Modeling of bending-torsion couplings in active-bending structures. Application to the design of elastic gridshell.

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par

Lionel du Peloux

acceptée sur proposition du jury:

Prof Name Surname, président du jury
Prof Name Surname, directeur de thèse
Prof Name Surname, rapporteur
Prof Name Surname, rapporteur
Prof Name Surname, rapporteur

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