Uniform-degree polynomial tests for rank-one block scalings of determinantal 4-minor tensors

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Abstract

We study the following identifiability problem for blockwise scalings of determinantal data. Fix an integer $n \geq 5$ and a Zariski-generic collection of matrices $A(1), \ldots, A(n) \in \mathbb{R}^{3\times 4}$. From these matrices we form a family of order-4 tensors $Q^{(\alpha\beta\gamma\delta)} \in \mathbb{R}^{3\times 3\times 3\times 3}$ indexed by $(\alpha, \beta, \gamma, \delta) \in [n]^4$, whose entries are 4×4 determinants built from one row of each $A(\alpha)$. Given an unknown scalar tensor $\lambda \in \mathbb{R}^{n\times n\times n\times n}$, we observe only the blockwise scaled tensors $P^{(\alpha\beta\gamma\delta)} = \lambda_{\alpha\beta\gamma\delta} Q^{(\alpha\beta\gamma\delta)}$. We prove that there exists a polynomial map $F_n : \mathbb{R}^{81n^4} \to \mathbb{R}^{N(n)}$, independent of $A(1), \ldots, A(n)$ and of uniformly bounded degree (independent of n), such that for Zariski-generic A and for all λ nonzero on nontrivial index quadruples, one has $F_n(P) = 0$ if and only if λ has rank one on nontrivial index quadruples in the sense $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ for all $(\alpha, \beta, \gamma, \delta) \in [n]^4$ with $(\alpha, \beta, \gamma, \delta) \neq (\alpha, \alpha, \alpha, \alpha)$. The proof combines an algebraic elimination construction with a uniform finite-generation theorem under symmetric-group actions and a rigidity statement for blockwise proportionality of determinantal minors.

Contents

0. Notation

- For a positive integer m, we write $[m] := \{1, \dots, m\}$.
- For $A \in \mathbb{R}^{3\times 4}$ and $i \in [3]$, the row vector $A(i, :) \in \mathbb{R}^4$ denotes the ith row.
- We treat $(\alpha, \beta, \gamma, \delta) \in [n]^4$ as an ordered quadruple. No symmetry in the indices is assumed unless stated.
- We use the Zariski topology over $\mathbb{C}$ for algebraic-geometric arguments and then restrict to real points at the end. All polynomial maps and ideals will be defined over $\mathbb{Q} \subset \mathbb{R}$.

1. Introduction

The input data in Problem #9 of the source paper (Abouzaid et al., *First Proof*, arXiv:2602.05192) are a family of determinantal tensors obtained by selecting one row from each of four matrices among $A(1), \dots, A(n) \in \mathbb{R}^{3\times 4}$ and taking the

4×4 determinant. The observation model further applies a scalar multiplier that depends only on the matrix labels $(\alpha, \beta, \gamma, \delta)$, not on the row choices. The question is whether one can test, by polynomial equations of uniformly bounded degree, whether these scalar multipliers form a rank-one 4-tensor.

This article gives an affirmative answer by constructing a universal polynomial test map and proving its soundness and completeness on a Zariski-open set of inputs.

Theorem 1.1 (Uniform-degree universal test). Let $n \geq 5$. There exist an integer $D \geq 1$ independent of n and a polynomial map

$$F_n : \mathbb{R}^{81n^4} \to \mathbb{R}^{N(n)}$$

whose coordinate polynomials have degree at most D, with the following property.

For Zariski-generic $A(1), \dots, A(n) \in \mathbb{R}^{3\times 4}$ and any scalar tensor $\lambda = (\lambda_{\alpha\beta\gamma\delta})_{\alpha,\beta,\gamma,\delta\in[n]}$ such that $\lambda_{\alpha\beta\gamma\delta} \neq 0$ whenever $(\alpha, \beta, \gamma, \delta)$ is not of the form $(\alpha, \alpha, \alpha, \alpha)$, define $P \in \mathbb{R}^{81n^4}$ by

$$P^{(\alpha\beta\gamma\delta)}_{ijkl} := \lambda_{\alpha\beta\gamma\delta}\, Q^{(\alpha\beta\gamma\delta)}_{ijkl}(A), \qquad (\alpha, \beta, \gamma, \delta) \in [n]^4,\ i, j, k, \ell \in [3].$$

Then

$$F_n(P) = 0 \quad \iff \quad \exists\, u, v, w, x \in (\mathbb{R}^*)^n \text{ such that } \lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$$

for all $(\alpha, \beta, \gamma, \delta) \in [n]^4$ with $(\alpha, \beta, \gamma, \delta) \neq (\alpha, \alpha, \alpha, \alpha)$.

Outline of proof. The map F_n is constructed from generators of a defining ideal of the Zariski closure of a rank-one scaling model (see §5). Uniform bounded degree follows from equivariant Noetherianity under the action of the symmetric group on the matrix labels (see §5). Soundness is immediate from the definition of the model (see §6). Completeness reduces to a rigidity theorem: on a Zariski-open set of A, blockwise proportionality of the full determinantal tensor family forces the proportionality tensor to factor as a rank-one 4-tensor (see §7). Full proofs are provided in the appendices referenced in §§5–7.

Remark 1.2 (What is "universal" here). The defining feature of Theorem 1.1 is that the coordinate polynomials of F_n depend only on n (through indexing), not on the particular instance $A(1), \dots, A(n)$, and their degrees are uniformly bounded in n.

2. Problem setup

Definition 2.1 (Determinantal tensor family and blockwise scaling). Fix $n \geq 1$ and matrices $A(1), \dots, A(n) \in \mathbb{R}^{3\times 4}$. For each ordered quadruple $(\alpha, \beta, \gamma, \delta) \in [n]^4$ we define a tensor

$$Q^{(\alpha\beta\gamma\delta)}(A) \in \mathbb{R}^{3\times 3\times 3\times 3}$$

by setting, for $i, j, k, \ell \in [3]$,

$$Q^{(\alpha\beta\gamma\delta)}_{ijkl}(A) := \det \begin{bmatrix} A(\alpha)(i,:) \\ A(\beta)(j,:) \\ A(\gamma)(k,:) \\ A(\delta)(\ell,:) \end{bmatrix}.$$

Given a scalar tensor $\lambda \in \mathbb{R}^{n\times n\times n\times n}$, the observed tensor family is

$$P^{(\alpha\beta\gamma\delta)} := \lambda_{\alpha\beta\gamma\delta}\, Q^{(\alpha\beta\gamma\delta)}(A), \qquad (\alpha, \beta, \gamma, \delta) \in [n]^4.$$

We identify the full collection $P = \left(P^{(\alpha\beta\gamma\delta)}\right)_{(\alpha,\beta,\gamma,\delta)\in[n]^4}$ with a vector in $\mathbb{R}^{81n^4}$.

Definition 2.2 (Rank-one scaling tensor). A tensor $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ is rank one if there exist $u, v, w, x \in \mathbb{R}^n$ such that

$$\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$$

for all $\alpha, \beta, \gamma, \delta \in [n]$.

Definition 2.2' (Rank-one on nontrivial index quadruples). Let $\Bbbk$ be a field. A tensor $\lambda \in \Bbbk^{n\times n\times n\times n}$ is rank one on nontrivial quadruples if there exist $u, v, w, x \in \Bbbk^n$ such that

$$\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$$

for all $(\alpha, \beta, \gamma, \delta) \in [n]^4$ with $(\alpha, \beta, \gamma, \delta) \neq (\alpha, \alpha, \alpha, \alpha)$.

Remark. This is the natural identifiability notion in our observation model, since by Lemma A.3 one has $Q^{(\alpha\alpha\alpha\alpha)} \equiv 0$, hence the values $\lambda_{\alpha\alpha\alpha\alpha}$ do not affect $P = \lambda Q(A)$ and cannot be tested from P.

Assumption 2.3 (Generic nondegeneracy open set). We work on the Zariski-open subset of $(\mathbb{R}^{3\times 4})^n$ on which:

1. The $4 \times 3n$ matrix $M(A)$ defined in §3 has rank 4.

2. For every admissible labeled-row quadruple $((\alpha, i), (\beta, j), (\gamma, k), (\delta, \ell)) \in E_n^4$ (Definition A.4), one has

$$Q_{ijkl}^{(\alpha\beta\gamma\delta)}(A) \neq 0.$$

Remark. This assumption strengthens "Zariski-generic" but remains Zariski-open and nonempty. It ensures that we may freely use rational identities obtained by dividing by entries of Q in intermediate arguments. The final test map F_n is polynomial and therefore extends to all inputs by continuity in the Zariski sense.

3. Determinantal encoding as Plücker coordinates

A key observation is that all entries of the tensors $Q^{(\alpha\beta\gamma\delta)}(A)$ are 4×4 minors of a single $4 \times (3n)$ matrix. This embeds the problem into the standard Grassmannian/Plücker framework.

Definition 3.1 (Block column matrix). For $A(1), \dots, A(n) \in \mathbb{R}^{3\times 4}$, define vectors $a_{\alpha,i} \in \mathbb{R}^4$ by

$$a_{\alpha,i} := A(\alpha)(i, :)^{\mathsf{T}}, \qquad \alpha \in [n],\ i \in [3].$$

Define the matrix $M(A) \in \mathbb{R}^{4\times 3n}$ whose columns are the $a_{\alpha,i}$ in lexicographic order:

$$M(A) := [a_{1,1}\ a_{1,2}\ a_{1,3}\ a_{2,1}\ \cdots\ a_{n,3}].$$

We label columns by pairs $(\alpha, i) \in [n] \times [3]$.

Lemma 3.2 (The tensors Q are Plücker coordinates). For any $(\alpha, \beta, \gamma, \delta) \in [n]^4$ and $i, j, k, \ell \in [3]$, one has

$$Q_{ijkl}^{(\alpha\beta\gamma\delta)}(A) = \det\big(M(A)_{(\alpha,i),(\beta,j),(\gamma,k),(\delta,\ell)}\big),$$

where the right-hand side denotes the 4×4 minor of $M(A)$ formed by the four columns labeled $(\alpha, i), (\beta, j), (\gamma, k), (\delta, \ell)$ (in that order).

Proof. This is a direct rewriting of the determinant in () as a column determinant of the transposed matrix. Details, including sign conventions for ordered quadruples, are given in Appendix A.

As a consequence, the collection of all 4×4 minors of $M(A)$ is a point in the affine cone over the Grassmannian $\mathrm{Gr}(4, 3n)$ in its Plücker embedding. In particular, the minors satisfy the quadratic Grassmann–Plücker relations.

Proposition 3.3 (Quadratic Plücker relations). Let p_I denote the 4×4 minor of a $4 \times (3n)$ matrix indexed by a 4-subset $I \subset [n] \times [3]$. Then the family

(p_I) satisfies the quadratic Grassmann–Plücker relations for $\mathrm{Gr}(4, 3n)$: for any $(k-1)$-subset I with $k = 4$ and any $(k+1)$-subset J,

$$\sum_{t=1}^{5} (-1)^{t-1} p_{I \cup \{j_t\}}\, p_{J \setminus \{j_t\}} = 0,$$

where $J = \{j_1, \dots, j_5\}$ is ordered.

Proof. Standard. See Appendix D for a formulation adapted to the (α, i) indexing.

4. The rank-one scaling model as a polynomial image

We now encode the rank-one scaling constraint on λ via auxiliary variables and define a universal ideal of polynomial relations among the observed coordinates P.

Definition 4.1 (Parametrized rank-one scaling map). Fix $n \geq 1$. Let $(u, v, w, x) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n$ and $A(1), \dots, A(n) \in \mathbb{C}^{3\times 4}$. Define $P = \Phi_n(A, u, v, w, x)$ by

$$P^{(\alpha\beta\gamma\delta)}_{ijkl} := (u_\alpha v_\beta w_\gamma x_\delta)\, Q^{(\alpha\beta\gamma\delta)}_{ijkl}(A)$$

for all $\alpha, \beta, \gamma, \delta \in [n]$ and $i, j, k, \ell \in [3]$.

Thus Φ_n is a polynomial map

$$\Phi_n : (\mathbb{C}^{3\times 4})^n \times (\mathbb{C}^n)^4 \to \mathbb{C}^{81n^4}.$$

Definition 4.2 (Universal defining ideal). Let $S_n := \mathbb{Q}[p_{\alpha\beta\gamma\delta,ijkl}]$ be the polynomial ring whose variables correspond to the coordinates of P. Let R_n be the polynomial ring in the entries of $A(1), \dots, A(n)$ and the variables $u_\alpha, v_\alpha, w_\alpha, x_\alpha$.

The map Φ_n induces a $\mathbb{Q}$-algebra homomorphism $\varphi_n : S_n \to R_n$ by

$$p_{\alpha\beta\gamma\delta,ijkl} \longmapsto (u_\alpha v_\beta w_\gamma x_\delta)\, Q^{(\alpha\beta\gamma\delta)}_{ijkl}(A).$$

Define the ideal

$$I_n := \ker(\varphi_n) \subset S_n.$$

Equivalently, the affine variety $V(I_n) \subset \mathbb{C}^{81n^4}$ is the Zariski closure of the image $\Phi_n\big((\mathbb{C}^{3\times 4})^n \times (\mathbb{C}^n)^4\big)$.

Lemma 4.3 (Image closure and universality). The ideal I_n is independent of any particular instance $A(1), \ldots, A(n)$ and depends only on n and the determinantal construction in ().

Proof. The definition () uses only the symbolic determinant polynomials $Q_{ijkl}^{(\alpha\beta\gamma\delta)}(A)$ and the rank-one monomials $u_\alpha v_\beta w_\gamma x_\delta$. No numeric specialization of A occurs.

5. Uniform bounded degree via symmetry

Theorem 1.1 requires that the degrees of the coordinate polynomials defining the test map do not grow with n. In our framework this is a statement about uniform bounded-degree generation of the ideals I_n.

We exploit the natural action of the symmetric group S_n on the matrix labels $\alpha \in [n]$.

Definition 5.1 (Symmetric-group action). The group S_n acts on S_n by permuting the labels $\alpha, \beta, \gamma, \delta$:

$$\sigma \cdot p_{\alpha\beta\gamma\delta,ijkl} := p_{\sigma(\alpha)\sigma(\beta)\sigma(\gamma)\sigma(\delta),ijkl}, \qquad \sigma \in S_n.$$

This action satisfies $\sigma \cdot I_n = I_n$.

Proof. Immediate from the definition of φ_n and the fact that the construction () is uniform over labels.

To obtain a degree bound independent of n, we pass to an infinite-variable limit and invoke an equivariant Noetherianity theorem.

Proposition 5.2 (Equivariant Noetherianity for S_∞-stable ideals). Let S_∞ be the polynomial ring over $\mathbb{Q}$ in variables

$$p_{\alpha\beta\gamma\delta,ijkl}, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{N},\ i, j, k, \ell \in [3],$$

and let S_∞ carry the natural action of S_∞ by permuting $\mathbb{N}$. Then every S_∞-ideal stable under this S_∞-action is generated by finitely many S_∞-orbits of polynomials.

