

# Modeling of bending-torsion couplings in active-bending structures. Application to the design of elastic gridshell.



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# 1 Geometry of smooth and discret space curves

## 1.1 Introduction

idée : Rapprocher l'étude des courbes continues et des discrétisées avec le concept de modèle de la «courbe polygone» de Leibniz et de ses successeurs, une infinité de côtés infiniment petits. [Del11, p.235]

Voir aussi que la geometry des space curves semble intimement liée à celle des surface. [Coo13, Del11]

Car les courbes n'étant que des polygones d'une infinité de côtés, et ne diffèrent entre elles que par la différence des angles que ces côtés infiniment petits font entre eux; il n'appartient qu'à l'Analyse des infiniment petits de déterminer la position de ces cotés pour avoir la courbure qu'ils forment [...] [Liebniz].

Attention à la terminologie smooth vs. continious :

A smooth curve is a curve which is a smooth function, where the word "curve" is interpreted in the analytic geometry context. In particular, a smooth curve is a continuous map  $f$  from a one-dimensional space to an  $n$ -dimensional space which on its domain has continuous derivatives up to a desired order. <sup>1</sup>

### 1.1.1 Goals and contributions

Dans ce chapitre, après un bref rappel sur le cadre mathématique d'étude des courbes paramétrique de l'espace, on présente les notions de courbures et de torsion géométrique associées au repère de Frenet. On montre ensuite le cas plus général d'un repère mobile

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<sup>1</sup>Definition of a smooth curve from mathworld : <http://mathworld.wolfram.com/SmoothCurve.html>

quelconque attaché à une courbe  $\gamma$ . On définit enfin la particularité d'un repère mobile adapté à une courbe, et on présente, en sus du repère de Frenet, une approche différente pour accrocher des repères le long d'une courbe (Bishop / RMF / Zéro-twisting frame)

Contributions : présentation et comparaison de différentes façons d'estimer la courbure discrete

### 1.1.2 Related work

(author?) [Bis75] [Bis75] (author?) (year?)

[Bis75] [BWR<sup>+</sup>08] [Hof08] [BELT14] [Fre52] [Del07] [FGSS14] [Gug89] [Klo86]

### 1.1.3 Overview

## 1.2 Parametric curves

### 1.2.1 Definition

Let  $I$  be an interval of  $\mathbb{R}$  and  $F: t \mapsto F(t)$  be a map of  $\mathcal{C}(I, \mathbb{R}^3)$ . Then  $\gamma = (I, F)$  is called a *parametric curve* and :

- The 2-uplet  $(I, F)$  is called a *parametrization* of  $\gamma$
- $\gamma = F(I) = \{F(t), t \in I\}$  is called the *graph* or *trace* of  $\gamma$
- $\gamma$  is said to be  $\mathcal{C}^k$  if  $F \in \mathcal{C}^k(I, \mathbb{R}^3)$

Note that for a given graph in  $\mathbb{R}^3$  they may be different possible parameterizations. Thereafter  $\gamma$  will simply refers to its graph  $F(I)$ .

### 1.2.2 Regularity

Let  $\gamma = (I, F)$  be a parametric curve, and  $t_0 \in I$  be a parameter.

- A point of parameter  $t_0$  is called *regular* if  $F'(t_0) \neq 0$ .  
The curve  $\gamma$  is called *regular* if  $\gamma$  is  $\mathcal{C}^1$  and  $F'(t) \neq 0, \forall t \in I$
- A point of parameter  $t_0$  is called *biregular* if  $F'(t_0)$  and  $F''(t_0)$  are not collinear  
The curve  $\gamma$  is called *biregular* if  $\gamma$  is  $\mathcal{C}^2$  and  $F'(t) \cdot F''(t) \neq 0, \forall t \in I$

### 1.2.3 Reparametrization

Let  $\gamma = (I, F)$  be a parametric curve of class  $\mathcal{C}^k$ ,  $J \in \mathbb{R}^3$  an interval, and  $\varphi: I \rightarrow J$  be a  $\mathcal{C}^k$  diffeomorphism. Let's define  $G = F \circ \varphi$ . Then :

- $G \in \mathcal{C}^k(J, \mathbb{R}^3)$
- $G(J) = F(I)$
- $\varphi$  is said to be an admissible *change of parameter* for  $\gamma$
- $(J, G)$  is said to be another *admissible parametrization* for  $\gamma$

### 1.2.4 Natural parametrization

Let  $\gamma$  be a space curve of class  $\mathcal{C}^1$ . A parametrization  $(I, F)$  of  $\gamma$  is called *natural* if  $\|F'(t)\| = 1, \forall t \in I$ . Thus :

- The curve is necessarily regular
- $F$  is strictly monotonic

### 1.2.5 Curve length

Let  $\gamma = (I, F)$  be a parametric curve of class  $\mathcal{C}^1$ . The length of  $\gamma$  is defined as :

$$L = \int_I \|F'(t)\| dt \quad (1.1)$$

Note that the length of  $\gamma$  is invariant under reparametrization.

### 1.2.6 Arc-length parametrization

Let  $\gamma = (I, F)$  be a regular parametric curve. Let  $t_0 \in I$  be a given parameter. The following map is said to be the *arc-length of origin  $t_0$*  of  $\gamma$  :

$$s: t \mapsto \int_{t_0}^t \|F'(u)\| du \quad , \quad s \in I \times \mathbb{R} \quad (1.2)$$

The arc-length  $s: I \rightarrow s(I)$  is an admissible change of parameter for  $\gamma$ . Indeed,  $s$  is a  $\mathcal{C}^1$  diffeomorphism because it is bijective ( $s' > 0$ ).

Let's define  $G = F \circ s^{-1}$  and  $J = s(I)$ . Thus  $(J, G)$  is a natural reparametrization of  $\gamma$  and  $\|G'(s)\| = 1, \forall s \in J$ . This parametrization is preferred because the natural parameter  $s$  traverses the image of  $\gamma$  at unit speed<sup>2</sup> ( $\|G'\| = 1$ ).

<sup>2</sup>Regular curves are also known as *unit-speed* curves.

Thereafter, for a regular curve  $\gamma : \gamma(t)$  will denote the point  $F(t)$  of parameter  $t \in I$  ; while  $\gamma(s)$  will denote the point  $G(s)$  of arc-length  $s \in J = [0, L]$ .

### 1.3 Frenet trihedron

The Frenet trihedron is a fundamental mathematical tool from the field of differential geometry to study local characterization of planar and non-planar space curves. It is a direct orthonormal basis attached to any point  $P$ , of parameter  $t \in I$ , on a parametric curve  $\gamma$ . This basis is composed of three unit vectors  $\{\mathbf{t}(t), \mathbf{n}(t), \mathbf{b}(t)\}$  called respectively the *tangent*, the *normal*, and the *binormal* unit vectors<sup>3</sup>.

Introduced by Frenet in 1847 in his thesis “Courbes à Double Courbure” [Fre52], it brings out intrinsic local properties of space curves : the curvature ( $\kappa$ ) which evaluates the deviance of  $\gamma$  from being a straight line (see §1.4.1) ; and the torsion ( $\tau_f$ ) which evaluates the deviance of  $\gamma$  from being a planar curve (see §1.4.2).

These quantities, also known as “generalized curvatures” in modern differential geometry, are essential to understand the geometry of space curves. As stated by the *Fundamental Theorem of Space Curves*<sup>4</sup>, a curve is fully determined by its curvature and torsion up to a solid movement in space (see §1.4.3).

#### 1.3.1 Tangent vector

The first component of the Frenet trihedron is called the *unit tangent vector*. Let  $\gamma = (I, F)$  be a regular parametric curve. Let  $t \in I$  be a parameter. The *unit tangent vector* is defined as :

$$\mathbf{t}(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|} \quad , \quad \|\mathbf{t}(t)\| = 1 \quad (1.3)$$

For an arc-length parametrized curve, this expression simply becomes :

$$\mathbf{t}(s) = \gamma'(s) \quad , \quad s \in [0, L] \quad (1.4)$$

In differential geometry, the unit tangente to the curve  $\gamma$  at point  $P_0$  is obtained as the limit of the (normalized) vector  $\overrightarrow{P_0 P}$ , as  $P$  approaches  $P_0$  on the path  $\gamma$  (fig. 1.1). For a regular curve, the left-sided and right-sided limits coincide as  $P^-$  and  $P^+$  approche  $P_0$  respectively from its left and right sides :

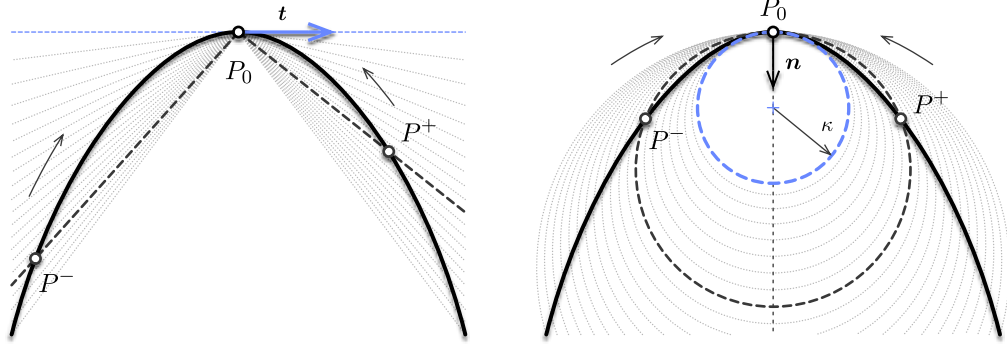
$$\mathbf{t}(P_0) = \lim_{P \rightarrow P_0} \frac{\overrightarrow{P_0 P}}{\|\overrightarrow{P_0 P}\|} = \lim_{P^- \rightarrow P_0} \frac{\overrightarrow{P_0 P^-}}{\|\overrightarrow{P_0 P^-}\|} = \lim_{P^+ \rightarrow P_0} \frac{\overrightarrow{P_0 P^+}}{\|\overrightarrow{P_0 P^+}\|} \quad (1.5)$$

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<sup>3</sup> Strictly speaking, the map  $\mathbf{t} : t \mapsto \mathbf{t}(t)$  is a *vector field* while  $\mathbf{t}(t)$  is a *vector* of  $\mathbb{R}^3$ . For the sake of simplicity, and if there is no ambiguity, these two notions will not be explicitly distinguished hereinafter.

<sup>4</sup>The full demonstration of this theorem is attributed to Darboux in [Del07, p.11].





(a) Curve's tangent.

(b) Curve's normal and osculating circle.

 Figure 1.1 – Differential definition of Frenet's trihedron at given point  $P_0$ .

