1. $u = 1/d = 2$

$S_0 = 14$

$r = 0.12 = 12\%$

$K = 20$

a) Possible Stock Prices at Expiration:

$S_5(5H, 0T) = S_0 u^5 = 14 \times 2^5 = 448$ $S_5(5)$

$S_5(4H, 1T) = S_0 u^4 d = 14 \times 16 \times 0.5 = 112$ $S_5(4)$

$S_5(3H, 2T) = S_0 u^3 d^2 = 14 \times 8 \times 0.25 = 28$ $S_5(3)$

$S_5(2H, 3T) = S_0 u^2 d^3 = 14 \times 4 \times 0.125 = 7$ $S_5(2)$

$S_5(1H, 4T) = S_0 u^1 d^4 = 14 \times 2 \times 0.0625 = 1.75$ $S_5(1)$

$S_5(0H, 5T) = S_0 d^5 = 14 \times 0.03125 = 0.4375$ $S_5(0)$

Call Option Payoff

$V_5(5) = \max(448 - 20, 0) = 428$

$V_5(4) = \max(112 - 20, 0) = 92$

$V_5(3) = \max(28 - 20, 0) = 8$

$V_5(2) = \max(7 - 20, 0) = 0$

$V_5(1) = \max(1.75 - 20, 0) = 0$

$V_5(0) = \max(0.4375 - 20, 0) = 0$

Risk - Neutral Probabilities

$$\tilde{p} = \frac{1+r-d}{u-d} = \frac{1+0.12-0.5}{2-0.5} = \frac{0.62}{1.5} = 0.4133$$

$$\tilde{q} = \frac{u-(1+r)}{u-d} = \frac{2-1.12}{2-0.5} = \frac{0.88}{1.5} = 0.5867$$

Traversing Backwards

$$V_4(4) = \frac{1}{1.12}(0.4133 \times 428 + 0.5867 \times 92) = 206.13$$

$$V_4(3) = \frac{1}{1.12}(0.4133 \times 0.92 + 0.5867 \times 8) = 38.14$$

$$V_4(2) = \frac{1}{1.12}(0.4133 \times 8 + 0.5867 \times 0) = 2.95$$

$$V_4(1) = \frac{1}{1.12}(0.4133 \times 0 + 0.5867 \times 0) = 0$$

$$V_4(0) = \frac{1}{1.12}(0.4133 \times 0 + 0.5867 \times 0) = 0$$

$$V_3(3) = \frac{1}{1.12}(0.4133 \times 206.13 + 0.5867 \times 38.14) = 96.05$$

$$V_3(2) = \frac{1}{1.12}(0.4133 \times 38.14 + 0.5867 \times 2.95) = 15.62$$

$$V_3(1) = \frac{1}{1.12}(0.4133 \times 2.95 + 0.5867 \times 0) = 1.09$$

$$V_3(0) = \frac{1}{1.12}(0.4133 \times 0 + 0.5867 \times 0) = 0$$

$$V_2(2) = \frac{1}{1.12}(0.4133 \times 96.05 + 0.5867 \times 15.62) = 43.64$$

$$V_2(1) = \frac{1}{1.12}(0.4133 \times 15.62 + 0.5867 \times 1.09) = 6.33$$

$$V_2(0) = \frac{1}{1.12}(0.4133 \times 1.09 + 0.5867 \times 0) = 0.40$$

$$V_1(1) = \frac{1}{1.12}(0.4133 \times 43.64 + 0.5867 \times 6.33) = 19.42$$

$$V_1(0) = \frac{1}{1.12}(0.4133 \times 6.33 + 0.5867 \times 0.40) = 2.55$$

$$V_0 = \frac{1}{1.12}(0.4133 \times 19.42 + 0.5867 \times 2.55) = 8.50$$

Value of European call option at $t=0$ is 8.50

b)