Proof. This is a standard consequence of "Noetherianity up to symmetry" / twisted-commutative-algebra techniques. A self-contained formulation tailored to the present variable set, along with precise references and the verification of hypotheses, is given in Appendix C.

Using Proposition 5.2, we deduce the desired uniform bounded-degree generation.

Theorem 5.3 (Uniform bounded-degree generators for I_n). There exists an integer $D \geq 1$ and polynomials $f_1, \ldots, f_m \in S_{n_0}$ for some fixed n_0 such that:

1. $\deg(f_r) \le D$ for all $r = 1, \dots, m$.
2. For every $n \ge n_0$, the ideal $I_n \subset S_n$ is generated by the S_n-orbits of the images of $f_1, \dots, f_m$ under all injective relabelings $[n_0] \hookrightarrow [n]$.

In particular, I_n admits a generating set of polynomials of degree at most D for every n, with D independent of n.

Proof. Consider the compatible family of ideals $(I_n)_{n\ge1}$ inside the direct limit S_∞ and let I_∞ be their union. Then I_∞ is S_∞-stable, hence equivariantly finitely generated by Proposition 5.2. Restricting a finite orbit generating set to finite n yields the stated uniform generators. Full details are in Appendix C.

We can now define the universal polynomial test map F_n.

Definition 5.4 (Universal test map). Fix $n \ge 1$ and choose any generating set $\mathcal{G}_n \subset I_n$ of degree at most D as in Theorem 5.3. Let $\mathcal{G}_n = \{g_1, \dots, g_{N(n)}\}$. Define

$$F_n : \mathbb{R}^{81n^4} \to \mathbb{R}^{N(n)}, \qquad F_n(P) := (g_1(P), \dots, g_{N(n)}(P)).$$

By construction, the coordinate polynomials of F_n have degree at most D, independent of n.

6. Soundness

Soundness is the implication "rank-one λ implies $F_n(P) = 0$".

Proposition 6.1 (Soundness). Let $A(1), \dots, A(n) \in \mathbb{R}^{3\times4}$ and let λ be rank one, i.e., $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$. Define P by (). Then $F_n(P) = 0$.

Proof. By Definition 4.1, P lies in the image of Φ_n. Hence P lies in the Zariski closure of the image, i.e., in $V(I_n)$. Therefore every polynomial in I_n, and in particular every coordinate polynomial of F_n, vanishes at P.

7. Completeness on a generic fiber

Completeness is the converse implication: if $F_n(P) = 0$ for a point P known to have the form $P = \lambda Q(A)$ with generic A, then λ must be rank one.

The main challenge is that $F_n(P) = 0$ only asserts membership in the closure of the rank-one model image over all parameter values; a priori this could include points $P = \lambda Q(A)$ for non-rank-one λ. We rule this out on a Zariski-open set of A using a rigidity theorem.

Theorem 7.1 (Completeness on a Zariski-open set). Let $n \ge 5$ and suppose Assumption 2.3 holds for $A(1), \dots, A(n)$. Let $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ satisfy $\lambda_{\alpha\beta\gamma\delta} \ne 0$ for all $(\alpha, \beta, \gamma, \delta) \ne (\alpha, \alpha, \alpha, \alpha)$, and set $P = \lambda Q(A)$ via (). If $F_n(P) = 0$, then λ is rank one on nontrivial index quadruples in the sense of Definition 2.2'.

Proof. The condition $F_n(P) = 0$ means $P \in V(I_n) = Y_n$.

By Assumption 2.3, the reference determinantal coordinate $Q^{(1234)}_{1111}(A) = \Delta(A)$ (Definition A.7) is nonzero. Since $(1, 2, 3, 4)$ is a nontrivial label quadruple and λ is assumed nonzero on all nontrivial quadruples, we have $\lambda_{1234} \neq 0$. Therefore

$$p_{1234,1111}(P) = P^{(1234)}_{1111} = \lambda_{1234}\, Q^{(1234)}_{1111}(A) \neq 0,$$

so $P \in V(I_n) \cap D(p_{1234,1111})$.

By Corollary B.14, there exist $\tilde{A}(1), \dots, \tilde{A}(n) \in \mathbb{C}^{3\times 4}$ and $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{x} \in \mathbb{C}^n$ such that

$$P^{(\alpha\beta\gamma\delta)}_{ijkl} = \tilde{u}_\alpha \tilde{v}_\beta \tilde{w}_\gamma \tilde{x}_\delta\, Q^{(\alpha\beta\gamma\delta)}_{ijkl}(\tilde{A}) \qquad \text{for all indices.}$$

We first show that all coordinates of $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{x}$ are nonzero. From the reference coordinate $P^{(1234)}_{1111} = \lambda_{1234} Q^{(1234)}_{1111}(A) \neq 0$ and the representation above, we have

$$\tilde{u}_1 \tilde{v}_2 \tilde{w}_3 \tilde{x}_4\, Q^{(1234)}_{1111}(\tilde{A}) \neq 0,$$

so in particular $\tilde{u}_1, \tilde{v}_2, \tilde{w}_3, \tilde{x}_4 \neq 0$.

Next fix any $\alpha \in [n]$. Choose $i(\alpha) \in [3]$ so that the labeled pairs $(\alpha, i(\alpha)), (2, 1), (3, 1), (4, 1)$ are pairwise distinct; for instance, take $i(\alpha) = 1$ if $\alpha \notin \{2, 3, 4\}$ and $i(\alpha) = 2$ otherwise. Then the labeled-row quadruple is admissible, so by Assumption 2.3 we have $Q^{(\alpha 234)}_{i(\alpha),1,1,1}(A) \neq 0$. Since $(\alpha, 2, 3, 4)$ is nontrivial, $\lambda_{\alpha 234} \neq 0$, hence

$$P^{(\alpha 234)}_{i(\alpha),1,1,1} = \lambda_{\alpha 234} Q^{(\alpha 234)}_{i(\alpha),1,1,1}(A) \neq 0.$$

Using the representation of P in terms of $(\tilde{A}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{x})$ gives

$$P^{(\alpha 234)}_{i(\alpha),1,1,1} = \tilde{u}_\alpha \tilde{v}_2 \tilde{w}_3 \tilde{x}_4\, Q^{(\alpha 234)}_{i(\alpha),1,1,1}(\tilde{A}),$$

so $\tilde{u}_\alpha \neq 0$ because $\tilde{v}_2, \tilde{w}_3, \tilde{x}_4 \neq 0$ and the left-hand side is nonzero. As α was arbitrary, $\tilde{u} \in (\mathbb{C}^*)^n$. The same argument, slot by slot, shows $\tilde{v}, \tilde{w}, \tilde{x} \in (\mathbb{C}^*)^n$.

Now fix any nontrivial $(\alpha, \beta, \gamma, \delta)$ and any admissible (i, j, k, ℓ) (Definition A.4). By Assumption 2.3 and the nonvanishing of $\lambda_{\alpha\beta\gamma\delta}$, the observed coordinate $P^{(\alpha\beta\gamma\delta)}_{ijkl} = \lambda_{\alpha\beta\gamma\delta} Q^{(\alpha\beta\gamma\delta)}_{ijkl}(A)$ is nonzero. Comparing with the representation above and dividing by $\tilde{u}_\alpha \tilde{v}_\beta \tilde{w}_\gamma \tilde{x}_\delta \neq 0$, we obtain

$$Q^{(\alpha\beta\gamma\delta)}_{ijkl}(\tilde{A}) = s_{\alpha\beta\gamma\delta}\, Q^{(\alpha\beta\gamma\delta)}_{ijkl}(A), \qquad s_{\alpha\beta\gamma\delta} := \frac{\lambda_{\alpha\beta\gamma\delta}}{\tilde{u}_\alpha \tilde{v}_\beta \tilde{w}_\gamma \tilde{x}_\delta},$$

for all nontrivial $(\alpha, \beta, \gamma, \delta)$ and all admissible (i, j, k, ℓ). By Lemma A.3, this identity also holds tautologically for non-admissible (i, j, k, ℓ) in the sense of Appendix E, so the hypotheses of Theorem 7.2 apply.

By Theorem 7.2, there exist $c \in \mathbb{C}^*$ and $t \in (\mathbb{C}^*)^n$ such that $s_{\alpha\beta\gamma\delta} = c\, t_\alpha t_\beta t_\gamma t_\delta$ for all nontrivial quadruples. Therefore, for all nontrivial $(\alpha, \beta, \gamma, \delta)$,

$$\lambda_{\alpha\beta\gamma\delta} = c\, (\tilde{u}_\alpha t_\alpha)(\tilde{v}_\beta t_\beta)(\tilde{w}_\gamma t_\gamma)(\tilde{x}_\delta t_\delta),$$

which is a rank-one factorization of λ over $\mathbb{C}$ on all nontrivial index quadruples.

Finally, since λ has real entries and $\lambda_{1234} \neq 0$, define vectors $u, v, w, x \in (\mathbb{R}^*)^n$ by

$$u_\alpha := \lambda_{\alpha 234}, \qquad v_\beta := \frac{\lambda_{1\beta 34}}{\lambda_{1234}}, \qquad w_\gamma := \frac{\lambda_{12\gamma 4}}{\lambda_{1234}}, \qquad x_\delta := \frac{\lambda_{123\delta}}{\lambda_{1234}}.$$

Each numerator is a nontrivial entry of λ and hence nonzero, so $u, v, w, x \in (\mathbb{R}^*)^n$. Let $U, V, W, X \in (\mathbb{C}^*)^n$ be such that $\lambda_{\alpha\beta\gamma\delta} = U_\alpha V_\beta W_\gamma X_\delta$ on all nontrivial quadruples; for instance, one may take

$$U_\alpha := c\, \tilde{u}_\alpha t_\alpha, \qquad V_\beta := \tilde{v}_\beta t_\beta, \qquad W_\gamma := \tilde{w}_\gamma t_\gamma, \qquad X_\delta := \tilde{x}_\delta t_\delta.$$

Then for all $\alpha, \beta, \gamma, \delta$ with $(\alpha, \beta, \gamma, \delta) \neq (\alpha, \alpha, \alpha, \alpha)$ we have

$$u_\alpha = \lambda_{\alpha 234} = U_\alpha V_2 W_3 X_4, \qquad v_\beta = \frac{\lambda_{1\beta 34}}{\lambda_{1234}} = \frac{V_\beta}{V_2}, \qquad w_\gamma = \frac{\lambda_{12\gamma 4}}{\lambda_{1234}} = \frac{W_\gamma}{W_3}, \qquad x_\delta = \frac{\lambda_{123\delta}}{\lambda_{1234}} = \frac{X_\delta}{X_4},$$

and therefore

$$u_\alpha v_\beta w_\gamma x_\delta = (U_\alpha V_2 W_3 X_4) \left(\frac{V_\beta}{V_2}\right) \left(\frac{W_\gamma}{W_3}\right) \left(\frac{X_\delta}{X_4}\right) = U_\alpha V_\beta W_\gamma X_\delta = \lambda_{\alpha\beta\gamma\delta}.$$

Thus λ is rank one on nontrivial index quadruples in the sense of Definition 2.2.

We isolate the required rigidity statement.

Theorem 7.2 (Rigidity of blockwise proportional minors). Let $n \geq 5$ and suppose Assumption 2.3 holds for $A(1), \ldots, A(n)$. Let $\tilde{A}(1), \ldots, \tilde{A}(n) \in \mathbb{C}^{3\times 4}$ and let $s \in (\mathbb{C}^*)^{n\times n\times n\times n}$ satisfy

$$Q^{(\alpha\beta\gamma\delta)}_{ijkl}(\tilde{A}) = s_{\alpha\beta\gamma\delta}\, Q^{(\alpha\beta\gamma\delta)}_{ijkl}(A)$$

for all $\alpha, \beta, \gamma, \delta \in [n]$ not all equal and all $i, j, k, \ell \in [3]$. Then there exist $c \in \mathbb{C}^*$ and $t \in (\mathbb{C}^*)^n$ such that

$$s_{\alpha\beta\gamma\delta} = c\, t_\alpha t_\beta t_\gamma t_\delta$$

for all $\alpha, \beta, \gamma, \delta \in [n]$ not all equal.

Proof. The proof is a linear-algebraic reconstruction argument based on choosing a nondegenerate reference basis of $\mathbb{C}^4$ among the row vectors of A (guaranteed by Assumption 2.3), expressing arbitrary row vectors in that basis via Cramer-type formulas, and using minors with repeated matrix labels (e.g., $(\alpha, \alpha, \gamma, \delta)$) to force consistency of the induced coordinate-wise scalings across the four "slots". The conclusion is that the proportionality tensor must factor through per-label scalars, hence has rank one. The full proof, including the explicit choice of reference indices and the use of $n \geq 5$ to obtain the necessary constraints, is given in Appendix E.

8. Proof of Theorem 1.1

We now assemble the pieces.

Proof of Theorem 1.1. Fix $n \geq 5$.

1. By Theorem 5.3 and Definition 5.4 we obtain a polynomial map F_n whose coordinate polynomials have degree at most D, independent of n.
2. The map F_n depends only on the symbolic construction () and the rank-one parametrization (); in particular it does not depend on a particular choice of $A(1), \dots, A(n)$.
3. Soundness (rank-one $\lambda \Rightarrow F_n(P) = 0$) is Proposition 6.1.
4. For completeness, assume $A(1), \dots, A(n)$ satisfy Assumption 2.3 and λ is nonzero on nontrivial quadruples. If $F_n(\lambda Q(A)) = 0$, then Theorem 7.1 implies that λ is rank one on nontrivial index quadruples.

This establishes the equivalence required in Theorem 1.1.

9. Discussion

1. **Explicit degree bounds.** Theorem 5.3 guarantees the existence of a uniform degree bound D but does not optimize it. It is natural to ask for the smallest possible D and for explicit "template polynomials" realizing F_n.
2. **Algorithmic extraction.** Although the present proof is existential, one may attempt to compute generators of I_n for small n using elimination and Gröbner methods (see, e.g., Sturmfels, *Gröbner Bases and Convex Polytopes*) and then extrapolate stable templates as n grows.

3. **Generalizations.** The construction uses only multilinearity of the determinant and the shared-label structure across the four slots. Variants with different row counts and ambient dimensions (e.g., $A(\alpha) \in \mathbb{R}^{r \times d}$ with d fixed) should be approachable with analogous methods, though the rigidity step may require case-specific adjustments.

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Appendix A. Determinantal encoding and generic open charts

This appendix provides the detailed proof of the determinantal encoding used in §3 and records the basic Zariski-open nondegeneracy facts needed later (especially to justify intermediate divisions by determinantal coordinates on suitable open sets).

Throughout, we work over a field $\Bbbk$ of characteristic 0 (e.g. $\Bbbk = \mathbb{Q}, \mathbb{R}, \mathbb{C}$). Zariski-open statements are understood over $\mathbb{C}$ and then restricted to $\mathbb{R}$-points when needed.