### 1.3.2 Normal vector

The second component of the Frenet trihedron is called the *unit normal vector*. It is constructed from  $\mathbf{t}'$  which is necessarily orthogonal to  $\mathbf{t}$ . Indeed :

$$\|\mathbf{t}\| = 1 \Rightarrow \mathbf{t}' \cdot \mathbf{t} = 0 \Leftrightarrow \mathbf{t}' \perp \mathbf{t} \quad (1.6)$$

Let  $\gamma = (I, F)$  be a biregular<sup>5</sup> parametric curve. Let  $t \in I$  be a parameter. The *unit normal vector* is defined as :

$$\mathbf{n}(t) = \frac{\mathbf{t}'(t)}{\|\mathbf{t}'(t)\|} = \frac{\gamma'(t) \times (\gamma''(t) \times \gamma'(t))}{\|\gamma'(t)\|^3}, \quad \|\mathbf{n}(t)\| = 1 \quad (1.7)$$

For an arc-length parametrized curve, this expression simply becomes :

$$\mathbf{n}(s) = \frac{\mathbf{t}'(s)}{\|\mathbf{t}'(s)\|} = \gamma'(s) \times (\gamma''(s) \times \gamma'(s)) \quad , \quad s \in [0, L] \quad (1.8)$$

In differential geometry, the unit normal to the curve  $\gamma$  at point  $P_0$  is obtained as the limit of the (normalized) vector  $\overrightarrow{P_0 P^+} - \overrightarrow{P_0 P^-}$ , as  $P^-$  and  $P^+$  appoche  $P_0$  respectively from its left and right sides (fig. 1.1) :

$$\mathbf{n}(P_0) = \lim \frac{\overrightarrow{P_0 P^+} - \overrightarrow{P_0 P^-}}{\left\| \overrightarrow{P_0 P^+} - \overrightarrow{P_0 P^-} \right\|} \quad (1.9)$$

Remark that the notion of *normal vector* is ambiguous for non-planar curves as there is an infinite number of possible normal vectors lying in the plane orthogonal to the curve's tangent. In practice, the tangent derivative is a convenient choice as it allows to extend the notion of curvature from planar to non-planar space curves. However, we will see (§1.5.6) that other kind of trihedron can be constructed regarding this choice and that one of them is especially suitable for the study of slender beams.

<sup>5</sup> Note that  $\mathbf{n}$  exists if only  $\gamma$  is biregular, that is  $\mathbf{t}'$  never vanishes or, equivalently,  $\gamma$  is never locally a straight line.

### 1.3.3 Binormal vector

The third vector of Frenet's trihedron is called the *unit binormal vector*. It is constructed from  $\mathbf{t}$  and  $\mathbf{n}$  to form an orthonormal direct basis of  $\mathbb{R}^3$ . Let  $\gamma = (I, F)$  be a biregular parametric curve. Let  $t \in I$  be a parameter. The *unit binormal vector* is defined as :

$$\mathbf{b}(t) = \mathbf{t}(t) \times \mathbf{n}(t) = \frac{\gamma'(t) \times \gamma''(t)}{\|\gamma'(t) \times \gamma''(t)\|} \quad , \quad \|\mathbf{b}(t)\| = 1 \quad (1.10)$$

For an arc-length parametrized curve, this expression simply becomes :

$$\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s) = \gamma'(s) \times \gamma''(s) \quad , \quad s \in [0, L] \quad (1.11)$$

### 1.3.4 Osculating plane

The tangent and normal unit vectors  $\{\mathbf{t}, \mathbf{n}\}$  form an orthonormal basis of the so-called *osculating plane*, whereas the binormal vector ( $\mathbf{b}$ ) is orthogonal to it. This plane is of prime importance because it is the plane in which the curve takes its curvature (see §1.4.1).

As reported in [Del07, p.45], the osculating plane seems to have been first introduced by Bernoulli as the plane passing through three infinitely near points on a curve<sup>6</sup>. Likewise, in modern differential geometry, the osculating plane is defined as the limit of the plane passing through the points  $P^-$ ,  $P_0$  and  $P^+$  while  $P^-$  and  $P^+$  approche  $P_0$  respectively from its left and right side (fig. 1.1).

Note that the normal unit vector and the binormal unit vector  $\{\mathbf{n}, \mathbf{b}\}$  define the so-called *normal plane*, while the normal tangent vector and the binormal unit vector  $\{\mathbf{t}, \mathbf{b}\}$  define the so-called *rectifying plane*. These planes are secondary for the present study.

## 1.4 Curves of double curvature

The study of space curves belongs to the field of differential geometry. According to [Del07, p.28], the terminology *curve of double curvature* is attributed to Pitot around 1724<sup>7</sup>. However, as stated in [Coo13, p.321], curvature and torsion were probably first thought by Monge around 1771<sup>8</sup>. It is also interesting to note that, at that time, “curvature” was

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<sup>6</sup> “Voco autem planum osculans, quod transit per tria curvae quaesitae puncta infinite sibi invicem propinqua” [Ber28, p.113].

<sup>7</sup> “Les Anciens ont nommé cette courbe Spirale ou Hélice ; parce que la formation sur le cylindre suit la même analogie que la formation d’une spirale ordinaire sur un plan; mais elle est bien différente de la spirale ordinaire, étant une des courbes à double courbure, ou une des lignes qu’on conçoit tracée sur la surface des Solides. Peut-être que ces sortes de courbes à double courbure, ou prises sur la surface des Solides, feront un jour l’objet des recherches des géomètres. Celle que nous venons d’examiner est, je crois, la plus simple de toutes. ” [Pit26, p.28]

<sup>8</sup> “On appelle point d’inflexion, dans une courbe plane, le point où cette ligne, après avoir été concave dans un sens, cesse de l’être pour devenir concave dans l’autre sens. Il est évident que dans ce point, la courbe perd sa courbure, et que les deux éléments consécutifs sont en ligne droite. Mais une courbe à double courbure peut perdre chacune de ses courbures en particulier, ou les perdre toutes deux dans le même point ; c’est-à-dire, qu’il peut arriver ou que trois éléments consécutifs d’une même courbe à double courbure se trouvent dans un même plan, ou que deux de ces éléments soient en ligne droite. Il suit de là que les

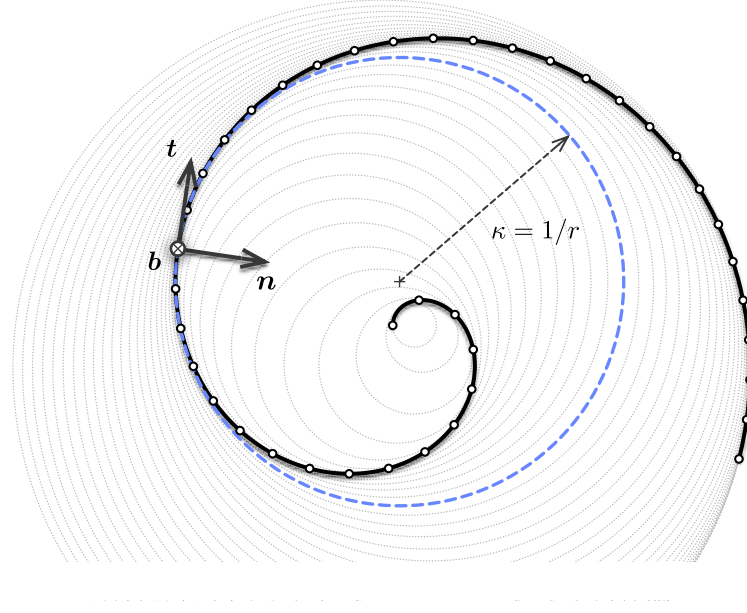


Figure 1.2 – Osculating circles for a spiral curve at different parameters.

also referred to as “flexure”, reflecting that the study of physical problems (e.g. *elastica*) and the study of curves of double curvature were intimately related to each other.

Space curves were historically understood as *curves of double curvature* by extension to the case of planar curves, where the curvature measures the deviance of a curve from being a straight line. The second curvature, nowadays known as the “torsion” or “second generalized curvature”, measures the deviance of a curve from being plane.

### 1.4.1 First invariant : the curvature

In differential geometry, the *osculating circle* is defined as the limit of the circle passing through the points  $P^-$ ,  $P_0$  and  $P^+$  while  $P^-$  and  $P^+$  approach  $P_0$  (fig. 1.1). This circle lies on the osculating plane and its radius is nothing but the inverse of the local curvature of a curve<sup>9</sup>. While the tangent gives the best local approximation of the curve as a straight line, the osculating circle gives the best local approximation of that curve as an arc.

courbes à double courbure peuvent avoir deux espèces d’inflexions; la première a lieu lorsque la courbe devient plane, et nous l’appellerons simple inflexion; la seconde, que nous appellerons double inflexion, a lieu lorsque la courbe devient droite dans un de ses points. ” [Mon09, p.363]

<sup>9</sup> As explained by Euler itself, at a given arc-length parameter ( $s$ ), the osculating plane is the plane in which a curve takes its curvature : “in quo bina fili elementa proxima in curvantur” [Eul75, p.364].

### Curvature

Let  $\gamma$  be a regular arc-length parametrized curve. Let  $s \in [0, L]$  be an arc-length parameter. The *curvature* is a positive scalar quantity defined as :

$$\kappa(s) = \|\mathbf{t}'(s)\| \geq 0 \quad , \quad \mathbf{t}'(s) = \kappa(s)\mathbf{n}(s) \quad (1.12)$$

The curvature is *independent* regarding the choice of parametrization. This makes the curvature an *intrinsic property* of a given curve and that is why it is also referred to as a *geometric invariant*. Following [GAS06, pp.203-204] it can be computed for any parametrization  $(I, F)$  of  $\gamma$  as :

$$\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3} \quad , \quad \mathbf{t}'(t) = \|\gamma'(t)\|\kappa(t)\mathbf{n}(t) \quad (1.13)$$

Note that in eq. (1.12) the prime denotes the derivative regarding the natural parameter  $(s)$  while it denotes the derivative regarding any parameter  $(t)$  in eq. (1.13). Consequently, the *speed* of the curve's parametrization appears in the latter equation :

$$v(t) = \frac{ds}{dt} = \|\gamma'(t)\| = s'(t) \quad (1.14)$$

The curvature measures how much a curve *bends instantaneously* in its osculating plane, that is how fast the tangent vector is rotating in the osculating plane around the binormal vector. In differential geometry this is expressed as :

$$\kappa(s) = \lim_{ds \rightarrow 0} \frac{\angle(\mathbf{t}(s), \mathbf{t}(s+ds))}{ds} = \lim_{ds \rightarrow 0} \frac{(\mathbf{t}(s+ds) - \mathbf{t}(s)) \cdot \mathbf{n}(s)}{ds} \quad (1.15)$$

This is equivalent as measuring how fast the osculating plane itself is rotating around the binormal vector. Consequently a curve is locally a *straight line* when its curvature vanishes ( $\kappa(s) = 0$ ).

