa_1^2 + \kappa_2^2}$ est un invariant (la courbure).

3.6.3 Frenet frame

Definition

The Frenet frame is a well-known particular adapted moving frame (§3.3). At any given regular point $\gamma(s_0)$ it is define as $\{\mathbf{t}(s_0), \mathbf{n}(s_0), \mathbf{b}(s_0)\}$ where :

$$\mathbf{t}(s_0) = \frac{\gamma^{'}(s_0)}{\|\gamma'(s_0)\|} \quad , \quad \mathbf{n}(s_0) = \frac{\mathbf{t'}(s_0)}{\kappa(s_0)} \quad , \quad \mathbf{b}(s_0) = \mathbf{t}(s_0) \times \mathbf{n}(s_0) \tag{3.15}$$

Governing equations

The Frenet frame satisfies the *Frenet-Serret* formulas, which govern the evolution of the frame along the curve γ :

$$\begin{bmatrix} \mathbf{t}^{'}(s) \\ \mathbf{n}^{'}(s) \\ \mathbf{b}^{'}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau_f(s) \\ 0 & -\tau_f(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{bmatrix} \tag{3.16}$$

One can remember the generic differential equations of an adapted moving frame attached to a curve, where :

$$\mathbf{d_3}(s) = \mathbf{t}(s) = \frac{\gamma^{'}(s)}{\|\gamma(s)\|} \quad , \quad \kappa_1(s) = 0 \quad , \quad \kappa_2(s) = \kappa(s) \quad , \quad \tau(s) = \tau_f(s) \tag{3.17}$$

Darboux vector

Consequently, the Darboux vector $(\mathbf{\Omega_f})$ of the Frenet frame is given by :

$$\mathbf{\Omega_f}(s) = \begin{bmatrix} \tau_f(s) \\ 0 \\ \kappa(s) \end{bmatrix} \tag{3.18}$$

Specific points

undefined when curvature vanishes

not related to mechanical torsion

3.6.4 Bishop frame

Definition

Different ways to frame a curve. The usual one is Frenet. But, it could not be as relevant as we want in our field of interest.

The Bishop frame is defined as a a well-known particular adapted moving frame (§3.3). At any given regular point $\gamma(s_0)$ it is define as $\{\mathbf{t}(s_0), \mathbf{n}(s_0), \mathbf{b}(s_0)\}$ where :

$$\mathbf{t}(s_0) = \frac{\gamma'(s_0)}{\|\gamma'(s_0)\|} \quad , \quad \mathbf{n}(s_0) = \frac{\mathbf{t}'(s_0)}{\kappa(s_0)} \quad , \quad \mathbf{b}(s_0) = \mathbf{t}(s_0) \times \mathbf{n}(s_0) \tag{3.19}$$

Governing equations

The Bishop frame evolution is governed by the following differential equations :

$$\begin{bmatrix} \mathbf{t}'(s) \\ \mathbf{u}'(s) \\ \mathbf{v}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_2(s) & -\kappa_1(s) \\ -\kappa_2(s) & 0 & 0 \\ \kappa_1(s) & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}(s) \\ \mathbf{u}(s) \\ \mathbf{v}(s) \end{bmatrix} \tag{3.20}$$

One can remember the generic differential equations of an adapted moving frame attached to a curve, where :

$$\mathbf{d_3}(s) = \mathbf{t}(s) = \frac{\gamma'(s)}{\|\gamma(s)\|} \quad , \quad \kappa_1(s) = 0 \quad , \quad \kappa_2(s) = \kappa(s) \quad , \quad \tau(s) = \tau_f(s) \tag{3.21}$$

Darboux vector

Consequently, the Darboux vector $(\mathbf{\Omega_b})$ of the Bishop frame is given by :

$$\mathbf{\Omega_b}(s) = \begin{bmatrix} 0 \\ \kappa_1(s) \\ \kappa_2(s) \end{bmatrix} \tag{3.22}$$

Specific points

well defined when curvature vanishes

related to mechanical torsion

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[Blo] Jules Bloomenthal. Calculation of Reference Frames along a Space Curve. 1:1–5.

4 Elastic rod : variational approch

4.1 Introduction

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Basile [BAV+10]

Basile [BWR+08]

Basile [AAP00]

Sina [Nab14]

4.2 Kinematic

Bishop

$\boldsymbol{x}$ only depends on arc-length along the curve : $\boldsymbol{x}(t)$

Implicit dependence of θ in $\boldsymbol{x}$: $\theta(t) \equiv \theta[x](t)$

Implicit dependence of $\boldsymbol{\omega}$ in $\boldsymbol{x}$ and θ : $\boldsymbol{\omega}(t) \equiv \boldsymbol{\omega}[\boldsymbol{x}, \theta](t)$

We denote τ the twist along the curve : $\tau(t) = \frac{\partial \theta}{\partial t} = \theta'(t)$

We will make the following mathematical assumptions :

$$\begin{array}{rcl} \boldsymbol{x}: \ [0,L] & \longrightarrow & \mathbb{R}^3 \\ t & \longmapsto & \boldsymbol{x}(t) \end{array} \qquad \boldsymbol{x} \in \mathcal{C}^\infty([0,L]^{\mathbb{R}^3}) \tag{4.1}$$

$$\begin{array}{rcl} \theta[\boldsymbol{x}]: \ [0,L] & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \theta[\boldsymbol{x}](t) \end{array} \qquad \theta[\boldsymbol{x}] \in \mathcal{C}^\infty([0,L]^{\mathbb{R}}) \tag{4.2}$$

$$\begin{array}{rcl} \boldsymbol{\omega}[\boldsymbol{x},\theta]: \ [0,L] & \longrightarrow & \mathbb{R}^2 \\ t & \longmapsto & \boldsymbol{\omega}[\boldsymbol{x},\theta](t) \end{array} \qquad \boldsymbol{\omega}[\boldsymbol{x},\theta] \in \mathcal{C}^\infty([0,L]^{\mathbb{R}^2}) \tag{4.3}$$

The dependence in the arc-length is denoted by the parameter $t \in [0, L]$ by the use of parenthesis $(.)$. The dependencies in functions are denoted by the use of brackets $[.]$.

