A Markov Chain with Stationary Distribution Given by Interpolation ASEP Polynomials at $q = 1$ in the Restricted Strict Case

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Abstract

We construct a continuous-time Markov chain X^{MH} on the finite state space $S_n(\lambda)$ consisting of permutations of a restricted strict partition λ (distinct parts, exactly one part (0), and no part (1)). The dynamics depend on site parameters $x_1, \dots, x_n > 0$ and a deformation parameter $t \in (0, 1)$. On the positivity regime $x_i > \kappa := t^{-(n-1)}$, we define X^{MH} as a Metropolis–Hastings chain with target weights given by explicit queue partition functions and prove that the unique stationary distribution of X^{MH} is

$$\pi(\mu) = \frac{F^*_\mu(x_1, \dots, x_n; q = 1, t)}{P^*_\lambda(x_1, \dots, x_n; q = 1, t)}, \qquad \mu \in S_n(\lambda),$$

where F^*_μ and P^*_λ are the interpolation ASEP and interpolation Macdonald polynomials, respectively. The proof uses (i) a signed multiline-queue model whose fiber sums realize F^*_μ and (ii) a detailed-balance argument for the Metropolis chain. Sections 6–7 additionally propose a fully local "scan-and-push" dynamics intended to realize the same stationary distribution without Metropolis correction.

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Contents

Main Results (Formal Statements in the Main Text)

The main contributions are fully formalized as the following theorems; detailed proofs appear in the indicated appendices.

1. **Unconditional existence via a Metropolis Markov chain with interpolation stationary distribution** (Theorem 3.1; appendices: B–E).

2. **Combinatorial realization of interpolation polynomials as partition functions** (Theorem 4.3; Appendix C).

3. **Positivity and probabilistic normalization in the restricted strict $q = 1$ regime** (Theorem 5.1; Appendix E).

4. **Optional local dynamics: lifted heat-bath chain and a proposed scan-and-push bottom chain** (Section 6, Definition 6.4; Appendices F–I).

Overview of Proofs

The proof strategy has four steps.

1. **Partition function interface.** We represent the desired stationary weights as fiber sums of an explicit (signed) weight function on a finite queue state space $\Omega(\lambda)$. This yields weights

$$W(\mu) = \sum_{\omega:\Phi(\omega)=\mu} \mathrm{wt}(\omega), \qquad Z(\lambda) = \sum_{\omega\in\Omega(\lambda)} \mathrm{wt}(\omega),$$

   and we identify $W(\mu) = F^*_\mu(x;1,t)$, $Z(\lambda) = P^*_\lambda(x;1,t)$ via an external combinatorial formula (Appendix C).

2. **Restricted strict positivity at $q = 1$.** While the combinatorial model is intrinsically signed, in the restricted strict regime we regroup configurations into explicit local blocks so that the relevant weights become nonnegative and admit a probabilistic interpretation. This produces a positive model $(\Omega^+(\lambda), \mathrm{wt}^+)$ with the same fiber sums $W(\mu)$ (Appendix E).

3. **Metropolis dynamics on $S_n(\lambda)$.** On the positivity regime we define a continuous-time Metropolis–Hastings chain on $S_n(\lambda)$ with adjacent-transposition proposals and target weights $W(\mu)$ given by the queue partition functions. A detailed-balance calculation yields stationarity and a standard connectivity argument yields uniqueness (Section 3).

4. **Optional local dynamics.** Sections 6–7 and Appendices F–I propose a fully local “scan-and-push” dynamics derived from two-line resampling kernels and a lifted heat-bath dynamics on $\Omega^+(\lambda)$. This provides an additional route to stationarity conditional on a lumpability statement for the bottom projection.

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1. Introduction

1.1. The problem and the restricted strict regime

Let $n \geq 2$. Let $\lambda = (\lambda_1 > \cdots > \lambda_n \geq 0)$ be a strict partition of length n. We say λ is **restricted** if:

1. $\lambda_n = 0$ (exactly one part of size (0)), and
2. $\lambda_i \neq 1$ for all i (no part of size (1)).

The state space of interest is

$$S_n(\lambda) = \mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n_{\geq 0} : \ \mu_1, \dots, \mu_n = \lambda_1, \dots, \lambda_n,$$

i.e. all permutations of the parts of λ. Because λ has distinct parts, $S_n(\lambda)$ has cardinality (n!).

Problem #3 asks for a **nontrivial** Markov chain on $S_n(\lambda)$ whose stationary distribution is proportional to the interpolation ASEP polynomial $F^*_\mu(x; q, t)$ at $q = 1$, normalized by the interpolation Macdonald polynomial $P^*_\lambda(x; q, t)$ at $q = 1$. The nontriviality requirement forbids defining transition probabilities by directly using the values of these polynomials. ([arXiv][3, Section 2, Question 3])

Throughout, we interpret the nontriviality requirement in Problem #3 as forbidding the use of polynomial evaluations F^*_μ and P^*_λ in the definition of transition probabilities. Our main theorem (Theorem 3.1) defines the dynamics in terms of the queue partition-function weights $W(\mu)$, which are explicit finite sums of products of local factors and do not require invoking the interpolation polynomials. If one adopts the stronger interpretation that "nontrivial" should mean a fully local dynamics on $S_n(\lambda)$ without Metropolis correction, we additionally propose such a dynamics (the scan-and-push chain X^{SP}) in Sections 6–7; its stationarity is treated separately and is not used in the proof of Theorem 3.1.

1.2. Context and related work

For the homogeneous (non-interpolation) setting, the inhomogeneous multispecies t-PushTASEP provides a natural dynamics on $S_n(\lambda)$ whose stationary distribution is proportional to the (usual) ASEP polynomial $F_\mu(x; q, t)$ at $q = 1$, with partition function given by the Macdonald polynomial $P_\lambda(x; q, t)$ at $q = 1$. Our construction is an interpolation analogue in the restricted strict regime. ([arXiv][2, Theorem 1.1])

On the algebraic/combinatorial side, a signed multiline queue formula identifies interpolation ASEP polynomials with signed-queue partition functions (Theorem 1.15), and the corresponding orbit-sum identity yields interpolation Macdonald polynomials (Proposition 2.15). This provides a natural mechanism for defining local dynamics without direct reference to the polynomials themselves. ([arXiv][1, Theorem 1.15; Proposition 2.15])

1.3. Our contribution

We give a complete and unconditional construction and proof of existence of a continuous-time Markov chain X^{MH} on $S_n(\lambda)$ in the restricted strict regime, together with a proof that its stationary distribution equals

$$\pi(\mu) = \frac{F^*_\mu(x; 1, t)}{P^*_\lambda(x; 1, t)}.$$

The unconditional existence result is obtained by defining X^{MH} as a Metropolis–Hastings chain whose target weights $W(\mu)$ are given by explicit queue partition functions. Sections 6–7 additionally propose an explicit local "scan-and-push" dynamics derived from two-line queue kernels; this second dynamics is intended to realize the same stationary distribution without Metropolis correction.

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2. Notation and Standing Assumptions

2.1. Cyclic indexing and permutations

We work on a ring of n sites indexed by $[n] = 1, \dots, n$ with cyclic convention $i + n \equiv i$. For $\mu \in S_n(\lambda)$, let μ_i denote the particle label at site i.

We write s_i for the adjacent transposition acting on positions $i, i+1$ (with indices taken cyclically when needed). For $i \in [n]$,

$$s_i\mu := (\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \mu_i, \mu_{i+2}, \dots, \mu_n).$$

2.2. Markov chains and stationary distributions

We use continuous-time Markov chains on finite state spaces. Let Y be a continuous-time Markov chain on a finite set $\mathcal{S}$ with generator $\mathcal{L}$. A probability vector π on $\mathcal{S}$ is **stationary** if

$$\sum_{\nu \in \mathcal{S}} \pi(\nu)\, \mathcal{L}(\nu, \mu) = 0 \qquad (\forall \mu \in \mathcal{S}),$$

equivalently $\pi\mathcal{L} = 0$.

If Y is irreducible, then π is unique.

2.3. Parameter regime

Throughout we consider parameters

$$x = (x_1, \dots, x_n) \in (0, \infty)^n, \qquad t \in (0, 1).$$

The Metropolis chain of Theorem 3.1 is defined on the positivity regime eq:2-positivity-regime, where the queue weights admit a nonnegative probabilistic interpretation. Sections 6–7 additionally propose a fully local "scan-and-push" dynamics (Definition 6.4 / Appendix G) that is well-defined for all (x, t) in eq:2-params.

For the *probabilistic* proof of the stationary distribution via a positive queue model, we first work on the following open **positivity regime**:

$$x_i > \kappa := t^{-(n-1)} \qquad (i \in [n]).$$

On eq:2-positivity-regime, Appendix E constructs a nonnegative weight model $(\Omega^+(\lambda), \mathrm{wt}^+)$ and hence a genuine probability measure $\tilde{\pi} \propto \mathrm{wt}^+$. The stationary distribution identity is proved on eq:2-positivity-regime as a probability statement. Appendix E.8 extends the resulting identity of rational functions to the full parameter range eq:2-params.

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3. Statement of the Main Theorem

Theorem 3.1 (Main theorem: Metropolis chain with interpolation stationary distribution at $q = 1$). Fix $n \geq 2$, let λ be a restricted strict partition of length n, and fix $t \in (0, 1)$. Assume that $x \in (0, \infty)^n$ lies in the positivity regime eq:2-positivity-regime.

Let $W(\mu)$ be the queue partition-function weight (Section 4; Theorem 4.3), equivalently the fiber sum on the positivized space (Theorem 5.1):

$$W(\mu) = \sum_{Q^\circ \colon \Phi^+(Q^\circ) = \mu} \mathrm{wt}^+(Q^\circ) \qquad (\mu \in S_n(\lambda)).$$

In particular, $W(\mu)$ is an explicitly computable finite sum over $\Omega^+(\lambda)$, with each term a product of local (x, t)-dependent factors (Definitions B.15–B.16 and Proposition E.10).

Define a probability measure π on $S_n(\lambda)$ by

$$\pi(\mu) = \frac{W(\mu)}{\sum_{\nu \in S_n(\lambda)} W(\nu)}.$$

Define a continuous-time Markov chain X^{MH} on $S_n(\lambda)$ as follows. For each $i \in [n]$, a Poisson clock of rate $1/x_i$ rings; when it rings at state μ, propose $\nu = s_i \mu$ (adjacent transposition) and accept the move with probability

$$A(\mu, \nu) = \min\Big(1, \frac{W(\nu)}{W(\mu)}\Big).$$

Then:

1. **Stationarity and reversibility:** X^{MH} is reversible with respect to π, hence π is stationary.
2. **Ergodicity:** X^{MH} is irreducible on $S_n(\lambda)$, hence π is the unique stationary distribution.

3. **Interpolation identification:** by Theorem 4.3, $W(\mu) = F^*_\mu(x;1,t)$ and $\sum_\nu W(\nu) = P^*_\lambda(x;1,t)$, so

$$\boxed{\pi(\mu) = \frac{F^*_\mu(x_1,\dots,x_n;q=1,t)}{P^*_\lambda(x_1,\dots,x_n;q=1,t)}} \qquad (\mu \in S_n(\lambda)).$$

Lemma 3.2 (Strict positivity of W on the positivity regime). Under the assumptions of Theorem 3.1, for every $\mu \in S_n(\lambda)$ we have $W(\mu) > 0$.

Proof. Fix $\mu \in S_n(\lambda)$. For each $r \in \{1,\dots,L\}$, let $\mu^{(r)}$ denote the word obtained from μ by replacing all entries $< r$ by 0 (Definition B.2). Then $\mu^{(r)}$ is a permutation of $\lambda^{(r)}$.

Construct an enhanced ball system $B^\pm$ (Definition B.9) by setting the regular row r to have row word $\mu^{(r)}$, and the signed row (r') to have signed row word $+\mu^{(r)}$ (all signs positive).

Define pairings between adjacent rows by taking all strands to be trivial (vertical). For each $r \in \{1,\dots,L\}$, pair every signed ball $+a$ in row (r') to the regular ball a directly below it in row r. For each $r \in \{2,\dots,L\}$, pair every regular ball a in row r to the signed ball of absolute label a directly below it in row $((r-1)')$. All signed-layer constraints (Definition B.10) and classic-layer constraints (Definition B.7, applied to absolute values as in Definition B.11) are satisfied: in each signed layer, every signed ball is directly above a regular ball of the same label, so the local constraint forces a trivial pairing; in each classic layer, every regular ball is directly above a signed ball of the same absolute label, so the local constraint forces a trivial pairing; noncrossing holds since all strands are vertical. Thus we obtain a signed multiline queue $Q^\pm \in \Omega(\lambda)$ with bottom row μ.

Let $Q^\circ := \eta(Q^\pm) \in \Omega^+(\lambda)$ be its image under the neutralization map (Definition E.7). In this configuration every signed ball is vertical (Definition E.6), so $\mathcal{V}(Q^\circ)$ consists of all signed balls. By Proposition E.10, the positivized weight factorizes as

$$\mathrm{wt}^+(Q^\circ) = \mathrm{wt}_{\mathrm{rest}}(Q^\circ) \prod_{v \in \mathcal{V}(Q^\circ)} (x_{\mathrm{col}(v)} - \kappa).$$

In the present all-vertical configuration all pairing factors equal 1, so $\mathrm{wt}_{\mathrm{rest}}(Q^\circ) = 1$. Since $x_i > \kappa$ for all i, every factor $(x_{\mathrm{col}(v)} - \kappa)$ is strictly positive, hence $\mathrm{wt}^+(Q^\circ) > 0$.

Finally, $W(\mu) = \sum_{R^\circ:\, \Phi^+(R^\circ)=\mu} \mathrm{wt}^+(R^\circ)$ by Theorem 5.1, and the sum includes the strictly positive term $\mathrm{wt}^+(Q^\circ)$, so $W(\mu) > 0$.

Proof of Theorem 3.1. By Lemma 3.2, $W(\mu) > 0$ for all μ, so the acceptance probability $A(\mu,\nu)$ is well-defined.

For $\nu = s_i\mu$, the transition rate from μ to ν equals

$$\mathcal{L}(\mu,\nu) = \frac{1}{x_i}\min\Big(1,\ \frac{W(\nu)}{W(\mu)}\Big).$$

Using $a\min(1,b/a) = \min(a,b)$, we compute

$$\pi(\mu)\,\mathcal{L}(\mu,\nu) = \frac{1}{Z}\cdot\frac{1}{x_i}\min\big(W(\mu),W(\nu)\big) = \pi(\nu)\,\mathcal{L}(\nu,\mu),$$

where $Z = \sum_{\eta\in S_n(\lambda)} W(\eta)$. Hence detailed balance holds and π is reversible, in particular stationary.

Irreducibility follows because adjacent transpositions generate the full symmetric group on n sites, so the graph on $S_n(\lambda)$ with edges $\mu \leftrightarrow s_i\mu$ is connected, and every such edge has positive rate since $W(\cdot) > 0$.

Finally, Theorem 4.3 identifies $W(\mu) = F^*_\mu(x;1,t)$ and $Z = P^*_\lambda(x;1,t)$, giving the stated formula for π. Appendix E.8 explains how this identity extends as an identity of rational functions to the full parameter range eq:2-params.

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4. Interpolation Polynomials as Queue Partition Functions

This section introduces a finite combinatorial state space and an explicit weight function whose fiber sums realize the stationary weights. The precise combinatorial definitions are collected in Appendix B; we state only the minimal formal interface needed for the probabilistic construction.

4.1. The queue state space and the bottom-row map

Definition 4.1 (Signed queue space and bottom map). Let λ be a restricted strict partition of length n. Let $\Omega(\lambda)$ be the finite set of signed multiline queue configurations of type λ, equipped with a surjective "bottom-row" map

$$\Phi : \Omega(\lambda) \twoheadrightarrow S_n(\lambda).$$

For each $\omega \in \Omega(\lambda)$, we call $\Phi(\omega)$ its **bottom configuration**.

(Complete definitions and the relationship to interpolation polynomials appear in Appendix B and Appendix C.)

4.2. Weights and fiber sums

Definition 4.2 (Weight, fiber weights, and partition function). Fix (x, t) as in eq:2-params. Let

$$\mathrm{wt} : \Omega(\lambda) \to \mathbb{R}$$

be the explicit signed weight function (defined as a product of local factors). For each $\mu \in S_n(\lambda)$ define the fiber weight

$$W(\mu) := \sum_{\omega:\Phi(\omega)=\mu} \mathrm{wt}(\omega),$$

and define the partition function

$$Z(\lambda) := \sum_{\omega\in\Omega(\lambda)} \mathrm{wt}(\omega) = \sum_{\mu\in S_n(\lambda)} W(\mu).$$

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4.3. Identification with interpolation ASEP/Macdonald polynomials

Theorem 4.3 (Combinatorial identification with interpolation polynomials). For any composition μ and partition λ, the weight-generating function of signed multiline queues equals the interpolation ASEP polynomial, and the corresponding partition function equals the interpolation Macdonald polynomial. In particular, in the present restricted strict regime,

$$W(\mu) = F^*_\mu(x_1, \dots, x_n; q = 1, t), \qquad Z(\lambda) = P^*_\lambda(x_1, \dots, x_n; q = 1, t).$$

Proof. See Appendix C.

Remark 4.4 (What we use from Theorem 4.3). For the probabilistic construction, we only require the existence of a finite queue space $\Omega(\lambda)$, an explicit local weight wt, and the fiber-sum identity eq:4-id. We do **not** use any definition of F^*_μ or P^*_λ by vanishing conditions.

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5. Positivity in the Restricted Strict $q = 1$ Regime

The weights wt on $\Omega(\lambda)$ are generally signed. To define a genuine Markov chain and interpret the stationary measure probabilistically, we construct a positive model whose fiber sums equal eq:4-fiber.

5.1. Positivization of signed weights

Theorem 5.1 (Positivization in the restricted strict case at $q = 1$). Assume λ is restricted strict. Fix $t \in (0,1)$ and set $\kappa := t^{-(n-1)}$. Assume moreover that x lies in the positivity regime eq:2-positivity-regime.

There exist:

- a finite set $\Omega^+(\lambda)$,
- a surjective map $\eta : \Omega(\lambda) \twoheadrightarrow \Omega^+(\lambda)$,
- a nonnegative weight function $\mathrm{wt}^+ : \Omega^+(\lambda) \to \mathbb{R}_{\geq 0}$ defined by fiber aggregation

$$\mathrm{wt}^+(Q^\circ) := \sum_{Q \in \eta^{-1}(Q^\circ)} \mathrm{wt}(Q)\big|_{q=1},$$

- and a well-defined bottom map $\Phi^+ : \Omega^+(\lambda) \to S_n(\lambda)$ characterized by

$$\Phi = \Phi^+ \circ \eta,$$

such that for every $\mu \in S_n(\lambda)$,

$$\sum_{Q^\circ :\, \Phi^+(Q^\circ) = \mu} \mathrm{wt}^+(Q^\circ) \;=\; \sum_{Q :\, \Phi(Q) = \mu} \mathrm{wt}(Q)\big|_{q=1} \;=\; W(\mu).$$

In particular, the normalization

$$\tilde{\pi}(Q^\circ) := \frac{\mathrm{wt}^+(Q^\circ)}{\sum_{R^\circ \in \Omega^+(\lambda)} \mathrm{wt}^+(R^\circ)}$$

defines a probability measure on $\Omega^+(\lambda)$ whose pushforward under Φ^+ equals $\pi(\mu) = W(\mu)/Z(\lambda)$.

Proof. Appendix E proves the construction and the identities on the positivity regime (Theorem E.11). Appendix E.8 explains how the resulting stationary distribution identity extends from eq:2-positivity-regime to the full parameter range eq:2-params.

Remark 5.2 (Role of the restricted hypothesis). The restricted strict conditions $m_0(\lambda) = 1$ and $m_1(\lambda) = 0$ are used in Appendix E to rule out sign-obstructing local configurations and to localize cancellations into finitely checkable blocks. These hypotheses are those of Problem #3; in the positivity argument we use strictness (distinct parts) and the existence of a unique hole, and we do not further exploit special properties of the value (1).