### Radius of curvature

The *radius of curvature* is defined as the inverse of the curvature ( $r = 1/\kappa$ ). From a geometric point of view, one can demonstrate that it is the radius of the osculating circle (fig. 1.2). Remark that where the curvature vanishes the radius of curvature goes to infinity ; that is the osculating circle becomes a line, a circle of infinite radius.

### Center of curvature

The *center of curvature* is defined as the center of the osculating circle (fig. 1.2). The locus of all the centers of curvature of a curve is called the *evolute*.

### Curvature binormal vector

Finally, following [BWR<sup>+</sup>08] we define the *curvature binormal vector*. Let  $\gamma$  be a biregular arc-length parametrized curve. Let  $s \in [0, L]$  be an arc-length parameter. The *curvature binormal vector* is defined as :

$$\kappa \mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{t}'(s) = \kappa(s) \cdot \mathbf{b}(s) \quad , \quad \|\kappa \mathbf{b}(s)\| = \kappa(s) \quad (1.16)$$

This vector will be useful as it embed all the necessary information on the curve's curvature. We will see (§1.5.6) that this vector is associated to the angular velocity of a specific adapted moving frame attached to the curve and called the *Bishop frame*.

### 1.4.2 Second invariant : the torsion

Let  $\gamma$  be a biregular arc-length parametrized curve. Let  $s \in [0, L]$  be an arc-length parameter. The *torsion* is a scalar quantity defined as :

$$\tau_f(s) = \mathbf{n}'(s) \cdot \mathbf{b}(s) = -\mathbf{b}'(s) \cdot \mathbf{n}(s) \quad (1.17)$$

The torsion is *independent* regarding the choice of parametrization. This makes the torsion an *intrinsic property* of a given curve and that is why it is also referred to as a *geometric invariant*. Following [GAS06, p.204] it can be computed for any parametrization  $(I, F)$  of  $\gamma$  as :

$$\tau_f(s) = \frac{(\gamma'(s) \times \gamma''(s)) \cdot \gamma'''(s)}{\|\gamma'(s) \times \gamma''(s)\|^2} \quad \text{when } \kappa(s) > 0 \quad (1.18)$$

The torsion measures how much a curve goes *instantaneously out of its plane*, that is to say how fast the normal or binormal vectors are rotating in the normal plane around the tangent vector. In differential geometry this is expressed as :

$$\tau_f(s) = \lim_{ds \rightarrow 0} \frac{\angle(\mathbf{n}(s), \mathbf{n}(s+ds))}{ds} = \lim_{ds \rightarrow 0} \frac{(\mathbf{n}(s+ds) - \mathbf{n}(s)) \cdot \mathbf{b}(s)}{ds} \quad (1.19)$$

This is equivalent as measuring how fast the osculating plane is rotating around the tangent vector. Consequently a curve is locally *plane* when its torsion vanishes ( $\tau_f(s) = 0$ ).

Remark that the *torsion* is denoted “ $\tau_f$ ” and not simply “ $\tau$ ” as the latter will be reserved to denote any angular velocity of a moving adapted frame around its tangent vector. Thus,  $\tau_f$  refers to the particular angular velocity of the Frenet trihedron around its tangent vector. This torsion, which is a geometric property of the curve, will be indifferently referred to as the *Frenet torsion* or the *geometric torsion*.

### 1.4.3 Fundamental theorem of space curves

These two *generalized curvatures*, respectively the curvature ( $\kappa$ ) and the torsion ( $\tau_f$ ), are *invariant* regarding the choice of parametrization and under *euclidean motions*. The

*Fundamental theorem of space curves* states that a curve is fully described, up to a Euclidean motion of  $\mathbb{R}^3$ , by its positive curvature ( $\kappa > 0$ ) and torsion ( $\tau$ ) [GAS06, p.229].

#### 1.4.4 Serret-Frenet formulas

The *Fundamental theorem of space curves* is somehow a consequence of the *Serret-Frenet formulas*, which is the first-order system of differential equations satisfied by the Frenet trihedron. Let  $\gamma$  be a biregular arc-length parametrized curve. Let  $s \in [0, L]$  be an arc-length parameter. Then, the Frenet trihedron satisfies the following formulas :

$$\begin{cases} \mathbf{t}'(s) = \kappa(s)\mathbf{n}(s) \\ \mathbf{n}'(s) = -\kappa(s)\mathbf{t}(s) + \tau_f(s)\mathbf{b}(s) \\ \mathbf{b}'(s) = -\tau_f(s)\mathbf{n}(s) \end{cases} \quad (1.20)$$

This system can be seen as the *equations of motion* of the Frenet trihedron moving along the curve  $\gamma$  at unit speed ( $\|\gamma'\| = 1$ ). Indeed, introducing its *angular velocity vector* also known as the *Darboux vector* ( $\mathbf{\Omega}_f$ ), the previous system is expressed as :

$$\begin{bmatrix} \mathbf{t}'(s) \\ \mathbf{n}'(s) \\ \mathbf{b}'(s) \end{bmatrix} = \mathbf{\Omega}_f(s) \times \begin{bmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{bmatrix} \quad \text{where} \quad \mathbf{\Omega}_f(s) = \begin{bmatrix} \tau_f(s) \\ 0 \\ \kappa(s) \end{bmatrix} \quad (1.21)$$

Because the Frenet trihedron satisfies a first-order system of differential equations of parameters  $\kappa$  and  $\tau_f$  it is possible, by integration, to reconstruct the trace of the moving frame and thus the curve, up to a constant of integration (a trihedron in this case).

Finally, those formulas can be generalized to any non unit-speed parametrization of a curve<sup>10</sup>. Let  $\gamma = (I, F)$  be a biregular parametric curve. Let  $t \in I$  be a parameter. Then the following *generalized Serret-Frenet formulas* hold :

$$\begin{cases} \mathbf{t}'(t) = v(t)\kappa(t)\mathbf{n}(t) \\ \mathbf{n}'(t) = -v(t)\kappa(t)\mathbf{t}(t) + v(t)\tau_f(s)\mathbf{b}(t) \\ \mathbf{b}'(t) = -v(t)\tau_f(t)\mathbf{n}(t) \end{cases} \quad (1.22)$$

Again, this system can be seen as the *equations of motion* of the Frenet trihedron moving along the curve  $\gamma$  at non unit-speed ( $v(t) = \|\gamma'(t)\|$ ). This time the *angular velocity vector* ( $\mathbf{\Omega}$ ) is distinct from the *Darboux vector* ( $\mathbf{\Omega}_f$ ) and the previous system is expressed as :

$$\begin{bmatrix} \mathbf{t}'(t) \\ \mathbf{n}'(t) \\ \mathbf{b}'(t) \end{bmatrix} = \mathbf{\Omega}(t) \times \begin{bmatrix} \mathbf{t}(t) \\ \mathbf{n}(t) \\ \mathbf{b}(t) \end{bmatrix} \quad \text{where} \quad \mathbf{\Omega}(t) = v(t) \begin{bmatrix} \tau_f(t) \\ 0 \\ \kappa(t) \end{bmatrix} \quad (1.23)$$

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<sup>10</sup>See [GAS06, p.203] for a complete proof.

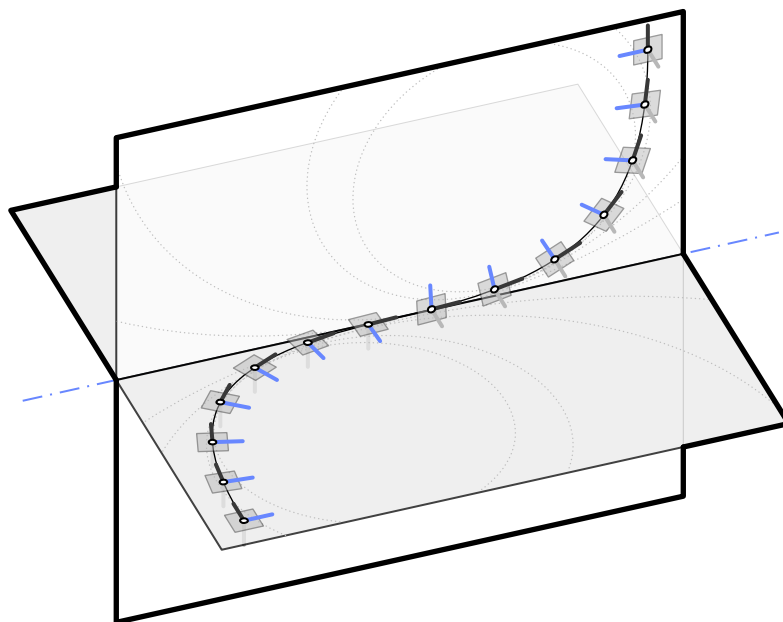


Figure 1.3 – Geometric torsion and rotation of the osculating plane. Note the existence of an inflexion point with a vanishing curvature and a discontinuity of both  $\tau_f$  and the osculating plane.

## 1.5 Curve framing

While the Frenet trihedron “has long been the standard vehicle for analysing properties of the curve invariant<sup>11</sup> under euclidean motions” [Bis75, p.1], a curve can be potentially framed with any arbitrary *moving frame*, understood as an *orthonormal basis field*. Thus, the Frenet frame is not the only way to frame a curve<sup>12</sup> and other frames may or may not exhibit some interesting properties.

In his paper [Bis75] Bishop establishes the differential equation that a moving frame must satisfy and remarks that, because of the orthonormality condition, the first derivatives of the frame components can be expressed in terms of themselves through a skew-symmetric coefficient matrix. For such a frame, the understanding of its motion along the curve is thus reduced to the knowledge of only three scalar coefficient functions. He remarks that most of the interesting properties that the Frenet frame exhibits are due to the fact that one of this coefficient function is vanishing everywhere on the curve (that is the frame is *rotation-minimizing* regarding one of its components) ; and that the Frenet frame is *adapted* to the curve (that is one of its components is nothing but the unit tangent vector).

In this section we introduce the *moving frame* notion and two properties of interest such a frame can exhibit in addition, that is : to be *adapted* to the curve ; and to be *rotation-minimizing* regarding a given direction. We then reconsider the case of the Frenet frame

<sup>11</sup>Namely the curvature ( $\kappa$ ) and the Frenet torsion ( $\tau_f$ ).

<sup>12</sup>Recall the title of Bishop’s paper : “There is more than one way to frame a curve” [Bis75].

regrading this mathematical framework. Finally, we introduce the *zero-twisting* frame also known as the *bishop* frame<sup>13</sup>. This tool will be fundamental for our futur study of slender beams.