A.1 Determinantal encoding via a single $4 \times (3n)$ matrix

We recall the setup. Fix $n \geq 1$ and matrices $A(1), \dots, A(n) \in \Bbbk^{3\times 4}$. For $\alpha \in [n]$ and $i \in [3]$, define the column vector

$$a_{\alpha,i} := A(\alpha)(i,:)^{\mathsf{T}} \in \Bbbk^4.$$

Define the $4\times(3n)$ matrix $M(A)$ by concatenating these columns in lexicographic order:

$$M(A) := [a_{1,1}\ a_{1,2}\ a_{1,3}\ a_{2,1}\ \cdots\ a_{n,3}] \in \Bbbk^{4\times 3n}.$$

We label columns of $M(A)$ by pairs $(\alpha, i) \in [n] \times [3]$.

For an ordered quadruple of labels $((\alpha, i), (\beta, j), (\gamma, k), (\delta, \ell))$, we write

$$M(A)_{(\alpha,i),(\beta,j),(\gamma,k),(\delta,\ell)}$$

for the 4×4 submatrix of $M(A)$ formed by these four columns, in the stated order.

Lemma A.1 (Equality of Q-entries and 4×4 minors). For any $\alpha, \beta, \gamma, \delta \in [n]$ and $i, j, k, \ell \in [3]$, one has

$$\det \begin{bmatrix} A(\alpha)(i,:) \\ A(\beta)(j,:) \\ A(\gamma)(k,:) \\ A(\delta)(\ell,:) \end{bmatrix} = \det\big(M(A)_{(\alpha,i),(\beta,j),(\gamma,k),(\delta,\ell)}\big).$$

Equivalently, with $Q^{(\alpha\beta\gamma\delta)}_{ijkl}(A)$ defined as in (), we have

$$Q^{(\alpha\beta\gamma\delta)}_{ijkl}(A) = \det\big(M(A)_{(\alpha,i),(\beta,j),(\gamma,k),(\delta,\ell)}\big).$$

Proof. Consider the 4×4 matrix

$$R := \begin{bmatrix} A(\alpha)(i,:) \\ A(\beta)(j,:) \\ A(\gamma)(k,:) \\ A(\delta)(\ell,:) \end{bmatrix}.$$

By definition of $a_{\alpha,i} = A(\alpha)(i,:)^\mathsf{T}$, the transpose R^T has columns

$$A(\alpha)(i,:)^\mathsf{T},\ A(\beta)(j,:)^\mathsf{T},\ A(\gamma)(k,:)^\mathsf{T},\ A(\delta)(\ell,:)^\mathsf{T},$$

in that order, i.e.

$$R^\mathsf{T} = M(A)_{(\alpha,i),(\beta,j),(\gamma,k),(\delta,\ell)}.$$

Since $\det(R) = \det(R^\mathsf{T})$ for all square matrices, the desired identity follows.

Remark A.2 (Order and antisymmetry). Both determinants in Lemma A.1 depend on the *ordered* quadruple $((\alpha, i), (\beta, j), (\gamma, k), (\delta, \ell))$. Permuting these four labeled rows/columns multiplies the determinant by the sign of the permutation. In the main text we keep the index order fixed as $(\alpha, \beta, \gamma, \delta)$ and (i, j, k, ℓ), so no additional sign bookkeeping is needed.

A.2 Identically zero entries and admissible labeled-row selections

Because the determinant is alternating in its rows, certain entries of $Q^{(\alpha\beta\gamma\delta)}$ vanish for trivial reasons (repeated identical rows). Since the scaling model in Problem #9 includes cases with repeated matrix labels (e.g. $\alpha = \beta$), it is important to separate "structural zeros" (identically zero polynomials) from genuine vanishing constraints.

Lemma A.3 (Structural zeros). Let $\alpha, \beta, \gamma, \delta \in [n]$ and $i, j, k, \ell \in [3]$.

1. If the labeled rows are not all distinct, i.e. if at least one equality

$$(\alpha, i) = (\beta, j), \quad (\alpha, i) = (\gamma, k), \quad (\alpha, i) = (\delta, \ell), \quad (\beta, j) = (\gamma, k), \ \dots$$

   holds, then

$$Q^{(\alpha\beta\gamma\delta)}_{ijkl}(A) \equiv 0$$

   as a polynomial in the entries of $A(1), \dots, A(n)$.

2. In particular, for any $\alpha \in [n]$ and any $i, j, k, \ell \in [3]$,

$$Q^{(\alpha\alpha\alpha\alpha)}_{ijkl}(A) \equiv 0.$$

Proof.

1. If two labeled rows coincide, say $(\alpha, i) = (\beta, j)$, then the first two rows of the 4×4 matrix in () are identical. The determinant of a matrix with two equal rows is 0. This holds identically as a polynomial identity, hence $Q^{(\alpha\beta\gamma\delta)}_{ijkl}(A) \equiv 0$.
2. If $(\alpha, \alpha, \alpha, \alpha)$ is the quadruple of matrix labels, then among the four labeled rows $(\alpha, i), (\alpha, j), (\alpha, k), (\alpha, \ell)$ there must be a repetition because $[3]$ has only three elements. Thus part 1 applies.

Definition A.4 (Admissible labeled-row quadruples). A labeled-row quadruple

$$((\alpha, i), (\beta, j), (\gamma, k), (\delta, \ell)) \in ([n] \times [3])^4$$

is **admissible** if the four pairs $(\alpha, i), (\beta, j), (\gamma, k), (\delta, \ell)$ are pairwise distinct.

Equivalently, admissible means that the determinant in () is not forced to vanish by Lemma A.3.

Lemma A.5 (Admissible determinants are nonzero polynomials). Fix an admissible labeled-row quadruple $((\alpha, i), (\beta, j), (\gamma, k), (\delta, \ell))$. The polynomial function

$$A \longmapsto Q^{(\alpha\beta\gamma\delta)}_{ijkl}(A)$$

is not the zero polynomial in the entries of $A(1), \dots, A(n)$.

Proof. Since the quadruple is admissible, the four labeled rows correspond to four *distinct* rows among the list

$$\{A(\alpha')(i', :) : \alpha' \in [n],\ i' \in [3]\}.$$

Hence we are free to assign values to these four rows independently.

Choose a specific specialization of A as follows:

- set $A(\alpha)(i, :) = e_1^\top$,
- set $A(\beta)(j, :) = e_2^\top$,
- set $A(\gamma)(k, :) = e_3^\top$,
- set $A(\delta)(\ell, :) = e_4^\top$,

where e_1, e_2, e_3, e_4 are the standard basis vectors of $\Bbbk^4$, and assign arbitrary values to all other rows of all $A(\alpha')$.

Under this specialization, the 4×4 matrix in () becomes the identity matrix, so its determinant equals $1 \neq 0$. Therefore $Q^{(\alpha\beta\gamma\delta)}_{ijkl}(A)$ attains a nonzero value at some point, and hence is not identically zero as a polynomial.

A.3 Zariski-open nondegeneracy sets

We record the elementary algebraic-geometry facts used to justify "genericity" assumptions. The ambient parameter space $(\Bbbk^{3\times 4})^n \cong \Bbbk^{12n}$ is an affine space, hence irreducible as an algebraic variety over $\Bbbk$.

Lemma A.6 (Nonvanishing loci are Zariski-open). Let $f \in \Bbbk[x_1, \dots, x_m]$ be a polynomial. Then the set

$$U_f := \{x \in \Bbbk^m : f(x) \neq 0\}$$

is Zariski-open. If $f \neq 0$ as a polynomial, then U_f is nonempty and Zariski-dense.

Proof. The complement of U_f is the algebraic set $V(f) = \{x : f(x) = 0\}$, hence Zariski-closed by definition. If $f \neq 0$, then $V(f) \subsetneq \Bbbk^m$ is a proper closed subset, so U_f is nonempty and dense.

We now package a convenient "generic open set" that excludes structural zeros and provides a fixed open chart on the Grassmannian.

Definition A.7 (A convenient Zariski-open set). Assume $n \geq 4$. Define $U_n \subset (\Bbbk^{3\times 4})^n$ to be the set of $A = (A(1), \dots, A(n))$ such that:

1. The “reference minor” is nonzero:

$$\Delta(A) := \det\left(M(A)_{(1,1),(2,1),(3,1),(4,1)}\right) \neq 0.$$

2. For every admissible labeled-row quadruple $((\alpha, i), (\beta, j), (\gamma, k), (\delta, \ell))$, one has

$$\det\left(M(A)_{(\alpha,i),(\beta,j),(\gamma,k),(\delta,\ell)}\right) \neq 0.$$

By Lemma A.1, condition 2 is equivalent to requiring $Q^{(\alpha\beta\gamma\delta)}_{ijkl}(A) \neq 0$ for every admissible labeled-row selection.

Proposition A.8 (U_n is Zariski-open, dense, and nonempty). For every $n \geq 4$, the set U_n in Definition A.7 is a nonempty Zariski-open dense subset of $(\Bbbk^{3\times4})^n$.

Proof. Each determinant appearing in Definition A.7 is a polynomial in the entries of $A(1), \ldots, A(n)$. By Lemma A.6, the nonvanishing condition for each determinant defines a Zariski-open set.

Since there are finitely many admissible labeled-row quadruples, the intersection defining U_n is a finite intersection of Zariski-open sets and hence Zariski-open.

To see nonemptiness (and density), it suffices to show that the product of all these determinant polynomials is not the zero polynomial. The reference minor $\Delta(A)$ is not the zero polynomial because we can choose $A(1), \ldots, A(4)$ so that $a_{1,1}, a_{2,1}, a_{3,1}, a_{4,1}$ are the standard basis vectors $e_1, \ldots, e_4$, giving $\Delta(A) = 1$.

Likewise, by Lemma A.5, every determinant polynomial corresponding to an admissible labeled-row quadruple is nonzero. Since the polynomial ring is an integral domain, the product of finitely many nonzero polynomials is nonzero. Therefore the common nonvanishing locus U_n is nonempty and Zariski-dense by Lemma A.6.

Remark A.9 (Genericity and structural zeros). In Problem #9, the index quadruple $(\alpha, \beta, \gamma, \delta)$ is allowed to have repeated labels as long as it is not fully identical. For such quadruples, some tensor entries $Q^{(\alpha\beta\gamma\delta)}_{ijkl}$ may be structurally zero when the labeled rows repeat (Lemma A.3). Any argument that divides by determinantal coordinates must therefore restrict to an open set where the specific denominators used are nonzero; Definition A.7 provides one convenient choice.

In particular, any “nonvanishing” hypothesis appearing later should be interpreted as applying only to admissible labeled-row selections (or to an explicitly specified finite list of determinantal coordinates), not to structurally zero entries.

Appendix B. Algebraic formulation, elimination, and open-set bookkeeping

This appendix makes precise the algebraic objects implicit in §§4–7 of the main text. We (i) record the coordinate-ring formulation of the rank-one scaling model, (ii) relate it to an elimination (projection) description via a graph ideal, and (iii) explain how nonvanishing hypotheses (e.g. “all relevant scalars are nonzero”) are handled by localization and saturation.

Unless stated otherwise, we work over an algebraically closed field $\Bbbk$ of characteristic 0 (typically $\Bbbk = \mathbb{C}$). All constructions are defined over $\mathbb{Q}$ and base-change to $\Bbbk$.

B.1 Coordinate rings and the universal model map

Fix $n \geq 1$. We use the same indexing conventions as in §0 and §2. The observed coordinates are denoted

$$p_{\alpha\beta\gamma\delta,ijkl}, \qquad \alpha,\beta,\gamma,\delta \in [n],\ i,j,k,\ell \in [3].$$

Let S_n be the polynomial ring in these variables:

$$S_n := \mathbb{Q}[p_{\alpha\beta\gamma\delta,ijkl}].$$

Let R_n be the polynomial ring in:

- the entries of $A(1),\dots,A(n) \in \Bbbk^{3\times 4}$, denoted $a_{\alpha,i,r}$ with $\alpha \in [n]$, $i \in [3]$, $r \in [4]$; and
- auxiliary scaling variables $u_\alpha, v_\alpha, w_\alpha, x_\alpha$ for $\alpha \in [n]$.

Thus

$$R_n := \mathbb{Q}[a_{\alpha,i,r}, u_\alpha, v_\alpha, w_\alpha, x_\alpha].$$

Define the determinantal polynomials

$$Q^{(\alpha\beta\gamma\delta)}_{ijkl}(A) := \det \begin{bmatrix} A(\alpha)(i,:) \\ A(\beta)(j,:) \\ A(\gamma)(k,:) \\ A(\delta)(\ell,:) \end{bmatrix} \in R_n$$

(as in ()).

The polynomial parametrization map Φ_n of Definition 4.1 induces a $\mathbb{Q}$-algebra homomorphism

$$\varphi_n : S_n \longrightarrow R_n$$

by

$$\varphi_n\big(p_{\alpha\beta\gamma\delta,ijkl}\big) := (u_\alpha v_\beta w_\gamma x_\delta)\, Q^{(\alpha\beta\gamma\delta)}_{ijkl}(A).$$

The ideal of polynomial relations among the p-coordinates of the model is

$$I_n := \ker(\varphi_n) \subset S_n.$$

This is the ideal used in Definition 4.2.

Lemma B.1 (Kernel ideal equals the ideal of the Zariski-closure of the image). Let $\Phi_n : (\Bbbk^{3\times 4})^n \times (\Bbbk^n)^4 \to \Bbbk^{81n^4}$ be the polynomial map of Definition 4.1, and let $Y_n \subset \Bbbk^{81n^4}$ denote the Zariski-closure of Φ_n's image. Then $Y_n = V(I_n)$, i.e., I_n is the vanishing ideal of Y_n.

Proof. The coordinate ring of the affine space $\Bbbk^{81n^4}$ is $S_n \otimes_{\mathbb{Q}} \Bbbk$. The comorphism of Φ_n is exactly $\varphi_n \otimes \mathrm{id}_{\Bbbk}$. By general facts on morphisms of affine varieties, the vanishing ideal of the Zariski-closure of the image equals the kernel of the comorphism. Concretely, a polynomial $f \in S_n$ vanishes on Φ_n's image if and only if f maps to 0 under φ_n; since I_n is radical after base change to $\Bbbk$ (by Nullstellensatz), this is equivalent to vanishing on the closure.

Remark B.2 (Universality). The ideal I_n is defined without specializing $A(1), \dots, A(n)$; it depends only on n and the symbolic determinantal construction. This is the precise algebraic meaning of "A-independence" in Theorem 1.1.

B.2 Elimination via a graph ideal

It is often convenient to represent I_n as an elimination ideal of a "graph ideal" in a larger polynomial ring.

Let T_n be the polynomial ring

$$T_n := S_n \otimes_{\mathbb{Q}} R_n ;\cong; \mathbb{Q}[p_{\alpha\beta\gamma\delta,ijkl}, a_{\alpha,i,r}, u_\alpha, v_\alpha, w_\alpha, x_\alpha].$$

Define the graph ideal

$$J_n := \Big\langle p_{\alpha\beta\gamma\delta,ijkl} - (u_\alpha v_\beta w_\gamma x_\delta)\, Q^{(\alpha\beta\gamma\delta)}_{ijkl}(A) \;\Big|\alpha, \beta, \gamma, \delta \in [n],\ i, j, k, \ell \in [3]\Big\rangle \subseteq T_n.$$

Let $\pi : \Bbbk^{\dim T_n} \to \Bbbk^{81n^4}$ denote the projection to the p-coordinates; then $\pi(V(J_n)) = \Phi_n\big((\Bbbk^{3\times 4})^n \times (\Bbbk^n)^4\big)$ set-theoretically.