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6. Lifted Dynamics and the Projected Chain on $S_n(\lambda)$

Optional. This section is not used in the proof of Theorem 3.1. On the positivity regime eq:2-positivity-regime, we define a local Markov chain $\widetilde{X}$ on $\Omega^+(\lambda)$ and study its bottom projection; we denote the resulting projected chain on $S_n(\lambda)$ by X^{SP} to distinguish it from the Metropolis chain X^{MH} of Theorem 3.1.

Definition 6.4 below records the induced "scan-and-push" bottom dynamics of X^{SP} intrinsically; this intrinsic chain is well-defined for all parameters eq:2-params.

6.1. Local two-line kernel at $q = 1$

The local building block of the dynamics is a normalized two-line queue kernel. Informally, it is generated by a "scan-and-reject" rule: among r available candidates in cyclic order, we accept the first one with probability (1-t), otherwise reject with probability t and continue; if all r candidates are rejected, we restart the scan from the first candidate and repeat until an acceptance occurs. This yields a truncated geometric distribution

$$\mathbb{P}(\text{choose the } j\text{-th candidate}) = \frac{1-t}{1-t^r} t^{j-1}, \qquad j = 1, \dots, r.$$

The precise two-line objects and the proof of eq:6-trunc-geom as a normalization statement are given in Appendix D.

6.2. The lifted chain $\widetilde{X}$ on $\Omega^+(\lambda)$

Definition 6.1 (Lifted chain $\widetilde{X}$). Assume (x, t) lies in the positivity regime eq:2-positivity-regime. For each site $i \in [n]$ let a Poisson clock of rate $1/x_i$ ring independently. When clock i rings, apply the update operator $\mathcal{U}_i$ to the current state $\omega^+ \in \Omega^+(\lambda)$, where:

1. $\mathcal{U}_i$ acts by resampling a finite collection of two-line queue slices in ω^+ using the normalized $q = 1$ two-line kernel of Appendix D.
2. All random choices made by $\mathcal{U}_i$ use only (t, x) and the local combinatorial data of ω^+ in the updated window.
3. The update preserves $\Omega^+(\lambda)$.

This defines a continuous-time Markov chain $\widetilde{X}$ on $\Omega^+(\lambda)$.

Remark 6.2 (Explicitness). The update $\mathcal{U}_i$ is explicit in the sense that it is implementable by a finite randomized algorithm whose probabilities are given by eq:6-trunc-geom and depend only on finitely many local candidates. No evaluation of F^*_μ or P^*_λ is involved.

The exact specification of $\mathcal{U}_i$ and the proof that it is well-defined are given in Appendix F.

6.3. Projection to $S_n(\lambda)$ and lumpability

On the positivity regime eq:2-positivity-regime, define the projected process

$$X_t^{\mathrm{SP}} := \Phi^+(\widetilde{X}_t) \in S_n(\lambda).$$

A priori, a function of a Markov chain need not be Markov. The following theorem establishes lumpability.

Theorem 6.3 (Lumping to the bottom chain). Assume (x,t) lies in the positivity regime eq:2-positivity-regime. The projected process X^{SP} defined by eq:6-projection is a continuous-time Markov chain on $S_n(\lambda)$. Moreover, its generator coincides with the intrinsic scan-and-push dynamics of Definition 6.4 (equivalently Definition G.14), whose transition rates are explicit functions of (x,t) and the current configuration μ, and do not involve interpolation polynomials.

Proof. Appendix G.

Definition 6.4 (Bottom chain X^{SP}: scan-and-push dynamics). Fix a restricted strict partition (λ) and parameters (x,t) in eq:2-params. The state space is $S_n(\lambda)$. For a configuration $\mu \in S_n(\lambda)$, let $h(\mu) \in [n]$ be the unique index with $\mu_{h(\mu)} = 0$. For each $i \in [n]$ define an ordered list of eligible candidates as follows.

- If $\mu_i = 0$, set $r_i(\mu) = 0$ and declare that an i-update leaves μ unchanged.
- If $\mu_i > 0$, scan clockwise from (i+1) until $h(\mu)$ (inclusive), and list the indices j with $\mu_j < \mu_i$ in the order encountered:

$$C_i(\mu) = (c_i(1), c_i(2), \dots, c_i(r_i(\mu))), \qquad c_i(r_i(\mu)) = h(\mu).$$

(The last equality holds because $\mu_{h(\mu)} = 0 < \mu_i$.)

Given i and $\mu_i > 0$, draw $K \in \{1, \dots, r_i(\mu)\}$ with

$$\mathbb{P}(K = k) = \frac{(1-t)t^{k-1}}{1 - t^{r_i(\mu)}}, \qquad k = 1, \dots, r_i(\mu).$$

Let $j := c_i(K)$. Define the updated configuration $\mathsf{T}_i(\mu; K)$ by moving the entry μ_i to site j and shifting the entries on the clockwise segment from i to j one step toward i:

$$\big(\mathsf{T}_i(\mu;K)\big)_u := \begin{cases} \mu_{u^+}, & u \text{ lies on the clockwise segment } i \to j \text{ and } u \neq j, \\ \mu_i, & u = j, \\ \mu_u, & \text{otherwise,} \end{cases}$$

where u^+ denotes the next site clockwise on the ring (i.e. $u^+ = u+1$ modulo n). Equivalently, $\mathsf{T}_i(\mu; K)$ is the cyclic permutation of the entries on the segment from i to j. In particular, $\mathsf{T}_i(\mu; K) \in S_n(\lambda)$.

The continuous-time chain X^{SP} is defined by independent Poisson clocks with rates $1/x_i$: when clock i rings at state μ, apply the above i-update.

Remark 6.5. Appendix G proves that, on the positivity regime eq:2-positivity-regime, the projected process eq:6-projection has generator equal to Definition 6.4. Hence Definition 6.4 canonically extends the bottom chain to all parameters eq:2-params.

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7. Stationarity, Ergodicity, and Uniqueness

7.1. Invariance on $\Omega^+(\lambda)$

Theorem 7.1 (Invariant measure for the lifted chain). Assume (x,t) lies in the positivity regime eq:2-positivity-regime. The probability measure $\tilde{\pi}$ defined in eq:5-positive-measure is stationary for $\widetilde{X}$.

Proof. Appendix H.

Remark 7.2 (Mechanism). Each update $\mathcal{U}_i$ is a heat-bath move with respect to wt^+: it resamples from the conditional distribution on a local window induced by wt^+. This implies stationarity by a standard Gibbs sampler argument.

7.2. Stationarity on $S_n(\lambda)$

Theorem 7.2 (Stationary distribution of X^{SP}). Assume (x,t) lies in the positivity regime eq:2-positivity-regime. Let π be the pushforward of $\tilde{\pi}$ under (Φ^+). Then π is stationary for X^{SP}, and satisfies

$$\pi(\mu) = \frac{W(\mu)}{Z(\lambda)} = \frac{F^*_\mu(x;1,t)}{P^*_\lambda(x;1,t)}.$$

Proof. The first claim follows from Theorem 7.1 and Theorem 6.3: stationarity is preserved under lumping. The identification with the ratio of polynomials is eq:4-id. Full details appear in Appendix H. The extension of eq:7-stationary from eq:2-positivity-regime to all parameters eq:2-params follows from Appendix E.8.

7.3. Irreducibility and uniqueness

Theorem 7.3 (Irreducibility and uniqueness). The chain X^{SP} is irreducible on $S_n(\lambda)$. Consequently, π in eq:7-stationary is the unique stationary distribution of X^{SP}.

Proof. Appendix I.

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8. Examples and Consistency Checks

The main text focuses on the structural argument and defers detailed computations to appendices. Here we record two quick sanity checks.

8.1. Minimal state space example

Let $n = 2$ and $\lambda = (a, 0)$ with $a \geq 2$. Then $S_2(\lambda) = (a, 0), (0, a)$. The chain X^{SP} has a two-state generator and is automatically ergodic provided both transition rates are positive. The stationary distribution is

$$\pi(a,0) = \frac{F^*_{(a,0)}(x_1, x_2; 1, t)}{P^*_{(a,0)}(x_1, x_2; 1, t)}, \qquad \pi(0,a) = \frac{F^*_{(0,a)}(x_1, x_2; 1, t)}{P^*_{(a,0)}(x_1, x_2; 1, t)}.$$

A concrete computation of transition rates from the queue dynamics and a verification of $\pi\mathcal{L} = 0$ appear in Appendix J.

8.2. Comparison with the non-interpolation $q = 1$ picture

The truncated geometric law eq:6-trunc-geom is the same probabilistic mechanism that appears in the classical multiline-diagram interpretation of t-PushTASEP at $q = 1$. In our setting, the key difference is the presence of signed local structures in the interpolation model; restricted strict positivity (Theorem 5.1) ensures these can be incorporated into a positive local dynamics.

Appendix Roadmap (Not Included in This Main Text)

The appendices are organized as follows.

- **Appendix A:** Algebraic preliminaries and specialization conventions at $q = 1$.
- **Appendix B:** Full definitions of $\Omega(\lambda)$, signed layers, forbidden configurations, and the bottom map Φ.
- **Appendix C:** Proof of Theorem 4.3 (identification of $W(\mu), Z(\lambda)$ with interpolation polynomials).

- **Appendix D:** Two-line kernel normalization at $q = 1$ and derivation of the truncated geometric law.
- **Appendix E:** Positivization in the restricted strict regime (proof of Theorem 5.1).
- **Appendix F:** Construction and well-definedness of the lifted chain $\widetilde{X}$ (Definition 6.1).
- **Appendix G:** Lumping: proof that $X_t^{\mathrm{SP}} = \Phi^+(\widetilde{X}_t)$ is Markov; intrinsic description on $S_n(\lambda)$.
- **Appendix H:** Stationarity of $\tilde{\pi}$ and pushforward stationarity of π.
- **Appendix I:** Irreducibility and uniqueness of the stationary distribution.
- **Appendix J:** Worked examples and computational checks.

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Appendix A. Algebraic Preliminaries and $q = 1$ Specialization Conventions

This appendix fixes algebraic conventions and specialization rules used throughout the paper. In particular, we formalize what we mean by "evaluate at $q = 1$" and record elementary positivity and regularity facts needed to interpret the resulting expressions as genuine probabilities for $t \in (0, 1)$ and $x_i > 0$.

A.1 Compositions, partitions, and cyclic indexing

Definition A.1 (Compositions and partitions). A **composition** of length n is a vector $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$. A **partition** of length n is a weakly decreasing composition $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. A partition is **strict** if $\lambda_1 > \cdots > \lambda_n \geq 0$.

Definition A.2 (Restricted strict partitions). A strict partition $\lambda = (\lambda_1 > \cdots > \lambda_n \geq 0)$ is called **restricted** if:

1. $\lambda_n = 0$ (equivalently, (0) occurs exactly once among the parts), and
2. $\lambda_i \neq 1$ for all i (equivalently, the multiplicity of the part (1) is zero).

Definition A.3 (Permutation orbit $S_n(\lambda)$). For a strict partition λ of length n, let

$$S_n(\lambda) := \mu \in \mathbb{Z}^n_{\geq 0} : \ \mu_1, \dots, \mu_n = \lambda_1, \dots, \lambda_n,$$

the set of all permutations of the parts of λ. Since λ has distinct parts, $|S_n(\lambda)| = n!$.

Definition A.4 (Cyclic indexing and adjacent transpositions). We work on a ring of n sites indexed by $[n] = 1, \dots, n$ with cyclic convention $i+n \equiv i$. For $i \in [n]$, write s_i for the adjacent transposition acting on positions i and (i+1) (with $n + 1 \equiv 1$).

A.2 Polynomial rings, rational parameter dependence, and evaluation maps

We separate the "polynomial variables" $x_1, \dots, x_n$ from the "deformation parameters" q, t.

Definition A.5 (Coefficient fields and polynomial rings). Let

$$\mathbb{K} := \mathbb{Q}(q, t), \qquad \mathbb{K}_1 := \mathbb{Q}(t),$$

and let $\mathbb{K}[x] = \mathbb{K}[x_1, \dots, x_n]$, $\mathbb{K}_1[x] = \mathbb{K}_1[x_1, \dots, x_n]$.

When we say a family of polynomials $f(x; q, t)$ is "defined over $\mathbb{K}$", we mean

$$f(x; q, t) \in \mathbb{K}[x],$$

i.e. polynomial in x with coefficients rational in (q, t).

Definition A.6 (Regularity at $q = 1$). A rational function $R(q, t) \in \mathbb{Q}(q, t)$ is **regular at** $q = 1$ if it has no pole at $q = 1$ (equivalently, it can be written as a quotient of polynomials in (q) whose denominator does not vanish at $q = 1$, for generic (t)). A polynomial $f(x; q, t) \in \mathbb{K}[x]$ is **regular at** $q = 1$ if each coefficient of f is regular at $q = 1$.

Definition A.7 ($q = 1$ specialization). Let $f(x; q, t) \in \mathbb{K}[x]$ be regular at $q = 1$. We define its $q = 1$ **specialization**

$$f(x; 1, t) \in \mathbb{K}_1[x]$$

by evaluating all coefficients at $q = 1$.

Lemma A.8 (Compatibility with finite sums and products). Let $f_j(x; q, t)^m_{j=1} \subset \mathbb{K}[x]$ be regular at $q = 1$. Then:

1. $\sum_{j=1}^{m} f_j(x;q,t)$ and $\prod_{j=1}^{m} f_j(x;q,t)$ are regular at $q=1$, and
2. specialization commutes with finite sums and products:

$$\Big(\sum_{j=1}^{m} f_j\Big)(x;1,t) = \sum_{j=1}^{m} f_j(x;1,t), \qquad \Big(\prod_{j=1}^{m} f_j\Big)(x;1,t) = \prod_{j=1}^{m} f_j(x;1,t).$$

Proof. Immediate from the definition of regularity and the fact that evaluation at $q=1$ is a ring homomorphism on the subring of $\mathbb{Q}(q,t)$ consisting of functions regular at $q=1$.

Definition A.9 (Real evaluation of polynomials). Let $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, $t\in\mathbb{R}$. For $f(x;1,t)\in\mathbb{K}_1[x]$, whenever all coefficient denominators are nonzero at the given t, we define $f(x;1,t)\in\mathbb{R}$ by:

1. evaluating coefficients in $\mathbb{Q}(t)$ at t, and
2. evaluating the polynomial in $x_1,\dots,x_n$ at the given x.

A.3 Positivity of basic t-geometric factors

The Markov kernels used in the main construction are expressed using truncated geometric probabilities, which require only the inequalities $x_i>0$ and $t\in(0,1)$.

Lemma A.10 (Truncated geometric distribution). Fix $t\in(0,1)$ and an integer $r\geq 1$. Define

$$p_r(j) := \frac{(1-t)t^{j-1}}{1-t^r}, \qquad j=1,\dots,r.$$

Then $p_r(j)\geq 0$ for all j, and $\sum_{j=1}^{r} p_r(j)=1$.

Proof. Nonnegativity is clear since $0<t<1$ implies $1-t>0$, $t^{j-1}\geq 0$, and $1-t^r>0$. For the sum:

$$\sum_{j=1}^{r} p_r(j) = \frac{1-t}{1-t^r}\sum_{j=0}^{r-1} t^j = \frac{1-t}{1-t^r}\cdot\frac{1-t^r}{1-t} = 1.$$

Lemma A.11 (Positivity of clock rates). If $x_i>0$ then the rate $1/x_i$ is well-defined and strictly positive. Consequently, for any finite family of independent Poisson clocks with rates $\{1/x_i\}_{i=1}^{n}$, the resulting continuous-time update scheme defines a valid continuous-time Markov chain on any finite state space.

Proof. If $x_i>0$ then $(1/x_i)\in(0,\infty)$ for each i, so each clock has a finite strictly positive rate. Let $\mathcal{S}$ be a finite state space, and suppose that for each $i\in[n]$ an update rule is specified (for example, by a deterministic map $U_i : \mathcal{S} \to \mathcal{S}$ or more generally by a Markov kernel $K_i(s, \cdot)$ on $\mathcal{S}$). Let N_i be independent Poisson processes with rates $1/x_i$, and let $T_1 < T_2 < \cdots$ be the ordered union of their jump times. Since $\sum_{i=1}^n 1/x_i < \infty$, the superposition $N := \sum_i N_i$ is a Poisson process of rate $\sum_{i=1}^n 1/x_i$, so almost surely it has finitely many jumps in every bounded time interval, and almost surely no two clocks ring at the same time. Define a càdlàg process X_t on $\mathcal{S}$ by holding the state constant on each interval $[T_k, T_{k+1})$ and, at time T_k, applying the update corresponding to the unique clock that rang at T_k: if that clock is i, then set $X_{T_k} := U_i(X_{T_k^-})$ in the deterministic case, or sample X_{T_k} from $K_i(X_{T_k^-}, \cdot)$ in the Markov-kernel case. This construction is non-explosive and has the Markov property by the independent increments of Poisson processes. Therefore it defines a valid continuous-time Markov chain on $\mathcal{S}$ (with generator given by the usual sum of rate-times-update contributions).

A.4 Normalization conventions for interpolation polynomials

This paper concerns two families of interpolation polynomials:

- interpolation ASEP polynomials $F^*_\mu(x; q, t)$ indexed by compositions $\mu \in \mathbb{Z}^n_{\geq 0}$, and
- interpolation Macdonald polynomials $P^*_\lambda(x; q, t)$ indexed by partitions λ.

Different normalizations appear in the literature. We fix the following convention, chosen to match the statement of the target stationary distribution and the combinatorial partition-function representation used later.

Convention A.12 (Normalization and identification by partition functions). We adopt the normalization in which:

1. There exists a finite "queue" state space $\Omega(\lambda)$ (depending on λ) and a weight function $\mathrm{wt}(\cdot; x, q, t)$ such that for each $\mu \in S_n(\lambda)$,

$$F^*_\mu(x; q, t) = \sum_{\omega:\Phi(\omega)=\mu} \mathrm{wt}(\omega; x, q, t),$$

   where Φ is the bottom-row map $\Omega(\lambda) \to S_n(\lambda)$, and

2. The partition function equals

$$P^*_\lambda(x; q, t) = \sum_{\omega\in\Omega(\lambda)} \mathrm{wt}(\omega; x, q, t) = \sum_{\mu\in S_n(\lambda)} F^*_\mu(x; q, t).$$

   The full construction of $\Omega(\lambda)$, Φ, and wt (including the "signed" nature before positivization) is given in Appendix B, and the proof of the identities eq:app-a-6–eq:app-a-7 is given in Appendix C.

Remark A.13 (Why this convention suffices). All probabilistic statements in the main text depend only on the ratios $F^*_\mu(x;1,t)/P^*_\lambda(x;1,t)$ and on the existence of a local weight model. Therefore, it is both natural and technically convenient to fix normalization through the partition-function representation eq:app-a-6–eq:app-a-7 rather than through vanishing/interpolation axioms.

A.5 Regularity of the $q = 1$ specialization in the weight model

For the probabilistic construction we require that the specialization at $q = 1$ is well-defined for the weights appearing in eq:app-a-6–eq:app-a-7. We record the needed regularity as an explicit standing hypothesis, verified later from the concrete formulas.

Hypothesis A.14 ($q = 1$ regularity of local weights). For each fixed λ and each $\omega \in \Omega(\lambda)$, the weight $\mathrm{wt}(\omega; x, q, t)$ is an element of $\mathbb{K}[x]$ that is regular at $q = 1$. Moreover, the specialization $\mathrm{wt}(\omega; x, 1, t)$ is well-defined for all $t \in (0, 1)$.

Under Hypothesis A.14 and Lemma A.8, the finite sums in eq:app-a-6–eq:app-a-7 are regular at $q = 1$, and thus $F^*_\mu(x;1,t)$ and $P^*_\lambda(x;1,t)$ are well-defined elements of $\mathbb{Q}(t)[x]$.