When computings energy gradients, it comes to differentiate energies regarding there dependencies in functions $\boldsymbol{x}$ and θ. Thus, we may recall that :

$$\begin{array}{rcl} \theta: \ \mathcal{C}^\infty([0,L]^{\mathbb{R}^3}) & \longrightarrow & \mathcal{C}^\infty([0,L]^{\mathbb{R}}) \\ \boldsymbol{x} & \longmapsto & \theta[\boldsymbol{x}] \end{array} \qquad \theta \in \mathcal{C}^\infty \tag{4.4}$$

$$\begin{array}{rcl} \boldsymbol{\omega}: \ \mathcal{C}^\infty([0,L]^{\mathbb{R}^3}) \times \mathcal{C}^\infty([0,L]^{\mathbb{R}}) & \longrightarrow & \mathcal{C}^\infty([0,L]^{\mathbb{R}^2}) \\ (\boldsymbol{x},\theta) & \longmapsto & \boldsymbol{\omega}[\boldsymbol{x},\theta] \end{array} \qquad \boldsymbol{\omega} \in \mathcal{C}^\infty \tag{4.5}$$

4.3 Energy

$$\mathcal{E}_p[\boldsymbol{x},\theta] = \mathcal{E}_{stretch}[\boldsymbol{x}] + \mathcal{E}_{bend}[\boldsymbol{x},\theta] + \mathcal{E}_{twist}[\theta] \tag{4.6}$$

$$\mathcal{E}_{stretch}[\boldsymbol{x}] = \frac{1}{2}\int_0^L K_s\left(\frac{\|\boldsymbol{x}'\|}{\|\bar{\boldsymbol{x}}'\|} - 1\right)^2 dt \tag{4.7}$$

$$\mathcal{E}_{bend}[\boldsymbol{x},\theta] = \frac{1}{2}\int_0^L (\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^T B\,(\boldsymbol{\omega} - \bar{\boldsymbol{\omega}}) dt \tag{4.8}$$

$$\mathcal{E}_{twist}[\theta] = \frac{1}{2}\int_0^L \beta(\theta' - \bar{\theta}')^2 dt \tag{4.9}$$

Energies could be seen as functional, i.e. functions that map vectors from a functional vector space to there underlying scalar field $\mathbb{R}$.

$$\begin{aligned} \mathcal{E}_{stretch}: \quad \mathcal{C}^\infty([0,L]^{\mathbb{R}^3}) &\longrightarrow \mathbb{R} \\ \boldsymbol{x} &\longmapsto \mathcal{E}_{stretch}[\boldsymbol{x}] \end{aligned} \tag{4.10}$$

$$\begin{aligned} \mathcal{E}_{bend}: \quad \mathcal{C}^\infty([0,L]^{\mathbb{R}^3}) \times \mathcal{C}^\infty([0,L]^{\mathbb{R}^3},[0,L]^{\mathbb{R}}) &\longrightarrow \mathbb{R} \\ (\boldsymbol{x},\theta) &\longmapsto \mathcal{E}_{bend}[\boldsymbol{x},\theta] \end{aligned} \tag{4.11}$$

$$\begin{aligned} \mathcal{E}_{twist}: \quad \mathcal{C}^\infty([0,L]^{\mathbb{R}^3},[0,L]^{\mathbb{R}}) &\longrightarrow \mathbb{R} \\ \theta &\longmapsto \mathcal{E}_{twist}[\theta] \end{aligned} \tag{4.12}$$

4.4 Inextensibility

4.5 Gradients

4.5.1 Momentum

Calcul du moment :

$$\mathcal{M} = -\frac{d\mathcal{E}_p}{d\theta} = -\frac{\partial \mathcal{E}_p}{\partial \theta} = -\frac{\partial \mathcal{E}_{bend}}{\partial \theta} - \frac{\partial \mathcal{E}_{twist}}{\partial \theta} \tag{4.13}$$

$$\tag{4.14}$$

Calcul de $\frac{\partial \mathcal{E}_{twist}}{\partial \theta}$

$$\begin{aligned} \mathcal{E}_{twist}[\theta + \lambda h_\theta] &= \frac{1}{2}\int_0^L \beta\Big((\theta+\lambda h_\theta)' - \bar{\theta}'\Big)^2 dt \qquad (4.15) \\ &= \mathcal{E}_{twist}[\theta] + \lambda \int_0^L \beta(\theta' - \bar{\theta}')h'_\theta dt + \lambda^2 \int_0^L \frac{\beta h'^2_\theta}{2}\, dt \\ &= \mathcal{E}_{twist}[\theta] + \lambda \left[\beta(\theta' - \bar{\theta}')h_\theta\right]_0^L - \lambda \int_0^L \Big(\beta(\theta'-\bar{\theta}')\Big)' h_\theta\, dt + o(\lambda) \\ &= \mathcal{E}_{twist}[\theta] + \lambda \int_0^L \Big[\beta(\theta'-\bar{\theta}')(\delta_L - \delta_0) - \Big(\beta(\theta'-\bar{\theta}')\Big)'\Big] h_\theta\, dt + o(\lambda) \\ &= \mathcal{E}_{twist}[\theta] + \lambda(D_{\mathcal{E}_{twist}} \cdot h_\theta) + o(\lambda) \end{aligned}$$