Lemma B.3 (Elimination description of I_n). With J_n as in (), one has

$$I_n = J_n \cap S_n$$

inside T_n.

Proof. Consider the canonical quotient map $q : T_n \to T_n/J_n$. By construction, the composite

$$S_n \hookrightarrow T_n \xrightarrow{q} T_n/J_n$$

identifies $S_n/(J_n \cap S_n)$ with the $\mathbb{Q}$-subalgebra of T_n/J_n generated by the p-coordinates. On the other hand, the map $\varphi_n : S_n \to R_n$ factors as

$$S_n \hookrightarrow T_n \twoheadrightarrow T_n/J_n \xrightarrow{\text{ev}} R_n,$$

where ev evaluates the p-variables according to the relations in J_n (i.e., replaces each $p_{\alpha\beta\gamma\delta,ijkl}$ by $(u_\alpha v_\beta w_\gamma x_\delta) Q^{(\alpha\beta\gamma\delta)}_{ijkl}(A)$). This shows that $\ker(\varphi_n) = J_n \cap S_n$.

Remark B.4 (Projection image vs. elimination variety). The set $\pi(V(J_n))$ is, in general, only constructible; its Zariski-closure is $V(J_n \cap S_n) = V(I_n)$ by Lemma B.3 and Lemma B.1. This is the standard "projection vs. elimination" phenomenon: elimination yields the ideal of the closure of a projection, not necessarily of the projection itself.

B.3 Nonvanishing constraints: localization and saturation

The main text imposes nonvanishing hypotheses, such as:

- the rank-one factors u, v, w, x have no zero coordinates (equivalently, the induced $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ is nonzero on all nontrivial quadruples); and
- certain determinantal coordinates are nonzero (to avoid divisions by 0 in intermediate rational reconstructions).

These are Zariski-open conditions. Algebraically, restricting to a principal open subset corresponds to localization; restricting the defining ideal of a closed set to that open subset corresponds to saturation.

We formalize the bookkeeping.

Definition B.5 (Products defining principal opens). Define the "scaling nonvanishing" product in R_n by

$$\Pi_{\text{scale}} := \prod_{\alpha=1}^{n} u_\alpha v_\alpha w_\alpha x_\alpha \in R_n.$$

Define the "observed nonvanishing" product in S_n by

$$\Pi_{\text{obs}} := \prod p_{\alpha\beta\gamma\delta,ijkl},$$

where the product ranges over a fixed finite set of indices whose nonvanishing will be required (e.g. all admissible labeled-row selections as in Appendix A, or a smaller list used in a specific reconstruction step).

When needed, we also define the corresponding product in T_n by viewing Π_{obs} as an element of T_n via the inclusion $S_n \hookrightarrow T_n$.

Lemma B.6 (Rank-one nonvanishing is equivalent to coordinatewise nonvanishing of factors). Let $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ be rank one over $\Bbbk$. Then the following are equivalent:

1. $u_\alpha, v_\alpha, w_\alpha, x_\alpha \neq 0$ for all $\alpha \in [n]$.
2. $\lambda_{\alpha\beta\gamma\delta} \neq 0$ for all $(\alpha, \beta, \gamma, \delta) \in [n]^4$.

In particular, condition 1 is sufficient for the nonvanishing requirement in Theorem 1.1 (which excludes only fully identical quadruples).

Proof. If some $u_{\alpha_0} = 0$, then $\lambda_{\alpha_0\beta\gamma\delta} = 0$ for all β, γ, δ, so 2 fails. Conversely, if all coordinates of u, v, w, x are nonzero, then every product $u_\alpha v_\beta w_\gamma x_\delta$ is nonzero, so 2 holds.

We next record the standard ideal-theoretic translation of "intersect with a principal open subset".

Lemma B.7 (Restriction to a principal open via saturation). Let S be a Noetherian ring, let $I \subset S$ be an ideal, and let $g \in S$. Write $D(g) = \{x \in \mathrm{Spec}(S) : g(x) \neq 0\}$. Then the ideal defining the locally closed subset $V(I) \cap D(g)$ inside $\mathrm{Spec}(S)$ is the saturation

$$I : g^\infty := \bigcup_{m \geq 1} (I : g^m).$$

Equivalently, the coordinate ring of $V(I) \cap D(g)$ is $S_g/(IS_g)$, and the contraction of IS_g back to S is $I : g^\infty$.

Proof. Standard commutative algebra: localization identifies the functions regular on $D(g)$, and the contraction of the localized ideal corresponds to saturation. See, e.g., Eisenbud, *Commutative Algebra*, for the equivalence between contraction from S_g and saturation by g.

Remark B.8 (How this is used in §§6–7). Whenever the main text informally "divides by" a determinantal coordinate, the corresponding argument is intended to take place on a principal open $D(g)$ where that coordinate is nonzero. Lemma B.7 ensures that the effect on ideals (and therefore on polynomial identities) can be tracked by saturation and localization without leaving the polynomial category.

B.4 A dense open subset of the closure lies in the image

We record a general fact used repeatedly: while Φ_n's image need not be Zariski-closed, it always contains a dense open subset of its closure. This is the precise “generic points are achieved by parameters” principle.

Proposition B.9 (Dense open subset contained in the image). Let X be an irreducible affine variety over $\Bbbk$ and $f : X \to \Bbbk^m$ a morphism. Let $Y := \overline{f(X)}$ be the Zariski-closure of the image, assumed irreducible. Then there exists a nonempty Zariski-open subset $Y^\circ \subset Y$ such that

$$Y^\circ \subset f(X).$$

Proof. By Chevalley’s theorem (see, e.g., Harris, *Algebraic Geometry: A First Course*), $f(X)$ is a constructible subset of $\Bbbk^m$. Since $Y = \overline{f(X)}$ is irreducible and $f(X)$ is dense in Y, write $f(X)$ as a finite union of locally closed sets:

$$f(X) = \bigcup_{r=1}^{M} (U_r \cap Z_r),$$

with $U_r \subset \Bbbk^m$ open and $Z_r \subset \Bbbk^m$ closed. At least one piece, say $U_{r_0} \cap Z_{r_0}$, has closure equal to Y. Then necessarily $Z_{r_0} \supset Y$, so $(U_{r_0} \cap Y)$ is a nonempty open subset of Y and is contained in $f(X)$. Taking $Y^\circ := U_{r_0} \cap Y$ proves the claim.

Applying Proposition B.9 to $f = \Phi_n$ and $Y = V(I_n)$ yields the following corollary.

Corollary B.10 (Generic preimages exist). There exists a nonempty Zariski-open subset $Y_n^\circ \subset V(I_n)$ such that every $P \in Y_n^\circ$ admits parameters (A, u, v, w, x) with

$$P = \Phi_n(A, u, v, w, x).$$

Proof. Take $X = (\Bbbk^{3\times 4})^n \times (\Bbbk^n)^4$ and $f = \Phi_n$ in Proposition B.9. By Lemma B.1, $Y = V(I_n)$ is the closure of the image.

Remark B.11 (Role in completeness arguments). Corollary B.10 is a “generic surjectivity” statement: it provides actual parameters for points in a dense open subset of the model closure. In the main text, whenever we require an explicit representation $P = (\tilde{u} \otimes \tilde{v} \otimes \tilde{w} \otimes \tilde{x}) \cdot Q(\tilde{A})$, the intent is to work on (possibly further) Zariski-open subsets where such representations exist. The passage from polynomial equalities (vanishing of F_n) to explicit parameter realizations is therefore justified on these open sets, and the open-set bookkeeping is handled via Lemma B.7.

B.5 Closedness on a reference principal open

For the completeness argument in §7 we need a pointwise (not merely generic) passage from membership in the model closure to the existence of actual parameters. This can be guaranteed on a suitable principal open subset.

Fix the reference observed coordinate

$$g := p_{1234,1111} \in S_n.$$

Write $D(g) \subset \Bbbk^{81n^4}$ for the principal open subset where $g \neq 0$.

Lemma B.12 (Ratio coordinates on $D(g)$). The map

$$\eta : \; D(g) \longrightarrow \mathbb{G}_m \times \Bbbk^{81n^4-1}, \qquad P \longmapsto \Big(g(P), \; \big(p_{\alpha\beta\gamma\delta,ijkl}(P)/g(P)\big)_{(\alpha,\beta,\gamma,\delta,i,j,k,\ell)\neq(1,2,3,4,1,1,1,1)} \Big)$$

is an isomorphism of varieties, with inverse given by

$$(t, (r_{\alpha\beta\gamma\delta,ijkl})) \longmapsto (p_{1234,1111} = t, \; p_{\alpha\beta\gamma\delta,ijkl} = t\, r_{\alpha\beta\gamma\delta,ijkl}).$$

Proof. This is the standard coordinate description of a principal open subset: $D(g)$ is isomorphic to $\operatorname{Spec}(S_{n,g})$, and $S_{n,g} \cong \Bbbk[g^{\pm 1}, p_{\alpha\beta\gamma\delta,ijkl}/g]$.

We now show that the model image is already Zariski-closed inside $D(g)$.

Proposition B.13 (No new points on $D(g)$). Let $Y_n := V(I_n)$ be the Zariski-closure of Φ_n's image. Then

$$Y_n \cap D(g) \;=\; \Phi_n\big((\Bbbk^{3\times 4})^n \times (\Bbbk^n)^4\big) \cap D(g).$$

Equivalently, Φ_n's image is Zariski-closed inside the principal open subset $D(g)$.

Proof. We work over the algebraic closure $\Bbbk$.

Step 1: pass to ratio coordinates. By Lemma B.12, it suffices to prove that the ratio set

$$\mathcal{R}_n := \Big\{ \big(p_{\alpha\beta\gamma\delta,ijkl}/p_{1234,1111}\big)_{(\alpha,\beta,\gamma,\delta,i,j,k,\ell)\neq(1,2,3,4,1,1,1,1)} \;:\; P \in \Phi_n\big((\Bbbk^{3\times 4})^n \times (\Bbbk^n)^4\big), \; p_{1234,1111}(P) \neq 0 \Big\}$$

is Zariski-closed in $\Bbbk^{81n^4-1}$. Indeed, under η we have

$$\eta\big(\Phi_n(\cdot) \cap D(g)\big) = \mathbb{G}_m \times \mathcal{R}_n,$$

and therefore

$$\eta(Y_n \cap D(g)) = \overline{\eta\big(\Phi_n(\cdot) \cap D(g)\big)} = \mathbb{G}_m \times \overline{\mathcal{R}_n}.$$

Thus $\overline{\mathcal{R}_n} = \mathcal{R}_n$ implies $Y_n \cap D(g) = \Phi_n(\cdot) \cap D(g)$.

Step 2: identify $\mathcal{R}_n$ with the image of a projective morphism on an affine chart. Let $\mathrm{Gr}(4, 3n)$ be the Grassmannian with Plücker embedding, and write $q_{\alpha\beta\gamma\delta,ijkl}$ for the Plücker coordinate corresponding to the ordered labeled-row quadruple $((\alpha, i), (\beta, j), (\gamma, k), (\delta, \ell))$ (equivalently, the corresponding 4×4 minor of a rank-4 matrix; cf. Appendix D). Let

$$q_{\mathrm{ref}} := q_{1234,1111}.$$

Consider the projective variety

$$\mathcal{X}_n := (\mathbb{P}^{n-1})^4 \times \mathrm{Gr}(4, 3n),$$

with homogeneous coordinates $[u] = [u_1 : \cdots : u_n]$, $[v]$, $[w]$, $[x]$, and Plücker coordinates $[q]$.

Define the morphism

$$\Psi_n : \mathcal{X}_n \longrightarrow \mathbb{P}^{81n^4-1}$$

by the coordinate rule

$$[p_{\alpha\beta\gamma\delta,ijkl}] = [u_\alpha v_\beta w_\gamma x_\delta \, q_{\alpha\beta\gamma\delta,ijkl}]_{(\alpha,\beta,\gamma,\delta,i,j,k,\ell)}.$$

This is well-defined because each output coordinate is multihomogeneous of degree 1 in each of the four $\mathbb{P}^{n-1}$ factors and degree 1 in the Plücker coordinates.

Since $\mathcal{X}_n$ is projective, $\Psi_n(\mathcal{X}_n)$ is a closed subset of $\mathbb{P}^{81n^4-1}$ (see, e.g., Harris, *Algebraic Geometry: A First Course*). Intersect with the affine chart

$$U := \{p_{1234,1111} \neq 0\} \subset \mathbb{P}^{81n^4-1},$$

which is isomorphic to $\Bbbk^{81n^4-1}$ via ratio coordinates

$$r_{\alpha\beta\gamma\delta,ijkl} := \frac{p_{\alpha\beta\gamma\delta,ijkl}}{p_{1234,1111}} \qquad ((\alpha, \beta, \gamma, \delta, i, j, k, \ell) \neq (1, 2, 3, 4, 1, 1, 1, 1)).$$

Then $\Psi_n(\mathcal{X}_n) \cap U$ is Zariski-closed in U, hence its image in the ratio affine space is Zariski-closed.

Finally, note that the affine ratio set $\mathcal{R}_n$ defined in Step 1 coincides with this closed set: for any affine parameters (A, u, v, w, x) with $p_{1234,1111} \neq 0$, take the corresponding Grassmannian point given by the row space of $M(A)$, and take the projective classes $[u], [v], [w], [x]$; conversely, any point of $\mathcal{X}_n$ lying over U admits a representative with $u_1 = v_2 = w_3 = x_4 = q_{\mathrm{ref}} = 1$, which yields affine ratio parameters. Therefore $\mathcal{R}_n$ is Zariski-closed, proving the claim.

Corollary B.14 (Pointwise preimages on $D(g)$). If $P \in V(I_n)$ satisfies $p_{1234,1111}(P) \neq 0$, then there exist parameters (A, u, v, w, x) such that $P = \Phi_n(A, u, v, w, x)$.

Proof. This is immediate from Proposition B.13.

Appendix C. Uniform bounded degree via symmetry and orbit-finite generation

This appendix supplies the details behind §5 of the main text. The goals are:

1. to place the ideals $I_n \subset S_n$ (Definition 4.2) into a compatible direct system as n varies;
2. to define the limit ideal $I_\infty \subset S_\infty$ and prove its stability under the diagonal action of the infinite symmetric group; and
3. to deduce a **uniform bounded-degree generating set** for I_n from an **equivariant Noetherianity** theorem ("Noetherianity up to symmetry").

Throughout, we work over $\mathbb{Q}$ and then base-change to an algebraically closed field $\Bbbk$ of characteristic 0 when invoking geometric statements. All rings below are commutative.

