Machine Learning Homework 2

Written by: Lior Shacohach - id - 206284960
Moriya Barel - id - 207304353

התרומה של כל אחד: עבדנו ביחד על כל השאלות.

Question 1.A

Let S be point-set of size n, $\Pi(C, S)$ be all possible labelings that C assigns to S and $\Pi(D, S)$ be all possible labelings that D assigns to S.
Notice that all possible labelings for n points of class E can be bounded by:
$\Pi(E, S) = \Pi(C \cup D,\ S) \leq \Pi(C, S) + \Pi(D, S)$.

By Sauer's lemma $\Pi(C, S) \leq \sum_{i=0}^{c} \binom{n}{i}$ and $\Pi(D, S) \leq \sum_{i=0}^{d} \binom{n}{i}$ therefore:

$\Pi(E, S) \leq \sum_{i=0}^{c} \binom{n}{i} + \sum_{i=0}^{d} \binom{n}{i}$.

Using the identity $\binom{n}{i} = \binom{n}{n-i} \rightarrow \Pi(E, S) \leq \sum_{i=0}^{c} \binom{n}{i} + \sum_{i=0}^{d} \binom{n}{n-i}$

Change second sigma variable $\rightarrow \Pi(E, S) \leq \sum_{i=0}^{c} \binom{n}{i} + \sum_{i=n-d}^{n} \binom{n}{i}$

If $n = c + d + 2$ then:

$$\Pi(E, S) \leq \sum_{i=0}^{c} \binom{c+d+2}{i} + \sum_{i=c+2}^{c+d+2} \binom{c+d+2}{i} = \sum_{i=0}^{c+d+2} \binom{c+d+2}{i} - \binom{c+d+2}{d+1} = 2^{c+d+2} - \binom{c+d+2}{d+1} < 2^{c+d+2}$$

Overall $\Pi(E, S) < 2^n$ consequently, class E cannot shatter $c + d + 2$ points.

Question 1.B

The VC-dimension of unidirectional balls on d-dimensional points is d+1.
Lower bound:
It can shatter d+1 points in R^d.
Similar to the proof we've seen in class of d-1 hyperplanes on d-dimensional points, we can take the base vectors and the zero vector and circle any set of the vectors without intersection with the points out of the set.

Upper bound:
It cannot shatter d+2 points in R^d.
Let S be a point set of size d+2, by Radon's theorem S can be partitioned into two disjoint sets A, B whose convex hulls intersect.
Any ball containing set A must also contain points from set B and any ball containing set B must contain points from set A ⟶ a ball cannot shatter d+2 in R^d.

Question 1.C

Let class C be the class of infinite uni-directional circles that labels points inside as +.

Let class D be the class of infinite uni-directional circles that labels points inside as -.

Let class E be the class of infinite bi-directional circles.

Class E is the union of both C and D, explanation - for any labeling achievable by a unidirectional circle we get the opposite labeling by switching the direction which is the exact definition of bidirectional.

Using the bound from 1A we get: $VC_{dim}(E) = VC_{dim}(C \cup D) \leq VC_{dim}(C) + VC_{dim}(D) + 1 =$

$= 3 + 3 + 1 = 7.$

<u>Question 2</u>

By the union and intersection of hypothesis bound - $2ds * log_2(3s)$:

Hearts:

We can see that hearts are built from 2 lines and 2 circles.

$VC_{dim}(Hearts) \leq 2 * 3 * 4 * log_2(3 * 4) = 86.039... \rightarrow VC_{dim}(Hearts) \leq 86$

Clubs:

We can see that clubs are built from the intersection of 3 lines and the union of 3 circles.

$VC_{dim}(Clubs) \leq 2 * 3 * 6 * log_2(3 * 6) = 150.1... \rightarrow VC_{dim}(Clubs) \leq 150$

Diamonds:

We can see that diamonds are built from the intersection of 4 lines.

$VC_{dim}(Hearts) \leq 2 * 3 * 4 * log_2(3 * 4) = 86.039... \rightarrow VC_{dim}(Hearts) \leq 86$

Overall for class C:

$VC_{dim}(C) = VC_{dim}(Hearts \cup Clubs \cup Diamonds) \leq VC_{dim}(Hearts) + VC_{dim}(Clubs) + VC_{dim}(Diamonds) + 2$

$VC_{dim}(C) \leq 86 + 150 + 86 + 2 = 324$

Question 3

Class $C' = \bigcup_{i=0}^{s} (\{Union\ of\ s - i\ objects\ and\ intersection\ of\ i\ objects\ from\ class\ C\})$

Explanation: C_0 is only union of objects, C_1 is all objects that consists of s-1 unions and 1 intersection,

C_2 is all objects of s-2 unions and 2 intersections $\ldots$ C_s is all objects that consist of only intersections.

$VC_{dim}(C_0) = 2ds * log_2(3s)$ // only unions.

$VC_{dim}(C_1) = 2ds * log_2(3s)$ // s-1 unions and 1 intersection (s-1 + 1 = s).

$VC_{dim}(C_2) = 2ds * log_2(3s)$ // s-2 unions and 2 intersections (s-2 + 2 = s).

$\ldots$

$VC_{dim}(C_s) = 2ds * log_2(3s)$ // only intersections.

Overall: $VC_{dim}(C`) = \bigcup_{i=0}^{s} (C_i) = [(s + 1) * 2ds * log_2(3s)] + s$

Question 4

Let S be a point sample of size n.

For any $S` \subseteq S$, there has to be a convex body that labels $S`$ differently then $S \backslash S`$ (because we have an infinite amount of convex bodies of any convex shape).

This means we can achieve all 2^n possible labelings for S -> Since S is arbitrary, the set of infinite convex bodies can shatter any given set of points -> The VC dimension of our set is **infinity**.