Lemma A.15 (Well-defined specialization of the partition functions). Assume Hypothesis A.14. Then for each $\mu \in S_n(\lambda)$, both $F^*_\mu(x;q,t)$ and $P^*_\lambda(x;q,t)$ are regular at $q = 1$, and

$$F^*_\mu(x;1,t) = \sum_{\omega:\Phi(\omega)=\mu} \mathrm{wt}(\omega; x, 1, t), \qquad P^*_\lambda(x;1,t) = \sum_{\omega\in\Omega(\lambda)} \mathrm{wt}(\omega; x, 1, t).$$

Proof. Apply Lemma A.8 to the finite sums defining F^*_μ and P^*_λ, using regularity of each summand at $q = 1$.

Remark A.16 (Verification of Hypothesis A.14). Hypothesis A.14 is verified in Appendix B and Appendix C by inspection of the explicit local weight factors in the queue model (they are rational in (q, t) with denominators that remain nonzero at $q = 1$ for $t \in (0, 1)$). Since this verification depends on the concrete combinatorial definitions, we defer it to those appendices.

A.6 Probability interpretation of the stationary ratio

Finally, we isolate the purely algebraic step that turns partition functions into a probability distribution once positivity is established.

Lemma A.17 (Normalization by a finite positive partition function). Let $\mathcal{S}$ be a finite set, and let $W : \mathcal{S} \to \mathbb{R}_{\geq 0}$ be nonnegative weights with total mass $Z := \sum_{s\in\mathcal{S}} W(s) > 0$. Then

$$\pi(s) := \frac{W(s)}{Z}$$

defines a probability distribution on $\mathcal{S}$.

Proof. Immediate.

Remark A.18 (Application in this paper). In the main text, $\mathcal{S} = S_n(\lambda)$ and $W(\mu) = F^*_\mu(x; 1, t)$ is realized as a fiber sum of a nonnegative weight model after positivization (Appendix E). The partition function $Z = P^*_\lambda(x; 1, t)$ is then strictly positive, and eq:app-a-9 yields the stationary distribution formula.

End of Appendix A

The remaining appendices supply: *i* the explicit queue model and weights (Appendix B), *ii* the identification of fiber sums with interpolation polynomials (Appendix C), (iii) probabilistic normalization of the local two-line kernel at $q = 1$ (Appendix D), and *iv* the restricted strict positivity mechanism (Appendix E) that enables a genuine Markov-chain interpretation.

Appendix B. Signed Multiline Queues and the Combinatorial Weight Model

This appendix defines the finite configuration space $\Omega(\lambda)$ of **signed multiline queues** associated with a partition λ, together with:

- a **bottom-row map** $\Phi : \Omega(\lambda) \to S_n(\lambda)$, and
- a **weight function** $\mathrm{wt}(\cdot; x, q, t) \in \mathbb{Q}(q, t)[x_1, \dots, x_n]$,

whose fiber sums realize the queue partition functions used in the main text. The identification of these partition functions with interpolation polynomials is proved in Appendix C; here we only define the combinatorial objects and their weights.

Throughout, we fix integers $n \geq 1$ and a partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_n \geq 0)$ with largest part $L := \lambda_1$. We work on a cylinder with n columns indexed cyclically by $[n] = 1, \dots, n$.

B.1 Type data and cylindrical words

Definition B.1 (Type vector of a partition). Let $L = \lambda_1$. Define the **type vector**

$$m(\lambda) := (m_0, m_1, \dots, m_L), \qquad m_r := |\{j \in [n] : \lambda_j = r\}|.$$

Thus $\sum_{r=0}^{L} m_r = n$.

Definition B.2 (Row-shape truncations $\lambda^{(r)}$). For each $r \in 1, \dots, L$, define the multiset (encoded as a length-(n) weakly decreasing word)

$$\lambda^{(r)} := \langle L^{m_L}, (L-1)^{m_{L-1}}, \dots, r^{m_r}, 0^{m_{r-1}+\cdots+m_0} \rangle.$$

Equivalently, $\lambda^{(r)}$ is obtained from λ by replacing all parts $< r$ by (0).

Definition B.3 (Permutations and signed permutations of $\lambda^{(r)}$). A **permutation of** $\lambda^{(r)}$ is a word $\mu = (\mu_1, \dots, \mu_n) \in 0, r, r+1, \dots, L^n$ whose multiset of entries equals $\lambda^{(r)}$. A **signed permutation of** $\lambda^{(r)}$ is a word $\alpha = (\alpha_1, \dots, \alpha_n) \in 0, \pm r, \pm(r+1), \dots, \pm L^n$ such that $(|\alpha_1|, \dots, |\alpha_n|)$ is a permutation of $\lambda^{(r)}$.

Definition B.4 (Cylindrical order). We regard the column set ([n]) as cyclic. For $i, j \in [n]$, the **clockwise interval** $(i, j) \subset [n]$ is

$$(i, j) := \begin{cases} i+1, i+2, \dots, j-1, & i < j, \\ i+1, \dots, n, 1, \dots, j-1, & i > j, \\ \emptyset, & i = j, \end{cases}$$

where indices are interpreted modulo n.

B.2 Ball systems and classic layers

Definition B.5 (Ball system of type λ). A **ball system** B of type λ is an $L \times n$ array whose rows are indexed from bottom to top by $r = 1, 2, \dots, L$ and whose columns are indexed by ([n]), such that:

1. Each entry is either (0) (empty) or a **regular ball** labeled by an integer in $1, \dots, L$.
2. For each row r, the length-n word obtained by reading that row left-to-right is a permutation of $\lambda^{(r)}$.

In particular, row r contains exactly $\sum_{a=r}^{L} m_a$ balls.

Definition B.6 (Strands and trivial pairings). A **strand** from column j in an upper row to column j' in the adjacent lower row is drawn as a shortest path that moves either straight down (if $j = j'$) or down and then rightward along the cylinder to reach j', and then down to the lower row.

A pairing is **trivial** if it connects two balls in the same column $j = j'$ (a vertical segment). A pairing is **nontrivial** otherwise.

Definition B.6.1 (Crossing of strands in a two-row layer). Fix two adjacent rows and consider a collection of strands drawn as in Definition B.6, connecting balls in the upper row to balls in the lower row. For two distinct

nontrivial strands $p_1 : j_1 \to j_1'$ and $p_2 : j_2 \to j_2'$ drawn in the same layer, fix an unrolling of the cylinder at a cut that avoids all strand endpoints in that layer and for which neither of the two strands wraps around the cut (so each strand moves weakly to the right in the unrolled picture). We say that p_1 and p_2 **cross** if, in the resulting linear order of columns, the endpoints form an inversion:

$$j_1 < j_2 \text{ and } j_1' > j_2' \qquad \text{or} \qquad j_2 < j_1 \text{ and } j_2' > j_1'.$$

In this right-moving setting, the inversion criterion is equivalent to the geometric fact that the two strand drawings intersect in the interior of the strip; see Lemma B.6.3.

Trivial (vertical) pairings are excluded from this definition.

Convention B.6.2 (Noncrossing admissibility). In every classic layer and signed layer, admissible pairing data are required to admit a shortest-strand drawing (Definition B.6) in which **no two strands in the same layer cross** in the sense of Definition B.6.1.

Lemma B.6.3 (Inversion forces an intersection for right-moving shortest strands). Consider two nontrivial strands in a fixed two-row layer, drawn by the rule of Definition B.6, and unroll the cylinder so that neither strand wraps around the cut (so each strand moves weakly to the right in the unrolled picture). If their endpoints satisfy $j_1 < j_2$ but $j_1' > j_2'$, then the two strand drawings intersect in the interior of the strip, and hence the strands cross in the sense of Definition B.6.1.

Proof. Under the chosen unrolling, nontriviality implies $j_1 < j_1'$ and $j_2 < j_2'$. Together with $j_1 < j_2$ and $j_1' > j_2'$, we obtain $j_1 < j_2 < j_2' < j_1'$. Write each strand as a down-right-down path, so p_k consists of a vertical segment in column j_k, a horizontal segment from j_k to j_k' at some intermediate height, and a vertical segment in column j_k'. Let h_1 (resp. h_2) be the height of the horizontal segment of p_1 (resp. p_2) in the unrolled picture. Since j_2 and j_2' lie strictly between j_1 and j_1', the horizontal segment of p_1 passes through both columns j_2 and j_2'.

- If $h_1 \le h_2$, then the vertical segment of p_2 in column j_2 reaches height h_1, so it meets the horizontal segment of p_1 at the interior point (j_2, h_1).
- If $h_2 \le h_1$, then the vertical segment of p_2 in column j_2' passes through height h_1, so it meets the horizontal segment of p_1 at the interior point (j_2', h_1).

In either case the drawings intersect in the interior of the strip, so the strands cross.

Lemma B.6.4 (Order-preserving criterion for noncrossing). Fix a two-row layer and unroll the cylinder at a cut so that **no strand in the layer wraps** (so every nontrivial strand moves weakly to the right in the unrolled picture). List the nontrivial strands in increasing order of their source columns:

$$p_k : j_k \to j'_k, \qquad j_1 < j_2 < \cdots < j_m.$$

If the destinations are weakly increasing,

$$j'_1 \le j'_2 \le \cdots \le j'_m,$$

then no two strands cross in the sense of Definition B.6.1. Equivalently, a crossing occurs if and only if there exist $k < \ell$ with $j'_k > j'_\ell$.

Proof.
If $j'_k > j'_\ell$ for some $k < \ell$, then $j_k < j_\ell$ but $j'_k > j'_\ell$, so the endpoints form an inversion and the corresponding strands cross by Definition B.6.1 (equivalently, by Lemma B.6.3 after the chosen unrolling). Conversely, if two strands cross, then by Definition B.6.1 their endpoints form an inversion, so for their indices $k < \ell$ we have $j'_k > j'_\ell$, contradicting weak monotonicity of the destination sequence.

Definition B.7 (Classic layer). Let $r \ge 2$. A set of pairings between row r (upper) and row (r-1) (lower) is a **classic layer** if:

1. Each ball in row r labeled a is paired to exactly one ball in row (r-1) labeled a.
2. Each pairing is drawn by a shortest strand, and wrapping around the cylinder is permitted.
3. (Local constraint) If a ball labeled a in row r has directly below it in row (r-1) a ball labeled (a'), then $a' \ge a$; and if $a' = a$, then these two balls must be trivially paired.

Definition B.8 (Multiline queue). Fix $\mu \in S_n(\lambda)$. A **multiline queue** of type μ is a ball system B of type λ together with, for each $r = 2, \dots, L$, a classic layer of pairings between row r and row (r-1), such that the bottom row (row (1)) equals μ as a word in $0, 1, \dots, L^n$. We write MLQ(μ) for the finite set of all such multiline queues.

B.3 Enhanced ball systems and signed layers

Signed multiline queues are defined on a $2L \times n$ "enhanced" array with alternating regular and signed rows.

Definition B.9 (Enhanced ball system of type λ). An **enhanced ball system** $B^\pm$ of type λ is a $2L \times n$ array whose rows are indexed (bottom to top) by

$$1,\ 1',\ 2,\ 2',\ \dots,\ L,\ L',$$

and whose columns are indexed by ([n]), such that:

1. Each **regular row** r contains only regular balls labeled in $1, \dots, L$, and its row word is a permutation of $\lambda^{(r)}$.
2. Each **signed row** (r') contains signed balls labeled in $\pm 1, \dots, \pm L$, and its row word is a signed permutation of $\lambda^{(r)}$.

A signed ball with positive (resp. negative) label is called a **positive** (resp. **negative**) ball.

Definition B.10 (Signed layer). Fix $r \in 1, \dots, L$. A set of pairings between row (r') (upper) and row r (lower) is a **signed layer** if:

1. Each signed ball labeled $\pm a$ in row (r') is paired to exactly one regular ball labeled a in row r.
2. Each pairing is drawn by a shortest strand that moves straight down or to the right, and **does not wrap around** the cylinder.
3. (Local constraints in the column directly below)
   - If $\alpha_j = +a$ in row (r') at column j, then the entry below in row r at column j is a regular ball labeled (a') with $a' \geq a$; and if $a' = a$, then the pairing must be trivial.
   - If $\alpha_j = -a$ in row (r') at column j, then the entry below in row r at column j is either empty, or a regular ball labeled (a') with $a \geq a'$.

Definition B.11 (Signed multiline queue). Fix $\mu \in S_n(\lambda)$. A **signed multiline queue** Q of type μ consists of:

1. An enhanced ball system $B^{\pm}$ of type λ, and
2. Pairings between adjacent rows satisfying:
   - For each $r = 1, \dots, L$, the pairings between row (r') and row r form a signed layer.
   - For each $r = 2, \dots, L$, if we ignore the signs on row ((r-1)‘), then the pairings between row r and row ((r-1)') form a classic layer (in the sense of Definition B.7 applied to absolute values).

Additionally, the bottom regular row (row (1)) is the word μ.

We write $\mathrm{MLQ}^{\pm}(\mu)$ for the finite set of all signed multiline queues of type μ.

Definition B.12 (Global configuration space and bottom map). Define the global state space

$$\Omega(\lambda) := \bigsqcup_{\mu \in S_n(\lambda)} \mathrm{MLQ}^{\pm}(\mu).$$

Define the **bottom map**

$$\Phi : \Omega(\lambda) \to S_n(\lambda), \qquad \Phi(Q) := (\text{bottom regular row of (Q)}).$$

—

B.4 Pairing orders and local statistics

Weights depend on local "counts" computed by placing strands in a deterministic order within each adjacent-row layer.

Definition B.13 (Classic-layer placement order and statistics). Fix a classic layer between an upper row r and a lower row (either (r-1) or ((r-1)'); the definition is identical after taking absolute values on the lower row). We place strands in the following order:

- read upper-row balls in **decreasing** label a, and within fixed a, from **right to left** (decreasing column index).

Consider a nontrivial pairing p that matches an upper ball labeled a in column j to a lower ball (of absolute label (a)) in column (j').

- Just before placing p, a lower-row ball is called **free for** p if it has not yet been matched by previously placed strands, and it is not reserved for a forced trivial pairing of the same label a in the same column.
- Let free(p) be the number of free lower-row balls at that moment.
- Let skip(p) be the number of free lower-row balls whose columns lie in the cyclic interval (j, j') from eq:app-b-3.
- (Clarification) Here "free lower-row balls" means free lower-row balls of absolute label a, i.e. the admissible candidate destinations for the a-strand at the moment p is placed in the deterministic order. Thus free(p) counts the number of admissible destinations and skip(p) counts how many admissible destinations lie strictly between the source column j and the chosen destination column j' in the relevant cyclic order.

Definition B.14 (Signed-layer placement order and statistics). Fix a signed layer between an upper signed row (r') and lower regular row r. We place strands in the following order:

- read signed balls in row (r') by **decreasing absolute label** (|a|), and within fixed (|a|), from **right to left**.

Consider a nontrivial pairing p that matches a signed ball labeled $\pm a$ in column j to a regular ball labeled a in column k, where (by definition of signed layers) $k > j$ and no wrapping occurs.

- Just before placing p, a lower-row ball is called **free for** p if it has not yet been matched by previously placed strands in this signed layer.
- Let skip(p) be the number of free lower-row balls in columns $j + 1, \dots, k - 1$.
- Let empty(p) be the number of empty positions (zeros) in row r in columns $j + 1, \dots, k - 1$.

B.5 Weight function

We now define the weight $\mathrm{wt}(Q; x, q, t)$ of a signed multiline queue $Q \in \Omega(\lambda)$. The weight factors into a **shifted ball weight** and a **pairing weight**.

Definition B.15 (Shifted ball weight). Let $Q \in \Omega(\lambda)$. For each signed row (r'), let $\alpha^{(r)} = (\alpha_1^{(r)}, \ldots, \alpha_n^{(r)})$ denote its signed word. Define the **row shifted ball weight**

$$\mathrm{wt}_{\mathrm{ball}}(r') := \Big(\prod_{i:\ \alpha_i^{(r)} > 0} x_i \Big) \cdot \Big(\prod_{i:\ \alpha_i^{(r)} < 0} \Big(-\frac{q^{r-1}}{t^{n-1}} \Big) \Big).$$

Define the **shifted ball weight of** Q by

$$\mathrm{wt}_{\mathrm{ball}}(Q) := \prod_{r=1}^{L} \mathrm{wt}_{\mathrm{ball}}(r').$$

Definition B.16 (Local pairing weights). Let p be a nontrivial pairing.

1. **Classic-layer pairing.** Suppose p belongs to a classic layer between an upper row r and a lower row, and it matches an upper ball labeled a in column j to a lower ball of absolute label a in column (j'). Define

$$\mathrm{wt}_{\mathrm{pair}}(p) := \frac{(1-t)t^{\mathrm{skip}(p)}}{1 - q^{a-r+1}t^{\mathrm{free}(p)}} \times \begin{cases} q^{a-r+1}, & j' < j \ \ \text{(wrap-around)}, \\ 1, & j' > j. \end{cases}$$

2. **Signed-layer pairing.** Suppose p belongs to a signed layer between (r') (upper) and r (lower), and it matches a signed ball labeled $\pm a$ in column j to a regular ball labeled a in column $k > j$. Define

$$\mathrm{wt}_{\mathrm{pair}}(p) := \begin{cases} (1-t)t^{\mathrm{skip}(p)+\mathrm{empty}(p)}, & \text{if the upper ball is } +a, \\ -(1-t)t^{\mathrm{skip}(p)+\mathrm{empty}(p)}, & \text{if the upper ball is } -a. \end{cases}$$

Definition B.17 (Pairing weight and total weight). For $Q \in \Omega(\lambda)$, define the **pairing weight**

$$\mathrm{wt}_{\mathrm{pair}}(Q) := \prod_{p \in \mathcal{P}_{\mathrm{nt}}(Q)} \mathrm{wt}_{\mathrm{pair}}(p),$$

where $\mathcal{P}_{\mathrm{nt}}(Q)$ is the finite set of all nontrivial pairings in Q. Define the **total weight**

$$\mathrm{wt}(Q;x,q,t) := \mathrm{wt}_{\mathrm{ball}}(Q)\,\mathrm{wt}_{\mathrm{pair}}(Q) \;\in\; \mathbb{Q}(q,t)[x_1,\dots,x_n].$$

Remark B.18 (Dependence on (x)). For a fixed signed queue Q, the x-dependence arises only through the factors x_i contributed by positive signed balls in eq:app-b-7; hence $\mathrm{wt}(Q;x,q,t)$ is a polynomial in x with coefficients rational in (q,t).

B.6 Partition functions and basic finiteness/regularity facts

Definition B.19 (Queue partition functions). For $\mu \in S_n(\lambda)$, define the **fiber partition function**

$$F^*_\mu(x;q,t) \;:=\; \sum_{Q\in \mathrm{MLQ}^\pm(\mu)} \mathrm{wt}(Q;x,q,t).$$

Define the **total partition function**

$$Z^*_\lambda(x;q,t) \;:=\; \sum_{\mu\in S_n(\lambda)} F^*_\mu(x;q,t) \;=\; \sum_{Q\in\Omega(\lambda)} \mathrm{wt}(Q;x,q,t).$$

Lemma B.20 (Finiteness). For each fixed λ, the sets $\mathrm{MLQ}^\pm(\mu)$ and $\Omega(\lambda)$ are finite. Consequently, the sums eq:app-b-13–eq:app-b-14 are finite sums in $\mathbb{Q}(q,t)[x]$.

Proof. Each row word in an enhanced ball system is a (signed) permutation of a fixed multiset of size n, hence admits finitely many possibilities. Pairings are matchings between finite sets of balls subject to local constraints, hence also finite in number.

Lemma B.21 (Regularity at $q = 1$ for $t \in (0,1)$). Fix $t \in (0,1)$. For every $Q \in \Omega(\lambda)$, the rational function $\mathrm{wt}(Q;x,q,t)$ is regular at $q = 1$. Hence each partition function $F^*_\mu(x;q,t)$ and $Z^*_\lambda(x;q,t)$ is regular at $q = 1$ in the sense of Appendix A.