C.1 The directed system (S_n, I_n) and the limit ideal I_∞

We recall the coordinate rings from Appendix B. For each $n \geq 1$ let

$$S_n := \mathbb{Q}[p_{\alpha\beta\gamma\delta,ijkl} \mid \alpha, \beta, \gamma, \delta \in [n],\ i, j, k, \ell \in [3]].$$

The ideal $I_n \subset S_n$ is defined as the kernel of the model comorphism $\varphi_n : S_n \to R_n$ (Definition 4.2, Appendix B.1).

For $m \geq n$, let $\iota_{n \hookrightarrow m} : [n] \hookrightarrow [m]$ denote the standard inclusion. It induces an injective ring homomorphism

$$\iota_{n \hookrightarrow m,*} : S_n \hookrightarrow S_m, \qquad p_{\alpha\beta\gamma\delta,ijkl} \longmapsto p_{\alpha\beta\gamma\delta,ijkl}.$$

We henceforth identify S_n with its image in S_m under this inclusion.

Lemma C.1 (Compatibility of kernels under inclusions). For all integers $m \geq n \geq 1$ one has

$$I_m \cap S_n = I_n.$$

Proof. Consider the parameter rings R_n and R_m from Appendix B.1. By construction, R_m is obtained from R_n by adjoining additional variables corresponding to the extra matrices $A(n+1), \dots, A(m)$ and extra coordinates of (u, v, w, x). In particular, there is a natural inclusion of $\mathbb{Q}$-algebras $R_n \hookrightarrow R_m$.

By definition, the map $\varphi_m : S_m \to R_m$ sends

$$p_{\alpha\beta\gamma\delta,ijkl} \longmapsto (u_\alpha v_\beta w_\gamma x_\delta)\, Q^{(\alpha\beta\gamma\delta)}_{ijkl}(A),$$

where the determinant uses only rows of $A(\alpha), A(\beta), A(\gamma), A(\delta)$. Restricting φ_m to the subring $S_n \subset S_m$ involves only variables with labels in $[n]$, hence coincides with φ_n followed by the inclusion $R_n \hookrightarrow R_m$. Thus the diagram

$$\begin{array}{ccc} & S_n \hookrightarrow S_m & \\ \varphi_n \downarrow & & \downarrow \varphi_m \\ & R_n \hookrightarrow R_m & \end{array}$$

commutes. Taking kernels on S_n gives

$$\ker(\varphi_n) = \ker(\varphi_m|_{S_n}) = \ker(\varphi_m) \cap S_n = I_m \cap S_n,$$

which is the claimed identity.

Definition C.2 (Direct limit ring and limit ideal). Define the direct limit polynomial ring

$$S_\infty := \varinjlim_n S_n = \mathbb{Q}[p_{\alpha\beta\gamma\delta,ijkl} \mid \alpha, \beta, \gamma, \delta \in \mathbb{N},\ i, j, k, \ell \in [3]],$$

where $\mathbb{N} = \{1, 2, 3, \dots\}$, with the evident inclusions $S_n \subset S_{n+1}$.

Define the limit ideal

$$I_\infty := \bigcup_{n \geq 1} I_n \ \subset\ S_\infty,$$

where we identify I_n with its image in S_∞ under $S_n \subset S_\infty$.

By Lemma C.1, the union is increasing and I_∞ is an ideal of S_∞.

C.2 Symmetry actions and stability

Let S_∞ act on the labels $\mathbb{N}$ diagonally.

Lemma C.3 (Stability under relabeling by finite permutations). Let $\mathfrak{S}_\infty$ denote the group of finitely supported permutations of $\mathbb{N}$. Define an action of $\mathfrak{S}_\infty$ on S_∞ by

$$\sigma \cdot p_{\alpha\beta\gamma\delta,ijkl} := p_{\sigma(\alpha)\sigma(\beta)\sigma(\gamma)\sigma(\delta),ijkl}, \qquad \sigma \in \mathfrak{S}_\infty.$$

Then I_∞ is $\mathfrak{S}_\infty$-stable.

Proof. Fix $f \in I_\infty$. Then $f \in I_n$ for some n and hence $f \in S_n$. Let $\sigma \in \mathfrak{S}_\infty$. Since σ has finite support, there exists $m \geq n$ such that $\sigma([n]) \subset [m]$. Restricting σ to $[m]$ yields a permutation $\sigma_m \in \mathfrak{S}_m$. By the S_m-symmetry of the defining map φ_m (Definition 5.1 in the main text), I_m is stable under σ_m, hence $\sigma_m \cdot f \in I_m \subset I_\infty$. But $\sigma_m \cdot f$ agrees with $\sigma \cdot f$ as elements of S_∞, so $\sigma \cdot f \in I_\infty$.

For later restriction arguments it is convenient to have explicit "truncation" maps.

Definition C.4 (Truncation homomorphism). For each $n \geq 1$, define a ring homomorphism $\pi_n : S_\infty \to S_n$ by

$$\pi_n(p_{\alpha\beta\gamma\delta,ijkl}) := \begin{cases} p_{\alpha\beta\gamma\delta,ijkl}, & \text{if } \alpha,\beta,\gamma,\delta \in [n], \\ 0, & \text{otherwise.} \end{cases}$$

This is well-defined because the $p_{\alpha\beta\gamma\delta,ijkl}$ are algebraically independent. The map π_n is a left inverse to the inclusion $S_n \hookrightarrow S_\infty$.

C.3 Noetherianity up to symmetry

We now state the external equivariant Noetherianity result required in §5. The specific variable set of S_∞ is a finite "colored" disjoint union of copies of $\mathbb{N}^4$ (one color for each $(i,j,k,\ell) \in [3]^4$). The group $\mathfrak{S}_\infty$ acts diagonally on the $\mathbb{N}$-coordinates and trivially on the finite color set.

Theorem C.5 (Equivariant Noetherianity for bounded-order tensor coordinate rings). Let $\Bbbk$ be a Noetherian commutative ring, let $d \geq 1$ be fixed, and let C be a finite set ("colors"). Consider the polynomial ring

$$\Bbbk[x_{\mathbf{a},c} \mid \mathbf{a} \in \mathbb{N}^d,\ c \in C],$$

equipped with the diagonal action of $\mathfrak{S}_\infty$ given by

$$\sigma \cdot x_{(a_1,\dots,a_d),c} := x_{(\sigma(a_1),\dots,\sigma(a_d)),c}.$$

Then every $\mathfrak{S}_\infty$-stable ideal is generated by finitely many $\mathfrak{S}_\infty$-orbits of polynomials.

Proof. This is a special case of "Noetherianity up to symmetry" for polynomial rings arising as coordinate rings of bounded-order tensors (also treated in the language of twisted commutative algebras). A proof in this setting appears in Draisma, "Noetherianity up to symmetry," in *Combinatorial Algebraic Geometry* (Lecture Notes in Mathematics 2108, Springer, 2014), and related Gröbner-theoretic methods for infinite-dimensional polynomial rings appear in Hillar and Sullivant.

Applied with $\Bbbk = \mathbb{Q}$, $d = 4$, and $|C| = 81$ (colors indexed by $(i, j, k, \ell) \in [3]^4$), Theorem C.5 implies Proposition 5.2 of the main text.

Corollary C.6 (Finite orbit generators for I_∞). There exist polynomials $f_1, \ldots, f_M \in I_\infty$ such that I_∞ is generated, as an ideal of S_∞, by the union of their $\mathfrak{S}_\infty$-orbits:

$$I_\infty = \langle\ \mathfrak{S}_\infty \cdot f_1\ \cup\ \cdots\ \cup\ \mathfrak{S}_\infty \cdot f_M\ \rangle.$$

Moreover, letting

$$D := \max_{1 \le r \le M} \deg(f_r),$$

the ideal I_∞ is generated by orbit elements of degree at most D.

Proof. By Lemma C.3 the ideal I_∞ is $\mathfrak{S}_\infty$-stable. Theorem C.5 therefore yields finitely many orbit generators $f_1, \ldots, f_M$. The degree bound is immediate from finiteness of the set $\{f_r\}$.

C.4 Restriction to finite n and uniform bounded-degree generators

We now derive the uniform bounded-degree statement for each finite $I_n \subset S_n$.

Proposition C.7 (Uniform bounded-degree generators for I_n). Let $f_1, \ldots, f_M \in I_\infty$ be as in Corollary C.6 and set $D = \max_r \deg(f_r)$. Let

$$n_0 := \max_{1 \le r \le M}\ \max\{\alpha, \beta, \gamma, \delta\ \mid\ p_{\alpha\beta\gamma\delta,ijkl} \text{ appears in } f_r\}.$$

Then $f_r \in S_{n_0}$ for all r. For every $n \ge n_0$, the ideal $I_n \subset S_n$ is generated by the set

$$\mathcal{G}_n := \bigcup_{r=1}^{M} (\mathfrak{S}_n \cdot f_r)\ \subset\ S_n,$$

and every element of $\mathcal{G}_n$ has degree at most D. In particular, I_n admits a generating set of degree $\le D$ with D independent of n.

Proof. By definition of n_0, each f_r involves only variables with labels in $[n_0]$, hence $f_r \in S_{n_0}$ and therefore $f_r \in S_n$ for all $n \ge n_0$.

First, $\langle \mathcal{G}_n \rangle \subset I_n$: since $f_r \in I_{n_0} \subset I_n$ and I_n is stable under $\mathfrak{S}_n$ (Definition 5.1 in the main text), every $\mathfrak{S}_n$-translate of f_r lies in I_n.

Conversely, let $g \in I_n$. Viewing g in S_∞ via the inclusion $S_n \subset S_\infty$, Lemma C.1 yields $g \in I_\infty \cap S_n$. By Corollary C.6, there exist polynomials $h_1, \dots, h_T \in S_\infty$ and orbit elements $q_1, \dots, q_T \in \bigcup_r (\mathfrak{S}_\infty \cdot f_r)$ such that

$$g = \sum_{t=1}^{T} h_t q_t$$

in S_∞.

Apply the truncation homomorphism $\pi_n : S_\infty \to S_n$ from Definition C.4. Since $g \in S_n$, we have $\pi_n(g) = g$. Thus

$$g = \pi_n(g) = \sum_{t=1}^{T} \pi_n(h_t)\, \pi_n(q_t)$$

in S_n. Each $\pi_n(q_t)$ is either 0 (if q_t involves some label $> n$) or equals q_t (if $q_t \in S_n$). Hence g lies in the ideal of S_n generated by those orbit elements q_t that belong to S_n.

It remains to identify which orbit elements $\mathfrak{S}_\infty \cdot f_r$ lie in S_n. Since $f_r \in S_{n_0}$ and $n \ge n_0$, an orbit element $\sigma \cdot f_r$ belongs to S_n if and only if σ maps every label in $[n_0]$ that actually appears in f_r into $[n]$. Any such restriction can be realized by some permutation in $\mathfrak{S}_n$ (extend the induced injection on the finite set of used labels to a bijection of $[n]$). Therefore, the set of orbit elements of f_r that lie in S_n is exactly $\mathfrak{S}_n \cdot f_r$. This shows $g \in \langle \mathcal{G}_n \rangle$.

Degree preservation under permutation is immediate, so all elements of $\mathcal{G}_n$ have degree $\le D$.

Corollary C.8 (Derivation of Proposition 5.2 and Theorem 5.3 in the main text). Theorem C.5 implies Proposition 5.2. Proposition C.7 implies Theorem 5.3 with the same uniform degree bound D.

Proof. Proposition 5.2 is precisely Theorem C.5 applied to the variable set of S_∞. Theorem 5.3 is Proposition C.7 rewritten in the notation of §5.

Appendix D. Plücker relations in block-indexed coordinates

This appendix reformulates the quadratic Grassmann–Plücker relations for $4 \times (3n)$ matrices in a form adapted to the block indexing $(\alpha, i) \in [n] \times [3]$ used throughout the paper. In particular, we express the relations directly in terms of the determinantal coordinates that appear as entries of the tensors $Q^{(\alpha\beta\gamma\delta)}(A)$.

We work over a field $\Bbbk$ of characteristic 0 (e.g. $\Bbbk = \mathbb{Q}, \mathbb{R}, \mathbb{C}$). All identities below are polynomial identities in the entries of the underlying matrices.

D.1 Ordered minors and sign conventions

We first record a convenient "ordered-minor" convention that avoids repeated sign bookkeeping when translating between the Grassmannian's Plücker coordinates (indexed by *subsets*) and the determinants used in this paper (indexed by *ordered tuples*).

Definition D.1 (Ordered 4×4 minors). Let $N \geq 4$ and let $M \in \Bbbk^{4 \times N}$ have columns $m_1, \ldots, m_N \in \Bbbk^4$.

For an ordered 4-tuple $I = (i_1, i_2, i_3, i_4)$ of pairwise distinct indices in $[N]$, define the ordered minor

$$\Delta_I(M) := \det \left[m_{i_1} \; m_{i_2} \; m_{i_3} \; m_{i_4} \right].$$

For a 4-element subset $I = \{i_1 < i_2 < i_3 < i_4\} \subset [N]$, define the corresponding Plücker coordinate

$$p_I(M) := \Delta_{(i_1, i_2, i_3, i_4)}(M).$$

Lemma D.2 (Alternating property). Let $I = (i_1, i_2, i_3, i_4)$ be an ordered 4-tuple of pairwise distinct indices and let $\pi \in S_4$. Then

$$\Delta_{(i_{\pi(1)}, i_{\pi(2)}, i_{\pi(3)}, i_{\pi(4)})}(M) = \operatorname{sgn}(\pi) \, \Delta_{(i_1, i_2, i_3, i_4)}(M).$$

Proof. This is the alternating property of the determinant under column permutations.

D.2 Quadratic Plücker relations for $k = 4$

We now state and prove the $k = 4$ Grassmann–Plücker quadrics in a form that is directly useful for our later specializations.

Proposition D.3 (Quadratic Grassmann–Plücker relations, $k = 4$). Let $M \in \Bbbk^{4 \times N}$ with $N \geq 5$. Fix pairwise distinct indices

$$a_1, a_2, a_3 \in [N], \qquad b_1, b_2, b_3, b_4, b_5 \in [N],$$

with $\{a_1, a_2, a_3\} \cap \{b_1, \dots, b_5\} = \emptyset$. Then the following quadratic identity holds:

$$\sum_{t=1}^{5} (-1)^{t-1} \Delta_{(a_1,a_2,a_3,b_t)}(M)\, \Delta_{(b_1,\dots,\widehat{b_t},\dots,b_5)}(M) = 0,$$

where $(b_1, \dots, \widehat{b_t}, \dots, b_5)$ denotes the ordered 4-tuple obtained by removing b_t.

Proof. Consider the 4×5 submatrix $B = [m_{b_1} \ \cdots \ m_{b_5}]$ of M. For $t \in [5]$ define

$$c_t := (-1)^{t-1} \Delta_{(b_1,\dots,\widehat{b_t},\dots,b_5)}(M).$$

Let $c = (c_1, \dots, c_5)^{\mathsf{T}} \in \Bbbk^5$. We claim that $Bc = 0 \in \Bbbk^4$.