### 1.5.1 Moving frame

Let  $\gamma$  be an arc-length parametrized curve. A map  $F$  which associates to each point of arc-length parameter  $s$  a direct orthonormal trihedron is said to be a *moving frame* :

$$\begin{aligned} F: [0, L] &\longrightarrow \mathcal{SO}_3(\mathbb{R}) \\ s &\longmapsto F(s) = \{\mathbf{e}_3(s), \mathbf{e}_1(s), \mathbf{e}_2(s)\} \end{aligned} \quad (1.24)$$

Consequently, a moving frame  $F$  attached to  $\gamma$  satisfies for all  $s \in [0, L]$  :

$$\begin{cases} \|\mathbf{e}_i(s)\| = 1 \\ \mathbf{e}_i(s) \cdot \mathbf{e}_j(s) = 0 \quad , \quad i \neq j \end{cases} \quad (1.25)$$

The term “moving frame” will refer indifferently to the map itself (denoted  $F = \{\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2\}$ ), or to a specific evaluation of the map (denoted  $F(s) = \{\mathbf{e}_3(s), \mathbf{e}_1(s), \mathbf{e}_2(s)\}$ ).

### Governing equations

Computing the derivatives of the previous relationships leads to the following system of differential equations that the frame must satisfy for all  $s \in [0, L]$  :

$$\begin{cases} \mathbf{e}'_i(s) \cdot \mathbf{e}_i(s) = 0 \\ \mathbf{e}'_i(s) \cdot \mathbf{e}_j(s) = -\mathbf{e}_i(s) \cdot \mathbf{e}'_j(s) \quad , \quad i \neq j \end{cases} \quad (1.26)$$

Thus, there exists 3 scalar functions  $(\tau, k_1, k_2)$  such that  $\{\mathbf{e}'_3, \mathbf{e}'_1, \mathbf{e}'_2\}$  can be expressed in the basis  $\{\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2\}$  :

$$\begin{cases} \mathbf{e}'_3(s) = k_2(s)\mathbf{e}_1(s) - k_1(s)\mathbf{e}_2(s) \\ \mathbf{e}'_1(s) = -k_2(s)\mathbf{e}_3(s) + \tau(s)\mathbf{e}_2(s) \\ \mathbf{e}'_2(s) = k_1(s)\mathbf{e}_3(s) - \tau(s)\mathbf{e}_1(s) \end{cases} \quad (1.27)$$

It is common to rewrite this first-order linear system of differential equations<sup>14,15</sup> as a matrix equation :

$$\begin{bmatrix} \mathbf{e}'_3(s) \\ \mathbf{e}'_1(s) \\ \mathbf{e}'_2(s) \end{bmatrix} = \begin{bmatrix} 0 & k_2(s) & -k_1(s) \\ -k_2(s) & 0 & \tau(s) \\ k_1(s) & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_3(s) \\ \mathbf{e}_1(s) \\ \mathbf{e}_2(s) \end{bmatrix} \quad (1.28)$$

---

<sup>13</sup>Named after Bishop who introduced it.

<sup>14</sup>In the case of a space curve, where  $\mathbf{e}_3$  is chosen to be the curve tangent unit vector and  $\mathbf{e}_1$  is chosen to be the curve normal unit vector, this set of equations is known as the *Serret-Frenet formulas*.

<sup>15</sup>In the case of a space curve drawn on a surface, where  $\mathbf{e}_3$  is chosen to be the curve tangent unit vector and  $\mathbf{e}_1$  is chosen to be the surface normal unit vector, this set of equations is known as the *Darboux-Ribaucour formulas*.



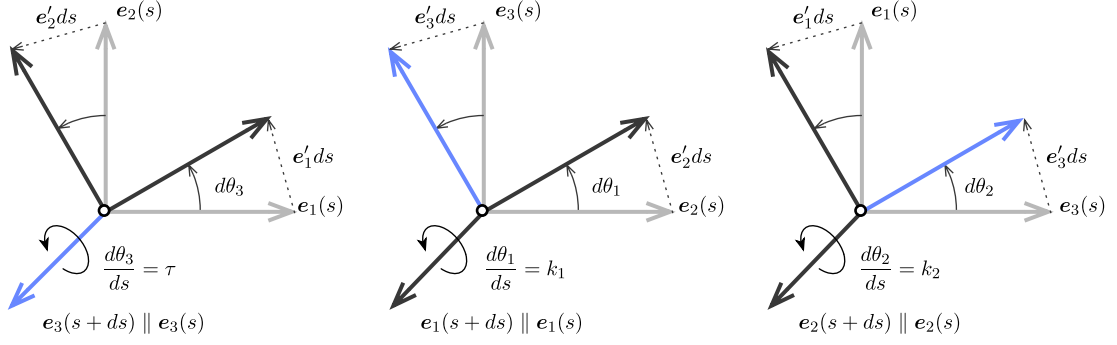


Figure 1.4 – Geometric interpretation of the angular velocity vector of a moving frame.

Since the progression of any moving frame along  $\gamma$  is ruled by a first-order system of differential equations, a unique triplet  $\{\tau, k_1, k_2\}$  leads to a set of moving frames equal to each other within a constant of integration<sup>16</sup>. Basically, with a given triplet  $\{\tau, k_1, k_2\}$ , one can propagate a given initial direct orthonormal trihedron (at  $s = 0$  for instance) through the whole curve by integrating the system of differential equations. In general, a moving frame will be fully determined by  $\tau$ ,  $\kappa_1$  and  $\kappa_2$  together with the initial condition  $\{e_3(s=0), e_1(s=0), e_2(s=0)\}$ .

### Angular velocity

This system can be seen as the *equations of motion* of the frame moving along the curve  $\gamma$  at unit speed ( $\|\gamma'\| = 1$ ). Indeed, introducing its *angular velocity vector* ( $\Omega$ ), the previous system is expressed as :

$$e'_i(s) = \Omega(s) \times e_i(s) \quad \text{avec} \quad \Omega(s) = \begin{bmatrix} \tau(s) \\ k_1(s) \\ k_2(s) \end{bmatrix} \quad (1.29)$$

This result is straightforward deduced from eq. (1.28). Note that the cross product reveals the skew-symmetric nature of the system, which could already be seen in eq. (1.28). Geometrically, decomposing the infinitesimal rotation of the moving frame around its directors between arc-length  $s$  and  $s + ds$  (fig. 1.4) shows that the scalar functions  $\tau$ ,  $k_1$  and  $k_2$  effectively correspond to the angular speed of the frame moving along  $\gamma$ , respectively around  $e_3$ ,  $e_1$  and  $e_2$  :

$$\frac{d\theta_3}{ds}(s) = \tau(s) \quad , \quad \frac{d\theta_1}{ds}(s) = k_1(s) \quad , \quad \frac{d\theta_2}{ds}(s) = k_2(s) \quad (1.30)$$

<sup>16</sup> This assumption reminds the *Fundamental theorem of space curves* (§1.4.3).

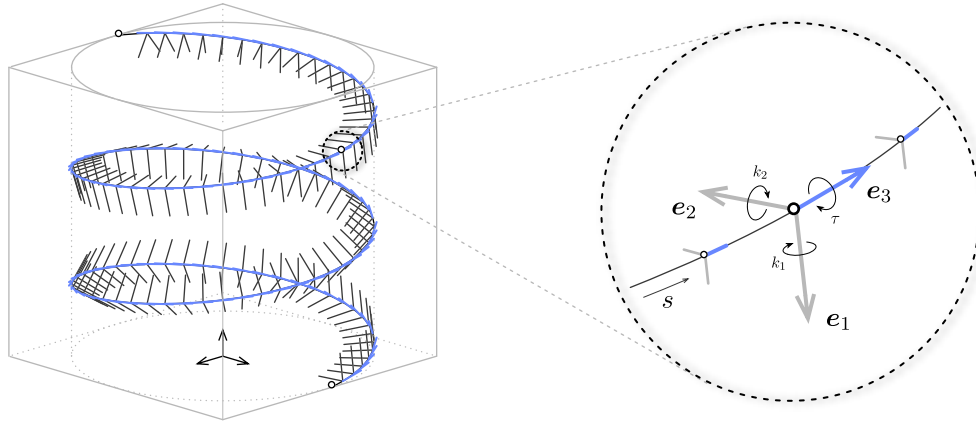


Figure 1.5 – Adapted moving frame  $F(s) = \{e_3(s), e_1(s), e_2(s)\}$  where  $e_3(s) = t(s)$ .

### 1.5.2 Adapted moving frame

Let  $F$  be a moving frame as defined in the previous section.  $F$  is said to be *adapted* to  $\gamma$  if at each point  $\gamma(s)$ ,  $e_3(s)$  is the unit tangent vector of  $\gamma$  (fig. 1.5) :

$$e_3(s) = t(s) = \frac{\gamma'(s)}{\|\gamma'(s)\|} \quad (1.31)$$

For an adapted frame, the components  $k_1$  and  $k_2$  of the angular velocity vector are related to the curvature of  $\gamma$ <sup>17</sup> :

$$\kappa(s) = \|e_3'(s)\| = \|k_2(s)e_1(s) + k_1(s)e_2(s)\| = \sqrt{k_1(s)^2 + k_2(s)^2} \quad (1.32)$$

Moreover, recalling the definition of the curvature binormal vector ( $\kappa b$ ) from eq. (1.16), it is easy to see that for an adapted moving frame the following relation holds :

$$\kappa b(s) = k_1(s)e_1(s) + k_2(s)e_2(s) \quad (1.33)$$

Consequently, the angular velocity vector of an adapted moving frame can be written as :

$$\Omega(s) = \kappa b(s) + \tau(s)t(s) \quad (1.34)$$

This last result is very interesting as it shows that any adapted moving frame will differ from each other only by their twisting speed as  $\Omega_\perp = \kappa b$  only depends on the curve.

<sup>17</sup>Faire le lien avec l'énergie de flexion, qui ne dépend donc que de la géométrie de la courbe dans le cas d'une isotropic rod  $\mathcal{E}_b = EI\kappa^2$ .

### 1.5.3 Rotation-minimizing frame

Following [FGSS14, WJZL08] we introduce the *rotation-minimizing frame* notion. A frame  $\{e_3, e_1, e_2\}$  is said to be *rotation-minimizing* regrading a given direction  $\mathbf{d}$  if :

$$\Omega(s) \cdot \mathbf{d}(s) = 0 \quad (1.35)$$

### 1.5.4 Parallel transport

The notion of *parallel transport* is somehow a generalization of the classical notion of collinearity in flat euclidean spaces (e.g.  $\mathbb{R}^2$  or  $\mathbb{R}^3$ ), to spaces that exhibit some non vanishing curvature (e.g. spheric or hyperbolic spaces)<sup>18</sup>.