With :

$$\frac{\partial \mathcal{E}_{twist}}{\partial \theta} \equiv D_{\mathcal{E}_t} = -\Big(\beta(\theta'-\bar{\theta}')\Big)' + \beta(\theta'-\bar{\theta}')(\delta_L - \delta_0) \quad , \quad \frac{\partial \mathcal{E}_{twist}}{\partial \theta} : [0;L] \longrightarrow \mathbb{R} \tag{4.16}$$

Calcul de $\frac{\partial \omega}{\partial \theta}$

On montre par des considérations géométriques que :

$$\begin{aligned}\boldsymbol{d_1}[\boldsymbol{x},\theta+\lambda h_\theta] &= \boldsymbol{d_1}[\boldsymbol{x},\theta] + \sin(\lambda h_\theta)\boldsymbol{d_2}[\boldsymbol{x},\theta] - (1-\cos(\lambda h_\theta)\boldsymbol{d_1}[\boldsymbol{x},\theta] \\ &= \boldsymbol{d_1}[\boldsymbol{x},\theta] + \lambda h_\theta \boldsymbol{d_2}[\boldsymbol{x},\theta] + o(\lambda)\end{aligned} \tag{4.17}$$

$$\begin{aligned}\boldsymbol{d_2}[\boldsymbol{x},\theta+\lambda h_\theta] &= \boldsymbol{d_2}[\boldsymbol{x},\theta] - \sin(\lambda h_\theta)\boldsymbol{d_1}[\boldsymbol{x},\theta] - (1-\cos(\lambda h_\theta)\boldsymbol{d_2}[\boldsymbol{x},\theta] \\ &= \boldsymbol{d_2}[\boldsymbol{x},\theta] - \lambda h_\theta \boldsymbol{d_1}[\boldsymbol{x},\theta] + o(\lambda)\end{aligned} \tag{4.18}$$

On en déduit pour le vecteur courbure matérielle :

$$\begin{aligned}\boldsymbol{\omega}[\boldsymbol{x},\theta+\lambda h_\theta] &= \begin{bmatrix} \boldsymbol{x}''\cdot\boldsymbol{d_1}[\boldsymbol{x},\theta+\lambda h_\theta] \\ \boldsymbol{x}''\cdot\boldsymbol{d_2}[\boldsymbol{x},\theta+\lambda h_\theta]\end{bmatrix} \\ &= \boldsymbol{\omega}[\boldsymbol{x},\theta] + \lambda \begin{bmatrix} \boldsymbol{x}''\cdot\boldsymbol{d_2}[\boldsymbol{x},\theta] \\ -\boldsymbol{x}''\cdot\boldsymbol{d_1}[\boldsymbol{x},\theta]\end{bmatrix}\cdot h_\theta + o(\lambda) \\ &= \boldsymbol{\omega}[\boldsymbol{x},\theta] + \lambda(D_{\boldsymbol{\omega}}\cdot h_\theta) + o(\lambda)\end{aligned} \tag{4.19}$$

With :

$$\frac{\partial \boldsymbol{\omega}}{\partial \theta} \equiv D_{\boldsymbol{\omega}} = -R_{\pi/2}\,\boldsymbol{\omega}[\boldsymbol{x},\theta] \quad , \quad R_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad , \quad \frac{\partial \boldsymbol{\omega}}{\partial \theta} : [0;L] \longrightarrow \mathbb{R}^2 \tag{4.20}$$

Calcul de $\frac{\partial \mathcal{E}_{bend}}{\partial \theta}$

$$\begin{aligned}\mathcal{E}_{bend}[\boldsymbol{x},\theta+\lambda h_\theta] &= \tfrac{1}{2}\int_0^L \Big(\boldsymbol{\omega}[\boldsymbol{x},\theta+\lambda h_\theta] - \bar{\boldsymbol{\omega}}[\boldsymbol{x},\theta]\Big)^T B \Big(\boldsymbol{\omega}[\boldsymbol{x},\theta+\lambda h_\theta] - \bar{\boldsymbol{\omega}}[\boldsymbol{x},\theta]\Big)dt \\ &= \mathcal{E}_{bend}[\boldsymbol{x},\theta] - \lambda\int_0^L \Big((\boldsymbol{\omega}-\bar{\boldsymbol{\omega}})^T B R_{\pi/2}\,\boldsymbol{\omega}\Big)\cdot h_\theta dt + o(\lambda) \\ &= \mathcal{E}_{bend}[\boldsymbol{x},\theta] + \lambda(D_{\mathcal{E}_{bend}}\cdot h_\theta) + o(\lambda)\end{aligned} \tag{4.21}$$

With :

$$\frac{\partial \mathcal{E}_{bend}}{\partial \theta} \equiv D_{\mathcal{E}_{bend}} = -(\boldsymbol{\omega}-\bar{\boldsymbol{\omega}})^T B R_{\pi/2}\,\boldsymbol{\omega} \quad , \quad \frac{\partial \mathcal{E}_{bend}}{\partial \theta} : [0;L] \longrightarrow \mathbb{R} \tag{4.22}$$

Calcul de $\mathcal{M}$

Finally:

$$\mathcal{M} = -\frac{d\mathcal{E}_p}{d\theta} = -(\boldsymbol{\omega}-\bar{\boldsymbol{\omega}})^T B R_{\pi/2}\,\boldsymbol{\omega} - \Big(\beta(\theta'-\bar{\theta}')\Big)' + \beta(\theta'-\bar{\theta}')(\delta_L-\delta_0) \tag{4.23}$$

ATTENTION : écrire ici que M est porté par d_3

4.5.2 Forces

Calcul des efforts:

$$\boldsymbol{\mathcal{F}} = -\frac{d\mathcal{E}_p}{d\boldsymbol{x}} = -\frac{\partial \mathcal{E}_p}{\partial \boldsymbol{x}} - \int_0^L \frac{\partial \mathcal{E}_p}{\partial \theta}\frac{\partial \theta}{\partial \boldsymbol{x}} \tag{4.24}$$

Calcul de $\frac{\partial \theta}{\partial \boldsymbol{x}}$ par le writh

[Ful78], [dV05], [Vau00], [Ber09]

On montre ici qu'en un point d'abscisse s, une variation de la centerline de $\lambda \boldsymbol{h_x}$ entraine une variation de θ intégrée sur la courbe Γ de $(\frac{\partial \theta[\boldsymbol{x}](s)}{\partial \boldsymbol{x}} \cdot \lambda \boldsymbol{h_x})$.