Indeed, for each row index $r \in [4]$, the rth entry of Bc equals

$$(Bc)_r = \sum_{t=1}^{5} B_{r,t} c_t = \sum_{t=1}^{5} B_{r,t} (-1)^{t-1} \det(B_{\hat{t}}),$$

where $B_{\hat{t}}$ denotes the 4×4 matrix obtained by deleting the tth column of B. Now form the 5×5 matrix

$$\widetilde{B}_r := \begin{bmatrix} B_{r,1} & B_{r,2} & B_{r,3} & B_{r,4} & B_{r,5} \\ B_{1,1} & B_{1,2} & B_{1,3} & B_{1,4} & B_{1,5} \\ B_{2,1} & B_{2,2} & B_{2,3} & B_{2,4} & B_{2,5} \\ B_{3,1} & B_{3,2} & B_{3,3} & B_{3,4} & B_{3,5} \\ B_{4,1} & B_{4,2} & B_{4,3} & B_{4,4} & B_{4,5} \end{bmatrix}.$$

The first row of $\widetilde{B}_r$ coincides with the $(r+1)$st row, so $\det(\widetilde{B}_r) = 0$. Expanding $\det(\widetilde{B}_r)$ along the first row yields exactly

$$\det(\widetilde{B}_r) = \sum_{t=1}^{5} B_{r,t} (-1)^{t-1} \det(B_{\hat{t}}) = (Bc)_r,$$

hence $(Bc)_r = 0$ for all r, so $Bc = 0$.

Finally, multilinearity of the determinant in the fourth column gives

$$0 = \det\left[m_{a_1}\ m_{a_2}\ m_{a_3}\ (Bc)\right] = \det\left[m_{a_1}\ m_{a_2}\ m_{a_3}\ \left(\sum_{t=1}^{5} c_t\, m_{b_t}\right)\right] = \sum_{t=1}^{5} c_t\, \Delta_{(a_1,a_2,a_3,b_t)}(M).$$

Substituting the definition of c_t yields the claimed identity.

Remark D.4 (Subset-indexed form). Proposition D.3 is equivalent to the usual Plücker relation indexed by a 3-subset $A = \{a_1, a_2, a_3\}$ and a 5-subset $B = \{b_1, \dots, b_5\}$:

$$\sum_{t=1}^{5} (-1)^{t-1} p_{A \cup \{b_t\}}(M)\, p_{B \setminus \{b_t\}}(M) = 0,$$

where $p_I(M)$ denotes the Plücker coordinate for a 4-subset I. The ordered form in Proposition D.3 is more convenient for translating directly to our ordered determinantal tensors.

D.3 Translation to the (α, i) labeling and to the tensors $Q(A)$

We now specialize to the matrix $M(A) \in \Bbbk^{4\times 3n}$ from §3 / Appendix A, whose columns are labeled by pairs $(\alpha, i) \in [n] \times [3]$.

Definition D.5 (Labeled-column set and labeled determinants). Fix $n \geq 1$ and set

$$E_n := [n] \times [3].$$

Let $A(1), \dots, A(n) \in \Bbbk^{3\times 4}$ and let $M(A)$ be the $4 \times 3n$ matrix from (), whose columns are indexed by E_n.

For an ordered 4-tuple of pairwise distinct labels

$$\mathbf{e} = (e_1, e_2, e_3, e_4) \in E_n^4, \qquad e_r = (\alpha_r, i_r),$$

define

$$\Delta_{\mathbf{e}}(A) := \det \begin{bmatrix} A(\alpha_1)(i_1, :) \\ A(\alpha_2)(i_2, :) \\ A(\alpha_3)(i_3, :) \\ A(\alpha_4)(i_4, :) \end{bmatrix} = \det\left(M(A)_{e_1,e_2,e_3,e_4}\right).$$

The equality of the two determinants is Appendix A, Lemma A.1.

Lemma D.6 (Identification with Q-entries). With $Q^{(\alpha\beta\gamma\delta)}_{ijkl}(A)$ defined by (), one has

$$\Delta_{((\alpha,i),(\beta,j),(\gamma,k),(\delta,\ell))}(A) = Q^{(\alpha\beta\gamma\delta)}_{ijkl}(A).$$

Proof. This is immediate from the definition () and ().

We can now state Plücker relations as identities among the $\Delta_{\mathbf{e}}(A)$, and hence among the entries of $Q(A)$.

Corollary D.7 (Plücker relations among labeled determinants). Let $n \geq 1$ and let $A(1), \dots, A(n) \in \Bbbk^{3\times 4}$. Fix pairwise distinct labels

$$e_1, e_2, e_3 \in E_n, \qquad f_1, f_2, f_3, f_4, f_5 \in E_n,$$

with $\{e_1, e_2, e_3\} \cap \{f_1, \dots, f_5\} = \emptyset$. Then

$$\sum_{t=1}^{5} (-1)^{t-1} \Delta_{(e_1,e_2,e_3,f_t)}(A)\, \Delta_{(f_1,\dots,\widehat{f_t},\dots,f_5)}(A) = 0.$$

Proof. Apply Proposition D.3 to the $4 \times 3n$ matrix $M(A)$, interpreting the labels e_r, f_t as column indices of $M(A)$. Then use Definition D.5.

Remark D.8 (Repeated matrix labels are allowed). Corollary D.7 requires the *pairs* $e_r, f_t \in E_n = [n] \times [3]$ to be distinct, but it does **not** require the first components (the matrix labels in $[n]$) to be distinct. Thus Corollary D.7 yields quadratic identities involving determinants in which some matrix labels repeat (e.g. $(\alpha, \alpha, \beta, \gamma)$), as long as the selected rows are distinct as labeled pairs. This is consistent with Appendix A, Lemma A.3 on structural zeros.

D.4 A concrete repeated-label Plücker identity (useful template)

We give an explicit Plücker relation that uses exactly five distinct matrix labels and includes repeated labels among the four-slot determinants. This is often the most convenient pattern when $n \geq 5$.

Example D.9 (A five-block Plücker quadric with repeated labels). Assume $n \geq 5$ and fix five distinct matrix labels

$$\alpha, \beta, \gamma, \delta, \epsilon \in [n].$$

Consider the eight distinct labeled columns

$$(\alpha, 1), (\alpha, 2), (\beta, 1) \quad \text{and} \quad (\alpha, 3), (\beta, 2), (\gamma, 1), (\delta, 1), (\epsilon, 1)$$

in $E_n = [n] \times [3]$. Set

$$(e_1, e_2, e_3) := ((\alpha, 1), (\alpha, 2), (\beta, 1)),$$

and

$$(f_1, f_2, f_3, f_4, f_5) := ((\alpha, 3), (\beta, 2), (\gamma, 1), (\delta, 1), (\epsilon, 1)).$$

Then Corollary D.7 yields the explicit identity

$$
\begin{aligned}
0 =;&\Delta_{((\alpha,1),(\alpha,2),(\beta,1),(\alpha,3))}(A)\,\Delta_{((\beta,2),(\gamma,1),(\delta,1),(\epsilon,1))}(A)\\
&-\Delta_{((\alpha,1),(\alpha,2),(\beta,1),(\beta,2))}(A)\,\Delta_{((\alpha,3),(\gamma,1),(\delta,1),(\epsilon,1))}(A)\\
&+\Delta_{((\alpha,1),(\alpha,2),(\beta,1),(\gamma,1))}(A)\,\Delta_{((\alpha,3),(\beta,2),(\delta,1),(\epsilon,1))}(A)\\
&-\Delta_{((\alpha,1),(\alpha,2),(\beta,1),(\delta,1))}(A)\,\Delta_{((\alpha,3),(\beta,2),(\gamma,1),(\epsilon,1))}(A)\\
&+\Delta_{((\alpha,1),(\alpha,2),(\beta,1),(\epsilon,1))}(A)\,\Delta_{((\alpha,3),(\beta,2),(\gamma,1),(\delta,1))}(A).
\end{aligned}
$$

Using Lemma D.6, each determinant can be rewritten as an entry of the tensor family $Q(A)$. For instance,

$$\Delta_{((\alpha,1),(\alpha,2),(\beta,1),(\alpha,3))}(A) = Q^{(\alpha\alpha\beta\alpha)}_{1213}(A),$$

and

$$\Delta_{((\beta,2),(\gamma,1),(\delta,1),(\epsilon,1))}(A) = Q^{(\beta\gamma\delta\epsilon)}_{2111}(A),$$

and similarly for the remaining terms in ().

Remark D.10 (Locality and uniformity). Every Plücker relation in Proposition D.3 (and hence every specialized relation in Corollary D.7) involves minors indexed by at most 5 column labels in one factor and at most 4 in the other, hence depends on at most 8 labeled columns overall. Consequently, when expressed in the block indexing (α, i), each such quadratic identity involves only finitely many matrix labels and row indices, with coefficients in $\{+1, -1\}$. This "finite template" nature is one source of uniformity in degree (always 2), although the construction of the universal test map F_n in the main text ultimately uses the stronger symmetry/finite-generation mechanism of Appendix C.

Appendix E. Rigidity: blockwise proportional minors force rank-one scaling

This appendix proves the rigidity statement used in §7, namely that **blockwise proportionality** of the determinantal tensor family $Q(A)$ forces the proportionality tensor to factor as a rank-one 4-tensor. Concretely, we prove the statement labeled as Theorem 7.2 in the main text.

We work over an algebraically closed field $\Bbbk$ of characteristic 0 (e.g. $\Bbbk = \mathbb{C}$). All equalities are algebraic identities; at the end one may restrict to $\mathbb{R}$-points.

Throughout we use the column-vector convention from Appendix A: for $\alpha \in [n]$ and $i \in [3]$,

$$b_{\alpha,i} := A(\alpha)(i,:)^{\mathsf{T}} \in \Bbbk^4, \qquad \tilde{b}_{\alpha,i} := \tilde{A}(\alpha)(i,:)^{\mathsf{T}} \in \Bbbk^4.$$

For an ordered quadruple of labeled rows $((\alpha, i), (\beta, j), (\gamma, k), (\delta, \ell))$, define the determinant

$$\Delta^{ijkl}_{\alpha\beta\gamma\delta}(A) := \det\,[b_{\alpha,i}\ b_{\beta,j}\ b_{\gamma,k}\ b_{\delta,\ell}].$$

By Appendix A (Lemma A.1 and Lemma D.6), this determinant equals the corresponding tensor entry

$$\Delta^{ijkl}_{\alpha\beta\gamma\delta}(A) = Q^{(\alpha\beta\gamma\delta)}_{ijkl}(A).$$

We emphasize that when two labeled pairs coincide (e.g. $(\alpha, i) = (\beta, j)$), the determinant in () is identically 0 (Appendix A, Lemma A.3). In the rigidity argument we always choose **admissible** labeled-row selections (pairwise distinct labeled pairs), so the determinants we divide by are nonzero on a suitable Zariski-open set (Appendix A, Proposition A.8).

E.1 Reference chart and the induced coordinate systems

Assumption E.1 (Reference chart). Fix $n \geq 5$. We assume (after relabeling the matrix indices if necessary) that $A = (A(1), \dots, A(n))$ lies in the Zariski-open set U_n of Appendix A (Definition A.7). In particular,

$$\Delta := \det\,[b_{1,1}\ b_{2,1}\ b_{3,1}\ b_{4,1}] \neq 0,$$

and every admissible 4×4 determinant formed from labeled columns of $M(A)$ is nonzero.

Moreover, we assume the proportionality tensor $s = (s_{\alpha\beta\gamma\delta})$ satisfies $s_{\alpha\beta\gamma\delta} \in \Bbbk^*$ for all $(\alpha, \beta, \gamma, \delta)$ not all equal.

Remark. These are Zariski-open conditions. The main text uses completeness only on a Zariski-open set, so strengthening “generic” to membership in U_n is permissible.

Definition E.2 (Reference bases and Cramer coordinates). Let

$$B := [b_{1,1}\ b_{2,1}\ b_{3,1}\ b_{4,1}] \in \Bbbk^{4\times 4}.$$

By Assumption E.1, B is invertible. For each labeled column $(\alpha, i) \in [n] \times [3]$, define the coordinate vector

$$x_{\alpha,i} := B^{-1} b_{\alpha,i} \in \Bbbk^4.$$

Similarly, define

$$\tilde{B} := [\tilde{b}_{1,1}\ \tilde{b}_{2,1}\ \tilde{b}_{3,1}\ \tilde{b}_{4,1}] \in \Bbbk^{4\times 4}, \qquad \tilde{x}_{\alpha,i} := \tilde{B}^{-1} \tilde{b}_{\alpha,i} \in \Bbbk^4,$$

whenever $\tilde{B}$ is invertible.

We will see below that $\tilde{B}$ is automatically invertible under the proportionality assumption.

Lemma E.3 (Cramer determinant formula for coordinates). Let $p \in [4]$. Let $B[p \leftarrow b_{\alpha,i}]$ denote the 4×4 matrix obtained from B by replacing its pth column with $b_{\alpha,i}$. Then

$$(x_{\alpha,i})_p = \frac{\det\left(B[p \leftarrow b_{\alpha,i}]\right)}{\det(B)}.$$

Analogously,

$$(\tilde{x}_{\alpha,i})_p = \frac{\det\left(\tilde{B}[p \leftarrow \tilde{b}_{\alpha,i}]\right)}{\det(\tilde{B})}.$$

Proof. This is Cramer's rule applied to the linear system $Bx_{\alpha,i} = b_{\alpha,i}$ and $\tilde{B}\tilde{x}_{\alpha,i} = \tilde{b}_{\alpha,i}$.

E.2 Blockwise proportionality induces diagonal scalings in Cramer coordinates

We now impose the proportionality hypothesis relating A and $\tilde{A}$.

Assumption E.4 (Blockwise proportionality of minors). Assume there exist $\tilde{A}(1), \dots, \tilde{A}(n) \in \Bbbk^{3\times 4}$ and a scalar tensor $s = (s_{\alpha\beta\gamma\delta}) \in (\Bbbk^*)^{n\times n\times n\times n}$ such that, for all $\alpha, \beta, \gamma, \delta \in [n]$ not all equal and all $i, j, k, \ell \in [3]$,

$$\Delta^{ijkl}_{\alpha\beta\gamma\delta}(\tilde{A}) = s_{\alpha\beta\gamma\delta}\, \Delta^{ijkl}_{\alpha\beta\gamma\delta}(A).$$

When the labeled-column selection is not admissible and both sides vanish structurally, () is interpreted as a tautology.

Lemma E.5 ($\tilde{B}$ is invertible and determines a base scaling). Under Assumptions E.1 and E.4, the matrix $\tilde{B}$ in () is invertible and satisfies

$$\det(\tilde{B}) = s_{1234} \det(B).$$

In particular, $\tilde{B}$ is invertible.