#### Relatively parallel fields

Following [Bis75], we define what is a (*relatively*) *parallel field*. Let  $\gamma$  be a regular arc-length parametrized curve. Let  $\mathbf{p}$  be a vector field along  $\gamma$ . The vector field  $\mathbf{p}$  is said to be *parallel* if its derivative is purely tangential, that is :

$$\mathbf{p}'(s) \times \mathbf{t}(s) = 0 \quad (1.36)$$

Consequently, for an adapted moving frame, the *normal fields*<sup>19</sup>  $\mathbf{e}_1$  and  $\mathbf{e}_2$  are both *relatively parallel* if and only if the frame angular velocity is itself a normal field, that is :

$$\Omega(s) = \Omega_{\perp}(s) = \kappa \mathbf{b}(s) \Leftrightarrow \Omega(s) \cdot \mathbf{t}(s) = 0 \Leftrightarrow \tau(s) = 0 \quad (1.37)$$

In other words, a *relatively parallel normal field* : “turns, only whatever amount is necessary for it to remain normal, so it is as close to being parallel as possible without losing normality” [Bis75].

#### Parallel transport of vectors along a curve

Reciprocally, it is possible to define the *parallel transport* of a vector along a curve  $\gamma$  as its propagation along  $\gamma$  at angular speed  $\kappa \mathbf{b}$ . An initial vector  $\mathbf{p}_0 = \mathbf{p}(s_0)$  is parallel transported at arc-length parameter  $s$  into the vector  $\mathbf{p}(s)$  by integrating the following first-order differential equation along  $\gamma$  :

$$\mathbf{p}'(s) = \kappa \mathbf{b}(s) \times \mathbf{p}(s) \quad (1.38)$$

Consequently, the resulting vector field  $\mathbf{p}$  is a parallel field. Note that a parallel field is not necessarily a normal field.

From the point of view of differential geometry, this means that the next vector  $\mathbf{p}(s + ds)$  is obtained by rotating the previous one  $\mathbf{p}(s)$  around the curve binormal  $\mathbf{b}(s)$  by an

<sup>18</sup> <https://www.youtube.com/watch?v=p1tfZD2Bm0w>

<sup>19</sup> A vector field  $\mathbf{p}$  is said to be *normal* along a curve  $\gamma$  if :  $\forall s \in [0, L], \mathbf{p} \cdot \mathbf{t} = 0$ .

infinitesimal angle  $d\theta(s) = \kappa(s)ds$ . Note that  $\mathbf{b}(s)$  has the same direction as  $\mathbf{t}(s) \times \mathbf{t}(s+ds)$ .

### Parallel transport of frames along a curve

Identically, the *parallel transport* of an adapted frame is defined as the parallel transport of its components along  $\gamma$ .

#### 1.5.5 Frenet frame

The Frenet frame is a well-known particular adapted moving frame. It is defined as the map that attach to any given point of  $\gamma$  the corresponding Frenet trihedron  $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$  where :

$$\mathbf{t}(s) = \frac{\gamma'(s)}{\|\gamma'(s)\|} \quad , \quad \mathbf{n}(s) = \frac{\mathbf{t}'(s)}{\kappa(s)} \quad , \quad \mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s) \quad (1.39)$$

### Governing equations

The Frenet frame satisfies the *Frenet-Serret formulas* (see §1.4.4), which govern the evolution of the frame along the curve  $\gamma$  :

$$\begin{bmatrix} \mathbf{t}'(s) \\ \mathbf{n}'(s) \\ \mathbf{b}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau_f(s) \\ 0 & -\tau_f(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{bmatrix} \quad (1.40)$$

Remember the generic system of differential equations of an adapted moving frame attached to a curve, established in eq. (1.28), where :

$$\mathbf{e}_3(s) = \mathbf{t}(s) \quad , \quad k_1(s) = 0 \quad , \quad k_2(s) = \kappa(s) \quad , \quad \tau(s) = \tau_f(s) \quad (1.41)$$

### Angular velocity

Consequently, the angular velocity vector ( $\mathbf{\Omega}_f$ ) of the Frenet frame, also known as the *Darboux vector* in this particular case, is given by :

$$\mathbf{\Omega}_f(s) = \begin{bmatrix} \tau_f(s) \\ 0 \\ \kappa(s) \end{bmatrix} = \kappa \mathbf{b}(s) + \tau_f(s) \mathbf{t}(s) \quad (1.42)$$

Remark that the Frenet frame satisfies  $\mathbf{\Omega}_f(s) \cdot \mathbf{n}(s) = 0$  and is thus a *rotation-minimizing* frame regarding the normal vector ( $\mathbf{n}$ ). The motion of this frame through the curve is known as “pitch-free”.

Note also that  $\mathbf{t}'(s)$  and  $\mathbf{b}'(s)$  are colinear to  $\mathbf{n}(s)$ . This means that the projection of  $\mathbf{t}(s)$  and  $\mathbf{b}(s)$  is conserved from one normal plane to another, that is  $\mathbf{t}$  and  $\mathbf{b}$  are parallel

transported along the vector field  $\mathbf{n}$ .

### Drawbacks and benefits

20 , 21 , 22

The Frenet frame is not continuously defined if  $\gamma$  is not  $\mathcal{C}^2$ . This is problematic for the study of slender beams as the centerline of a beam subject to punctual external forces and moments or to material discontinuities will not be  $\mathcal{C}^2$  but only piecewise  $\mathcal{C}^2$ . In that case, the centerline tangent will be continuously defined everywhere but the curvature will be subject to discontinuities, that is  $\mathbf{t}'$  will not be continuously defined.

Moreover, even if  $\gamma$  is  $\mathcal{C}^2$ , the Frenet frame is not defined where the curvature vanishes, which obviously is an admissible configuration for a beam centerline. This issue can be partially addressed by parallel transporting the normal vector along the straight regions of the curve. Thus, the extended frame will still satisfy the governing equations exposed in eq. (1.40). However, if the osculating planes are not parallels on both sides of a region of null curvature, torsion will be subject to a discontinuity and so the Frenet frame<sup>23</sup> (fig. 1.3). Again, if the region of null curvature is not a point, that is the region is not an inflexion point but a locus where the curve is locally a straight line, the change in torsion on both sides of the region can be accommodated by a continuous rotation from one end to the other.

One benefit of the Frenet frame is that, when transported along a *closed curve*, the frame at the end of the curve will align back with the frame at the beginning of the curve, that is the frame will returns to its initial value after a complete turn. During its trip, the frame will make a total twist of  $\int_0^L \tau_f(s)ds = 0[2\pi]$  around the tangent vector.

A second benefit is that any adapted frame can be obtained by a rotation of the Frenet frame around the unit tangent vector [Gug89, p.2].

### 1.5.6 Bishop frame

A *Bishop frame*<sup>24</sup> denoted  $\{\mathbf{t}, \mathbf{u}, \mathbf{v}\}$ , also known as *zero-twisting* or *parallel-transported* frame, is an adapted moving frame that has no tangential angular velocity :

$$\boldsymbol{\Omega} \cdot \mathbf{t} = \tau = \mathbf{u}' \cdot \mathbf{v} = -\mathbf{u} \cdot \mathbf{v}' = 0 \quad (1.43)$$

Because a Bishop frame is an adapted frame, it can be defined relatively to the Frenet frame by a rotation around the unit tangent vector. A Bishop frame is a frame that cancels

<sup>20</sup>une perturbation de la courbe dans le sens de la courbure engendre une variation de longueur de la courbe proportionnelle à l'inverse de la courbure (au premier ordre) + schéma.

<sup>21</sup>une perturbation de la courbe dans le sens de la binormale (en tout point) préserve la longueur de la courbe au 1er ordre : c'est un déplacement qui conserve l'hypothèse d'inextensibilité au premier ordre.

<sup>22</sup>Examiner la question de la fermeture sur une boucle fermée. Schéma.

<sup>23</sup>This is also highlighted in [Blo90, WJZL08].

<sup>24</sup>Introduce as *relatively parallel adapted frame* in [? ].

out the rotational movement of the Frenet frame around the tangent vector. At arc-length parameter  $s$ , the Frenet frame has continuously rotated around its tangent vector of a cumulative angle :  $\int_0^s \tau_f(t)dt$ . Thus, any Bishop frame will be obtained, within a constant rotation angle  $\theta_0$ , through a rotation of the Frenet frame around the tangent vector by an angle :

$$\theta(s) = - \int_0^s \tau_f(t)dt + \theta_0(s) \quad (1.44)$$

Consequently, a Bishop frame can be expressed relatively to the Frenet frame as :

$$\begin{cases} \mathbf{u} = \cos \theta \mathbf{n} + \sin \theta \mathbf{b} \\ \mathbf{v} = -\sin \theta \mathbf{n} + \cos \theta \mathbf{b} \end{cases} \quad (1.45)$$

### Governing equations

The Bishop frame satisfies the following system of differential equations, which govern the evolution of the frame along the curve  $\gamma$  :

$$\begin{bmatrix} \mathbf{t}'(s) \\ \mathbf{u}'(s) \\ \mathbf{v}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) \sin \theta(s) & -\kappa(s) \cos \theta(s) \\ -\kappa(s) \sin \theta(s) & 0 & 0 \\ \kappa(s) \cos \theta(s) & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}(s) \\ \mathbf{u}(s) \\ \mathbf{v}(s) \end{bmatrix} \quad (1.46)$$

One can remember the generic differential equations of an adapted moving frame attached to a curve, where<sup>25</sup> :

$$k_1(s) = \kappa(s) \sin \theta(s) \quad , \quad k_2(s) = \kappa(s) \cos \theta(s) \quad , \quad \tau(s) = 0 \quad (1.48)$$

### Angular velocity

Consequently, the angular velocity vector ( $\boldsymbol{\Omega}_b$ ) of the Bishop frame is given by :

$$\boldsymbol{\Omega}_b(s) = \begin{bmatrix} 0 \\ \kappa(s) \sin \theta(s) \\ \kappa(s) \cos \theta(s) \end{bmatrix} = \kappa \mathbf{b}(s) \quad (1.49)$$

Remark that the Bishop frame satisfies  $\boldsymbol{\Omega}_b(s) \cdot \mathbf{t}(s) = 0$  and is thus *rotation-minimizing* regarding the tangent vector. The motion of this frame through the curve is known as “roll-free”.

Because the motion of this frame is described by an angular velocity vector that is nothing

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25

$$\begin{aligned} \tau &= \mathbf{u}' \cdot \mathbf{v} = (\boldsymbol{\Omega}_f \times \mathbf{u} + \theta' \mathbf{v}) \cdot \mathbf{v} = \tau_f - \tau_f = 0 \\ k_1 &= -\mathbf{t}' \cdot \mathbf{v} = -\kappa \mathbf{n} \cdot \mathbf{v} = \kappa \sin \theta \\ k_2 &= \mathbf{t}' \cdot \mathbf{u} = \kappa \mathbf{n} \cdot \mathbf{u} = \kappa \cos \theta \end{aligned} \quad (1.47)$$

but the curvature binormal vector ( $\Omega_b = \kappa b$ ), it can be interpreted in terms of *parallel transport* as defined in §1.5.4. Thus, given an initial frame at arc-length parameter  $s = 0$ , the Bishop frame at any arc-length parameter ( $s$ ) is obtained by parallel transporting the initial frame  $\{t(0), u(0), v(0)\}$  along the curve from 0 to  $s$ .