Proof. The only denominators in $\mathrm{wt}(Q;x,q,t)$ arise from the classic-layer factors $(1-q^{a-r+1}t^{\mathrm{free}(p)})^{-1}$ in eq:app-b-9. For any nontrivial pairing p, we have $\mathrm{free}(p) \geq 1$ (a pairing cannot be placed without at least one available lower-row ball), so at $q = 1$ the denominator becomes $1 - t^{\mathrm{free}(p)} \neq 0$ since $t \in (0,1)$. All other factors are Laurent monomials in q (and powers of t), hence regular at $q = 1$. Finally, finite sums preserve regularity.

Remark B.22 (Strict and restricted strict specializations). When λ is strict, each multiplicity $m_r \in 0, 1$, and the orbit $S_n(\lambda)$ has cardinality (n!). Under the additional restriction $\lambda_n = 0$ and $\lambda_i \neq 1$, the combinatorics of admissible sign patterns and the cancellation structure at $q = 1$ simplify substantially; this is exploited in Appendix E to obtain a nonnegative weight model.

End of Appendix B

Appendix C proves that the partition functions $F^*_\mu(x; q, t)$ and $Z^*_\lambda(x; q, t)$ coincide with the relevant interpolation polynomials under the normalization adopted in Appendix A.

Appendix C. Identification of the Queue Partition Functions with Interpolation Polynomials

This appendix justifies the identification (used throughout the main text) between:

- the **combinatorial** partition functions defined from signed multiline queues in Appendix B, and
- the **algebraic** interpolation polynomials from the Hecke-algebraic theory.

Concretely, we show that the signed-MLQ generating function equals the **interpolation ASEP polynomial**, and that its orbit-sum equals the **interpolation Macdonald polynomial**. (arXiv)

C.0 Notational warning

Appendix B used the symbol $F^*_\mu(x; q, t)$ for the **signed-MLQ weight-generating function**. In the broader literature, the **interpolation ASEP polynomial** is typically denoted $f^*_\mu(x; q, t)$. For clarity in this appendix only, we write:

- $\mathcal{F}^*_\mu(x; q, t)$ for the **combinatorial** signed-MLQ generating function (Appendix B), and
- $f^*_\mu(x; q, t)$ for the **algebraic** interpolation ASEP polynomial (Definition C.7 below).

Theorem C.12 proves $\mathcal{F}^*_\mu = f^*_\mu$, so the overloading of notation in the main text is harmless.

C.1 Interpolation points

We recall the specialization points at which interpolation polynomials are characterized by vanishing.

Definition C.1 (The statistic $k_i(\mu)$). Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$. For each $i \in [n]$, define

$$k_i(\mu) \ := \ |\{j < i : \ \mu_j > \mu_i\}| \ + \ |\{j > i : \ \mu_j \geq \mu_i\}|.$$

This is the standard statistic used to define the interpolation evaluation points. (arXiv)

Definition C.2 (Interpolation point $\tilde{\mu}$). Define

$$\tilde{\mu} \ := \ (q^{\mu_1} t^{-k_1(\mu)}, \ \dots, \ q^{\mu_n} t^{-k_n(\mu)}) \ \in \ \mathbb{Q}(q,t)^n.$$

When evaluating a polynomial $g(x_1, \dots, x_n)$ at $\tilde{\mu}$, we substitute $x_i = q^{\mu_i} t^{-k_i(\mu)}$. (arXiv)

C.2 Interpolation Macdonald polynomials

We use the Knop–Sahi characterization via vanishing.

Theorem C.3 (Interpolation Macdonald polynomials; Knop–Sahi). For each partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$, there exists a unique inhomogeneous symmetric polynomial

$$P^*_\lambda(x; q, t) \ \in \ \mathbb{Q}(q,t)[x_1, \dots, x_n]^{S_n}$$

such that:

1. The coefficient of the monomial symmetric function m_λ in P^*_λ is (1), and
2. $P^*_\lambda(\tilde{\nu}) = 0$ for every partition $\nu \neq \lambda$ with $|\nu| \leq |\lambda|$.

Moreover, the top homogeneous component of P^*_λ is the (homogeneous) Macdonald polynomial P_λ. (arXiv)

Remark C.4. In this paper we only require the existence/uniqueness of P^*_λ and the compatibility with Hecke-algebraic constructions recalled below.

C.3 Hecke operators and reduced words

The interpolation ASEP polynomials are defined by applying Hecke operators to nonsymmetric interpolation Macdonald polynomials.

Definition C.5 (Hecke operators on $\mathbb{Q}(q,t)[x]$). For $1 \leq i \leq n-1$, let s_i act on polynomials by swapping variables $x_i \leftrightarrow x_{i+1}$. Define the Hecke operator T_i by

$$(T_i f)(x) \ := \ t\, f(x) \ - \ \frac{t x_i - x_{i+1}}{x_i - x_{i+1}} (f(x) - f(s_i x)).$$

These operators satisfy the type-A_{n-1} Hecke relations:

$$(T_i - t)(T_i + 1) = 0, \qquad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \qquad T_i T_j = T_j T_i \ \ (|i-j| > 1).$$

(arXiv)

Definition C.6 (Hecke operator T_σ). If $\sigma \in S_n$ has a reduced decomposition $\sigma = s_{i_1} \cdots s_{i_\ell}$, define

$$T_\sigma \ := \ T_{i_1} \cdots T_{i_\ell}.$$

By the braid relations in eq:app-c-5, this definition is independent of the choice of reduced decomposition. (arXiv)

C.4 Interpolation ASEP polynomials

Fix a partition λ of length n. Let $E^*_\lambda(x; q, t)$ denote the nonsymmetric interpolation Macdonald polynomial of index λ (Knop–Sahi). The precise axioms of E^*_λ are standard and can be found in the literature; for our purposes, it suffices that these polynomials exist and are stable under the Hecke action. (arXiv)

Definition C.7 (Interpolation ASEP polynomials). Let λ be a partition of length n, and let $\mu \in S_n(\lambda)$. Let $\sigma_\mu \in S_n$ be the shortest permutation such that $\sigma_\mu(\lambda) = \mu$. Define the **interpolation ASEP polynomial**

$$f^*_\mu(x; q, t) \ := \ T_{\sigma_\mu} \cdot E^*_\lambda(x; q, t).$$

In particular, $f^*_\lambda = E^*_\lambda$. (arXiv)

Remark C.8 (Top homogeneous component). The top homogeneous component of f^*_μ is the (homogeneous) ASEP polynomial f_μ. (arXiv)

C.5 From interpolation ASEP to interpolation Macdonald

The symmetric interpolation Macdonald polynomial is recovered by summing the interpolation ASEP polynomials over the orbit.

Proposition C.9 (Orbit-sum identity). For any partition λ of length n,

$$P^*_\lambda(x; q, t) \ = \sum_{\mu \in S_n(\lambda)} f^*_\mu(x; q, t).$$

(arXiv)

Proof. This identity is stated explicitly as Proposition 2.15 in the reference. We include a brief sketch for completeness: the orbit-sum $g := \sum_{\mu \in S_n(\lambda)} f^*_\mu$ lies in the Hecke-stable space generated by $E^*_\mu : \mu \in S_n(\lambda)$, satisfies the interpolation vanishing conditions that characterize P^*_λ, and is symmetric by the Hecke relations (equivalently $T_i g = tg$ for all i). The normalization is fixed by comparing top homogeneous components. ([arXiv][1, Proposition 2.15])

C.6 The signed-MLQ generating functions

We now recall the combinatorial partition functions defined in Appendix B.

Definition C.10 (Combinatorial partition functions; recall). Fix λ and the signed multiline queue state space $\Omega(\lambda)$ from Appendix B. For $\mu \in S_n(\lambda)$, define

$$\mathcal{F}^*_\mu(x; q, t) \ := \sum_{Q \in \mathrm{MLQ}^{\pm}(\mu)} \mathrm{wt}(Q; x, q, t),$$

and define the orbit-sum (total partition function)

$$Z^*_\lambda(x; q, t) \ := \sum_{\mu \in S_n(\lambda)} \mathcal{F}^*_\mu(x; q, t) \ = \sum_{Q \in \Omega(\lambda)} \mathrm{wt}(Q; x, q, t).$$

This matches eq:app-b-13–eq:app-b-14 after the temporary notational change $F \mapsto \mathcal{F}$. (arXiv)

Lemma C.11 (Agreement with the standard signed-MLQ model). The definitions of:

- signed multiline queues (enhanced ball systems, classic layers, signed layers), and
- the weight function $\mathrm{wt} = \mathrm{wt}_{\mathrm{ball}} \mathrm{wt}_{\mathrm{pair}}$,

used in Appendix B are equivalent to the standard definitions in the signed-MLQ model for interpolation polynomials, i.e. the shifted ball-weight factors ((5))–((6)) and the local pairing weights ((4)) and ((7)). (arXiv)

Proof. This is a direct comparison of Definitions B.9–B.17 with the standard signed-MLQ definitions and weight factors cited above. In particular, the classic-layer pairing weights eq:app-b-9 coincide with ((4)), the signed-layer pairing weights eq:app-b-10 coincide with ((7)), and the shifted ball-weight eq:app-b-7–eq:app-b-8 coincides with ((5))–((6)). (arXiv)

C.7 Identification theorem

We can now identify the combinatorial partition functions with the interpolation polynomials.

Theorem C.12 (Signed-MLQ formula for interpolation polynomials). Let λ be a partition of length n, and let $\mu \in S_n(\lambda)$. Then:

1. The interpolation ASEP polynomial equals the signed-MLQ generating function:

$$f^*_\mu(x;q,t) \;=\; \mathcal{F}^*_\mu(x;q,t).$$

2. The interpolation Macdonald polynomial equals the signed-MLQ partition function:

$$P^*_\lambda(x;q,t) \;=\; Z^*_\lambda(x;q,t).$$

   (arXiv)

Proof. By Lemma C.11, our combinatorial objects and weights coincide with the signed-MLQ model of the reference (Definitions 1.4, 1.6, 1.11 and the pairing-weight formula (4)). Therefore the equality eq:app-c-11 is exactly Theorem 1.15 of the reference, which identifies interpolation ASEP polynomials with signed-MLQ weight-generating functions. In particular, $\mathcal{F}^*_\mu(x;q,t) = f^*_\mu(x;q,t)$. ([arXiv][1, Theorem 1.15])

For eq:app-c-12, combine Proposition C.9 with eq:app-c-11 and the definition of Z^*_λ as an orbit-sum:

$$Z^*_\lambda(x;q,t) = \sum_{\mu \in S_n(\lambda)} \mathcal{F}^*_\mu(x;q,t) = \sum_{\mu \in S_n(\lambda)} f^*_\mu(x;q,t) = P^*_\lambda(x;q,t).$$

This completes the proof. ([arXiv][1, Theorem 1.15; Proposition 2.15])

End of Appendix C

Appendix D will use the explicit locality of the two-line layer weights at $q = 1$ to define and verify the Markov transition kernel used in the main text, and Appendix E will establish the restricted-strict positivity mechanism needed for a genuine probabilistic interpretation.

Appendix D. The $q = 1$ Two-line Kernel: Normalization and the Truncated Geometric Law

This appendix proves the probabilistic normalization used in Section 6.1 of the main text: at $q = 1$, the local two-line queue weights in a **classic layer** define a bona fide probability kernel whose one-step choice rule is a **truncated**

geometric distribution. This is the mechanism underlying the explicit "scan-and-reject" update used in the lifted dynamics.

We work in the setting of Appendix B. In particular, "classic layers" refer to adjacent-row pairings of the type used in Definition B.7, and the local classic-layer pairing weight is given by eq:app-b-9.

D.1 A single-label two-line slice and candidate ordering

For normalization, it is enough to isolate the randomness associated with a **single label** a in a classic layer.

Definition D.1 (Two-line single-label slice at label (a)). Fix a classic layer between an upper row and a lower row (as in Appendix B). Fix a label $a \geq 1$. Let:

- $U_a \subset [n]$ be the set of columns in the **upper row** that carry a ball of label a, and
- $L_a \subset [n]$ be the set of columns in the **lower row** that carry a ball of label a.

In a classic layer, we must produce a perfect matching between U_a and L_a (each upper (a)-ball is paired to exactly one lower (a)-ball). We call such a matching an a**-pairing**.

In addition, certain pairings may be *forced* by the local constraint "if the ball directly below has the same label then the pairing must be trivial." In such cases there is no randomness; we treat forced trivial pairings as deterministic choices of probability (1).

Definition D.2 (Cyclic order of candidates to the right). Fix a column $j \in [n]$. For any subset $C \subset [n] \setminus j$, define the **rightward cyclic order from** j to be the list obtained by scanning

$$j+1,\ j+2,\ \dots,\ n,\ 1,\ 2,\ \dots,\ j-1$$

and recording those indices that lie in C, in that order. (This allows wrap-around.)

If $C = c_1, \dots, c_r$ listed in this order, we write $c_{(j)}(k)$ for the k-th element c_k, $k = 1, \dots, r$.

Definition D.3 (Free candidates, skip, and free counts at $q = 1$). Consider constructing an a-pairing sequentially by processing upper a-balls in a fixed deterministic order (e.g. right-to-left in U_a, consistent with Definition B.13). At a given step, let $C \subseteq L_a$ be the set of **free** lower a-balls not yet matched.

For the current upper a-ball at column j, we define:

- $r := |C|$, the number of free candidates;
- order C by the rightward cyclic order from j (Definition D.2): $C = c_{(j)}(1), \ldots, c_{(j)}(r)$;
- if we choose the candidate $c_{(j)}(k)$, we define

$$\mathrm{skip} := k - 1, \qquad \mathrm{free} := r.$$

Remark D.4 (Relation to Appendix B statistics). In Appendix B, the classic-layer weight eq:app-b-9 involves skip(p) and free(p). In the signed-MLQ model, these are computed relative to the candidate set relevant to the current pairing (here: free lower a-balls). In this appendix we use the explicit formulation eq:app-d-1, which is the one needed for normalization at $q = 1$.

D.2 $q = 1$ specialization of the classic pairing weight

Recall the classic-layer pairing weight from Appendix B:

$$\mathrm{wt}_{\mathrm{pair}}(p) = \frac{(1-t)t^{\mathrm{skip}(p)}}{1 - q^{a-r+1}t^{\mathrm{free}(p)}} \times (\text{wrap factor}),$$

where the wrap factor equals q^{a-r+1} if the strand wraps and 1 otherwise; see eq:app-b-9.

At $q = 1$, the wrap factor becomes 1 and $q^{a-r+1} = 1$. Therefore:

Lemma D.5 (Classic pairing weight at $q = 1$). For any nontrivial classic-layer pairing p, the $q = 1$ specialization of eq:app-b-9 is

$$\mathrm{wt}_{\mathrm{pair}}(p)\big|_{q=1} = \frac{(1-t)t^{\mathrm{skip}(p)}}{1 - t^{\mathrm{free}(p)}}.$$

In particular, at $q = 1$ the weight depends on the pairing only through the pair $(\mathrm{skip}(p), \mathrm{free}(p))$.

Proof. Substitute $q = 1$ into eq:app-b-9 and note that the wrap factor becomes 1.

Using eq:app-d-1, if the chosen candidate is the k-th free lower a-ball to the right (cyclic) from the upper column j, then skip $= k - 1$ and free $= r$, hence

$$\mathrm{wt}_{\mathrm{pair}}(p)\big|_{q=1} = \frac{(1-t)t^{k-1}}{1 - t^{r}}.$$

—

D.3 The truncated geometric law and its scan-and-reject interpretation

Equation eq:app-d-3 is exactly the truncated geometric distribution.

Proposition D.6 (Truncated geometric distribution for a single choice). Fix $t \in (0, 1)$ and an integer $r \geq 1$. Define

$$p_r(k) := \frac{(1-t)t^{k-1}}{1-t^r}, \qquad k = 1, \dots, r.$$

Then $p_r(k)_{k=1}^r$ is a probability distribution.

Moreover, it admits the following **scan-and-reject** interpretation: scan candidates in order $k = 1, 2, \dots, r$; at each candidate accept with probability (1-t), otherwise reject with probability t and continue; if all r candidates are rejected, restart the scan from $k = 1$ and repeat until an acceptance occurs. The resulting selection law is eq:app-d-4.

Proof. The fact that $\sum_{k=1}^r p_r(k) = 1$ is Lemma A.10. For the scan-and-reject interpretation, consider a single scan through the ordered candidates. The probability that the first accepted candidate is $k \in 1, \dots, r$ equals $t^{k-1}(1-t)$, and the probability that all r candidates are rejected is t^r. Since scans are independent and we restart after a full rejection, the probability that the algorithm outputs k is

$$\mathbb{P}(K = k) = \sum_{m \geq 0} (t^r)^m \, t^{k-1}(1-t) = \frac{(1-t)t^{k-1}}{1-t^r} = p_r(k), \qquad k = 1, \dots, r.$$

This is exactly eq:app-d-4.

Combining Proposition D.6 with Lemma D.5 shows:

Corollary D.7 (Local probabilistic meaning of $\mathrm{wt}_{\mathrm{pair}}\big|_{q=1}$). Fix a step in the construction of an a-pairing, with r free lower a-candidates. If the chosen candidate is the k-th free lower a-ball to the right (cyclic) of the current upper column, then

$$\mathrm{wt}_{\mathrm{pair}}(p)\big|_{q=1} = p_r(k),$$

the truncated geometric probability of selecting that candidate.

D.4 Sequential construction and global normalization of a classic layer at $q = 1$

We now prove that the product of local factors eq:app-d-2 over all nontrivial pairings in a classic layer yields a globally normalized probability measure on the set of admissible pairings.

To state this cleanly, we fix the row words and the deterministic placement order.

Definition D.8 (Admissible classic-layer pairings and $q = 1$ layer weight). Fix two adjacent rows forming a classic layer (Definition B.7), and fix their row words (including their labels). Let $\mathcal{M}$ be the finite set of admissible complete pairing configurations P between the two rows satisfying the classic-layer constraints.

For $P \in \mathcal{M}$, define its $q = 1$ **layer weight**

$$\mathrm{wt}^{(1)}_{\mathrm{layer}}(P) := \prod_{p \in \mathcal{P}_{\mathrm{nt}}(P)} \frac{(1-t)t^{\mathrm{skip}(p)}}{1-t^{\mathrm{free}(p)}},$$

where the product ranges over the set of nontrivial pairings in P and $(\mathrm{skip}(p), \mathrm{free}(p))$ are computed in the deterministic placement order (with forced trivial pairings contributing no factor, i.e. weight 1).

Theorem D.9 (Normalization of the classic-layer kernel at $q = 1$). With notation as in Definition D.8, we have

$$\sum_{P \in \mathcal{M}} \mathrm{wt}^{(1)}_{\mathrm{layer}}(P) = 1.$$

Equivalently, $P \mapsto \mathrm{wt}^{(1)}_{\mathrm{layer}}(P)$ defines a probability distribution on $\mathcal{M}$.

Proof. We argue by sequential conditioning (equivalently, by induction on the number of nontrivial choices).

Fix a deterministic placement order for upper balls (as in Definition B.13). Consider the first upper ball in this order for which the pairing is not forced to be trivial; let this upper ball have label a and lie in column j. At that moment, let $C \subseteq L_a$ be the set of free lower a-candidates; set $r := |C| \geq 1$. Any admissible complete pairing $P \in \mathcal{M}$ must choose exactly one candidate in C for this upper ball, say the k-th candidate in the rightward cyclic order. By Corollary D.7, the local factor contributed by this choice is precisely $p_r(k)$, with $\sum_{k=1}^{r} p_r(k) = 1$.

For each $k \in 1, \dots, r$, let $\mathcal{M}_k \subseteq \mathcal{M}$ be the subset of admissible pairings in which the first nontrivial choice selects the k-th candidate. Then $\mathcal{M}$ is the disjoint union $\bigsqcup_{k=1}^{r} \mathcal{M}_k$, and for any $P \in \mathcal{M}_k$,

$$\mathrm{wt}^{(1)}_{\mathrm{layer}}(P) = p_r(k) \cdot \mathrm{wt}^{(1)}_{\mathrm{layer}}(P \text{ restricted to the remaining choices}),$$

because the remaining factors correspond to subsequent steps under the same deterministic order after removing the chosen candidate from the free set.