$H : t \mapsto \begin{cases} 0, & t < 0 \\ 1, & t \geqslant 0 \end{cases}$ est la fonction de Heaviside.

$\delta_{t_0} : t \mapsto \delta(t - t_0)$ est la distribution de dirac centrée en t_0.

ATTENTION : ici il y a qqch à expliquer entre θ et ψ (il y a une question de signe à détailler).

$$\Delta\psi_{\lambda \boldsymbol{h_x}}[\boldsymbol{x}](s) = \psi[\boldsymbol{x} + \lambda \boldsymbol{h_x}](s) - \psi[\boldsymbol{x}](s) \tag{4.25}$$

$$\begin{aligned}
\Delta\psi_{\lambda \boldsymbol{h_x}}[\boldsymbol{x}](s) &= \int_0^s \frac{\boldsymbol{x}' \times (\boldsymbol{x} + \lambda \boldsymbol{h_x})'}{1 + \boldsymbol{x}' \cdot (\boldsymbol{x} + \lambda \boldsymbol{h_x})'} \cdot \Big(\boldsymbol{x}'' + (\boldsymbol{x} + \lambda \boldsymbol{h_x})''\Big) dt \\
&= \int_0^s \frac{\lambda \boldsymbol{x}' \times \boldsymbol{h_x}'}{2(1 + \frac{\lambda \boldsymbol{x}' \cdot \boldsymbol{h_x}'}{2})} \cdot (2\boldsymbol{x}'' + \lambda \boldsymbol{h_x}'') dt \\
&= \int_0^s \frac{\lambda}{2}(\boldsymbol{x}' \times \boldsymbol{h_x}')\Big(1 - \frac{\lambda \boldsymbol{x}' \cdot \boldsymbol{h_x}'}{2} + o(\lambda)\Big)(2\boldsymbol{x}'' + \lambda \boldsymbol{h_x}'') dt \\
&= \int_0^s \Big(-\boldsymbol{k_b} \cdot \lambda \boldsymbol{h}_{\boldsymbol{x}}' + \frac{\lambda^2}{2}(\boldsymbol{x}' \times \boldsymbol{h_x}') \cdot \boldsymbol{h}_{\boldsymbol{x}}''\Big)\Big(1 - \frac{\lambda \boldsymbol{x}' \cdot \boldsymbol{h_x}'}{2} + o(\lambda)\Big) dt \\
&= -\int_0^s -\boldsymbol{k_b} \cdot \lambda \boldsymbol{h}_{\boldsymbol{x}}' dt + o(\lambda) \\
&= \Big[-\boldsymbol{k_b} \cdot \lambda \boldsymbol{h_x}\Big]_0^s + \int_0^s \boldsymbol{k}_{\boldsymbol{b}}' \cdot \lambda \boldsymbol{h_x} dt + o(\lambda) \\
&= \int_0^L \Big((1 - H)\boldsymbol{k}_{\boldsymbol{b}}' - (\delta_s - \delta_0)\boldsymbol{k_b}\Big) \cdot \lambda \boldsymbol{h_x} dt + o(\lambda) \\
&= \lambda(\boldsymbol{D}_\psi(s) \cdot \boldsymbol{h_x}) + o(\lambda)
\end{aligned} \tag{4.26}$$

With :

$$\frac{\partial \theta}{\partial \boldsymbol{x}}(s) \equiv -\boldsymbol{D}_\psi(s) = (\delta_s - \delta_0)\boldsymbol{k_b} - (1 - H)\boldsymbol{k}_{\boldsymbol{b}}' \quad , \quad \frac{\partial \theta}{\partial \boldsymbol{x}}(s) : [0; L] \longrightarrow \mathbb{R}^3 \tag{4.27}$$

Calcul de $\frac{\partial \omega}{\partial x}$

Quand $\boldsymbol{x}$ varie de $\lambda \boldsymbol{h_x}$, le repère materiel est tourné de sorte que, en décomposant sur le repère de base :

$$\boldsymbol{d_1}[\boldsymbol{x} + \lambda \boldsymbol{h_x}, \theta] = \boldsymbol{d_1}[\boldsymbol{x}, \theta] + \Delta\psi_{\lambda \boldsymbol{h_x}}[\boldsymbol{x}]\boldsymbol{d_2}[\boldsymbol{x}, \theta] + \Delta\phi_{\lambda \boldsymbol{h_x}}[\boldsymbol{x}]\boldsymbol{d_3}[\boldsymbol{x}, \theta] + o(\lambda) \tag{4.28}$$

$$\boldsymbol{d_2}[\boldsymbol{x} + \lambda \boldsymbol{h_x}, \theta] = \boldsymbol{d_2}[\boldsymbol{x}, \theta] - \Delta\psi_{\lambda \boldsymbol{h_x}}[\boldsymbol{x}]\boldsymbol{d_1}[\boldsymbol{x}, \theta] - \Delta\phi_{\lambda \boldsymbol{h_x}}[\boldsymbol{x}]\boldsymbol{d_3}[\boldsymbol{x}, \theta] + o(\lambda) \tag{4.29}$$

On montre aisément que la contribution en $\Delta\psi$ autour de $\boldsymbol{d_1}$ et $\boldsymbol{d_2}$ est d'un ordre supérieur à celle autour de $\boldsymbol{d_3} = \boldsymbol{x}'$ (ce qui se comprend bien dans le cas d'une variation en hélice).