Proof. Apply () to $(\alpha, \beta, \gamma, \delta) = (1, 2, 3, 4)$ and $(i, j, k, \ell) = (1, 1, 1, 1)$:

$$\det(\tilde{B}) = \Delta^{1111}_{1234}(\tilde{A}) = s_{1234}\, \Delta^{1111}_{1234}(A) = s_{1234} \det(B).$$

By Assumption E.1, $\det(B) \neq 0$, and by Assumption E.4, $s_{1234} \neq 0$, hence $\det(\tilde{B}) \neq 0$.

We now derive the key structural consequence: each block label α acts diagonally on the Cramer coordinates, with different diagonal entries corresponding to different "slots".

Definition E.6 (Slot-wise scaling factors). For each $\alpha \in [n]$, define four scalars

$$\rho_1(\alpha) := \frac{s_{\alpha 234}}{s_{1234}}, \qquad \rho_2(\alpha) := \frac{s_{1\alpha 34}}{s_{1234}}, \qquad \rho_3(\alpha) := \frac{s_{12\alpha 4}}{s_{1234}}, \qquad \rho_4(\alpha) := \frac{s_{123\alpha}}{s_{1234}}.$$

Since all $s_{\cdot}$ appearing above are nonzero by Assumption E.4, each $\rho_p(\alpha) \in \Bbbk^*$.

For each α, define the diagonal matrix

$$D(\alpha) := \operatorname{diag}(\rho_1(\alpha), \rho_2(\alpha), \rho_3(\alpha), \rho_4(\alpha)) \in \Bbbk^{4\times 4}.$$

Lemma E.7 (Diagonal action in Cramer coordinates). Under Assumptions E.1 and E.4, for all $\alpha \in [n]$ and $i \in [3]$ one has

$$\tilde{x}_{\alpha,i} = D(\alpha)\, x_{\alpha,i}.$$

Proof. Fix $\alpha \in [n]$ and $i \in [3]$.

We prove the coordinate identity for each $p \in [4]$ using Lemma E.3 and the proportionality hypothesis ().

For $p = 1$, the matrix $B[1 \leftarrow b_{\alpha,i}]$ has columns $(b_{\alpha,i}, b_{2,1}, b_{3,1}, b_{4,1})$, hence

$$\det\left(B[1 \leftarrow b_{\alpha,i}]\right) = \Delta^{i111}_{\alpha 234}(A), \qquad \det\left(\tilde{B}[1 \leftarrow \tilde{b}_{\alpha,i}]\right) = \Delta^{i111}_{\alpha 234}(\tilde{A}).$$

By (),

$$\det\left(\tilde{B}[1 \leftarrow \tilde{b}_{\alpha,i}]\right) = s_{\alpha 234} \det\left(B[1 \leftarrow b_{\alpha,i}]\right).$$

Dividing by () and applying Lemma E.3 yields

$$(\tilde{x}_{\alpha,i})_1 = \frac{\det(\tilde{B}[1 \leftarrow \tilde{b}_{\alpha,i}])}{\det(\tilde{B})} = \frac{s_{\alpha 234}}{s_{1234}} \frac{\det(B[1 \leftarrow b_{\alpha,i}])}{\det(B)} = \rho_1(\alpha)\,(x_{\alpha,i})_1.$$

The same argument with $p = 2, 3, 4$ uses the determinants

$$\Delta^{1i11}_{1\alpha 34}(A), \quad \Delta^{11i1}_{12\alpha 4}(A), \quad \Delta^{111i}_{123\alpha}(A),$$

and yields

$$(\tilde{x}_{\alpha,i})_p = \rho_p(\alpha)\,(x_{\alpha,i})_p.$$

Collecting the four coordinate identities gives ().

E.3 Bilinear constraints force a rank-one structure across slots

We now use the fact that the proportionality constants $s_{\alpha\beta\gamma\delta}$ do not depend on row indices (i, j, k, ℓ) to deduce strong constraints on the functions $\rho_p(\alpha)$. The idea is to isolate 2×2 determinants in selected coordinate pairs.

For $p < q$ in $[4]$, write $\pi_{pq} : \Bbbk^4 \to \Bbbk^2$ for the coordinate projection onto components p and q. For $v \in \Bbbk^4$, write

$$v^{(pq)} := \pi_{pq}(v) \in \Bbbk^2.$$

Lemma E.8 (Two-dimensional reduction with fixed basis columns). Let $x, y \in \Bbbk^4$. Then

$$\det\left[x\ y\ e_3\ e_4\right] = \det\left[x^{(12)}\ y^{(12)}\right],$$

where $e_3, e_4 \in \Bbbk^4$ are the standard basis vectors. More generally, for any distinct $r, s \in [4]$ and complement $\{p, q\} = [4] \setminus \{r, s\}$, one has

$$\det\left[x\ y\ e_r\ e_s\right] = \pm \det\left[x^{(pq)}\ y^{(pq)}\right],$$

where the sign depends only on the ordered pair (r, s).

Proof. With e_3, e_4 as the last two columns, the matrix is block lower-triangular:

$$\begin{bmatrix} x_1 & y_1 & 0 & 0 \\ x_2 & y_2 & 0 & 0 \\ x_3 & y_3 & 1 & 0 \\ x_4 & y_4 & 0 & 1 \end{bmatrix},$$

so the determinant equals $\det\left[\begin{smallmatrix} x_1 & y_1 \\ x_2 & y_2 \end{smallmatrix}\right]$. The general case follows by permuting coordinates and tracking the sign via Lemma D.2.

We now apply Lemma E.8 to determinants in the $x_{\alpha,i}$ coordinate system.

Lemma E.9 (Cross constraint for (ρ_1, ρ_2)). Under Assumptions E.1 and E.4, for all $\alpha, \beta \in [n]$ one has

$$\rho_1(\alpha)\rho_2(\beta) = \rho_2(\alpha)\rho_1(\beta).$$

Proof. Fix $\alpha, \beta \in [n]$. By Assumption E.1 we may choose row indices $i, j \in [3]$ such that the labeled pairs (α, i) and (β, j) are distinct, and such that the corresponding 4×4 minors with labels $(\alpha, \beta, 3, 4)$ and fixed row indices for labels 3 and 4 equal to 1 are nonzero. Concretely, consider the minors

$$\Delta^{ij11}_{\alpha\beta34}(A) = \det\left[b_{\alpha,i}\ b_{\beta,j}\ b_{3,1}\ b_{4,1}\right] \quad \text{and} \quad \Delta^{ij11}_{\alpha\beta34}(\tilde{A}) = \det\left[\tilde{b}_{\alpha,i}\ \tilde{b}_{\beta,j}\ \tilde{b}_{3,1}\ \tilde{b}_{4,1}\right],$$

where $b_{\alpha,i}$ denotes the (α, i)-column of $M(A)$ and similarly for $\tilde{b}$.

By Assumption E.4, we have

$$\Delta^{ij11}_{\alpha\beta34}(\tilde{A}) = s_{\alpha\beta34}\, \Delta^{ij11}_{\alpha\beta34}(A).$$

In the coordinate system of Lemma E.8, write $u_{\alpha,i} := x^{(12)}_{\alpha,i} \in \mathbb{k}^2$ and $u_{\beta,j} := x^{(12)}_{\beta,j} \in \mathbb{k}^2$. The determinant $\Delta^{ij11}_{\alpha\beta34}(A)$ is, up to the fixed nonzero factor $\Delta^{1111}_{1234}(A)$, equal to the 2×2 determinant $\det[u_{\alpha,i}\ u_{\beta,j}]$; the analogous statement holds for $\tilde{A}$. More precisely, Lemma E.8 yields

$$\frac{\Delta^{ij11}_{\alpha\beta34}(A)}{\Delta^{1111}_{1234}(A)} = \det\left[u_{\alpha,i}\ u_{\beta,j}\right], \qquad \frac{\Delta^{ij11}_{\alpha\beta34}(\tilde{A})}{\Delta^{1111}_{1234}(\tilde{A})} = \det\left[\tilde{u}_{\alpha,i}\ \tilde{u}_{\beta,j}\right],$$

where $\tilde{u}_{\alpha,i} := \tilde{x}^{(12)}_{\alpha,i}$.

Using Lemma E.8 again, $\tilde{u}_{\alpha,i} = (D(\alpha)x_{\alpha,i})^{(12)}$ and $\tilde{u}_{\beta,j} = (D(\beta)x_{\beta,j})^{(12)}$, and $D(\alpha)$ acts diagonally by $(\rho_1(\alpha), \rho_2(\alpha))$ on the $(1,2)$-coordinates. Hence

$$\det\left[\tilde{u}_{\alpha,i}\ \tilde{u}_{\beta,j}\right] = \rho_1(\alpha)\rho_2(\beta)\, u_{\alpha,i,1}u_{\beta,j,2} - \rho_2(\alpha)\rho_1(\beta)\, u_{\alpha,i,2}u_{\beta,j,1},$$

while

$$\det\left[u_{\alpha,i}\ u_{\beta,j}\right] = u_{\alpha,i,1}u_{\beta,j,2} - u_{\alpha,i,2}u_{\beta,j,1}.$$

Combining these identities with $\Delta^{ij11}_{\alpha\beta34}(\tilde{A}) = s_{\alpha\beta34}\Delta^{ij11}_{\alpha\beta34}(A)$ and $\Delta^{1111}_{1234}(\tilde{A}) = s_{1234}\Delta^{1111}_{1234}(A)$ gives

$$\rho_1(\alpha)\rho_2(\beta)\, u_{\alpha,i,1}u_{\beta,j,2} - \rho_2(\alpha)\rho_1(\beta)\, u_{\alpha,i,2}u_{\beta,j,1} = \frac{s_{\alpha\beta34}}{s_{1234}}\left(u_{\alpha,i,1}u_{\beta,j,2} - u_{\alpha,i,2}u_{\beta,j,1}\right)$$

for all such i, j. To justify the spanning claim, fix $\alpha \in [n]$ and choose distinct $i_1, i_2 \in [3]$ such that the labeled pairs $(\alpha, i_1), (\alpha, i_2), (3, 1), (4, 1)$ are pairwise distinct. This is always possible because each label has three available rows and at most one of them can coincide with $(3,1)$ or $(4,1)$. Then the labeled-row quadruple is admissible, so by Assumption E.1 we have

$$\Delta^{i_1 i_2 11}_{\alpha\alpha 34}(A) = \det\left[b_{\alpha,i_1}\ b_{\alpha,i_2}\ b_{3,1}\ b_{4,1}\right] \neq 0.$$

Dividing by $\Delta^{1111}_{1234}(A) = \det(B) \neq 0$ and applying Lemma E.8 gives

$$\det\left[u_{\alpha,i_1}\ u_{\alpha,i_2}\right] \neq 0,$$

so $u_{\alpha,i_1}, u_{\alpha,i_2}$ form a basis of $\Bbbk^2$, and hence $\{u_{\alpha,1}, u_{\alpha,2}, u_{\alpha,3}\}$ spans $\Bbbk^2$. Since $\{u_{\alpha,1}, u_{\alpha,2}, u_{\alpha,3}\}$ spans $\Bbbk^2$ for each α (by Assumption E.1 and Lemma E.8), the bilinear identity above holds for all $u, v \in \Bbbk^2$ in place of $u_{\alpha,i}, u_{\beta,j}$. Comparing coefficients of the monomials $u_1 v_2$ and $u_2 v_1$ yields

$$\rho_1(\alpha)\rho_2(\beta) = \rho_2(\alpha)\rho_1(\beta),$$

as claimed.

Lemma E.10 (Cross constraints for (ρ_1, ρ_3) and (ρ_1, ρ_4)). Under Assumptions E.1 and E.4, for all $\alpha, \beta \in [n]$ one has

$$\rho_1(\alpha)\rho_3(\beta) = \rho_3(\alpha)\rho_1(\beta), \qquad \rho_1(\alpha)\rho_4(\beta) = \rho_4(\alpha)\rho_1(\beta).$$

Proof. The proof is identical to Lemma E.9 after replacing the fixed basis columns as follows.

- For the pair (ρ_1, ρ_3), use determinants with labels $(\alpha, \beta, 2, 4)$ and fixed row indices for labels 2 and 4 equal to 1. In Cramer coordinates, the columns corresponding to labels 2 and 4 become e_2 and e_4, so by Lemma E.8 the determinant reduces (up to a fixed sign) to a 2×2 determinant in the $(1,3)$-coordinates.
- For the pair (ρ_1, ρ_4), use determinants with labels $(\alpha, \beta, 2, 3)$ (with fixed row indices 1 for labels 2 and 3), reducing (up to sign) to the $(1,4)$-coordinate determinant.

In both cases, U_n implies the corresponding projected coordinate sets span $\Bbbk^2$, so the same bilinear comparison yields the stated cross relations.

E.4 Common diagonal factor and factorization of s

The cross relations imply that each $\rho_p(\alpha)$ is proportional to $\rho_1(\alpha)$ with a constant independent of α.

Proposition E.11 (Common diagonal factor). Under Assumptions E.1 and E.4, there exist constants $c_1, c_2, c_3, c_4 \in \Bbbk^*$ and a function $t : [n] \to \Bbbk^*$ such that for all $\alpha \in [n]$ and all $p \in [4]$,

$$\rho_p(\alpha) = c_p\, t_\alpha.$$

Equivalently, there exists a fixed diagonal matrix

$$D_0 := \operatorname{diag}(c_1, c_2, c_3, c_4) \in \Bbbk^{4\times 4}$$

such that

$$D(\alpha) = t_\alpha\, D_0 \qquad \text{for all } \alpha \in [n].$$

Proof. Set $t_\alpha := \rho_1(\alpha)$ (which is nonzero by Definition E.6). Set $c_1 := 1$.

By Lemma E.9 with $\beta = 1$ and using $\rho_1(1) = s_{1234}/s_{1234} = 1$, we obtain

$$\rho_2(\alpha)\rho_1(1) = \rho_1(\alpha)\rho_2(1) \quad \Longrightarrow \quad \rho_2(\alpha) = \rho_2(1)\,\rho_1(\alpha).$$

Thus we can set $c_2 := \rho_2(1)$.

Similarly, Lemma E.10 with $\beta = 1$ gives

$$\rho_3(\alpha) = \rho_3(1)\,\rho_1(\alpha), \qquad \rho_4(\alpha) = \rho_4(1)\,\rho_1(\alpha),$$

so we may set $c_3 := \rho_3(1)$ and $c_4 := \rho_4(1)$.

This yields (). The diagonal matrix form () follows immediately from Definition E.6 and ().

We can now prove the desired rank-one factorization of s.

Theorem E.12 (Rigidity of blockwise proportional minors). Assume $n \geq 5$. Let $A(1), \dots, A(n) \in \Bbbk^{3\times 4}$ satisfy Assumption E.1, and let $\tilde{A}(1), \dots, \tilde{A}(n) \in \Bbbk^{3\times 4}$ and $s \in (\Bbbk^*)^{n\times n\times n\times n}$ satisfy Assumption E.4. Then there exist $c \in \Bbbk^*$ and $t \in (\Bbbk^*)^n$ such that

$$s_{\alpha\beta\gamma\delta} = c\, t_\alpha t_\beta t_\gamma t_\delta$$

for all $(\alpha, \beta, \gamma, \delta) \in [n]^4$ not all equal.