Therefore,

$$\sum_{P\in\mathcal{M}} \mathrm{wt}^{(1)}_{\mathrm{layer}}(P) = \sum_{k=1}^{r} p_r(k) \sum_{P\in\mathcal{M}_k} \mathrm{wt}^{(1)}_{\mathrm{layer}}(\text{remaining part of } P).$$

But the inner sum is (1) by the induction hypothesis applied to the remaining steps (there are strictly fewer nontrivial choices after fixing the first one). Thus

$$\sum_{P\in\mathcal{M}} \mathrm{wt}^{(1)}_{\mathrm{layer}}(P) = \sum_{k=1}^{r} p_r(k) = 1,$$

completing the induction.

Corollary D.9S (Strict case: deterministic normalization). Assume λ is strict. Fix a classic layer with fixed row words. If an admissible pairing configuration exists, then it is unique, and its $q = 1$ weight equals 1. In particular, the classic-layer kernel at $q = 1$ is a point mass and is normalized.

Proof. Fix a label a. Under strictness, the upper row contains at most one ball of label a and the lower row contains at most one ball of label a. Therefore any admissible pairing must pair these two balls (if both are present), and the a-strand is uniquely determined (it may be trivial or nontrivial depending on whether the balls lie in the same column). Repeating over all labels shows the entire pairing configuration is unique whenever it exists.

Trivial pairings contribute weight 1 by definition. If p is a nontrivial classic-layer pairing, then at the moment p is placed there is exactly one admissible lower candidate of label a, so $\mathrm{free}(p) = 1$ and $\mathrm{skip}(p) = 0$. By eq:app-d-5,

$$\mathrm{wt}_{\mathrm{pair}}(p)\big|_{q=1} = \frac{(1-t)t^{\mathrm{skip}(p)}}{1-t^{\mathrm{free}(p)}} = \frac{(1-t)}{1-t} = 1.$$

Therefore the product defining the layer weight at $q = 1$ equals 1.

D.5 Kernel viewpoint and compatibility with local heat-bath updates

Theorem D.9 can be rephrased as: the $q = 1$ specialization of the classic-layer weights defines a stochastic kernel on pairings.

Definition D.10 (Classic-layer pairing kernel at $q = 1$). Fix two adjacent rows forming a classic layer, with fixed row words. Define a kernel $\mathbf{K}^{(1)}$ on the set $\mathcal{M}$ of admissible pairing configurations by

$$\mathbf{K}^{(1)}(P) := \mathrm{wt}^{(1)}_{\mathrm{layer}}(P),$$

viewed as a probability distribution on $\mathcal{M}$.

By Theorem D.9 (and, in the strict case, Corollary D.9S), $\mathbf{K}^{(1)}$ is well-defined and normalized.

Proposition D.11 (Heat-bath property for classic layers at $q = 1$). Let $\mathbb{P}$ be any probability measure on a larger configuration space that factors as "(outside data) $\times$ (classic-layer pairing data)", and suppose the conditional weight of the pairing data given the outside data is proportional to $\mathrm{wt}^{(1)}_{\mathrm{layer}}$. Then resampling the pairing data using $\mathbf{K}^{(1)}$ is a heat-bath move and leaves $\mathbb{P}$ invariant.

Proof. This is the standard Gibbs sampler invariance: a heat-bath move that replaces a block by a draw from its conditional distribution preserves the global measure.

Remark D.12 (Use in the main construction). In Appendix F and Appendix H we define the lifted dynamics by repeatedly applying local resampling operations that are, after positivization, of the above heat-bath form. Theorem D.9 (and, in the strict case, Corollary D.9S) supplies the normalization needed to interpret the local resampling probabilities concretely via the truncated geometric law eq:app-d-4.

D.6 What this appendix does *not* address

Signed layers (Definition B.10) have local weights eq:app-b-10 that are not probabilistically normalized and may carry negative signs. The conversion of the signed structure into a nonnegative probabilistic model in the restricted strict regime is carried out in Appendix E; Appendix D concerns only the **classic-layer** portion of the local kernel at $q = 1$.

End of Appendix D

Appendix E will use the restricted strict assumptions to eliminate or absorb signed contributions and thereby produce a genuinely nonnegative weight model compatible with the $q = 1$ heat-bath kernels normalized here.

Appendix E. Positivization at $q = 1$ in the Restricted Strict Regime

This appendix proves the "positivization" statement used in the main text: in the restricted strict regime, the signed multiline queue (signed-MLQ) partition function at $q = 1$ can be rewritten as a sum of **nonnegative** weights over a modified configuration space. The construction is explicit and purely combinatorial: we group together signed-MLQ configurations that differ only by sign choices at certain **trivially paired signed balls**, and sum their weights.

The key observation is that for strict partitions, the signed-layer matchings are label-forced; therefore the sign of a trivially paired signed ball affects **only one scalar factor** in the total weight, and the grouping factorizes.

E.1 Standing assumptions and the positivity regime

Throughout Appendix E we assume:

Assumption E.1 (Restricted strict input). Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ be a **restricted strict** partition, meaning:

1. λ is strict: all positive parts are distinct;
2. $m_0(\lambda) = 1$ (exactly one zero part);
3. $m_1(\lambda) = 0$ (no part of size (1)).

Then $S_n(\lambda)$ consists of compositions μ which are permutations of λ; in particular each μ has exactly one (0) and all positive entries are distinct.

We also fix the probabilistic parameter range for the positivity statement:

Definition E.2 (Positivity regime). Fix $t \in (0, 1)$. Define the constant

$$\kappa := t^{-(n-1)} > 0.$$

We say that $x = (x_1, \dots, x_n) \in (0, \infty)^n$ lies in the **positivity regime** if

$$x_i > \kappa \qquad \text{for all } i \in [n].$$

Remark E.3 (Why a regime is needed). At $q = 1$, a negative signed ball contributes a fixed negative scalar $-\kappa$ to the shifted ball-weight. If a negative signed ball is **trivially paired** in its signed layer, no additional negative sign appears in pairing weights, so the contribution can be negative. The grouping below replaces $+, -$-choices at such sites by the aggregate factor $x_i - \kappa$, which is nonnegative precisely under eq:app-e-2.

E.2 Forced matchings in strict layers

We recall from Appendix B that a signed MLQ has alternating signed and classic layers, with signed layers pairing a signed row (r') to the regular row r, and classic layers pairing a regular row r to the signed row ((r-1)') (pairings in classic layers are defined on absolute values).

The following strictness consequence is the engine of the positivity argument.

Lemma E.4 (Uniqueness of label matchings in strict partitions). Assume λ is strict. Then in every row word $\lambda^{(r)}$ (and hence in every regular row (r) and signed row (r')) each positive label a appears with multiplicity at most (1). Consequently, in any layer (signed or classic), for each $a > 0$ there is at most one ball of (absolute) label a in the upper row and at most one in the lower row, so any admissible a-pairing is forced to match those two balls.

Proof. Strictness means $m_a(\lambda) \in 0, 1$ for all $a \geq 1$. The truncation $\lambda^{(r)}$ replaces parts $< r$ by (0) and preserves the multiplicities of parts $\geq r$, so it remains true that each $a \geq r$ appears at most once. In a layer pairing equal (absolute) labels, so if there is at most one a on each side, the matching for that label is unique (when it exists).

As a consequence, in the strict regime, the combinatorial freedom in a signed MLQ lies in **where** the labels are placed in each row (subject to local admissibility), not in choosing among multiple partners for the same label.

E.3 The only source of sign-indefiniteness at $q = 1$

We isolate the sign contribution in signed rows.

Definition E.5 (Signed-row local weights at $q = 1$). Consider a signed row (r') and a column $i \in [n]$.

- A **positive** signed ball in (r', i) contributes the factor x_i.
- A **negative** signed ball in (r', i) contributes the factor $-\kappa$, with $\kappa = t^{-(n-1)}$.

For a signed-layer nontrivial pairing p initiated by a signed ball (in row (r')) and terminating at a regular ball (in row (r)), the pairing factor is

$$\mathrm{wt}_{\mathrm{pair}}(p) = \begin{cases} (1-t)t^{\mathrm{skip}(p)+\mathrm{empty}(p)} & \text{if the signed ball is positive,} \\ -(1-t)t^{\mathrm{skip}(p)+\mathrm{empty}(p)} & \text{if the signed ball is negative,} \end{cases}$$

and trivial signed-layer pairings contribute (1).

(Here skip and empty are the signed-layer statistics as in Appendix B / Definition B.14.)

From eq:app-e-3, we see:

- If a negative signed ball initiates a **nontrivial** signed pairing, its two minus signs (ball-weight and pairing-weight) cancel, producing a positive factor $\kappa(1-t)t^{\mathrm{skip}(p)+\mathrm{empty}(p)}$.
- If a negative signed ball is **trivially paired**, only the ball-weight $-\kappa$ appears, producing a negative factor.

Thus the sign-indefiniteness at $q = 1$ is concentrated exactly at negative signed balls that are trivially paired in signed layers.

This motivates the following notion.

Definition E.6 (Vertical signed balls). In a signed layer between a signed row (r') (upper) and a regular row r (lower), a signed ball of absolute value a is called **vertical** if it is directly above the unique regular ball of label a in the row below (hence, by Lemma E.4, its pairing is forced to be trivial).

We write $\mathcal{V}(Q^{\pm})$ for the set of all vertical signed balls in a signed MLQ $Q^{\pm}$, across all signed layers.

E.4 Neutralization and the positivized configuration space

We now define the "positivized" configuration space by forgetting the sign of all vertical signed balls, and replacing the two sign choices by a single aggregated nonnegative factor.

Definition E.7 (Neutralization map). Let $\Omega(\lambda)$ denote the set of signed MLQs of shape/type λ (Appendix B). Define the **neutralization map**

$$\eta : \Omega(\lambda) \longrightarrow \Omega^{+}(\lambda)$$

as follows: given $Q^{\pm} \in \Omega(\lambda)$, $\eta(Q^{\pm})$ is obtained by **forgetting the sign** of every vertical signed ball in $\mathcal{V}(Q^{\pm})$, while leaving all other data unchanged (row placements, all pairings, and the signs of non-vertical signed balls).

Elements $Q^{\circ} \in \Omega^{+}(\lambda)$ will be called **semi-signed MLQs**.

For a semi-signed MLQ Q°, we denote by $\mathcal{V}(Q^{\circ})$ its set of neutral (formerly vertical) signed balls, and we write $\mathrm{col}(v) \in [n]$ for the column of a neutral ball v.

Lemma E.8 (Fibers of η). Let $Q^{\circ} \in \Omega^{+}(\lambda)$. Then the fiber $\eta^{-1}(Q^{\circ})$ consists exactly of all choices of signs $\pm$ for each $v \in \mathcal{V}(Q^{\circ})$, with all other data fixed. In particular,

$$|\eta^{-1}(Q^{\circ})| = 2^{|\mathcal{V}(Q^{\circ})|}.$$

Proof. By construction, η only forgets signs on vertical signed balls. Conversely, if v is vertical then the sign flip $+a \leftrightarrow -a$ preserves admissibility in the signed layer because the ball below has label a, which satisfies both the positive constraint ("below label $\geq a$ and if equal then trivial") and the negative constraint ("below label $\leq a$ or empty"), and the pairing is fixed trivial. Classic layers depend only on absolute values. Therefore each independent choice of signs on $\mathcal{V}(Q^\circ)$ produces a valid preimage, and every preimage differs only in these choices.

E.5 Definition and factorization of the positivized weight

We define the $q = 1$ weight on $\Omega^+(\lambda)$ by summing over the fiber of η.

Definition E.9 (Positivized weight). Fix $q = 1$ and parameters (x, t). For $Q^\circ \in \Omega^+(\lambda)$, define

$$\mathrm{wt}^+(Q^\circ) := \sum_{Q^\pm \in \eta^{-1}(Q^\circ)} \mathrm{wt}(Q^\pm)\Big|_{q=1},$$

where $\mathrm{wt}(Q^\pm)$ is the signed-MLQ weight from Appendix B.

We now show that wt^+ factorizes explicitly.

Proposition E.10 (Fiber-sum factorization). Let $Q^\circ \in \Omega^+(\lambda)$. Then

$$\mathrm{wt}^+(Q^\circ) = \mathrm{wt}_{\mathrm{rest}}(Q^\circ) \prod_{v \in \mathcal{V}(Q^\circ)} (x_{\mathrm{col}(v)} - \kappa), \qquad \kappa = t^{-(n-1)}.$$

Here $\mathrm{wt}_{\mathrm{rest}}(Q^\circ)$ is the product of all weight factors in $\mathrm{wt}(Q^\pm)|_{q=1}$ that do **not** depend on the sign choices of neutral balls; equivalently, it is the weight of any representative $Q^\pm \in \eta^{-1}(Q^\circ)$ with the ball-weight factors for neutral balls removed.

Proof. Fix Q°. By Lemma E.8, the fiber $\eta^{-1}(Q^\circ)$ is obtained by independently assigning a sign to each $v \in \mathcal{V}(Q^\circ)$.

For any $v \in \mathcal{V}(Q^\circ)$, the signed-layer pairing initiated by v is trivial, hence contributes no factor from eq:app-e-3. Therefore, the only dependence of $\mathrm{wt}(Q^\pm)|_{q=1}$ on the chosen sign of v comes from the signed-row ball weight in the column $\mathrm{col}(v)$:

$$\mathrm{sign}(v) = + \ \Rightarrow\ \text{factor } x_{\mathrm{col}(v)}, \qquad \mathrm{sign}(v) = - \ \Rightarrow\ \text{factor } (-\kappa).$$

All other factors (classic-layer pairing weights, and signed-layer factors from nontrivial pairings) are independent of these sign choices, because they depend only on placements, absolute labels, and the signs of non-vertical balls, all of which are fixed in Q°.

Hence the sum over the fiber factorizes as a product of two-term sums:

$$\begin{aligned}\mathrm{wt}^{+}(Q^{\circ}) &= \mathrm{wt}_{\mathrm{rest}}(Q^{\circ}) \prod_{v \in \mathcal{V}(Q^{\circ})} \left(x_{\mathrm{col}(v)} + (-\kappa)\right) \\ &= \mathrm{wt}_{\mathrm{rest}}(Q^{\circ}) \prod_{v \in \mathcal{V}(Q^{\circ})} (x_{\mathrm{col}(v)} - \kappa),\end{aligned}$$

which is eq:app-e-7.

E.6 Nonnegativity and preservation of fiber sums

We now conclude the main statements needed in the body of the paper.

Theorem E.11 (Positivization in the restricted strict regime at $q = 1$). Assume λ is restricted strict (Assumption E.1), and assume (x, t) lies in the positivity regime (Definition E.2). Then:

1. (**Nonnegativity**) For every $Q^{\circ} \in \Omega^{+}(\lambda)$,

$$\mathrm{wt}^{+}(Q^{\circ}) \geq 0.$$

2. (**Exact fiber-sum preservation**) Fix $\mu \in S_n(\lambda)$. Let $\Omega(\mu) \subset \Omega(\lambda)$ be the set of signed MLQs of type μ, and $\Omega^{+}(\mu) \subset \Omega^{+}(\lambda)$ the set of semi-signed MLQs of type μ. Then

$$\sum_{Q^{\circ} \in \Omega^{+}(\mu)} \mathrm{wt}^{+}(Q^{\circ}) = \sum_{Q^{\pm} \in \Omega(\mu)} \mathrm{wt}(Q^{\pm}) \Big|_{q=1}.$$

   In particular, the $q = 1$ interpolation-ASEP weight admits the nonnegative representation

$$F^{*}_{\mu}(x; 1, t) = \sum_{Q^{\circ} \in \Omega^{+}(\mu)} \mathrm{wt}^{+}(Q^{\circ}) \ \geq 0.$$

   Proof.

(1) By Proposition E.10, $\mathrm{wt}^{+}(Q^{\circ})$ is a product of $\mathrm{wt}_{\mathrm{rest}}(Q^{\circ})$ and factors $(x_{\mathrm{col}(v)} - \kappa)$. In the positivity regime eq:app-e-2, each $(x_{\mathrm{col}(v)} - \kappa) \geq 0$.

It remains to see $\mathrm{wt}_{\mathrm{rest}}(Q^{\circ}) \geq 0$. Every factor contributing to $\mathrm{wt}_{\mathrm{rest}}(Q^{\circ})$ is nonnegative at $q = 1$, $t \in (0, 1)$, and $x_i > 0$:

- classic-layer factors are positive (Appendix D shows their normalization at $q = 1$, hence in particular positivity);
- for any nontrivial signed-layer pairing, the product of the signed-row factor and the signed pairing factor is nonnegative:

$$(+):\ x_i\cdot(1-t)t^{\mathrm{skip}(p)+\mathrm{empty}(p)}\geq 0,\qquad (-):\ (-\kappa)\cdot\big(-(1-t)t^{\mathrm{skip}(p)+\mathrm{empty}(p)}\big)=\kappa(1-t)t^{\mathrm{skip}(p)+\mathrm{empty}(p)}\geq 0.$$

Thus $\mathrm{wt}_{\mathrm{rest}}(Q^\circ)\geq 0$, proving eq:app-e-9.

(2) The neutralization map η preserves the bottom row, and the sets $\{\eta^{-1}(Q^\circ) : Q^\circ\in\Omega^+(\mu)\}$ form a disjoint partition of $\Omega(\mu)$. Therefore

$$\begin{aligned}\sum_{Q^\circ\in\Omega^+(\mu)}\mathrm{wt}^+(Q^\circ)&=\sum_{Q^\circ\in\Omega^+(\mu)}\ \sum_{Q^\pm\in\eta^{-1}(Q^\circ)}\mathrm{wt}(Q^\pm)\Big|_{q=1}\\&=\sum_{Q^\pm\in\Omega(\mu)}\mathrm{wt}(Q^\pm)\Big|_{q=1},\end{aligned}$$

which is eq:app-e-10. The identification with F^*_μ is the definition of the signed-MLQ fiber sum at $q=1$, yielding eq:app-e-11.

E.7 Discussion and how Appendix E is used downstream

1. **What changes compared to the original model.** We do not alter the signed-MLQ weight on $\Omega(\lambda)$. We only reorganize the sum by grouping configurations into fibers of η, producing a new space $\Omega^+(\lambda)$ with nonnegative weights.
2. **Why strictness matters.** The factorization in Proposition E.10 relies on the independence of sign choices at vertical balls, which is immediate once matchings are label-forced (Lemma E.4). For partitions with repeated parts, the signed-layer pairing structure can depend on the placement order among equal labels, and the same "forget signs and factorize" argument requires a more delicate cluster/inclusion–exclusion analysis (not pursued here).
3. **Role in the Markov construction.** Theorem E.11 supplies the nonnegative configuration-space model needed to interpret the $q=1$ weights probabilistically. In later appendices, local heat-bath updates are defined on $\Omega^+(\lambda)$ and then projected to a Markov chain on $S_n(\lambda)$; Appendix D supplies the normalization for the classic-layer resampling kernels used in those updates.

E.8 Extension to all $x>0$ by rational continuation

Theorem E.11 (and hence Theorem 5.1 in the main text) is proved under the positivity regime $x_i>\kappa$, which ensures that wt^+ is nonnegative and can be normalized to a probability measure. The bottom chain X^{SP} on $S_n(\lambda)$, however, is defined for all $x \in (0,\infty)^n$ (its rates only involve the factors $1/x_i$). We now explain why the stationary distribution identity extends from the open set eq:2-positivity-regime to the full parameter range eq:2-params.

Lemma E.12 (Rational dependence of the stationary distribution on x). Fix $t \in (0,1)$. Let $\mathcal{L}(x)$ be the generator of the bottom chain X^{SP} from Appendix G. Then the unique stationary distribution $\pi(\cdot; x, t)$ of X^{SP} is a vector of rational functions in $x_1, \dots, x_n$ (with coefficients in $\mathbb{Q}(t)$).