Par propriété d'orthogonalité des vecteurs du repère mobile (cf § sur le curve framing) on montre que :

$$\begin{aligned} \boldsymbol{d_1} \cdot \boldsymbol{d_3} = 0 &\Rightarrow \boldsymbol{d_1}[\boldsymbol{x} + \lambda \boldsymbol{h_x}, \theta] \cdot (\boldsymbol{x} + \lambda \boldsymbol{h_x})' = 0 \\ &\Rightarrow \Delta\phi_{\lambda \boldsymbol{h_x}}[\boldsymbol{x}] = \boldsymbol{d_1}[\boldsymbol{x} + \lambda \boldsymbol{h_x}, \theta] \cdot \boldsymbol{x}' = \lambda(-\boldsymbol{d_1}[\boldsymbol{x}, \theta] \cdot \boldsymbol{h_x}') + o(\lambda) \end{aligned} \tag{4.30}$$

$$\begin{aligned} \boldsymbol{d_2} \cdot \boldsymbol{d_3} = 0 &\Rightarrow \boldsymbol{d_1}[\boldsymbol{x} + \lambda \boldsymbol{h_x}, \theta] \cdot (\boldsymbol{x} + \lambda \boldsymbol{h_x})' = 0 \\ &\Rightarrow \Delta\phi_{\lambda \boldsymbol{h_x}}[\boldsymbol{x}] = \boldsymbol{d_2}[\boldsymbol{x} + \lambda \boldsymbol{h_x}, \theta] \cdot \boldsymbol{x}' = \lambda(-\boldsymbol{d_2}[\boldsymbol{x}, \theta] \cdot \boldsymbol{h_x}') + o(\lambda) \end{aligned} \tag{4.31}$$

On en déduit le calcul de la variation de $\boldsymbol{\omega}$ pour une vairation de la centerline de $\lambda \boldsymbol{h_x}$ qui vaut :

$$\begin{aligned} \boldsymbol{\omega}[\boldsymbol{x} + \lambda \boldsymbol{h_x}, \theta] &= \begin{bmatrix} (\boldsymbol{x} + \lambda \boldsymbol{h_x})'' \cdot \boldsymbol{d_1}[\boldsymbol{x} + \lambda \boldsymbol{h_x}, \theta] \\ (\boldsymbol{x} + \lambda \boldsymbol{h_x})'' \cdot \boldsymbol{d_2}[\boldsymbol{x} + \lambda \boldsymbol{h_x}, \theta] \end{bmatrix} \\ &= \begin{bmatrix} (\boldsymbol{x} + \lambda \boldsymbol{h_x})'' \cdot (\boldsymbol{d_1}[\boldsymbol{x}] + (-\lambda \frac{\partial\theta}{\partial \boldsymbol{x}}(s) \cdot \boldsymbol{h_x} + o(\lambda))\boldsymbol{d_2}[\boldsymbol{x}] - (\lambda \boldsymbol{d_1}[\boldsymbol{x}] \cdot \boldsymbol{h_x} + o(\lambda)) \cdot \boldsymbol{x}'') \\ (\boldsymbol{x} + \lambda \boldsymbol{h_x})'' \cdot (\boldsymbol{d_2}[\boldsymbol{x}] - (-\lambda \frac{\partial\theta}{\partial \boldsymbol{x}}(s) \cdot \boldsymbol{h_x} + o(\lambda))\boldsymbol{d_1}[\boldsymbol{x}] - (\lambda \boldsymbol{d_2}[\boldsymbol{x}] \cdot \boldsymbol{h_x} + o(\lambda)) \cdot \boldsymbol{x}'') \end{bmatrix} \\ &= \boldsymbol{\omega}[\boldsymbol{x}, \theta] + \lambda \begin{bmatrix} \boldsymbol{d_1}[\boldsymbol{x}, \theta]^T \\ \boldsymbol{d_2}[\boldsymbol{x}, \theta]^T \end{bmatrix} \cdot \boldsymbol{h_x}'' - \Big(\lambda \frac{\partial\theta}{\partial \boldsymbol{x}}(s) \cdot \boldsymbol{h_x}\Big) \begin{bmatrix} \boldsymbol{d_2}[\boldsymbol{x}, \theta] \cdot \boldsymbol{x}'' \\ -\boldsymbol{d_1}[\boldsymbol{x}, \theta] \cdot \boldsymbol{x}'' \end{bmatrix} \\ &\qquad + \lambda(\boldsymbol{x}'' \cdot \boldsymbol{x}') \begin{bmatrix} \boldsymbol{d_1}[\boldsymbol{x}, \theta] \cdot \boldsymbol{h_x}' \\ \boldsymbol{d_2}[\boldsymbol{x}, \theta] \cdot \boldsymbol{h_x}'' \end{bmatrix} + o(\lambda) \end{aligned}$$