$$\Delta_n = \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)}$$

$t = 0$: $\Delta_0 = \dfrac{19.42 - 2.55}{28 - 7}$ $\leftarrow$ $S_1(H) = 14u = 14 \times 2 = 28$
$S_1(T) = 14d = 14 \times 0.5 = 7$

$$= \frac{16.87}{21} = 0.8$$

$t = 1$: $\Delta_1(H) = \dfrac{43.64 - 6.33}{56 - 14}$ $\leftarrow$ $V_2(2)$ & $V_2(1)$

$$= \frac{37.31}{42} = 0.89$$

$\Delta_1(T) = \dfrac{6.33 - 0.40}{14 - 3.5}$ $\leftarrow$ $V_2(1)$ & $V_2(0)$

$$= \frac{5.93}{10.5} = 0.56$$

2. **Put Option Payoffs**

$V_5(5) = \max(21-448, 0) = 0$

$V_5(4) = \max(21-112, 0) = 0$

$V_5(3) = \max(21-28, 0) = 0$

$V_5(2) = \max(21-7, 0) = 14$

$V_5(1) = \max(21-1.75, 0) = 19.25$

$V_5(0) = \max(21-0.4375, 0) = 20.56$

Traversing Backwards

$V_4(4) = \frac{1}{1.12}(0.4133 \times 0 + 0.5867 \times 0) = 0$

$V_4(3) = \frac{1}{1.12}(0.4133 \times 0 + 0.5867 \times 0) = 0$

$V_4(2) = \frac{1}{1.12}(0.4133 \times 0 + 0.5867 \times 14) = 7.33$

$V_4(1) = \frac{1}{1.12}(0.4133 \times 14 + 0.5867 \times 19.25) = 15.25$

$V_4(0) = \frac{1}{1.12}(0.4133 \times 19.25 + 0.5867 \times 20.56) = 17.88$

$V_3(3) = \frac{1}{1.12}(0.4133 \times 0 + 0.5867 \times 0) = 0$

$V_3(2) = \frac{1}{1.12}(0.4133 \times 0 + 0.5867 \times 7.33) = 3.85$

$V_3(1) = \frac{1}{1.12}(0.4133 \times 7.33 \times 0.5867 \times 15.25) = 11.91$

$V_3(0) = \frac{1}{1.12}(0.4133 \times 15.25 \times 0.5867 \times 17.88) = 15.48$

$$V_2(2) = \frac{1}{1.12}(0.4133 \times 0 + 0.5867 \times 3.85) = 2.02$$

$$V_2(1) = \frac{1}{1.12}(0.4133 \times 3.85 + 0.5867 \times 11.91) = 8.28$$

$$V_2(0) = \frac{1}{1.12}(0.4133 \times 11.91 + 0.5867 \times 15.48) = 12.36$$

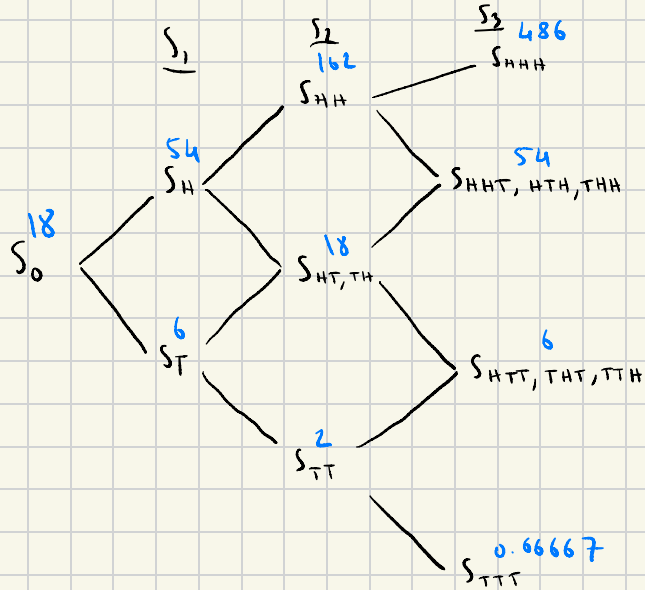
$$V_1(1) = \frac{1}{1.12}(0.4133 \times 2.02 + 0.5867 \times 8.28) = 5.39$$

$$V_1(0) = \frac{1}{1.12}(0.4133 \times 8.28 + 0.5867 \times 12.36) = 9.51$$

$$V_0 = \frac{1}{1.12}(0.4133 \times 5.39 + 0.5867 \times 9.51) = 7.12$$

Value of European put option at $t=0$ is 7.12

3.