Proof. Write the stationarity equations as

$$\pi(x,t)\,\mathcal{L}(x) = 0, \qquad \sum_{\mu \in S_n(\lambda)} \pi(\mu; x, t) = 1.$$

The entries of $\mathcal{L}(x)$ are rational functions of x (indeed, linear combinations of $1/x_i$ with coefficients depending only on t and the current state). Since X^{SP} is irreducible for every $x \in (0,\infty)^n$ (Appendix I), the left kernel of $\mathcal{L}(x)$ is one-dimensional, and the augmented linear system above has a unique solution.

Choose an ordering of $S_n(\lambda)$ and fix a reference state μ_0. Replace the equation indexed by μ_0 in $\pi\mathcal{L} = 0$ by the normalization $\sum \pi = 1$, obtaining a square linear system $A(x)\,\pi(x,t)^\top = b$ whose entries are rational in x. The matrix $A(x)$ is invertible for all $x > 0$ because the solution is unique. By Cramer's rule, each coordinate $\pi(\mu; x, t)$ is a ratio of determinants of matrices with rational entries in x, hence is rational in x.

Corollary E.13 (Continuation from the positivity regime). Assume that for all x in the open positivity regime eq:2-positivity-regime we have

$$\pi(\mu; x, t) = \frac{F^*_\mu(x; 1, t)}{P^*_\lambda(x; 1, t)}.$$

Then the same identity holds as an identity of rational functions in x. In particular, since $\pi(\mu; x, t)$ is well-defined for every $x \in (0,\infty)^n$, the rational function $F^*_\mu(x;1,t)/P^*_\lambda(x;1,t)$ has no pole on $(0,\infty)^n$, and the identity holds for all $x \in (0,\infty)^n$.

Proof. By Lemma E.12, the left-hand side is rational in x. Since $F^*_\mu(\cdot; 1, t)$ and $P^*_\lambda(\cdot; 1, t)$ are (inhomogeneous) polynomials in x at fixed t, the right-hand side is also rational in x. Two rational functions that agree on a nonempty open set agree identically in the field $\mathbb{Q}(t)(x_1, \dots, x_n)$.

Finally, in the construction of Lemma E.12 each coordinate $\pi(\mu; x, t)$ is represented as a quotient of determinants with a common denominator $\det A(x)$, and $\det A(x) \neq 0$ for all $x \in (0,\infty)^n$ because $A(x)$ is invertible for every $x > 0$. Therefore $\pi(\mu; x, t)$ has no pole on $(0,\infty)^n$. Since it equals $F^*_\mu(x;1,t)/P^*_\lambda(x;1,t)$

as a rational function, the latter is also regular on $(0, \infty)^n$, and the identity holds for all $x \in (0, \infty)^n$ by evaluation.

End of Appendix E

Appendix F will use this positivized model to define explicit local resampling moves (and a projected Markov chain on $S_n(\lambda)$) and verify stationarity with respect to the induced measure proportional to $F^*_\mu(x; 1, t)$.

Appendix F. Construction and Well-definedness of the Lifted Markov Chain on $\Omega^+(\lambda)$

This appendix specifies the lifted Markov chain $\widetilde{X}$ announced in Definition 6.1 of the main text. We present:

1. the precise state space $\Omega^+(\lambda)$ used for the probabilistic construction at $q = 1$,
2. the local update operators $\mathcal{U}_{i_{i \in [n]}}$ and their sampling rules, and
3. a proof that these updates define a well-posed continuous-time Markov chain on the finite set $\Omega^+(\lambda)$.

The stationarity of the measure $\tilde{\pi}(\omega^+) \propto \mathrm{wt}^+(\omega^+)$ is proved in Appendix H. The fact that the bottom projection of $\widetilde{X}$ is Markov and yields the desired chain on $S_n(\lambda)$ is proved in Appendix G.

F.1 The positive queue space and canonical bottom map

We work at $q = 1$ and under the positivity assumptions of Appendix E.

Assumption F.1 (Parameter regime for probabilistic dynamics). Fix $t \in (0, 1)$ and set $\kappa := t^{-(n-1)}$. Throughout Appendices F–H we assume that x lies in the positivity regime of Appendix E, i.e.

$$x_i > \kappa \qquad (i \in [n]).$$

Under this assumption the weight model $(\Omega^+(\lambda), \mathrm{wt}^+)$ is nonnegative (Theorem E.11) and can be normalized to a probability measure. Appendix E.8 explains why the final stationary distribution statement extends to all $x > 0$.

Definition F.2 (Positive queue space). Let $\Omega(\lambda)$ be the signed-MLQ space of Appendix B and $\eta : \Omega(\lambda) \twoheadrightarrow \Omega^+(\lambda)$ the neutralization map of Appendix E. Elements of $\Omega^+(\lambda)$ are called **semi-signed MLQs**.

Definition F.3 (Canonical section and bottom map on $\Omega^+(\lambda)$). Fix a canonical section

$$\rho : \Omega^+(\lambda) \to \Omega(\lambda)$$

defined by assigning the **positive** sign to every neutral ball (i.e. every vertical signed ball whose sign was forgotten by η), and leaving all other data unchanged. By construction, $\eta \circ \rho = \mathrm{id}$ on $\Omega^+(\lambda)$.

Define the bottom map on $\Omega^+(\lambda)$ by

$$\Phi^+ := \Phi \circ \rho : \Omega^+(\lambda) \to S_n(\lambda),$$

where $\Phi : \Omega(\lambda) \to S_n(\lambda)$ is the bottom map of Appendix B.

Remark F.4 (Independence of ρ). The bottom row does not depend on the signs of vertical signed balls, hence Φ^+ is independent of the choice of section ρ. The notation Φ^+ is therefore canonical.

F.2 Local resampling primitives

The lifted dynamics are built from two types of local primitives:

1. **Classic-layer resampling kernels** at $q = 1$, normalized by Appendix D, and
2. **Neutral factors** (from Appendix E) that contribute nonnegative multiplicative weights but do not introduce additional local randomness.

To define local updates, we will resample certain classic-layer pairing data while keeping the complement fixed.

F.2.1 Classic-layer pairing kernels at $q = 1$ Let $r \in 2, \dots, L$. Consider the classic layer between the regular row r (upper) and the signed row ((r-1)') (lower) in a (semi-)signed MLQ. For a fixed outside configuration, the admissible pairing configurations in this layer form a finite set $\mathcal{M}$, and at $q = 1$ the local weight of a pairing configuration $P \in \mathcal{M}$ is the product of factors

$$\frac{(1-t)t^{\mathrm{skip}(p)}}{1-t^{\mathrm{free}(p)}}$$

over nontrivial pairings p in that layer, computed in a fixed deterministic placement order (Appendix D / Definition D.8).

Definition F.5 (Classic-layer heat-bath kernel at $q = 1$). Fix a semi-signed MLQ $Q^\circ \in \Omega^+(\lambda)$, a layer index $r \in 2, \dots, L$, and a specification of all data outside the classic layer $(r \to (r-1)')$. Let $\mathcal{M}_r(Q^\circ)$ denote the (finite) set of admissible pairing configurations in this layer consistent with the fixed outside data.

Define the probability kernel

$$\mathbf{K}_r^{(1)}(Q^\circ;\cdot) : \mathcal{M}_r(Q^\circ) \to [0,1]$$

by

$$\mathbf{K}_r^{(1)}(Q^\circ;P) := \prod_{p\in\mathcal{P}_{\mathrm{nt}}(P)} \frac{(1-t)t^{\mathrm{skip}(p)}}{1-t^{\mathrm{free}(p)}}.$$

By Appendix D (Theorem D.9; in the strict case, Corollary D.9S), this is normalized:

$$\sum_{P\in\mathcal{M}_r(Q^\circ)} \mathbf{K}_r^{(1)}(Q^\circ;P) = 1.$$

Remark F.6 (Sampling rule). Appendix D shows $\mathbf{K}_r^{(1)}$ can be sampled sequentially via the truncated geometric law: at each nontrivial choice with r candidates, choose the k-th candidate with probability $\frac{(1-t)t^{k-1}}{1-t^r}$.

F.2.2 Neutral vertical factors Neutralization in Appendix E eliminates the signs of vertical signed balls and replaces the two sign choices by a single nonnegative factor $(x_i - \kappa)$ at column i. These factors do not require resampling and are absorbed into the global weight wt^+. In particular:

- our local updates will never “toggle” neutral vertical balls, and
- all randomness in the lifted chain comes from classic-layer pairing choices at $q = 1$.

F.3 Definition of the local update operator $\mathcal{U}_i$ (restricted strict case)

In the restricted strict regime (Assumption E.1) the lifted chain must **move the bottom word** $\mu = \Phi^+(Q^\circ) \in S_n(\lambda)$; resampling only pairing data cannot change μ because Φ (and hence Φ^+) reads the bottom regular row. In this subsection we therefore define $\mathcal{U}_i$ as a finite composition of local two-line resampling kernels along the active corridor determined by site i (Sections F.3.2–F.3.3).

F.3.1 The i-interval and eligible destinations Let $Q^\circ \in \Omega^+(\lambda)$ and write $\mu = \Phi^+(Q^\circ)$. Because λ is restricted strict, μ contains a **unique** 0; denote its position by

$$h(\mu) \in [n] \qquad \text{where} \qquad \mu_{h(\mu)} = 0.$$

For $i \in [n]$ define the clockwise cyclic interval from i to the hole

$$I_i(\mu) := [\, i \to h(\mu)\,] := \{i, i+1, \dots, h(\mu)\} \subset [n],$$

with indices taken modulo n (and interpreted as the unique simple path on the directed n-cycle).

Let $a := \mu_i$. If $a = 0$ we set $\mathcal{U}_i(Q^\circ) = Q^\circ$ (no move). Otherwise, let

$$C_i(\mu) = \{\, j \in I_i(\mu) \setminus \{i\} : \mu_j < a \,\} \ \cup \ \{h(\mu)\}$$

and list it in clockwise order

$$C_i(\mu) = \{c_i(1), c_i(2), \dots, c_i(r_i(\mu))\}, \qquad r_i(\mu) := |C_i(\mu)|.$$

(These are exactly the "eligible destinations" used in Appendix G.)

For each $k \in [r_i(\mu)]$ define the corresponding bottom word

$$\mathsf{T}_i(\mu; k) \in S_n(\lambda)$$

by the insertion map of Appendix G.4 (Definition G.10): move the letter $\mu_i = a$ to column $c_i(k)$, shifting every entry on the clockwise segment from i to $c_i(k)$ one step toward i, equivalently performing the cyclic permutation described in eq:app-g-17.

F.3.2 Local two-line resampling operators For each adjacent-row layer ℓ in the semi-signed MLQ (either a classic layer $r \to (r-1)'$ or a signed layer $r' \to r$), and each column index $i \in [n]$, we define a finite two-line window $W_{i,\ell}$ consisting of the two rows of ℓ restricted to the minimal set of columns needed to determine the admissible destinations and local statistics (skip/free or skip/empty) for the unique strand segment whose decision is triggered at column i in layer ℓ.

Let $\mathcal{P}_{i,\ell}$ be the partition of $\Omega^+(\lambda)$ obtained by fixing all queue data outside $W_{i,\ell}$. On each cell $C \in \mathcal{P}_{i,\ell}$, the weight wt^+ factorizes as

$$\mathrm{wt}^+(Q) = \mathrm{wt}^+_{\mathrm{out}}(C) \cdot \mathrm{wt}^+_{\mathrm{in}}(Q|_{W_{i,\ell}}),$$

where $\mathrm{wt}^+_{\mathrm{out}}(C)$ depends only on the fixed outside data and $\mathrm{wt}^+_{\mathrm{in}}$ depends only on the local two-line slice in $W_{i,\ell}$.

Define the local resampling kernel $R_{i,\ell}$ to be the heat-bath kernel on $\mathcal{P}_{i,\ell}$ with respect to wt^+; i.e., given Q, the new configuration $R_{i,\ell}(Q)$ is obtained by resampling the slice $Q|_{W_{i,\ell}}$ from its conditional distribution proportional to $\mathrm{wt}^+_{\mathrm{in}}$, keeping the complement fixed.

By construction, each $R_{i,\ell}$ preserves the lifted measure $\tilde{\pi}(Q) \propto \mathrm{wt}^+(Q)$.

F.3.3 Definition of the site update $\mathcal{U}_i$ as a finite composition Fix $i \in [n]$. Given a state $Q^\circ \in \Omega^+(\lambda)$ with bottom word $\mu := \Phi^+(Q^\circ)$, define a finite ordered list of layers $\ell_1, \ell_2, \dots, \ell_m$ to be resampled, together with corresponding windows W_{i,ℓ_j}, by the following deterministic rule: start from the bottom and follow the unique active corridor determined by the letter $a := \mu_i$ and the strand constraints up to height a, recording exactly those adjacent-row layers in which the corridor makes a nontrivial rightward choice. All other layers contribute no randomness and need not be resampled.

Define the site-update kernel by

$$\mathcal{U}_i := R_{i,\ell_m} \circ R_{i,\ell_{m-1}} \circ \cdots \circ R_{i,\ell_1}.$$

This is a well-defined Markov kernel on $\Omega^+(\lambda)$ because it is a finite composition of well-defined Markov kernels. Since each R_{i,ℓ_j} preserves $\tilde{\pi}$, Lemma H.5.1 implies that $\mathcal{U}_i$ preserves $\tilde{\pi}$ as well.

Lemma F.10 (Bottom marginal of the site update $\mathcal{U}_i$). Assume λ is restricted strict and work in the positivity regime $x_i > \kappa = t^{-(n-1)}$. Fix $i \in [n]$ and $Q^\circ \in \Omega^+(\lambda)$, and write $\mu := \Phi^+(Q^\circ)$ and $a := \mu_i$.

If $a = 0$, then $\Phi^+(\mathcal{U}_i(Q^\circ)) = \mu$ almost surely.

Assume $a > 0$ and form the ordered eligible list $C_i(\mu) = (c_i(1), \dots, c_i(r_i(\mu)))$ as in Definition G.8. Then:

1. (**Support**) The bottom outcome of the update lies in the scan-and-push set:

$$\Phi^+(\mathcal{U}_i(Q^\circ)) \in \{\, \mathsf{T}_i(\mu; k) : k = 1, 2, \dots, r_i(\mu) \,\} \quad \text{almost surely.}$$

2. (**Truncated geometric law**) For each $k = 1, \dots, r_i(\mu)$,

$$\mathbb{P}(\Phi^+(\mathcal{U}_i(Q^\circ)) = \mathsf{T}_i(\mu; k) \mid Q^\circ) = \frac{(1-t)t^{k-1}}{1-t^{r_i(\mu)}}.$$

In particular, the bottom marginal depends only on μ (not on the chosen lift Q°).

Proof.
By definition of $\mathcal{U}_i$, the only randomness comes from the finite list of two-line resamplings $R_{i,\ell_1}, \dots, R_{i,\ell_m}$ along the active corridor of the letter a.

Each time the corridor encounters an eligible destination in clockwise order, the local two-line slice contributes a Bernoulli accept/reject decision: accept with probability $(1-t)$ and otherwise reject with probability t and continue to the next eligible destination. This is exactly the scan-and-reject mechanism of Appendix D (Proposition D.6), hence the index K of the first accepted eligible destination has the truncated geometric law

$$\mathbb{P}(K = k) = \frac{(1-t)t^{k-1}}{1-t^{r_i(\mu)}}, \qquad k = 1, \dots, r_i(\mu).$$

Conditional on $K = k$, the deterministic strand constraints force the bottom row to be the cyclic permutation $\mathsf{T}_i(\mu; k)$ (Definition G.10), so the support statement follows as well.

F.4 Well-definedness of $\mathcal{U}_i$

By construction, each $\mathcal{U}_i$ is a finite composition of local two-line heat-bath kernels R_{i,ℓ_j} on the finite state space $\Omega^+(\lambda)$.

Lemma F.12 (Well-definedness of local resampling kernels). For each site $i \in [n]$ and each layer ℓ in the active-corridor list of Section F.3.3, $R_{i,\ell}$ is a Markov kernel on $\Omega^+(\lambda)$ and preserves $\tilde{\pi}$.

Proof. Each $R_{i,\ell}$ is, by definition, a heat-bath kernel on the partition $\mathcal{P}_{i,\ell}$ of Section F.3.2. Since $\Omega^+(\lambda)$ is finite, the corresponding conditional distributions are well-defined. Stationarity of heat-bath kernels (Corollary H.5) yields $\tilde{\pi} R_{i,\ell} = \tilde{\pi}$.

Proposition F.13 (Each $\mathcal{U}_i$ is a stochastic kernel). For each $i \in [n]$, $\mathcal{U}_i$ defines a Markov transition kernel on $\Omega^+(\lambda)$; i.e. $\mathcal{U}_i(Q^\circ, \widetilde{Q}^\circ) \geq 0$ and $\sum_{\widetilde{Q}^\circ \in \Omega^+(\lambda)} \mathcal{U}_i(Q^\circ, \widetilde{Q}^\circ) = 1$ for all Q°.

Proof. By eq:F9-new, $\mathcal{U}_i$ is a finite composition of kernels R_{i,ℓ_j}. A finite composition of Markov kernels is a Markov kernel, proving stochasticity. By Lemma F.12, each factor preserves $\tilde{\pi}$, and Lemma H.5.1 implies that their composition also preserves $\tilde{\pi}$.

Remark F.14 (Sampling viewpoint). Lemma F.10 identifies the bottom marginal of $\mathcal{U}_i$ as a truncated geometric choice of destination $c_i(k)$. One may therefore implement $\mathcal{U}_i$ by: i sampling k via $\frac{(1-t)t^{k-1}}{1-t^{r_i(\mu)}}$, and ii sampling the remaining local slices along the active corridor conditionally on the chosen destination. The proofs below use only the kernel definition eq:F9-new, not a specific implementation.

F.5 The continuous-time lifted chain $\widetilde{X}$

We now assemble the local kernels into a continuous-time chain.

Definition F.15 (Generator of the lifted chain). Define a continuous-time Markov chain $\widetilde{X}$ on $\Omega^+(\lambda)$ by independent Poisson clocks $N_{i}{}_{i=1}^{n}$ with rates

$$\mathrm{rate}(i) := \frac{1}{x_i}.$$

When clock i rings, update the configuration by applying $\mathcal{U}_i$.

Equivalently, for any function $f : \Omega^+(\lambda) \to \mathbb{R}$, the generator $\widetilde{\mathcal{L}}$ is

$$(\widetilde{\mathcal{L}}f)(Q^\circ) = \sum_{i=1}^{n} \frac{1}{x_i} \left(\sum_{Q^{\circ\prime} \in \Omega^+(\lambda)} \mathbb{P}(\mathcal{U}_i(Q^\circ) = Q^{\circ\prime})\, f(Q^{\circ\prime}) - f(Q^\circ) \right).$$

Theorem F.16 (Well-posedness of $\widetilde{X}$). The update scheme of Definition F.15 defines a well-posed continuous-time Markov chain on the finite state space $\Omega^+(\lambda)$.

Proof. Each $\mathcal{U}_i$ is a stochastic kernel by Proposition F.13, and the total jump rate out of any state is $\sum_{i=1}^{n} 1/x_i < \infty$. Since $\Omega^+(\lambda)$ is finite, standard continuous-time Markov chain theory yields existence and uniqueness of the process with generator eq:app-f-10.

F.6 Nontriviality (no dependence on interpolation polynomials)

Finally, we record the sense in which the lifted dynamics are nontrivial.

Proposition F.17 (Algorithmic explicitness). The transition mechanism of $\widetilde{X}$ depends only on:

- the parameters (x, t),
- the current combinatorial configuration $Q^\circ \in \Omega^+(\lambda)$, and
- the truncated geometric sampling rule $\frac{(1-t)t^{k-1}}{1-t^{r}}$ used for classic-layer resampling,

and does not require evaluation of interpolation ASEP or interpolation Macdonald polynomials.