$$\begin{aligned} \boldsymbol{\omega}[\boldsymbol{x} + \lambda \boldsymbol{h_x}, \theta] &= \begin{bmatrix} (\boldsymbol{x} + \lambda \boldsymbol{h_x})'' \cdot \boldsymbol{d_1}[\boldsymbol{x} + \lambda \boldsymbol{h_x}, \theta] \\ (\boldsymbol{x} + \lambda \boldsymbol{h_x})'' \cdot \boldsymbol{d_2}[\boldsymbol{x} + \lambda \boldsymbol{h_x}, \theta] \end{bmatrix} && POUET && POUET \\ &= \begin{bmatrix} (\boldsymbol{x} + \lambda \boldsymbol{h_x})'' \cdot (\boldsymbol{d_1}[\boldsymbol{x}] + (-\lambda \frac{\partial\theta}{\partial \boldsymbol{x}}(s) \cdot \boldsymbol{h_x} + o(\lambda))\boldsymbol{d_2}[\boldsymbol{x}]) \\ (\boldsymbol{x} + \lambda \boldsymbol{h_x})'' \cdot (\boldsymbol{d_2}[\boldsymbol{x}] - (-\lambda \frac{\partial\theta}{\partial \boldsymbol{x}}(s) \cdot \boldsymbol{h_x} + o(\lambda))\boldsymbol{d_1}[\boldsymbol{x}]) \end{bmatrix} && POUET && POUET \\ &\quad - \begin{bmatrix} -(\lambda \boldsymbol{d_1}[\boldsymbol{x}] \cdot \boldsymbol{h_x} + o(\lambda)) \cdot \boldsymbol{x}'') \\ -(\lambda \boldsymbol{d_2}[\boldsymbol{x}] \cdot \boldsymbol{h_x} + o(\lambda)) \cdot \boldsymbol{x}'') \end{bmatrix} && POUET && POUET \end{aligned} \tag{4.32}$$

$$g + h = i \tag{4.33}$$

$$\begin{aligned} a &= b + c - d \\ &\quad + e - f \end{aligned} \tag{4.34}$$

$$\tag{4.35}$$

$$\begin{aligned} H_c = \frac{1}{2n} \sum_{l=0}^{n} (-1)^l (n-l)^{p-2} \sum_{l_1+\cdots+l_p=l} \prod_{i=1}^{p} \binom{n_i}{l_i} \\ \cdot \left[(n-l) - (n_i - l_i)\right]^{n_i - l_i} \cdot \left[(n-l)^2 - \sum_{j=1}^{p} (n_i - l_i)^2\right]. \end{aligned} \tag{4.36}$$

$$\boldsymbol{\omega}[\boldsymbol{x}, \theta] + \lambda \begin{bmatrix} \boldsymbol{d_1}[\boldsymbol{x}]^T \\ \boldsymbol{d_2}[\boldsymbol{x}]^T \end{bmatrix} \cdot \boldsymbol{h_x}'' - \left(\lambda \frac{\partial \theta}{\partial \boldsymbol{x}}(s) \cdot \boldsymbol{h_x}\right) \begin{bmatrix} \boldsymbol{d_2}[\boldsymbol{x}, \theta] \cdot \boldsymbol{x}'' \\ -\boldsymbol{d_1}[\boldsymbol{x}, \theta] \cdot \boldsymbol{x}'' \end{bmatrix} \tag{4.37}$$

$$- \lambda (\boldsymbol{x}'' \cdot \boldsymbol{x}') \begin{bmatrix} \boldsymbol{d_1}[\boldsymbol{x}, \theta] \cdot \boldsymbol{h_x}' \\ \boldsymbol{d_2}[\boldsymbol{x}, \theta] \cdot \boldsymbol{h_x}'' \end{bmatrix} \tag{4.38}$$

$$+ o(\lambda) \tag{4.39}$$

4.6 Discretization

4.7 Connection

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5 Elastic rod : a novel element from kirchhoff equations

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5.1 Introduction

Dans ce chapitre, après un bref rappel sur le cadre mathématique d'étude des courbes paramétrique de l'espace, on présente les notions de courbures et de torsion géométrique associées au repère de fraient. On montre ensuite le cas plus général d'un repère mobile quelconque attaché à une courbe gamma. On définit enfin la particularité d'un repère mobile adapté à un courbe, et on présente, en sus du repère de Frenet, une approche différente pour accrocher des repères le long d'une courbe (Bishop / RMF / Zéro-twisting frame)

Connection Part II

6 Calculus of variations

6.1 Introduction

In this appendix we drawback essential mathematical concepts for the calculus of variations. Recall how the notion of energy, gradients are extended to function spaces.

[AMR02]

6.2 Spaces

6.2.1 Normed space

A *normed space* $V(\mathbb{K})$ is a vector space V over the scalar field $\mathbb{K}$ with a norm $\|.\|$.

A *norm* is a map $\|.\| : V \times V \longmapsto \mathbb{K}$ which satisfies :

$$\forall x \in V, \qquad \|x\| = 0_{\mathbb{K}} \Rightarrow x = 0_V \tag{6.1}$$

$$\forall x \in V, \forall \lambda \in \mathbb{K}, \qquad \|\lambda x\| = |\lambda| \, \|x\| \tag{6.2}$$

$$\forall (x, y) \in V^2, \qquad \|x + y\| \leqslant \|x\| + \|y\| \tag{6.3}$$

6.2.2 Inner product space

A *inner product space* or *pre-hilbert space* $E(\mathbb{K})$ is a vector space E over the scalar field $\mathbb{K}$ with an inner product.

An *inner product* is a map $\langle\, ;\rangle : E \times E \longmapsto \mathbb{K}$ which is bilinear, symmetric, positive-definite :

$$
\begin{aligned}
&\forall (x,y,z) \in E^3, \forall(\lambda,\mu) \in \mathbb{K}^2, && \langle \lambda x \ + \mu y \,;z\rangle = \lambda\langle x\,;z\rangle + \mu\langle y\,;z\rangle && (6.4)\\
& && \langle x\,;\lambda y + \mu z\rangle = \lambda\langle x\,;y\rangle + \mu\langle x\,;z\rangle\\
&\forall (x,y) \in E^2, && \langle x\,;y\rangle = \langle y\,;x\rangle && (6.5)\\
&\forall x \in E, && \langle x\,;x\rangle \geqslant 0_{\mathbb{K}} && (6.6)\\
&\forall x \in E, && \langle x\,;x\rangle = 0_{\mathbb{K}} \Rightarrow x = 0_E && (6.7)
\end{aligned}
$$

Moreover, an inner product naturally induces a norm on E defined by :

$$\forall x \in E, \quad \|x\| = \sqrt{\langle x\,;x\rangle} \tag{6.8}$$

Thus, an inner product vector space is also naturally a normed vector space.