### Drawbacks and benefits

26 27

One of the main benefits of the Bishop frame is that its generative method : “is immune to degeneracies in the curvature vector” [Blo90]. Although we first expressed the construction of the Bishop frame relatively to the Frenet frame (which exists wherever  $\gamma$  is biregular), the existence of the Bishop frame, understood in terms of parallel transport, is guaranteed wherever the curvature binormal ( $\kappa b = t \times t'$ ) is defined. To be continuously defined over  $[0, L]$ , a Bishop frame only needs the curvature binormal vector to be piecewise continuously defined over  $[0, L]$ , which only requires that  $\gamma'$  is  $C^0$  and that  $\gamma''$  is piecewise  $C^0$ . Obviously, those weaker existence conditions are profitables to bypass the drawbacks of the Frenet frame regarding the modeling of slender beams listed in §1.5.5.

Strictly speaking, a Bishop frame is not a reference frame as it is defined within an initial condition. However, we will see later that strains in a beam are modeled as a rate of change in the Bishop frame, and consequently the initial condition will disappear in the equations.

Unlike the Frenet frame, when transported along a *closed curve*, the Bishop frame at the end of the curve will not necessarily align back with the frame at the beginning of the curve<sup>28</sup>. Even if the frame returns to its initial value after a complete turn, it may returns in its position after several complete turns ( $2k\pi$ ) around the curve tangent. During its trip, the frame will make a total twist of  $\int_0^L \tau_f(s)ds = \alpha[2\pi]$  around the tangent vector. This difference of angle is related to the concept of *holonomy*.

Remark also that Frenet and Bishop frames coincide for planar curves ( $\tau_f = 0$ ), within a constant rotation around the unit tangent vector.

### 1.5.7 Comparison between Frenet and Bishop frames

#### Application A : circular helix

$$\begin{cases} \rho = a \\ z = b\theta \end{cases} \quad (1.50)$$

<sup>26</sup>[Gug89, Klo86, Blo90, WJZL08, PFL95, Men13]

<sup>27</sup>“Regarder la méthode de la bi-reflexion pour le calcul du repère de bishop” [WJZL08, p.6]

<sup>28</sup>“it is possible for closed curves to have parallel transport frames that do not match up after one full circuit of the curve” [HM95]

**Application B : conical helix (spiral)**

$$\begin{cases} \rho = ae^{k\theta} \\ z = \rho \cot \alpha \end{cases} \quad (1.51)$$

soit pour une spirale dont on connaît

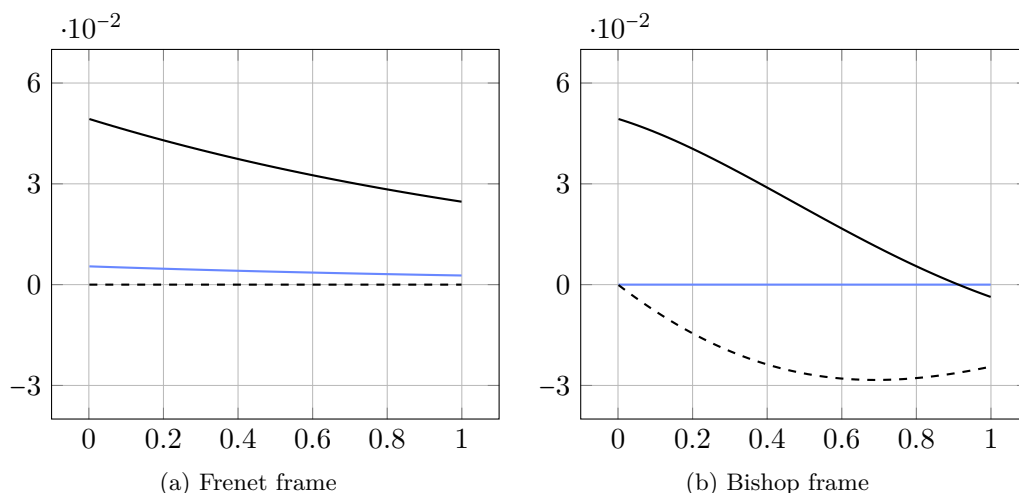


Figure 1.6 – Comparison between Frenet and Bishop frame velocity for a spirale curve.

## 1.6 Discret curves

### 1.6.1 Definition and arc-length parametrization

### 1.6.2 Discret curvature : an equivocal concept

Curvature is defined from the osculating circle, which is the best approximation of a curve by a circle. We can define such a circle and it's radius will be the curvature at that point. Problem : there are several ways to define such a circle.

#### Definition

Très intéressant de constater que cette vision 3 verticies vs. 2 edges est déjà présente dès le début dans l'histoire de la compréhension de la courbure.

Pour Euler, le rayon de courbure est le rapport de l'élément d'arc sur l'angle de contingence entre deux tangentes infiniment proches. Par ailleurs, la définition du plan osculateur n'est pas tout à fait la même que chez Bernoulli, plan passant par trois points consécutifs, puis- qu'Euler dit que ce plan contient deux éléments successifs. Il le définit aussi en disant que c'est le plan où la courbe s'incurve. Pour le dire de façon un peu différente : la tangente contient un élément, c'est le lieu où la courbe est droite, la plan osculateur représente l'étape suivante, c'est le lieu où la courbe est arc de cercle. Nous ne pensons



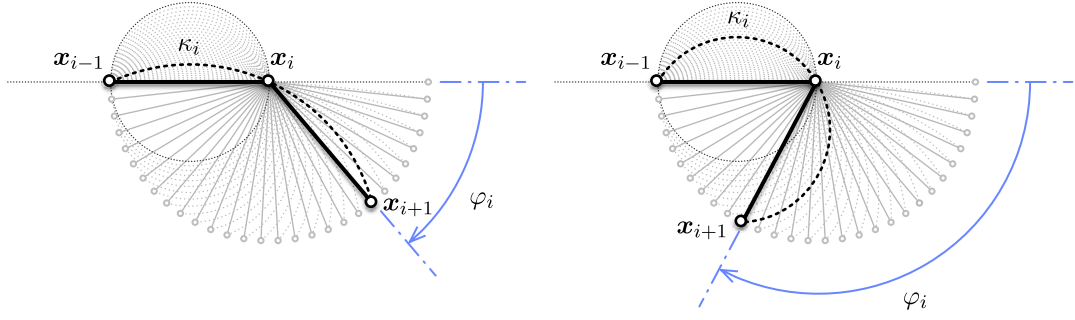


Figure 1.7 – Variation of the vertex-based discrete curvature.

pas trahir Euler en faisant cette présen- tation : cela justifie que, pour lui, il est naturel de se placer sur le plan osculateur pour calculer le rayon de courbure. [Del07]

[Hof08]

The edge osculating circle. The vertex osculating circle. La localité est meilleur dans le cas du vertex-based discret osculating circle. Pour des anlgés élevés, le edge-based discret osculating circle est plus pertinent. La courbure tend vers l'infini quand les 2 edges deviennent colineaires.

La définition du plan osculateur est univoque dans le cas discret : c'est localement le plan défini par 2 edges consécutifs.

Ce n'est pas le cas de la courbure qui perd son côté intrinsèque.

courbure discrete dans le cas général

### Vertex-based osculating circle

$$\kappa_1 = \frac{2 \sin(\varphi_i)}{\|e_{i-1} + e_i\|}, \quad (1.52)$$

Problème de convergence lorsque l'angle tend vers pi et que les segments ont même longueur. En pratique peut probable.

### Edge-based osculating circle

$$\kappa_2 = \frac{\tan(\varphi_i/2) + \tan(\varphi_{i+1}/2)}{\|e_i\|} \quad (1.53)$$

Bon comportement lorsque l'angle tend vers pi. Demande que les 3 segments soient coplanaires.

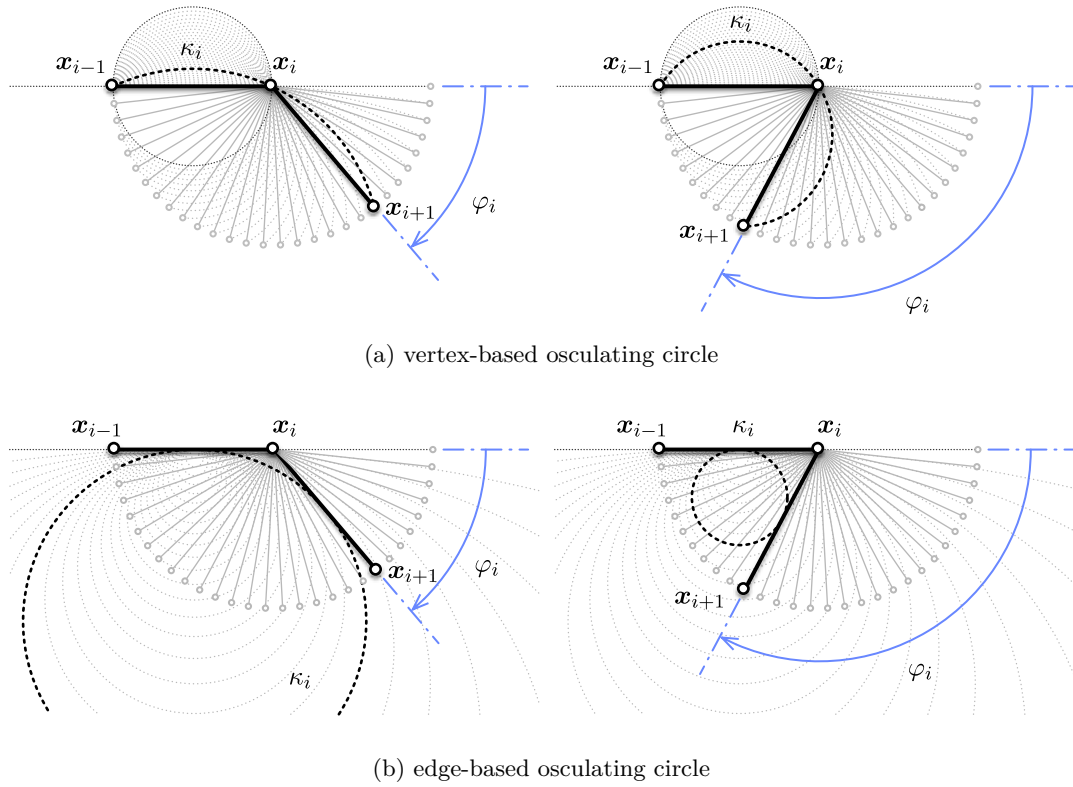


Figure 1.8 – Definition of the osculating circle for discrete curves.