Proof. The update kernels are defined entirely in terms of local classic-layer statistics skip, free and the normalized kernels of Appendix D. No polynomial evaluations appear in the definition of $\mathcal{U}_i$ or in the generator eq:app-f-10.

End of Appendix F

Appendix G proves that the projected process $X_t^{\mathrm{SP}} := \Phi^+(\widetilde{X}_t)$ is Markov on $S_n(\lambda)$ and admits an intrinsic description, while Appendix H proves that $\tilde{\pi}(Q^\circ) \propto \mathrm{wt}^+(Q^\circ)$ is stationary for $\widetilde{X}$.

Appendix G. Lumping: The Bottom Projection is Markov and Defines a Chain on $S_n(\lambda)$

This appendix proves Theorem 6.3 from the main text: the bottom projection of the lifted process $\widetilde{X}$ is itself a (continuous-time) Markov chain on $S_n(\lambda)$, and admits an intrinsic description with explicit transition rates depending only on (x, t) and the current bottom configuration.

The proof is organized as follows.

1. We recall a standard **strong lumpability** criterion for continuous-time Markov chains (Section G.1).
2. We apply it to the lifted generator $\widetilde{\mathcal{L}}$ of Appendix F and the bottom map $\Phi^+ : \Omega^+(\lambda) \to S_n(\lambda)$ (Section G.2).
3. We identify the induced bottom transition kernel of each site-update $\mathcal{U}_i$ and show it depends only on the bottom configuration (Sections G.3–G.4).
4. We record an intrinsic "scan-and-push" sampling rule for the bottom kernel, expressed using the truncated geometric distribution from Appendix D (Section G.5).

Throughout, we work under the standing assumptions of the main text and Appendices E–F: λ is restricted strict, $t \in (0, 1)$, $x \in (0, \infty)^n$, and $\Omega^+(\lambda)$ carries the nonnegative weight wt^+.

G.1 Strong lumpability for continuous-time Markov chains

We begin with a general criterion ensuring that a projection of a continuous-time chain is Markov.

Definition G.1 (Fibers of a projection). Let Ω and $\mathcal{S}$ be finite sets, and let $\Phi : \Omega \to \mathcal{S}$ be surjective. For $s \in \mathcal{S}$, define the fiber

$$\Omega_s := \Phi^{-1}(s) \subseteq \Omega.$$

Lemma G.2 (Strong lumpability criterion; generator form). Let $\widetilde{X}$ be a continuous-time Markov chain on Ω with generator $\widetilde{\mathcal{L}}$. Let $\Phi : \Omega \to \mathcal{S}$ be surjective. Assume that for every $s, s' \in \mathcal{S}$ and every $\omega, \omega' \in \Omega_s$,

$$\sum_{\eta \in \Omega_{s'}} \widetilde{\mathcal{L}}(\omega, \eta); =; \sum_{\eta \in \Omega_{s'}} \widetilde{\mathcal{L}}(\omega', \eta).$$

Then the projected process $Y_t := \Phi(\widetilde{X}_t)$ is a continuous-time Markov chain on $\mathcal{S}$, with generator $\mathcal{L}$ given by

$$\mathcal{L}(s, s') := \sum_{\eta \in \Omega_{s'}} \widetilde{\mathcal{L}}(\omega, \eta), \qquad \text{for any } \omega \in \Omega_s.$$

Well-definedness follows from eq:app-g-2.

Proof. Fix $s \in \mathcal{S}$ and $\omega \in \Omega_s$. For any bounded $f : \mathcal{S} \to \mathbb{R}$, define $\tilde{f} := f \circ \Phi$ on Ω. Then, using $\widetilde{\mathcal{L}}$ and partitioning Ω into fibers,

$$(\widetilde{\mathcal{L}}\tilde{f})(\omega) = \sum_{\eta \in \Omega} \widetilde{\mathcal{L}}(\omega, \eta)\big(f(\Phi(\eta)) - f(\Phi(\omega))\big) = \sum_{s' \in \mathcal{S}} \big(f(s') - f(s)\big) \sum_{\eta \in \Omega_{s'}} \widetilde{\mathcal{L}}(\omega, \eta).$$

By eq:app-g-2, the inner sum depends only on $s = \Phi(\omega)$, not on ω, hence defines a linear operator $\mathcal{L}$ on functions on $\mathcal{S}$ by the coefficients eq:app-g-3. Therefore $\Phi(\widetilde{X}_t)$ has generator $\mathcal{L}$ and is Markov.

Remark G.3. Condition eq:app-g-2 is called **strong lumpability** (or the statement that Φ is a Markov function for $\widetilde{X}$). It is stronger than "the projection is Markov under stationarity": it guarantees the Markov property for all initial conditions on Ω.

G.2 From the lifted update $\mathcal{U}_i$ to a kernel on $S_n(\lambda)$

Recall that the (semi-signed) bottom map $\Phi^+ : \Omega^+(\lambda) \to S_n(\lambda)$ reads the bottom regular row. Given a lifted transition kernel $\mathcal{U}_i$ on $\Omega^+(\lambda)$, define its induced bottom transition kernel K_i by

$$K_i(\mu, \nu) \;:=\; \mathbb{P}\big(\Phi^+(\widetilde{Q}^\circ) = \nu \;\big|\; Q^\circ\big), \qquad \text{for any } Q^\circ \in \Omega^+(\lambda) \text{ with } \Phi^+(Q^\circ) = \mu.$$

A priori this could depend on the choice of Q° in the fiber $\mathrm{MLQ}^+(\mu)$; the next proposition shows it does not, in the restricted strict setting.

Proposition G.5 (Lumpability of the lifted heat–bath updates). Assume λ is restricted strict and $x_i > \kappa = t^{-(n-1)}$. For each $i \in [n]$, the quantity $K_i(\mu, \nu)$ in eq:G1-new is well-defined (independent of the choice of $Q^\circ \in \mathrm{MLQ}^+(\mu)$). Moreover, K_i coincides with the scan-and-push kernel of Section G.5 (Definition G.14):

$$K_i(\mu,\nu) \;=\; \begin{cases} 1, & \mu_i = 0 \text{ and } \nu = \mu, \\ \dfrac{(1-t)t^{k-1}}{1-t^{r_i(\mu)}}, & \mu_i > 0 \text{ and } \nu = \mathsf{T}_i(\mu;k) \text{ for some } k \in [r_i(\mu)], \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Fix $\mu \in S_n(\lambda)$ and $i \in [n]$. If $\mu_i = 0$, then Lemma F.10 gives $\Phi^+(\mathcal{U}_i(Q^\circ)) = \mu$ almost surely for every lift Q° with bottom word μ, hence $K_i(\mu, \mu) = 1$.

Now assume $\mu_i = a > 0$. Let $Q^\circ \in \Omega^+(\lambda)$ be any lift with $\Phi^+(Q^\circ) = \mu$. By Lemma F.10, the support of $\Phi^+(\mathcal{U}_i(Q^\circ))$ is exactly $\{\mathsf{T}_i(\mu;k) : k = 1, \dots, r_i(\mu)\}$ and

$$\mathbb{P}(\Phi^+(\widetilde{Q}^\circ) = \mathsf{T}_i(\mu;k) \mid Q^\circ) = \frac{(1-t)t^{k-1}}{1-t^{r_i(\mu)}}.$$

This depends only on μ, not on the chosen lift Q°, so K_i is well-defined and equals eq:G2-new.

Theorem G.6 (Induced bottom generator). Assume (x,t) lies in the positivity regime eq:2-positivity-regime. Let $\widetilde{\mathcal{L}}$ be the lifted generator $\widetilde{\mathcal{L}}f = \sum_{i=1}^n \frac{1}{x_i}(\mathcal{U}_i f - f)$ on $\Omega^+(\lambda)$. Then the projected process $\mu_t = \Phi^+(\widetilde{X}_t)$ is a continuous-time Markov chain on $S_n(\lambda)$ with generator

$$\mathcal{L}g(\mu) \;=\; \sum_{i=1}^n \frac{1}{x_i}\Big(\sum_{\nu \in S_n(\lambda)} K_i(\mu,\nu)\, g(\nu) - g(\mu)\Big),$$

where K_i is given by eq:G2-new. Equivalently, $\mathcal{L}$ is the scan-and-push generator of Definition G.14.

Proof. We verify strong lumpability (Lemma G.2) for the surjection $\Phi^+ : \Omega^+(\lambda) \to S_n(\lambda)$ and the lifted generator $\widetilde{\mathcal{L}}$. Fix bottom states $\mu, \nu \in S_n(\lambda)$ and two lifted states $Q^\circ, Q^{\circ\prime} \in \Omega^+(\lambda)$ with $\Phi^+(Q^\circ) = \Phi^+(Q^{\circ\prime}) = \mu$. Using $\widetilde{\mathcal{L}} = \sum_{i=1}^n \frac{1}{x_i}(\mathcal{U}_i - I)$, we have

$$\sum_{R^\circ : \Phi^+(R^\circ) = \nu} \widetilde{\mathcal{L}}(Q^\circ, R^\circ) = \sum_{i=1}^n \frac{1}{x_i}\Big(\sum_{R^\circ : \Phi^+(R^\circ)=\nu} \mathcal{U}_i(Q^\circ, R^\circ) - \mathbf{1}\{\nu = \mu\}\Big).$$

For each fixed i, the fiber-sum $\sum_{R^\circ:\,\Phi^+(R^\circ)=\nu} \mathcal{U}_i(Q^\circ, R^\circ)$ is exactly the conditional probability that an i-update sends the bottom word to ν, namely

$$\sum_{R^\circ:\,\Phi^+(R^\circ)=\nu} \mathcal{U}_i(Q^\circ, R^\circ) = \mathbb{P}\big(\Phi^+(\mathcal{U}_i(Q^\circ)) = \nu \bigm| Q^\circ\big) = K_i(\mu, \nu).$$

By Proposition G.5, this quantity depends only on the bottom word μ, and hence is the same for $Q^{\circ\prime}$ as for Q°. Therefore the right-hand side above depends only on μ, so the strong lumpability condition eq:app-g-2 holds. Applying Lemma G.2 yields that $\mu_t = \Phi^+(\widetilde{X}_t)$ is Markov with generator $\mathcal{L}$ given by eq:G3-new. Finally, substituting the explicit kernel eq:G2-new shows that $\mathcal{L}$ agrees with the scan-and-push generator of Definition G.14.

G.3 Bottom-determining randomness in a site update

This section formalizes the idea that a site update $\mathcal{U}_i$ draws a **finite collection of truncated-geometric random choices** (Appendix D), and that the induced change in the bottom row depends only on $\mu = \Phi^+(Q^\circ)$ and these choices.

The key point is that, under the restricted strict assumptions, the data of a semi-signed MLQ Q° can be decomposed into:

- a bottom word $\mu = \Phi^+(Q^\circ)$, and
- "fiber variables" ξ which parameterize higher-row placements and pairing choices.

The update $\mathcal{U}_i$ uses a finite random seed $\mathbf{Z}$ (a list of truncated-geometric draws) and produces a new state:

$$Q^{\circ\prime} = \mathcal{U}_i(Q^\circ; \mathbf{Z}).$$

We will show:

- the resulting bottom word $\Phi^+(Q^{\circ\prime})$ is a deterministic function of $(\mu, \mathbf{Z})$, and
- the law of $\mathbf{Z}$ depends only on μ (in particular, the candidate counts in each truncated-geometric draw depend only on μ).

To state this, we introduce the notion of **candidate lists** at the bottom level.

Definition G.7 (Hole position and cyclic interval). For $\mu \in S_n(\lambda)$, let $h(\mu) \in [n]$ be the unique index with $\mu_{h(\mu)} = 0$. For $i \in [n]$, define the cyclic interval from i to the hole by

$$[i! \to! h(\mu)] := i, i+1, \dots, h(\mu) \quad \text{(cyclic indices)},$$

and its interior

$$(i! \to !h(\mu)] := [i! \to !h(\mu)] \setminus i.$$

Definition G.8 (Eligible candidates). Fix $\mu \in S_n(\lambda)$ and $i \in [n]$. If $\mu_i = 0$, set $C_i(\mu) := \emptyset$. Otherwise, define the **eligible candidate set**

$$C_i(\mu) := j \in (i! \to !h(\mu)] : \mu_j < \mu_i.$$

Because $\mu_{h(\mu)} = 0 < \mu_i$, we always have $h(\mu) \in C_i(\mu)$ whenever $\mu_i > 0$. Hence $C_i(\mu) \neq \emptyset$ in that case.

We order $C_i(\mu)$ by cyclic distance from i: write

$$C_i(\mu) = c_i(1), c_i(2), \ldots, c_i(r_i),$$

where $c_i(1)$ is the first eligible index encountered when scanning clockwise from (i+1), and $c_i(r_i) = h(\mu)$. Here $r_i := |C_i(\mu)|$.

The following random choice rule will appear in both the bottom chain and the lifted chain.

Definition G.9 (Truncated-geometric choice among eligible candidates). If $\mu_i > 0$, define a random index $J_i(\mu) \in C_i(\mu)$ by

$$\mathbb{P}\big(J_i(\mu) = c_i(k)\big) = \frac{(1-t)t^{k-1}}{1-t^{r_i(\mu)}} \qquad (k = 1, \ldots, r_i(\mu)).$$

If $\mu_i = 0$, define $J_i(\mu)$ to be a cemetery value and interpret it as "no move."

This is exactly the truncated geometric distribution from Appendix D (Proposition D.6).

G.4 A canonical bottom update map and a coupling refinement

We now record an explicit scan-and-push map for the induced bottom update, and we give a coupling reformulation of the lifted update.

Definition G.10 (Insertion map into the selected eligible position). Fix $\mu \in S_n(\lambda)$ and $i \in [n]$.

- If $\mu_i = 0$, define $\mathsf{T}_i(\mu) = \mu$.
- If $\mu_i > 0$, define $\mathsf{T}_i(\mu; k) \in S_n(\lambda)$ for $k \in 1, \ldots, r_i(\mu)$ as follows.

Let $j_k := c_i(k)$ be the k-th eligible candidate. Let the cyclic list of sites from i to j_k (inclusive) be

$$i = j_0,\ j_1,\ \ldots,\ j_m = j_k.$$

Define $\mathsf{T}_i(\mu; k)$ by "removing" μ_i and inserting it at j_k, shifting the intervening values one step clockwise:

$$\big(\mathsf{T}_i(\mu;k)\big)_{j_\ell} := \begin{cases} \mu_{j_{\ell+1}}, & \ell = 0, 1, \dots, m-1, \\ \mu_{j_0}, & \ell = m, \end{cases} \qquad \text{and } \big(\mathsf{T}_i(\mu;k)\big)_u := \mu_u \text{ for } u \notin \{j_0, \dots, j_m\}.$$

Because this is a cyclic permutation of the entries on $j_0, \dots, j_m$, it preserves the multiset of entries; hence $\mathsf{T}_i(\mu; k) \in S_n(\lambda)$.

We then define the random bottom update

$$\mathsf{T}_i(\mu) := \mathsf{T}_i(\mu; K), \qquad \mathbb{P}(K = k) = \frac{(1-t)t^{k-1}}{1 - t^{r_i(\mu)}}.$$

Remark G.11 (Locality on the ring). The update T_i modifies only the cyclic segment from i to the chosen eligible site j_k, and leaves the complement unchanged.

We now connect the lifted update to this bottom map.

Lemma G.12 (Bottom update depends only on μ and the truncated-geometric seed). Fix $i \in [n]$. There exists a coupling of the lifted update $\mathcal{U}_i$ with a single truncated-geometric draw $K \in 1, \dots, r_i$ such that for every $Q^\circ \in \Omega^+(\lambda)$ with $\Phi^+(Q^\circ) = \mu$,

$$\Phi^+(\mathcal{U}_i(Q^\circ)) = \mathsf{T}_i(\mu; K) \qquad \text{almost surely.}$$

Proof. Fix $i \in [n]$ and $Q^\circ \in \Omega^+(\lambda)$ with bottom word $\mu := \Phi^+(Q^\circ)$. If $\mu_i = 0$, then Proposition G.5 gives $\Phi^+(\mathcal{U}_i(Q^\circ)) = \mu$ almost surely, and the coupling is trivial. Assume $\mu_i > 0$, so $r_i(\mu) \geq 1$.

By Proposition G.5, for each $k \in \{1, \dots, r_i(\mu)\}$,

$$\mathbb{P}\big(\Phi^+(\mathcal{U}_i(Q^\circ)) = \mathsf{T}_i(\mu; k) \bigm| Q^\circ\big) = \frac{(1-t)t^{k-1}}{1 - t^{r_i(\mu)}} \; > \; 0.$$

Define a coupling as follows. First sample $K \in \{1, \dots, r_i(\mu)\}$ from the truncated geometric law eq:app-g-15. Then sample a state $\widetilde{Q}^\circ$ from the conditional distribution of the kernel $\mathcal{U}_i(Q^\circ, \cdot)$ given the event $\Phi^+(\widetilde{Q}^\circ) = \mathsf{T}_i(\mu; K)$. By the law of total probability, the marginal law of $\widetilde{Q}^\circ$ is exactly $\mathcal{U}_i(Q^\circ, \cdot)$, and by construction $\Phi^+(\widetilde{Q}^\circ) = \mathsf{T}_i(\mu; K)$ almost surely. This is the desired coupling.

G.5 Intrinsic description of the bottom chain X^{SP}

We now record the bottom chain as a self-contained Markov process on $S_n(\lambda)$ that does not reference queue fibers.

Combine Theorem G.6 with Lemma G.12: when the clock at site i rings, the bottom configuration μ is updated to $\mathsf{T}_i(\mu)$, where the eligible candidate index is sampled from the truncated geometric law.

Definition G.14 (Bottom chain X^{SP}: scan-and-push dynamics). Fix λ restricted strict and parameters (x, t). Define a continuous-time Markov chain X^{SP} on $S_n(\lambda)$ by independent Poisson clocks $N_{i}{}_{i=1}^{n}$ with rates $1/x_i$. When clock i rings and the current state is μ,

- if $\mu_i = 0$, set μ unchanged;
- if $\mu_i > 0$, form the ordered eligible candidate list $c_i(1), \ldots, c_i(r_i)$ as in eq:app-g-14, draw K with $\mathbb{P}(K = k) = \frac{(1-t)t^{k-1}}{1-t^{r_i(\mu)}}$, and replace μ by $\mathsf{T}_i(\mu; K)$ given by eq:app-g-17.

This defines the intrinsic chain on $S_n(\lambda)$.

Proposition G.15 (Explicit transition rates). Let X^{SP} be the chain of Definition G.14. Its generator $\mathcal{L}$ is given by

$$\mathcal{L}(\mu, \nu) = \sum_{i=1}^{n} \frac{1}{x_i} \mathbf{1}\{\nu \neq \mu\} \sum_{k=1}^{r_i(\mu)} \mathbf{1}\{\nu = \mathsf{T}_i(\mu; k)\} \frac{(1-t)t^{k-1}}{1-t^{r_i(\mu)}},$$

and $\mathcal{L}(\mu, \mu) = -\sum_{\nu \neq \mu} \mathcal{L}(\mu, \nu)$.

Proof. This is a direct translation of the clock dynamics and the conditional jump distribution in Definition G.14.

Remark G.16 (Nontriviality). The transition rates eq:app-g-20 are explicit and use only:

- comparisons of labels $\mu_j < \mu_i$ to form $C_i(\mu)$,
- cyclic scanning to order candidates, and
- truncated-geometric probabilities (Appendix D).

In particular, they do not require evaluating F^*_μ or P^*_λ, meeting the nontriviality requirement of the target problem.

End of Appendix G

Appendix H proves that the lifted measure $\tilde{\pi}(Q^\circ) \propto \mathrm{wt}^+(Q^\circ)$ is stationary for $\widetilde{X}$, and therefore the pushforward $\pi(\mu) = F^*_\mu(x; 1, t)/P^*_\lambda(x; 1, t)$ is stationary for the bottom chain X^{SP}. Appendix I establishes irreducibility (and hence uniqueness of π).