a) $E_1[S_2](H)$ – expected value of S_2 based on information available at step 1 (i.e., S_1 is H)

$$= \left(\frac{1}{3}\right)(162) + \left(\frac{2}{3}\right)(18) = 66$$

b) $E_1[S_2](T)$

$= (\frac{1}{3})(18) + (\frac{2}{3})(2) = \frac{22}{3} \approx 7.33$

c) $E_1[S_3](T)$

$= \frac{1}{3}E[S_3 \mid S_2 = 18] + \frac{2}{3}E[S_3 \mid S_2 = 2]$

$= \frac{1}{3}[(\frac{1}{3})(54) + (\frac{2}{3})(6)] + \frac{2}{3}[(\frac{1}{3})(6) + (\frac{2}{3})(0.\overset{\frac{2}{3}}{6666}7)]$

$= \frac{1}{3}[22] + \frac{2}{3}[\frac{22}{9}] = \frac{242}{27} \approx 8.96$

4. a) $E_1[S_2 + S_3](H) = E_1[S_2](H) + E_1[S_3](H)$

$E_1[S_2](H) = \frac{1}{3}(40) + \frac{2}{3}(10) = 20$

$E_1[S_3](H) = \frac{1}{3}E[S_3 \mid S_2 = 40] + \frac{2}{3}E[S_3 \mid S_2 = 10]$

$= \frac{1}{3}\left(\frac{1}{3}(80) + \frac{2}{3}(20)\right) + \frac{2}{3}\left(\frac{1}{3}(20) + \frac{2}{3}(5)\right)$

$= \frac{1}{3}(40) + \frac{2}{3}(10) = 20$

$\Rightarrow E_1[S_2](H) + E_1[S_3](H) = 40$

$E_1[S_2 + S_3](H)$:

- HHH $\rightarrow S_2 + S_3 = 120$ w/ prob $= \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$
- HHT $\rightarrow S_2 + S_3 = 60$ w/ prob $= \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$
- HTH $\rightarrow S_2 + S_3 = 30$ w/ prob $= \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$
- HTT $\rightarrow S_2 + S_3 = 15$ w/ prob $= \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$

$$E_1[S_2+S_3](H) = \frac{1}{9}(120) + 2\cdot\frac{2}{9}(60+30) + \frac{4}{9}(15)$$

$$= 40 = 40 \text{, as required}$$

$$E_1[S_2+S_3](T) = E_1[S_2](T) + E_1[S_3](T)$$

$$E_1[S_2](T) = \frac{1}{3}(10) + \frac{2}{3}(2.5) = 5$$

$$E_1[S_3](T) = \frac{1}{3}E[S_3 \mid S_2 = 10] + \frac{2}{3}E[S_3 \mid S_2 = 2.5]$$

$$= \frac{1}{3}\left(\frac{1}{3}(20) + \frac{2}{3}(5)\right) + \frac{2}{3}\left(\frac{1}{3}(5) + \frac{2}{3}(1.25)\right)$$

$$= \frac{1}{3}(10) + \frac{2}{3}(2.5)$$

$$= 5$$

$$\Rightarrow E_1[S_2](T) + E_1[S_3](T) = 10$$

$E_1[S_2+S_3](T)$:

- THH : $S_2 + S_3 = 30$ w/ prob $= \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9}$
- THT : $S_2 + S_3 = 15$ w/ prob $= \frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9}$
- TTH : $S_2 + S_3 = 7.5$ w/ prob $= \frac{2}{3}\cdot\frac{1}{3} = \frac{2}{9}$
- TTT : $S_2 + S_3 = 3.75$ w/ prob $= \frac{2}{3}\cdot\frac{2}{3} = \frac{4}{9}$

$$E_1[S_2+S_3](T) = \frac{1}{9}(30) + 2\cdot\frac{2}{9}(15+7.5) + \frac{4}{9}(3.75)$$

$$= \frac{90}{9} = 10 = 10 \text{, as required}$$

b) $E_2[S_3](HH) = \frac{1}{3}(\overset{S_2 = 40}{40} \times 2) + \frac{2}{3}(40 \times \frac{1}{2}) = \frac{1}{3}(80) + \frac{2}{3}(20) = 40$

$E_2[S_3](HT) = \frac{1}{3}(\overset{S_2 = 10}{10} \times 2) + \frac{2}{3}(10 \times \frac{1}{2}) = \frac{1}{3}(20) + \frac{2}{3}(5) = 10$

$E_2[S_3](TH) = 10$ just like above

$E_2[S_3](TT) = \frac{1}{3}(\overset{S_2 = 2.5}{2.5} \times 2) + \frac{2}{3}(2.5 \times \frac{1}{2}) = \frac{1}{3}(5) + \frac{2}{3}(1.25) = 2.5$

$E_1[E_2[S_3]](H)$:

$S_1(H) = 20$ and

- HH w/ prob $p = \frac{1}{3}$ and $E_2[S_3](HH) = 40$
- HT w/ prob $q = \frac{2}{3}$ and $E_2[S_3](HT) = 10$

$\Rightarrow E_1[E_2[S_3]](H) = \frac{1}{3}(40) + \frac{2}{3}(10) = 20$

$E_1[E_2[S_3]](T)$:

$S_1(T) = 5$ and

- TH w/ prob $p = \frac{1}{3}$ and $E_2[S_3](TH) = 10$
- TT w/ prob $q = \frac{2}{3}$ and $E_2[S_3](TT) = 2.5$

$\Rightarrow E_1[E_2[S_3]](T) = \frac{1}{3}(10) + \frac{2}{3}(2.5) = 5$

We already computed that $E_1[S_3](H) = 20$ and $E_1[S_3](T) = 5$ from a)

5. a) M_n martingale if, for every n

$$M_n = E_n[M_{n+1}]$$

$$S_{n+1} = S_n \cdot u \quad \text{or} \quad S_{n+1} = S_n \cdot d$$

$$\Rightarrow E_n[S_{n+1}] = p S_n u + q S_n d = S_n(pu + qd)$$

$$= S_n\left(\tfrac{1}{3} \cdot 2 + \tfrac{2}{3} \cdot \tfrac{1}{2}\right) = S_n$$

$\Rightarrow S_n = E_n[S_{n+1}]$ i.e., a martingale

b) $M_{n+1} = \dfrac{S_{n+1}}{(1+r)^{n+1}}$

$$\Rightarrow E_n[M_{n+1}] = \frac{1}{(1+r)^{n+1}}[p S_n u + q S_n d]$$

$$= \frac{S_n}{(1+r)^{n+1}}[pu + qd]$$

$$= \frac{S_n}{(1+r)^{n+1}}\left[\tfrac{1}{2} 2 + \tfrac{1}{2}\tfrac{1}{2}\right]$$

$$= \frac{S_n}{(1+r)^n} \cdot \frac{1}{(1+r)}\left[\tfrac{5}{4}\right]$$

$$= \frac{S_n}{(1+r)^n} \cdot \tfrac{4}{5} \cdot \tfrac{5}{4} = \frac{S_n}{(1+r)^n}$$

$$\Rightarrow M_n = \frac{S_n}{(1+r)^n} = E_n[M_{n+1}] = \frac{S_n}{(1+r)^n}$$

c) $pu + qd = \tfrac{1}{2} \cdot 2 + \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{5}{4}$