### Osculating circle for an arc-length parametrized curve

$$\kappa_3 = \frac{2 \tan(\varphi_i/2)}{l} \quad , \quad l = \|\mathbf{e}_i\| = cst \quad (1.54)$$

Qu'on peut réécrire pour des segments de longueur variable, avec la moyenne

$$\kappa_3 = \frac{2 \tan(\varphi_i/2)}{l_i} \quad , \quad l_i = \frac{\|\mathbf{e}_{i-1}\| \|\mathbf{e}_i\|}{\|\mathbf{e}_{i-1}\| + \|\mathbf{e}_i\|} \quad (1.55)$$

Unlike the smooth case we can not reparameterize a curve. A discrete curve is parameterized by arc-length or it is not [Hof08, p. 10].

Cette condition est extrêmement exigeante  $\|\mathbf{e}_i\| = cst$ . Elle est tenable pour des modèles de poutre non connectées (où le pas de discrétisation peut-être choisi uniforme) mais pour en cas de connexion. Ce point n'est pas éclairci dans les articles de Audoly.

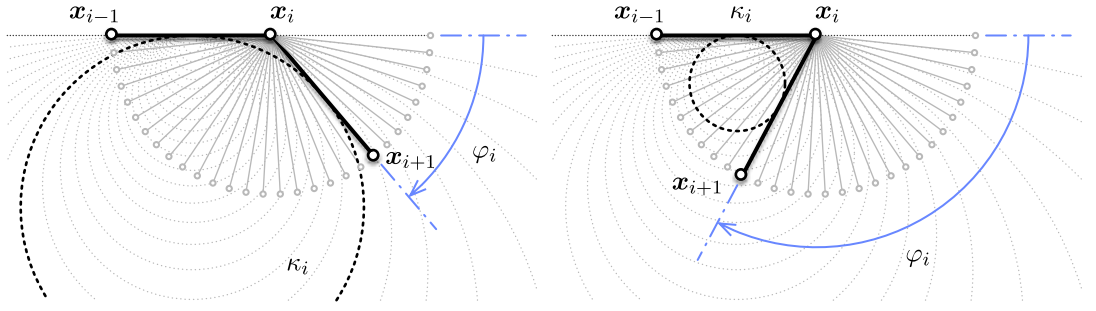
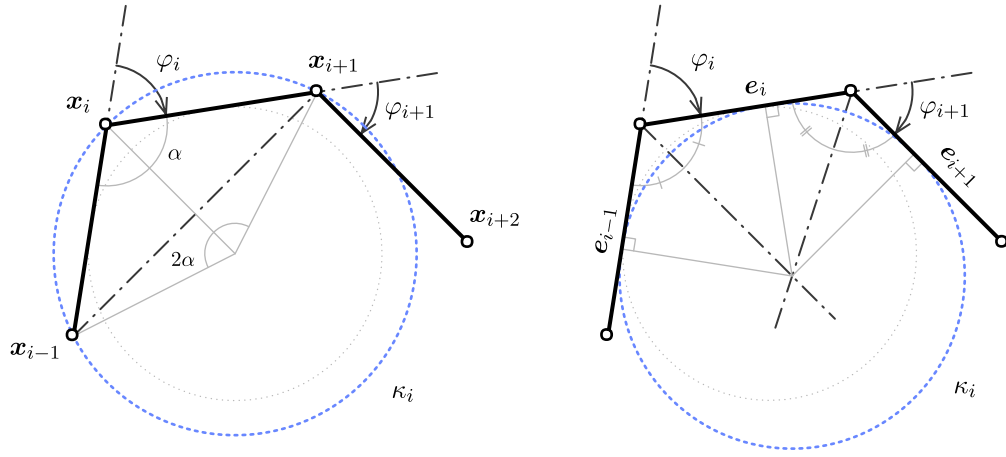


Figure 1.9 – Variation of the edge-based discrete curvature.



(a) vertex-based osculating circle

(b) edge-based osculating circle

Figure 1.10 – Definition of the osculating circle for discrete curves.

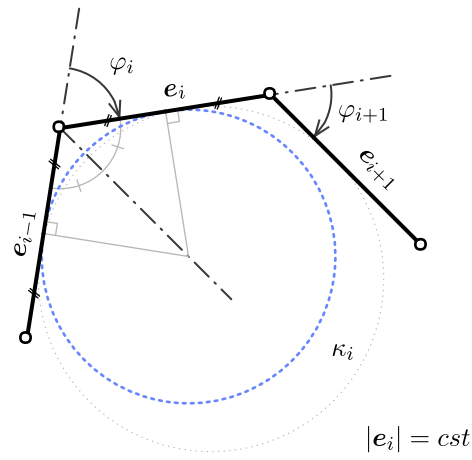


Figure 1.11 – Another definition of the osculating circle for arc-length parametrized curves.

### 1.6.3 Variability of discrete curvature regarding $\alpha$

Qu'on réécrit en posant  $\|e_{i-1}\| = \alpha\|e_i\|$ ,  $\alpha \geq 0$  :

$$\kappa_1 = \frac{2 \sin(\varphi_i)}{\|e_i\|(1 + \alpha^2 + 2\alpha \cos(\varphi_i))^{1/2}}, \quad \kappa_2 = \frac{4 \tan(\varphi_i/2)}{\|e_i\|(1 + \alpha)} \quad (1.56)$$

$$\frac{\kappa_1}{\kappa_2}(\alpha) = \frac{\kappa_1}{\kappa_2}(1/\alpha) = \frac{1 + \alpha}{(1 + \alpha^2 + 2\alpha \cos(\varphi_i))^{1/2}} \cos^2(\varphi_i/2) \quad (1.57)$$

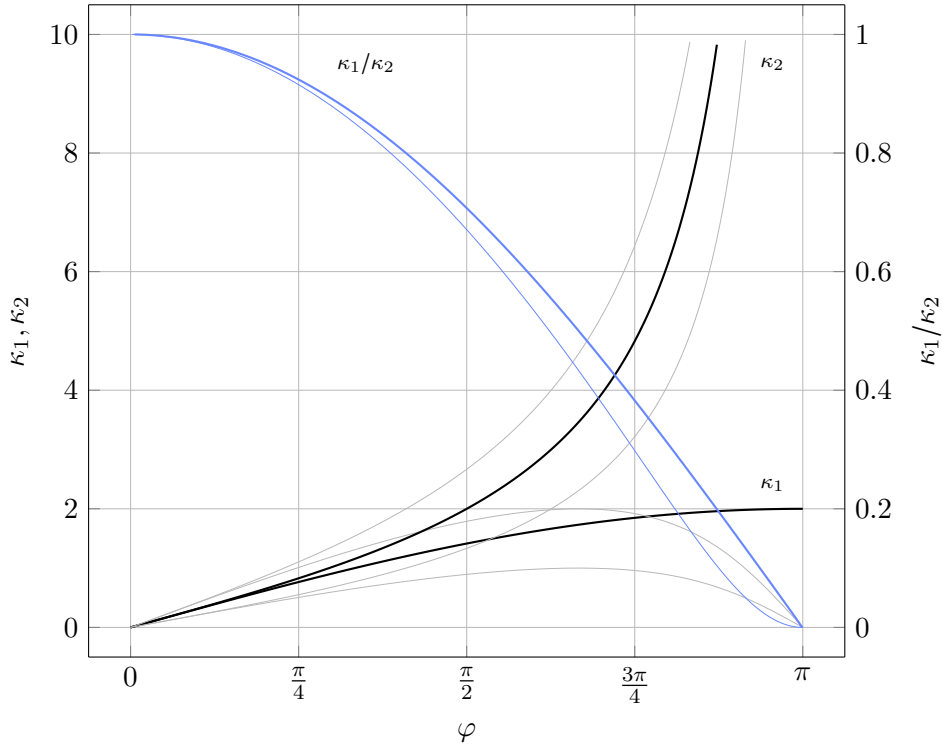


Figure 1.12 – Discrete curvature comparison for  $\alpha \in [0.5, 2]$

### 1.6.4 Convergence benchmark $\kappa_1$ vs. $\kappa_2$

**Straight line**

**Circle**

Smooth curve settings:

$$\mathcal{E} = \int_0^l \kappa^2 ds = \kappa \pi, \quad l = \pi r, \quad \kappa = \frac{1}{r} \quad (1.58)$$

Discrete curve :

$$\varphi_N = \frac{\pi}{N}, \quad |e| = 2r \sin \frac{\varphi}{2}, \quad l_N = N|e| = 2Nr \sin \frac{\varphi}{2} = l \frac{\sin \frac{\varphi}{2}}{\frac{\varphi}{2}} \quad (1.59)$$

Discrete bending energies :

$$\mathcal{E}_1 = \mathcal{E} \frac{\sin \frac{\varphi}{2}}{\frac{\varphi}{2}}, \quad \mathcal{E}_2 = \mathcal{E} \frac{\sin \frac{\varphi}{2}}{\frac{\varphi}{2} \cos^2 \frac{\varphi}{2}}, \quad (1.60)$$

Remarque that ratios are independent of scale change (independent of R)

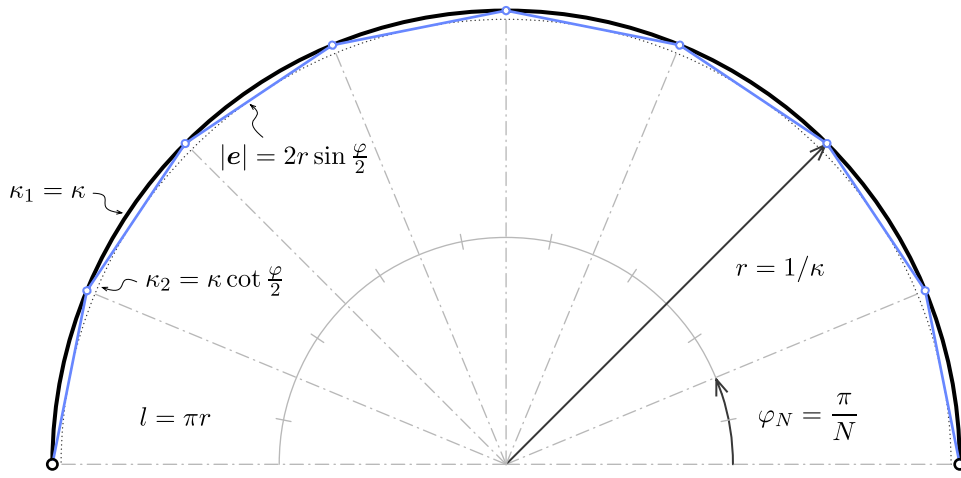


Figure 1.13 – Another definition of the osculating circle for arc-length parametrized curves.