6.2.3 Euclidean space

An *Euclidean space* $\mathcal{E}(\mathbb{R})$ is a finite-dimensional real vector space with an inner product. Thus, distances and angles between vectors could be defined and measured regarding to the norm associated with the chosen inner product.

An Euclidean space is nothing but a finite-dimensional real pre-hilbert space.

6.2.4 Banach space

A *Banach space* $\mathcal{B}(\mathbb{K})$ is a complete normed vector space, which means that it is a normed vector space in which every Cauchy sequence of $\mathcal{B}$ converges in $\mathcal{B}$ for the given norm.

Thus, a Banach space is a vector space with a metric that allows the computation of vector length and distance between vectors and is complete in the sense that a Cauchy sequence of vectors always converges to a well defined limit in that space.

6.2.5 Hilbert space

A *Hilbert space* is an inner product vector space $\mathcal{H}(\mathbb{K})$ such that the natural norm induced by the inner product turns $\mathcal{H}$ into a complete metric space (i.e. every Cauchy sequence of $\mathcal{H}$ converges in $\mathcal{H}$).

The Hilbert space concept is a generalization of the Euclidean space concept. In physics it's common to encounter Hilbert spaces as infinite-dimensional function spaces.

Hilbert spaces are Banach spaces, but the converse does not hold generally.

For example, $\mathcal{L}^2([a,b])$ is an infinite-dimensional Hilbert space with the canonical inner product $\langle f\,;g\rangle = \int_a^b fg$.

Note that $\mathcal{L}^2$ is the only Hilbert space among the $\mathcal{L}^p$ spaces.

6.3 Derivative

The well known notion of function derivative in $\mathbb{R}^{\mathbb{R}}$ can be extended to maps between Banach spaces. This is useful in physics when formulating problems as variational problems, usually in terms of energy minimization. Indeed, energy is generally defined over a functional vector space and not simply over the real line.

In this case, the research of minimal values of a potential energy rests on the calculus of variations of the energy function compared to variations to other functions defining the problem (geometry, materials, boundary conditions, ...).

Mathematical concepts extended well-known notions of derivative, jacobian and hessian in Euclidean spaces (typically $\mathbb{R}^2$ or $\mathbb{R}^3$) for Banach functional spaces.

6.3.1 Fréchet derivative - strong

Differentiability

Let $\mathcal{B}_V$ and $\mathcal{B}_W$ be two Banach spaces and $U \subset \mathcal{B}_V$ an open subset of $\mathcal{B}_V$. Let $f : u \longmapsto f(u)$ be a function of $U^{\mathcal{B}_W}$. f is said to be *Fréchet differentiable* at $u \in U$ if there exists a continious linear operator $Df_u \in \mathcal{L}(\mathcal{B}_V, \mathcal{B}_W)$ such that :

$$\lim_{\|h\| \to 0} \frac{\|f(u+h) - f(u) - Df_u(h)\|_{\mathcal{B}_W}}{\|h\|_{\mathcal{B}_V}} = 0 \tag{6.9}$$

Or, equivalently :

$$f(u+h) = f(u) + Df_u(h) + o(\|h\|) \quad , \quad \lim_{\|h\| \to 0} \frac{\|o(\|h\|)\|_{\mathcal{B}_W}}{\|h\|} = 0 \tag{6.10}$$

Note that $df = Df_u(h)$ is called the differential of f at point u and represents the change in the function f for a perturbation h from u.

In the literature, it is common to found the following notation : $df = Df_u(h) = Df(u)h$, for the differential of f, which means nothing but $Df(u)$ is linear regarding h. This notation can be ambiguous as far as the linearity of $Df(u)$ in h is denoted as a product which is not explicitly defined.

Derivability

If f is Fréchet differentiable at $u \in U$, the continous linear operator $Df_u \in \mathcal{L}(\mathcal{B}_V, \mathcal{B}_W)$ is called the Fréchet derivative of f at u and is also denoted :

$$f'(u) = Df(u) = Df_u \tag{6.11}$$

f is said to be $\mathcal{C}^1$ in the sens of Fréchet if f is Fréchet differentiable for all $u \in U$ and the function $Df : u \longmapsto f'(u)$ of $U^{\mathcal{L}(\mathcal{B}_V, \mathcal{B}_W)}$ is continuous.

Higer derivatives

Because the differential of f is a linear map from $\mathcal{B}_V$ to $\mathcal{L}(\mathcal{B}_V, \mathcal{B}_W)$ it is possible to look for the differentiability of Df. If it exists, it is denoted $D^2 f$ and maps $\mathcal{B}_V$ to $\mathcal{L}(\mathcal{B}_V, \mathcal{L}(\mathcal{B}_V, \mathcal{B}_W))$.

6.3.2 Gâteaux derivative - weak

Differentiability

Let $\mathcal{B}_V$ and $\mathcal{B}_W$ be two Banach spaces and $U \subset \mathcal{B}_V$ an open subset of $\mathcal{B}_V$. Let $f : u \longmapsto f(u)$ be a function of $U^{\mathcal{B}_W}$. f is said to be *Gâteaux differentiable* at $u \in U$ if there exists a continious linear operator $Df_u \in \mathcal{L}(\mathcal{B}_V, \mathcal{B}_W)$ such that :

$$\forall h \in \mathcal{U}, \quad \lim_{\lambda \to 0} \frac{f(u + \lambda h) - f(u)}{\lambda} = \frac{d}{d\lambda} f(u + \lambda h)\bigg|_{\lambda = 0} = Df_u(h) \tag{6.12}$$

Or, equivalently :

$$\forall h \in \mathcal{U}, \quad f(u + \lambda h) = f(u) + \lambda D_{f,u}(h) + o(|\lambda|) \quad , \quad \lim_{\lambda \to 0} \frac{\|o(|\lambda|)\|_{\mathcal{B}_W}}{\lambda} = 0 \tag{6.13}$$

Note that $df = Df_u(h)$ is called the differential of f at point u.