$\Rightarrow S_n \neq E_n[S_{n+1}] \Rightarrow$ not a martingale, unless $S_n = 0$

d) S_n : $E_n[S_{n+1}] = p S_n u + q S_n d$

$$= S_n(pu + qd) = S_n\left(\tfrac{2}{3}\cdot 2 + \tfrac{1}{3}\cdot\tfrac{1}{2}\right)$$

$$= S_n\left(\tfrac{4}{3} + \tfrac{1}{6}\right) = S_n\left(\tfrac{9}{6}\right) = S_n\left(\tfrac{3}{2}\right)$$

$\Rightarrow$ clearly not a martingale

$$M_n = \frac{S_n}{(1+r)^n} \Rightarrow M_{n+1} = \frac{S_{n+1}}{(1+r)^{n+1}}$$

$$E_n[M_{n+1}] = \frac{S_n}{(1+r)^n}\cdot\frac{1}{(1+r)}[pu + qd]$$

$$= \frac{S_n}{(1+r)^n}\cdot\frac{4}{5}\cdot\left(\tfrac{2}{3}\cdot 2 + \tfrac{1}{3}\cdot\tfrac{1}{2}\right)$$

$$= \frac{S_n}{(1+r)^n}\cdot\frac{4}{5}\cdot\left(\tfrac{9}{6}\right)$$

$$= \frac{S_n}{(1+r)^n}\cdot\frac{\cancel{36}^{\,7}}{\cancel{30}}\,\frac{6}{5}$$

$\Rightarrow$ clearly not a martingale

e) Risk-neutral measure is defined so that the discounted asset price process is a martingale.

$\frac{S_n}{(1+r)^n}$ is a martingale only when $(p, q) = (\frac{1}{2}, \frac{1}{2})$

$\ulcorner$ When $(p, q) = (\frac{1}{3}, \frac{2}{3})$

$$\frac{S_n}{(1+r)^n}\cdot\frac{4}{5}\cdot\left(\tfrac{1}{3}\cdot 2 + \tfrac{2}{3}\cdot\tfrac{1}{2}\right) = \frac{S_n}{(1+r)^n}\cdot\frac{4}{5}\cdot(1)$$

$\Rightarrow$ not a martingale $\lrcorner$

Only $(\frac{1}{2}, \frac{1}{2})$ is a risk-neutral measure.

e) Alternate approach:

A risk-neutral measure pair (p, q) is one such that:

$$\frac{pu + qd}{1+r} = 1 \Rightarrow pu + qd = 1+r \Leftrightarrow pu + qd = \frac{5}{4}$$

$$\Leftrightarrow p = \frac{1}{2}\left(\frac{5}{4} - \frac{1}{2}q\right)$$

From all three pairs (i.e., $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{3}, \frac{2}{3})$, $(\frac{2}{3}, \frac{1}{3})$), only $(\frac{1}{2}, \frac{1}{2})$ satisfies the above.

$\Rightarrow (\frac{1}{2}, \frac{1}{2})$ is a risk-neutral measure.

6. $$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$$

$$\frac{X_{n+1}}{(1+r)^{n+1}} = \frac{\Delta_n S_{n+1}}{(1+r)^{n+1}} + \frac{X_n - \Delta_n S_n}{(1+r)^n}$$

$$E_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right] = E_n\left[\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}} + \frac{X_n - \Delta_n S_n}{(1+r)^n}\right]$$

linearity $\longrightarrow$ $$= E_n\left[\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}}\right] + E_n\left[\frac{X_n - \Delta_n S_n}{(1+r)^n}\right]$$

taking out what is known $\longrightarrow$ $$= \Delta_n E_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] + \frac{X_n - \Delta_n S_n}{(1+r)^n}$$

since the discounted stock price is a martingale under the risk-neutral measure $\longrightarrow$ $$= \Delta_n \frac{S_n}{(1+r)^n} + \frac{X_n - \Delta_n S_n}{(1+r)^n}$$

$$= \frac{X_n}{(1+r)^n} \qquad \blacksquare$$