In particular, this proves Theorem 7.2 of the main text.

Proof. By Proposition E.11, there exist $t_\alpha \in \Bbbk^*$ and a fixed diagonal matrix D_0 such that

$$\tilde{x}_{\alpha,i} = D(\alpha)x_{\alpha,i} = t_\alpha D_0 x_{\alpha,i} \qquad \text{for all } \alpha \in [n],\ i \in [3].$$

Fix any label quadruple $(\alpha, \beta, \gamma, \delta) \in [n]^4$ not all equal. Choose row indices $i, j, k, \ell \in [3]$ so that the labeled pairs $(\alpha, i), (\beta, j), (\gamma, k), (\delta, \ell)$ are pairwise distinct. This is always possible because:

- no label occurs four times (since "not all equal" excludes $(\alpha, \alpha, \alpha, \alpha)$), and
- each label provides three distinct rows.

By Assumption E.1 (membership in U_n), the corresponding determinant $\Delta^{ijkl}_{\alpha\beta\gamma\delta}(A)$ is nonzero.

Now compute both sides of () in Cramer coordinates. Using $b_{\alpha,i} = Bx_{\alpha,i}$ and $\tilde{b}_{\alpha,i} = \tilde{B}\tilde{x}_{\alpha,i}$, we have

$$\Delta^{ijkl}_{\alpha\beta\gamma\delta}(A) = \det(B)\, \det\left[x_{\alpha,i}\ x_{\beta,j}\ x_{\gamma,k}\ x_{\delta,\ell}\right],$$

and similarly

$$\Delta^{ijkl}_{\alpha\beta\gamma\delta}(\tilde{A}) = \det(\tilde{B})\, \det\left[\tilde{x}_{\alpha,i}\ \tilde{x}_{\beta,j}\ \tilde{x}_{\gamma,k}\ \tilde{x}_{\delta,\ell}\right].$$

Using () and () and canceling $\det(B) \neq 0$ yields

$$\det\left[\tilde{x}_{\alpha,i}\ \tilde{x}_{\beta,j}\ \tilde{x}_{\gamma,k}\ \tilde{x}_{\delta,\ell}\right] = \frac{s_{\alpha\beta\gamma\delta}}{s_{1234}}\, \det\left[x_{\alpha,i}\ x_{\beta,j}\ x_{\gamma,k}\ x_{\delta,\ell}\right].$$

Substitute () into the left-hand side of (). Since D_0 is common to all four columns, we may factor it out as a left multiplication:

$$\det\left[t_\alpha D_0 x_{\alpha,i}\ \ t_\beta D_0 x_{\beta,j}\ \ t_\gamma D_0 x_{\gamma,k}\ \ t_\delta D_0 x_{\delta,\ell}\right] = t_\alpha t_\beta t_\gamma t_\delta\ \det(D_0)\ \det\left[x_{\alpha,i}\ x_{\beta,j}\ x_{\gamma,k}\ x_{\delta,\ell}\right].$$

Therefore () becomes

$$t_\alpha t_\beta t_\gamma t_\delta\ \det(D_0)\ \det\left[x_{\alpha,i}\ x_{\beta,j}\ x_{\gamma,k}\ x_{\delta,\ell}\right] = \frac{s_{\alpha\beta\gamma\delta}}{s_{1234}}\, \det\left[x_{\alpha,i}\ x_{\beta,j}\ x_{\gamma,k}\ x_{\delta,\ell}\right].$$

Since $\det\left[x_{\alpha,i}\ x_{\beta,j}\ x_{\gamma,k}\ x_{\delta,\ell}\right] \neq 0$ (because $\Delta^{ijkl}_{\alpha\beta\gamma\delta}(A) \neq 0$ and $\det(B) \neq 0$), we may cancel it and obtain

$$s_{\alpha\beta\gamma\delta} = \left(s_{1234}\ \det(D_0)\right) t_\alpha t_\beta t_\gamma t_\delta.$$

Setting $c := s_{1234} \det(D_0) \in \Bbbk^*$ proves the desired factorization.

Remark E.13 (Where genericity enters). The only uses of genericity are:

1. the existence of an invertible reference minor $\det(B) \neq 0$ (Assumption E.1), and
2. the availability of admissible labeled-row selections with nonzero determinants, to guarantee that the relevant 2×2 projected determinants span $\mathbb{k}^2$ (used in Lemmas E.9–E.10) and to allow the cancellation step in the final factorization (Theorem E.12).

Both conditions define Zariski-open subsets of $(\mathbb{k}^{3\times 4})^n$, and Assumption E.1 is one convenient explicit choice (Appendix A, Proposition A.8).

Appendix F. Worked examples and computational sanity checks

This appendix is optional and is not used in the logical proof of the main theorem. Its purpose is to provide (i) small worked identities that reflect the rigidity mechanism in Appendix E and (ii) practical sanity checks one may run on synthetic instances.

F.1 Quadratic rank-one identities in the proportionality tensor

The rigidity argument in Appendix E produces "cross" identities that are characteristic of rank-one factorization. It is useful to state these explicitly as low-degree constraints in the proportionality tensor s.

Example F.1 (Cross identities implied by a rank-one factorization). Let $n \geq 5$ and let $s \in (\mathbb{k}^*)^{n\times n\times n\times n}$. Suppose

$$s_{\alpha\beta\gamma\delta} = c\, t_\alpha t_\beta t_\gamma t_\delta$$

for some $c \in \mathbb{k}^*$ and $t \in (\mathbb{k}^*)^n$. Fix distinct labels $1, 2, 3, 4 \in [n]$. Then for all $\alpha, \beta \in [n]$ one has the quadratic identities

$$s_{\alpha 234}\, s_{1\beta 34} - s_{1\alpha 34}\, s_{\beta 234} = 0,$$

$$s_{\alpha 234}\, s_{12\beta 4} - s_{12\alpha 4}\, s_{\beta 234} = 0,$$

$$s_{\alpha 234}\, s_{123\beta} - s_{123\alpha}\, s_{\beta 234} = 0.$$

Explanation. Substituting $s_{\alpha\beta\gamma\delta} = c\, t_\alpha t_\beta t_\gamma t_\delta$ into the first identity gives

$$(c\, t_\alpha t_2 t_3 t_4)(c\, t_1 t_\beta t_3 t_4) - (c\, t_1 t_\alpha t_3 t_4)(c\, t_\beta t_2 t_3 t_4) = 0$$

by commutativity of multiplication. The other two identities are analogous. These are exactly the cross relations that appear in Appendix E when translating $\rho_p(\alpha)\rho_q(\beta) = \rho_q(\alpha)\rho_p(\beta)$ into the s-coordinates.

F.2 A concrete nondegenerate numerical instance

The following explicit construction provides a convenient family of "very generic" A for which all admissible 4×4 determinants are nonzero, and for which the predicted factorization of s can be checked numerically by hand.

Example F.2 (A Vandermonde-type instance with explicit scaling). Let $n = 5$. For each labeled pair $(\alpha, i) \in [5] \times [3]$ define a distinct scalar

$$\tau_{\alpha,i} := 10\alpha + i.$$

Define column vectors $b_{\alpha,i} \in \mathbb{Q}^4$ by

$$b_{\alpha,i} := \begin{bmatrix} 1 \\ \tau_{\alpha,i} \\ \tau_{\alpha,i}^2 \\ \tau_{\alpha,i}^3 \end{bmatrix}, \qquad (\alpha, i) \in [5] \times [3],$$

and define $A(\alpha) \in \mathbb{Q}^{3\times 4}$ by setting its ith row to $b_{\alpha,i}^{\mathsf{T}}$:

$$A(\alpha)(i, :) := b_{\alpha,i}^{\mathsf{T}}.$$

Then for any admissible labeled-row quadruple $((\alpha, i), (\beta, j), (\gamma, k), (\delta, \ell))$, the determinant

$$\det \left[b_{\alpha,i}\ b_{\beta,j}\ b_{\gamma,k}\ b_{\delta,\ell}\right]$$

is a Vandermonde determinant in the four distinct parameters $\tau_{\alpha,i}, \tau_{\beta,j}, \tau_{\gamma,k}, \tau_{\delta,\ell}$, hence is nonzero. In particular, A lies in the explicit Zariski-open set U_5 of Appendix A.

Now choose a fixed diagonal matrix and per-label scalars

$$D_0 := \mathrm{diag}(2, 3, 5, 7), \qquad t_\alpha := \alpha \ \ (\alpha \in [5]).$$

Define $\tilde{b}_{\alpha,i} := t_\alpha D_0 b_{\alpha,i}$ and $\tilde{A}(\alpha)(i, :) := \tilde{b}_{\alpha,i}^{\mathsf{T}}$. For any admissible quadruple,

$$\det \left[\tilde{b}_{\alpha,i}\ \tilde{b}_{\beta,j}\ \tilde{b}_{\gamma,k}\ \tilde{b}_{\delta,\ell}\right] = \det(D_0)\, t_\alpha t_\beta t_\gamma t_\delta \, \det \left[b_{\alpha,i}\ b_{\beta,j}\ b_{\gamma,k}\ b_{\delta,\ell}\right].$$

Thus the proportionality tensor is

$$s_{\alpha\beta\gamma\delta} = \det(D_0)\, t_\alpha t_\beta t_\gamma t_\delta, \qquad \det(D_0) = 2 \cdot 3 \cdot 5 \cdot 7 = 210.$$

As a concrete check, take $(\alpha, \beta, \gamma, \delta) = (1, 2, 3, 4)$ and $(i, j, k, \ell) = (1, 1, 1, 1)$. Then

$$Q^{(1234)}_{1111}(A) = \det \left[b_{1,1}\ b_{2,1}\ b_{3,1}\ b_{4,1}\right] = \prod_{1 \le r < s \le 4} (\tau_s - \tau_r),$$

where $(\tau_1, \tau_2, \tau_3, \tau_4) = (11, 21, 31, 41)$, hence

$$Q^{(1234)}_{1111}(A) = (10)(20)(30)(10)(20)(10) = 12{,}000{,}000.$$

The predicted scaling factor is

$$s_{1234} = \det(D_0)\, t_1 t_2 t_3 t_4 = 210 \cdot (1 \cdot 2 \cdot 3 \cdot 4) = 5040,$$

so

$$Q^{(1234)}_{1111}(\tilde{A}) = s_{1234}\, Q^{(1234)}_{1111}(A) = 5040 \cdot 12{,}000{,}000 = 60{,}480{,}000{,}000,$$

which matches the direct determinant computation by multilinearity.

F.3 Suggested Monte Carlo sanity checks (soundness and rigidity)

The map F_n in the main theorem is constructed existentially (via generators of I_n), and the full ideal I_n is too large to compute directly for realistic n. Nevertheless, there are simple sanity checks one can run on synthetic instances to validate the key mechanisms: (i) soundness (rank-one scaling produces points in $V(I_n)$) and (ii) rigidity (blockwise proportionality forces rank one).

Remark F.3 (A minimal computational checklist). Fix $n \in \{5, 6\}$.

1. **Generate a nondegenerate A.** Use the Vandermonde construction of Example F.2, or sample random rational entries until a chosen finite list of admissible determinants is nonzero.

2. **Generate a rank-one scaling tensor.** Sample random $u, v, w, x \in (\mathbb{Q}^*)^n$ and set $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$.

3. **Form the scaled observations.** Compute $Q(A)$ from () and set $P = \lambda Q(A)$.

4. **Check internal consistency of the blockwise scaling model.** For each fixed $(\alpha, \beta, \gamma, \delta)$, compute the ratios

$$r_{ijkl}^{(\alpha\beta\gamma\delta)} := \frac{P_{ijkl}^{(\alpha\beta\gamma\delta)}}{Q_{ijkl}^{(\alpha\beta\gamma\delta)}(A)}$$

for admissible (i, j, k, ℓ) (avoiding structural zeros). All such ratios should agree and equal $\lambda_{\alpha\beta\gamma\delta}$.

5. **Verify rank-one of the recovered λ.** Form a flattening of λ (e.g. λ as an $n^2 \times n^2$ matrix via (α, β) vs. (γ, δ)) and check that all 2×2 minors vanish numerically (over $\mathbb{Q}$ or modulo a large prime). This confirms the rank-one condition in Definition 2.2.
6. **Negatives (non-rank-one).** Sample a random λ that is not rank one (e.g. as a random sum of two rank-one tensors), form $P = \lambda Q(A)$, and verify that the recovered λ fails the 2×2 minor test.
7. **Optional: restricted elimination tests.** If using a CAS (e.g. Macaulay2/Singular), restrict attention to a small, symmetry-closed subset of indices (for example: five matrix labels and a fixed finite set of admissible row index patterns), compute the corresponding restricted elimination ideal, and evaluate its generators on P from steps 2–3. One should observe vanishing for rank-one λ and nonvanishing for generic non-rank-one λ.

A compact pseudocode sketch is:

```
Input: n   {5,6}
Construct A by Example F.2 (or random until nondegenerate)
Compute Q(A)

Sample u,v,w,x   (Q*)^n
Set  _{  } = u_  v_  w_  x_
Set P =    Q(A)

Recover ^_{   } by dividing P^{(   )}_{ijkl} / Q^{(   )}_{ijkl}(A) on admissible (i,j,k, )
Check ^ is consistent across (i,j,k, )
Check rank-one of ^ via 2×2 minors of a flattening
```

These checks do not compute F_n itself; rather, they validate the soundness and rigidity phenomena that underpin the proof.

F.4 Practical reductions for elimination computations (optional)

Even on small n, the full coordinate ring S_n contains $81n^4$ variables, and the elimination ideal I_n is typically far beyond direct Gröbner computation. Nonetheless, the proof suggests practical reductions that preserve the relevant structure.

Remark F.4 (Reducing the computational footprint). The following reductions often make small-scale elimination feasible while still probing the geometry of $V(I_n)$.

1. **Locality in labels.** Many structural identities (Plücker quadrics and the rigidity constraints) only involve finitely many matrix labels at a time. One may therefore fix a small subset of labels (e.g. five labels) and work entirely inside the induced subring.
2. **Admissible index patterns only.** Structural zeros (Appendix A, Lemma A.3) introduce irrelevant components. Restricting to admissible labeled-row selections avoids these degeneracies and stabilizes saturation/localization steps.
3. **Saturation by a controlled product.** When dividing by determinants in intermediate reasoning, one should instead saturate by the product of the corresponding coordinates (Appendix B, Lemma B.7). In computation this often removes parasitic components.
4. **Symmetry reduction.** Use the action of S_n to compute orbit representatives of polynomial generators. In practice, one computes generators on a minimal set of labels and then applies permutations to generate further relations.
5. **Modular computations.** Gröbner computations over $\mathbb{Q}$ are expensive. Performing computations modulo several large primes and reconstructing over $\mathbb{Q}$ (when needed) is often substantially faster.

These techniques are not needed for the theoretical proof, but they help in producing illustrative low-degree relations on restricted index sets and in validating experimental claims about degrees and generators.