Derivability

If f is Gâteaux differentiable at $u \in U$, the continous linear operator $Df_u \in \mathcal{L}(\mathcal{B}_V, \mathcal{B}_W)$ is called the Gâteaux derivative of f at u and is also denoted :

$$f'(u) = Df(u) = Df_u \tag{6.14}$$

f is said to be $\mathcal{C}^1$ in the sens of Gâteaux if f is Gâteaux differentiable for all $u \in U$ and the function $Df : u \longmapsto f'(u)$ of $U^{\mathcal{L}(\mathcal{B}_V, \mathcal{B}_W)}$ is continuous.

The Gâteaux derivative is a weaker form of derivative than the Fréchet derivative : if f is Fréchet differentiable, then it is also Gâteaux differentiable, and its Fréchet and Gâteaux derivatives agree, but the converse does not hold generally.

6.3.3 Useful properties

Let $\mathcal{B}_V$, $\mathcal{B}_W$ and $\mathcal{B}_Z$ be three Banach spaces. Let $f, g : \mathcal{B}_V \longmapsto \mathcal{B}_W$ and $h : \mathcal{B}_W \longmapsto \mathcal{B}_Z$ be three Gâteaux differentiable functions. Then, the following useful properties holds :

$$D(f + g)(u) = Df(u) + Dg(u) \tag{6.15}$$

$$D(h \circ f)(u) = Dh(f(u)) \circ Df(u) \tag{6.16}$$

6.3.4 Partial derivative

6.4 Gradient

The *gradient vector* denotes the differential or total derivative of a differentiable function F in the special case it is a map from $\mathbb{R}^n$ to $\mathbb{R}$. Thus, it satisfies the following relationships :

$$\nabla F(x) = F'(x) = DF_x \tag{6.17}$$

$$DF_x(h) = \langle \nabla F(x)\,;h\rangle \tag{6.18}$$

$$F(x+h) = F(x) + \langle \nabla F(x)\,;h\rangle + o(\|h\|) \tag{6.19}$$

If the matrix notation is adopted, the previous relations leads to :

$$F(X+H) = F(X) + \nabla F(u)^T H + o(\|H\|) \quad , \quad \nabla F(x) = \begin{bmatrix} \frac{\partial F}{\partial x_1} \\ \vdots \\ \frac{\partial F}{\partial x_n} \end{bmatrix} \in (R)^n \tag{6.20}$$

6.5 Jacobian

Let f be a differentiable function from $\mathbb{R}^n$ to $\mathbb{R}^p$. The differential of such a fonction is a linear application from $\mathbb{R}^n$ to $\mathbb{R}^p$ which could be represented with the following matrix called the *jacobian matrix* :

$$Df_x = \boldsymbol{J}_f(x) = \frac{df}{dx}(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1}, & \cdots, & \frac{\partial f}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial x_1} & \cdots & \frac{\partial f_p}{\partial x_n} \end{bmatrix} \in \mathcal{M}_{p,n}(\mathbb{R}) \tag{6.21}$$

Thus, with the matrix notation, the Taylor expansion takes the following form :

$$F(X+H) = F(X) + \boldsymbol{J}_f^X H + o(\|H\|) \tag{6.22}$$

In the cas $p = 1$, the jacobian matrix of the functional F is nothing but the gradient transpose itself :

$$DF_x = \boldsymbol{J}_F(x) = \frac{dF}{dx} = \begin{bmatrix} \frac{\partial F}{\partial x_1}, & \cdots, & \frac{\partial F}{\partial x_n} \end{bmatrix} = \nabla F^T \tag{6.23}$$

6.6 Hessian

Let F be a differentiable function from $\mathbb{R}^n$ to $\mathbb{R}$. The second order differential of such a fonction is a linear application from $\mathbb{R}^n$ to $\mathbb{R}^n$ which could be represented with the following matrix called the *hessian matrix* :

$$D^2F_x = \boldsymbol{H}_F(x) = \frac{d^2F}{dx}(x) = \begin{bmatrix} \frac{\partial F_1^2}{\partial x_1^2} & \frac{\partial F_1^2}{\partial x_1 \partial x_2} & \cdots & \frac{\partial F_1^2}{\partial x_1 \partial x_n} \\ \frac{\partial F_1^2}{\partial x_2 \partial x_1} & \frac{\partial F_1^2}{\partial x_2^2} & \cdots & \frac{\partial F_1^2}{\partial x_2 \partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial F_p^2}{\partial x_n \partial x_1} & \frac{\partial F_p^2}{\partial x_n \partial x_2} & \cdots & \frac{\partial F_p^2}{\partial x_n^2} \end{bmatrix} \in \mathcal{M}_{n,n}(\mathbb{R}) \tag{6.24}$$

Thus, with the matrix notation, the Taylor expansion takes the following form :

$$F(X+H) = F(X) + \boldsymbol{J}_F^X H + \frac{1}{2} H^T \boldsymbol{H}_F^X H + o(\|H\|) \tag{6.25}$$

6.7 Functional

A *functional* is a map from a vector space $E(\mathbb{K})$ into its underlying scalar field $\mathbb{K}$. Here $\mathcal{E}_p[\boldsymbol{x}, \theta]$ is a functional depending over $\boldsymbol{x}$ and θ.

Bibliography

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