qsmldkqsmldk s qsd qsd sqd qs dqs=dlk qs=ldk sq

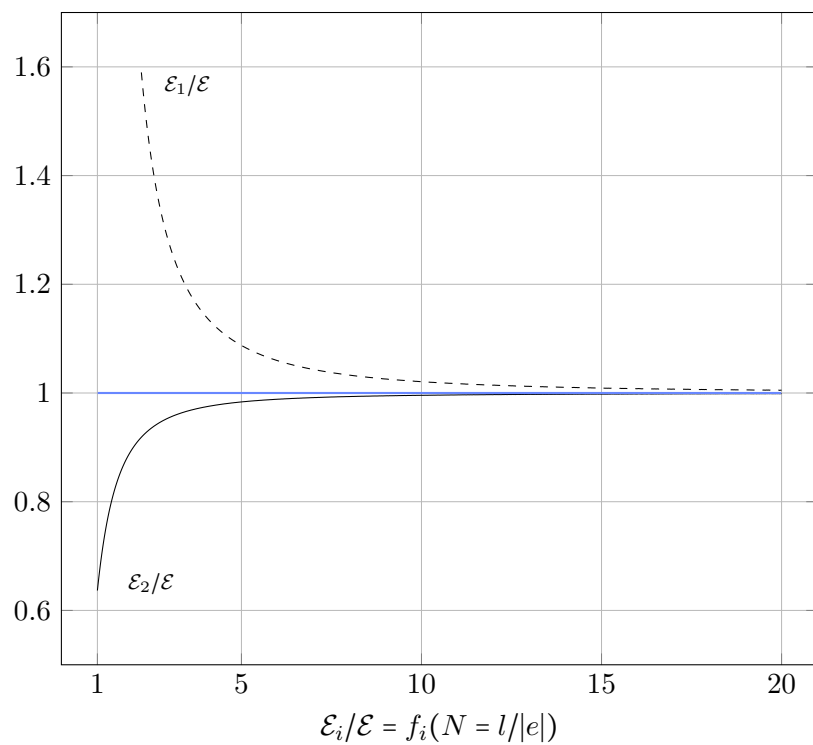


Figure 1.14 – Discrete curvature comparison for  $\alpha \in [0.5, 2]$

## Elastica

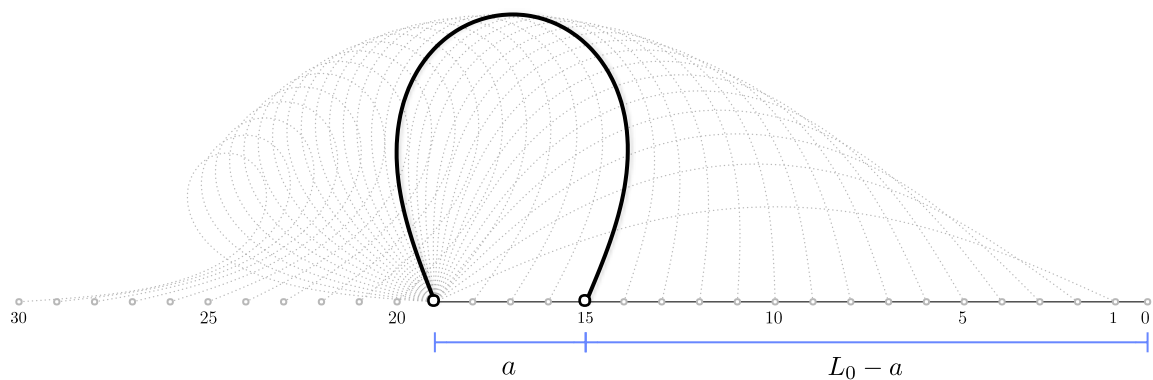


Figure 1.15 – Another definition of the osculating circle for arc-length parametrized curves.

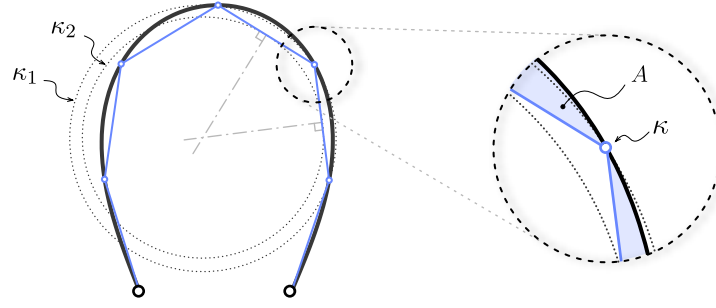


Figure 1.16 – Another definition of the osculating circle for arc-length parametrized curves.

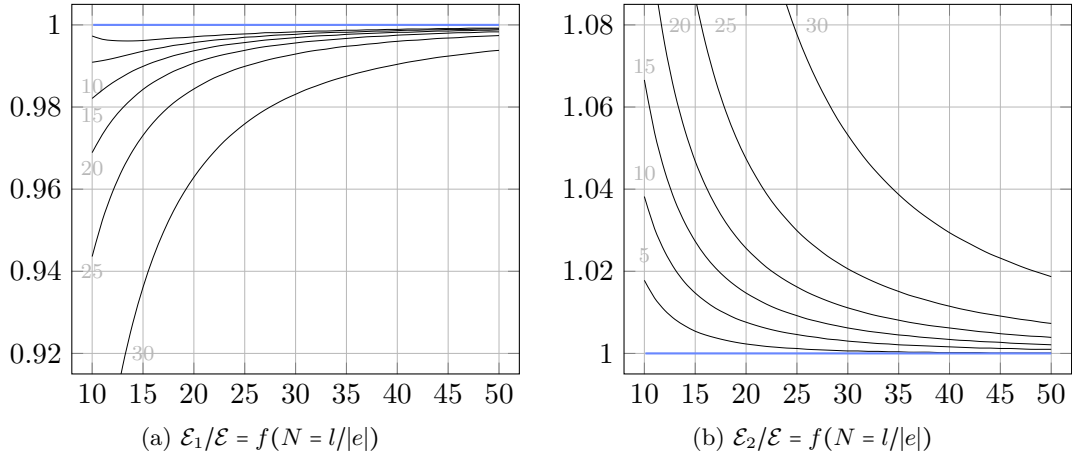


Figure 1.17 – Bending energy representativity

### 1.6.5 Edge versus vertex based tangent vector

Problème de définition. Facile de définir une tangente sur un edge. Mais une infinité de tangentes possibles à chaque vertex.

So in case of an arc-length parameterized curve the vertex tangent vector points in the same direction as the averaged edge tangent vectors [Hof08, p. 12].

Nous verrons que le cercle 3 points, en plus de mieux représenter l'énergie d'une courbe discrete dans les cas typiques, offre un choix de tangente non ambigu.

## Bibliography

[BELT14] J. Bluth, Alain Ehrlicher, Baptiste Lefevre, and Frédéric Tayeb. Modélisation de poutres en torsion. Technical report, Na, Paris, 2014.

[Ber28] Johannis Bernoulli. *Johannis Bernoulli... Opera omnia, tam antea sparsim*

## Bibliography

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- edita, quam hactenus inedita*. Number IV. sumptibus Marci-Michaelis Bousquet, 1728.
- [Bis75] Richard L. Bishop. There is more than one way to frame a curve. *Mathematical Association of America*, 1975.
- [Blo90] Jules Bloomenthal. Calculation of reference frames along a space curve. In Andrew S Glassner, editor, *Graphics Gems*, volume 1, pages 567–571. Academic Press Professional, Inc., San Diego, CA, USA, 1990.
- [BWR<sup>+</sup>08] Miklós Bergou, Max Wardetzky, Stephen Robinson, Basile Audoly, and Eitan Grinspun. Discrete elastic rods. *ACM SIGGRAPH*, pages 1–12, 2008.
- [CES07] Daniel Carroll, K Emek, and Ivan Sterling. Improving Frenet’s frame using Bishop’s frame. 2007.
- [Coo13] Julian Lowell Coolidge. *A history of geometrical methods*. Dover Books on Mathematics. Dover Publications, 2013.
- [Del07] Jean Delcourt. *Analyse et géométrie : les courbes gauches de Clairaut à Serret*. PhD thesis, Université de Paris VI, 2007.
- [Del11] Jean Delcourt. Analyse et géométrie, histoire des courbes gauches de Clairaut à Darboux. *Archive for History of Exact Sciences*, 65(3):229–293, 2011.
- [Eul75] Leonhard Euler. Continens analysin, pro incuruatione fili in fingulis punctis inueniendia. *Novi Commentarii academiae scientiarum Petropolitanae*, 19:340–370, 1775.
- [FGSS14] Rida T. Farouki, Carlotta Giannelli, Maria Lucia Sampoli, and Alessandra Sestini. Rotation-minimizing osculating frames. *Computer Aided Geometric Design*, 31(1):27–42, 2014.
- [Fre52] Frédéric Frenet. Sur les courbes à double courbure. *Journal de Mathématiques Pures et Appliquées*, 17:437–447, 1852.
- [GAS06] Alfred Gray, Elsa Abbena, and Simon Salamon. *Modern differential geometry of curves and surfaces with Mathematica*. CRC Press, third edition, 2006.
- [Gug89] H. Guggenheimer. Computing frames along a trajectory. *Computer Aided Geometric Design*, 6(1):77–78, 1989.
- [HM95] Andrew J Hanson and Hui Ma. Parallel Transport Approach to Curve Framing. Technical report, 1995.
- [Hof08] Tim Hoffmann. Discrete Differential Geometry of Curves and Surfaces, 2008.
- [Klo86] Fopke Klok. Two moving coordinate frames for sweeping along a 3D trajectory. *Computer Aided Geometric Design*, 3(3):217–229, 1986.
- [McC13] Andrew Grady McCormick. Discrete Differential Geometry and Physics of Elastic Curves. 2013.



- [Men13] Toni Menninger. Frenet curves and successor curves : generic parametrizations of the helix and slant helix. Technical report, 2013.
- [Mon09] Gaspard Monge. *Application de l'analyse à la géométrie*. 1809.
- [PFL95] Tim Poston, Shiaofen Fang, and Wayne Lawton. Computing and approximating sweeping surfaces based on rotation minimizing frames. In *Proceedings of the 4th International Conference on CAD/CG*, pages 1–8, 1995.
- [Pit26] Henri Pitot. Quadrature de la moitié d'une courbe des arcs appelée la compagne de la cycloïde. *Mémoires de l'Académie Royale des Sciences (1724)*, pages 107–113, 1726.
- [WJZL08] Wenping Wang, Bert Jüttler, Dayue Zheng, and Yang Liu. Computation of rotation minimizing frames. *ACM Transactions on Graphics*, 27(1):1–18, 2008.