Appendix H. Stationarity of the Lifted Measure and Pushforward Stationarity on $S_n(\lambda)$

This appendix proves the invariance statements used in the main text:

1. the lifted probability measure $\tilde{\pi}$ on $\Omega^+(\lambda)$ (defined from the nonnegative weight wt^+) is stationary for the lifted continuous-time chain $\widetilde{X}$; and
2. the pushforward $\pi = (\Phi^+)_*\tilde{\pi}$ is stationary for the bottom chain X^{SP} on $S_n(\lambda)$.

The proofs are purely Markov-chain arguments (heat-bath / Gibbs sampler invariance) together with the lumpability result of Appendix G and the fiber-sum identities from Appendices C and E.

H.1 The lifted weight measure and its bottom pushforward

We fix a restricted strict partition λ of length n, and parameters (x, t) in a regime where the nonnegative weight model of Appendix E is available.

Definition H.1 (Lifted normalization and lifted measure). Let $\Omega^+(\lambda)$ be the semi-signed queue state space of Appendix E, equipped with a nonnegative weight function $\mathrm{wt}^+ : \Omega^+(\lambda) \to \mathbb{R}_{\geq 0}$. Define

$$Z^+(x,t) := \sum_{Q^\circ \in \Omega^+(\lambda)} \mathrm{wt}^+(Q^\circ)$$

and assume $Z^+(x,t) > 0$. Define the probability measure $\tilde{\pi}$ on $\Omega^+(\lambda)$ by

$$\tilde{\pi}(Q^\circ) := \frac{\mathrm{wt}^+(Q^\circ)}{Z^+(x,t)}.$$

Definition H.2 (Bottom weights and bottom pushforward). Let $\Phi^+ : \Omega^+(\lambda) \to S_n(\lambda)$ be the bottom map (Appendix F). For $\mu \in S_n(\lambda)$ define the bottom fiber weight

$$W^+(\mu) := \sum_{Q^\circ:\ \Phi^+(Q^\circ)=\mu} \mathrm{wt}^+(Q^\circ),$$

so that $Z^+(x,t) = \sum_{\mu \in S_n(\lambda)} W^+(\mu)$. Define the pushforward probability measure π on $S_n(\lambda)$ by

$$\pi(\mu) := \sum_{Q^\circ:\ \Phi^+(Q^\circ)=\mu} \tilde{\pi}(Q^\circ) = \frac{W^+(\mu)}{Z^+(x,t)}.$$

—

H.2 A general heat-bath invariance lemma on a finite state space

We record the standard Gibbs-sampler fact used repeatedly below.

Definition H.3 (Heat-bath kernel associated with a partition). Let Ω be a finite set and let $\tilde{\pi}$ be a probability measure on Ω with $\tilde{\pi}(\omega) > 0$ for all $\omega \in \Omega$. Let $\mathcal{P} = \{C_\alpha\}_{\alpha \in A}$ be a partition of Ω into nonempty cells.

Define the heat-bath kernel $K_{\mathcal{P}}$ on Ω by

$$K_{\mathcal{P}}(\omega, \eta) := \mathbf{1}\{\eta \in C(\omega)\} \, \frac{\tilde{\pi}(\eta)}{\tilde{\pi}(C(\omega))}, \qquad \tilde{\pi}(C) := \sum_{\xi \in C} \tilde{\pi}(\xi),$$

where $C(\omega)$ denotes the unique cell of $\mathcal{P}$ containing ω.

Lemma H.4 (Reversibility of heat-bath kernels). For all $\omega, \eta \in \Omega$,

$$\tilde{\pi}(\omega) \, K_{\mathcal{P}}(\omega, \eta) = \tilde{\pi}(\eta) \, K_{\mathcal{P}}(\eta, \omega).$$

Proof. If ω and η lie in different cells then both sides are zero. If they lie in the same cell C then, by eq:h-5,

$$\tilde{\pi}(\omega) \, K_{\mathcal{P}}(\omega, \eta) = \tilde{\pi}(\omega) \, \frac{\tilde{\pi}(\eta)}{\tilde{\pi}(C)} = \tilde{\pi}(\eta) \, \frac{\tilde{\pi}(\omega)}{\tilde{\pi}(C)} = \tilde{\pi}(\eta) \, K_{\mathcal{P}}(\eta, \omega).$$

Corollary H.5 (Stationarity). The measure $\tilde{\pi}$ is stationary for $K_{\mathcal{P}}$, i.e. $\tilde{\pi} K_{\mathcal{P}} = \tilde{\pi}$.

Proof. Sum eq:h-6 over $\omega \in \Omega$.

Lemma H.5.1 (Closure of stationarity under composition). Let Ω be a finite set and let π be a probability measure on Ω. If K_1 and K_2 are Markov kernels on Ω such that $\pi K_1 = \pi$ and $\pi K_2 = \pi$, then $\pi(K_1 K_2) = \pi$. More generally, if $\{K_j\}_{j=1}^m$ satisfy $\pi K_j = \pi$ for all j, then $\pi(K_1 K_2 \cdots K_m) = \pi$.

Proof. Immediate from associativity: $\pi(K_1 K_2) = (\pi K_1) K_2 = \pi K_2 = \pi$. The general case follows by induction.

H.3 Conditional factorization for classic-layer pairing data at $q = 1$

We now relate the abstract heat-bath lemma to the concrete queue weights.

Recall that for a semi-signed MLQ Q°, the weight $\mathrm{wt}^+(Q^\circ)$ is obtained by *i* summing over neutral-sign fibers (Appendix E), and *ii* specializing to $q = 1$. In particular, wt^+ is a product of nonnegative local factors, and dependence on **classic-layer pairing data** appears only through the classic-layer pairing weights (Appendix D).

We formalize this as follows.

Definition H.6 (Classic-layer pairing datum). Fix $r \in 2, \dots, L$. For $Q^\circ \in \Omega^+(\lambda)$, let $\mathsf{P}_r(Q^\circ)$ denote the complete pairing configuration in the classic layer between regular row r (upper) and signed row ((r-1)') (lower), computed on absolute values as in Appendix B.

Define the map

$$\mathsf{P}_r : \Omega^+(\lambda) \to \mathcal{M}_r$$

where $\mathcal{M}_r$ denotes the finite set of admissible classic-layer pairing configurations consistent with the fixed row words of row r and row ((r-1)').

Lemma H.7 (Classic-layer conditional distribution is the $q = 1$ kernel). Fix $r \in 2, \dots, L$. Fix any values of all queue data outside the classic-layer pairing variable P_r; equivalently, fix a cell of the partition

$$\mathcal{P}_r := \{Q^\circ \in \Omega^+(\lambda) : \text{outside fixed}\}.$$

consisting of all states $Q^\circ \in \Omega^+(\lambda)$ that agree on every component except possibly P_r. Then, under the lifted measure $\tilde{\pi}$, the conditional distribution of P_r given the outside data is proportional to the $q = 1$ classic-layer weight of Appendix D:

$$\tilde{\pi}(\mathsf{P}_r = P \mid \text{outside}) = \mathbf{K}_r^{(1)}(\text{outside}; P).$$

where $\mathbf{K}_r^{(1)}$ is the normalized classic-layer kernel (Appendix D / Definition D.10).

Proof. Fix outside data. Consider two states $Q^\circ, Q^{\circ\prime}$ that agree outside the layer but have different classic-layer pairings $P = \mathsf{P}_r(Q^\circ)$ and $P' = \mathsf{P}_r(Q^{\circ\prime})$. By construction of wt^+ (Appendix E), every factor in wt^+ that does not belong to the classic layer $(r \to (r-1)')$ is identical in Q° and $Q^{\circ\prime}$, because all other data are fixed.

Therefore the ratio

$$\frac{\mathrm{wt}^+(Q^\circ)}{\mathrm{wt}^+(Q^{\circ\prime})}$$

depends only on the ratio of the classic-layer pairing weights at $q = 1$, i.e. the product over nontrivial pairings of the factors $\frac{(1-t)t^{\text{skip}}}{1-t^{\text{free}}}$ (Appendix D). Thus the conditional distribution of P_r given outside is proportional to $\mathrm{wt}^{(1)}_{\text{layer}}(P)$ in the notation of Appendix D (Definition D.8). By Theorem D.9 (and, in the strict case, Corollary D.9S), $\mathrm{wt}^{(1)}_{\text{layer}}$ is already normalized to sum to 1 over admissible P, and therefore it equals the conditional distribution. This is exactly eq:h-9.

H.4 Stationarity of $\tilde{\pi}$ for the lifted chain

Recall the positive weight wt^+ on $\Omega^+(\lambda)$ (Appendix E) and the corresponding probability measure

$$\tilde{\pi}(Q^\circ) = \frac{\mathrm{wt}^+(Q^\circ)}{Z^+(x,t)}, \qquad Z^+(x,t) = \sum_{Q^\circ \in \Omega^+(\lambda)} \mathrm{wt}^+(Q^\circ).$$

For each $i \in [n]$, Appendix F.3 defines $\mathcal{U}_i$ as a finite composition of local heat-bath kernels R_{i,ℓ_j} associated with fixed two-line partitions $\mathcal{P}_{i,\ell_j}$.

Theorem H.8 (Invariance of $\tilde{\pi}$ under each $\mathcal{U}_i$). Assume the positivity regime $x_i > \kappa$ so that $\tilde{\pi}$ is well-defined. Then for every $i \in [n]$,

$$\tilde{\pi}\,\mathcal{U}_i \;=\; \tilde{\pi},$$

i.e. $\tilde{\pi}$ is stationary for the discrete-time chain with one-step transition $\mathcal{U}_i$.

Proof. Fix $i \in [n]$. By construction in Appendix F.3, the site update $\mathcal{U}_i$ is a finite composition of local heat-bath kernels R_{i,ℓ_j} associated with fixed two-line partitions $\mathcal{P}_{i,\ell_j}$. Each R_{i,ℓ_j} preserves $\tilde{\pi}$ by Corollary H.5, hence Lemma H.5.1 implies that the composition $\mathcal{U}_i$ also preserves $\tilde{\pi}$.

Corollary H.9 (Stationarity for the continuous-time lifted chain). Let $(\widetilde{X}_t)_{t\geq 0}$ be the continuous-time Markov chain on $\Omega^+(\lambda)$ with generator

$$\widetilde{\mathcal{L}}f(Q^\circ) \;:=\; \sum_{i=1}^{n} \frac{1}{x_i}\Big(\sum_{\widetilde{Q}^\circ \in \Omega^+(\lambda)} \mathcal{U}_i(Q^\circ, \widetilde{Q}^\circ)\, f(\widetilde{Q}^\circ) - f(Q^\circ)\Big).$$

Then $\tilde{\pi}$ is stationary for $\widetilde{X}_t$.

Proof. By Theorem H.8, each $\mathcal{U}_i$ leaves $\tilde{\pi}$ invariant, so $\sum_{Q^\circ} \tilde{\pi}(Q^\circ)(\mathcal{U}_i f)(Q^\circ) = \sum_{Q^\circ} \tilde{\pi}(Q^\circ) f(Q^\circ)$ for every bounded f. Summing with coefficients $1/x_i$ yields $\sum_{Q^\circ} \tilde{\pi}(Q^\circ)(\widetilde{\mathcal{L}}f)(Q^\circ) = 0$, i.e. $\tilde{\pi}\widetilde{\mathcal{L}} = 0$.

H.5 Stationarity of the pushforward measure on $S_n(\lambda)$

Define the pushforward of $\tilde{\pi}$ under Φ^+ by

$$\pi(\mu) \;:=\; \sum_{Q^\circ \in \mathrm{MLQ}^+(\mu)} \tilde{\pi}(Q^\circ), \qquad \mu \in S_n(\lambda).$$

Equivalently, π is the law of $\mu = \Phi^+(\widetilde{X})$ when $\widetilde{X} \sim \tilde{\pi}$.

Theorem H.10 (Stationarity of π for the projected chain). Assume λ is restricted strict and $x_i > \kappa$. Let $\mathcal{L}$ be the bottom generator eq:G3-new (scan-and-push). Then $\pi\mathcal{L} = 0$, i.e. π is stationary for the bottom process $\mu_t = \Phi^+(\widetilde{X}_t)$.

Proof. Let $g : S_n(\lambda) \to \mathbb{R}$ and set $f = g \circ \Phi^+ : \Omega^+(\lambda) \to \mathbb{R}$. Stationarity of $\tilde{\pi}$ for $\widetilde{\mathcal{L}}$ (Corollary H.9) gives

$$0 = \sum_{Q^\circ \in \Omega^+(\lambda)} \tilde{\pi}(Q^\circ)\,(\widetilde{\mathcal{L}}f)(Q^\circ) = \sum_{i=1}^{n} \frac{1}{x_i} \sum_{Q^\circ} \tilde{\pi}(Q^\circ)\Big((\mathcal{U}_i f)(Q^\circ) - f(Q^\circ)\Big).$$

Condition on $\mu = \Phi^+(Q^\circ)$ and use Proposition G.5, which identifies the induced kernel K_i and shows it depends only on μ:

$$(\mathcal{U}_i f)(Q^\circ) = \sum_{\widetilde{Q}^\circ} \mathcal{U}_i(Q^\circ, \widetilde{Q}^\circ)\, g(\Phi^+(\widetilde{Q}^\circ)) = \sum_{\nu \in S_n(\lambda)} K_i(\mu,\nu)\, g(\nu).$$

Therefore

$$0 = \sum_{\mu \in S_n(\lambda)} \pi(\mu) \sum_{i=1}^{n} \frac{1}{x_i} \Big(\sum_{\nu} K_i(\mu,\nu) g(\nu) - g(\mu)\Big) = \sum_{\mu \in S_n(\lambda)} \pi(\mu)\,(\mathcal{L}g)(\mu),$$

which is exactly $\pi\mathcal{L} = 0$.

Corollary H.11 (Stationary distribution as normalized fiber weights). In the positivity regime $x_i > \kappa$,

$$\pi(\mu) \;=\; \frac{W^+(\mu)}{Z^+(x,t)} \;=\; \frac{W(\mu)}{Z(\lambda)}, \qquad \mu \in S_n(\lambda),$$

where $W^+(\mu) = \sum_{Q^\circ :\, \Phi^+(Q^\circ) = \mu} \mathrm{wt}^+(Q^\circ)$, $Z^+(x,t) = \sum_{\mu \in S_n(\lambda)} W^+(\mu)$, and $W(\mu) = \sum_{Q^\pm :\, \Phi(Q^\pm) = \mu} \mathrm{wt}(Q^\pm)\big|_{q=1}$, $Z(\lambda) = \sum_{\mu \in S_n(\lambda)} W(\mu)$. The second equality follows from the exact fiber-sum preservation in Theorem E.11.

Proof. By definition,

$$\pi(\mu) = \sum_{Q^\circ :\, \Phi^+(Q^\circ) = \mu} \tilde{\pi}(Q^\circ) = \frac{1}{Z^+(x,t)} \sum_{Q^\circ :\, \Phi^+(Q^\circ) = \mu} \mathrm{wt}^+(Q^\circ) = \frac{W^+(\mu)}{Z^+(x,t)}.$$

Theorem E.11 gives $W^+(\mu) = W(\mu)$ for every μ, and summing over μ yields $Z^+(x,t) = Z(\lambda)$.

H.6 Identification with interpolation polynomials at $q = 1$

We complete the link to the target stationary distribution formula.

Proposition H.12 (Bottom weights equal interpolation ASEP weights; partition function equals interpolation Macdonald). For every $\mu \in S_n(\lambda)$,

$$W^+(\mu) = F^*_\mu(x_1, \dots, x_n; q = 1, t).$$

and

$$Z^+(x, t) = P^*_\lambda(x_1, \dots, x_n; q = 1, t).$$

Proof. By Theorem E.11 (exact fiber-sum preservation), the total $\Omega^+(\lambda)$-fiber sum equals the original signed-MLQ $q = 1$ fiber sum:

$$W^+(\mu) = \sum_{Q^\circ : \Phi^+(Q^\circ)=\mu} \mathrm{wt}^+(Q^\circ) = \sum_{Q^\pm : \Phi(Q^\pm)=\mu} \mathrm{wt}(Q^\pm)\big|_{q=1}.$$

By Appendix C (Theorem C.12), the right-hand side equals the interpolation ASEP polynomial $F^*_\mu(x; 1, t)$ under our normalization convention. This proves eq:h-13. Summing eq:h-13 over μ and using Appendix C again yields eq:h-14.

Corollary H.13 (Explicit stationary distribution formula). The stationary measure π of Theorem H.10 satisfies

$$\pi(\mu) = \frac{F^*_\mu(x_1, \dots, x_n; q = 1, t)}{P^*_\lambda(x_1, \dots, x_n; q = 1, t)}.$$

Proof. Combine eq:h-4 with Proposition H.12.

End of Appendix H

Appendix I proves irreducibility of the bottom chain X^{SP} on $S_n(\lambda)$, implying uniqueness of the stationary distribution π from Corollary H.13.

Appendix I. Irreducibility of the Bottom Chain on $S_n(\lambda)$

This appendix proves that the bottom chain X^{SP} constructed in Appendix G is **irreducible** on the finite state space $S_n(\lambda)$ (under the restricted strict assumptions). Consequently, the stationary distribution identified in Appendix H is **unique**.

We work with:

- a restricted strict partition λ of length n (Assumption E.1),
- parameters $t \in (0,1)$ and $x = (x_1, \dots, x_n) \in (0,\infty)^n$,
- the bottom chain X^{SP} on $S_n(\lambda)$ defined in Appendix G (Definition G.14),
- the induced per-site move $\mathsf{T}_i(\mu; k)$ (Definition G.10) with the truncated geometric choice (Definition G.9).

I.1 Hole position and the cyclic word representation

Fix $\mu \in S_n(\lambda)$. Because λ is restricted strict, μ contains exactly one zero entry.

Definition I.1 (Hole position and successor map). Let $\sigma : [n] \to [n]$ be the cyclic successor $\sigma(i) = i+1$ (indices modulo (n)). Define the **hole position** $h(\mu) \in [n]$ by

$$\mu_{h(\mu)} = 0.$$

Definition I.2 (Cyclic word after the hole). Define the length-((n-1)) word of positive labels (the cyclic order of the nonzero entries after the hole)

$$w(\mu) := \big(\mu_{\sigma(h(\mu))},\ \mu_{\sigma^2(h(\mu))},\ \dots,\ \mu_{\sigma^{n-1}(h(\mu))}\big).$$

Since μ is a permutation of λ and λ is strict, the entries of $w(\mu)$ are distinct and enumerate $\lambda \setminus 0$.

Remark I.3 (Reconstruction). The pair $(h(\mu), w(\mu))$ uniquely determines μ: place (0) at position $h(\mu)$ and then place the entries of $w(\mu)$ clockwise starting at $\sigma(h(\mu))$.

I.2 The "jump-to-hole" move is always available and has positive rate

Recall that when site i rings in the bottom chain (Definition G.14), if $\mu_i > 0$ then an eligible target is selected from the ordered list $c_i(1), \dots, c_i(r_i)$ with truncated geometric probabilities

$$\mathbb{P}(K = k) = \frac{(1-t)t^{k-1}}{1 - t^{r_i(\mu)}}.$$

The last eligible target $c_i(r_i)$ is the hole $h(\mu)$ (Definition G.8).

Definition I.4 (Jump-to-hole transition). For $\mu \in S_n(\lambda)$ and $i \in [n]$ with $\mu_i > 0$, define the **jump-to-hole update**

$$\mathsf{J}_i(\mu) := \mathsf{T}_i(\mu; r_i(\mu)),$$

i.e. the bottom update at site i with the chosen eligible target equal to the hole.

Lemma I.5 (Positive rate for jump-to-hole). Fix $i \in [n]$. If $\mu_i > 0$, then $\mathsf{J}_i(\mu) \neq \mu$ is a valid jump of the chain and occurs with strictly positive rate:

$$\mathrm{rate}(\mu \to \mathsf{J}_i(\mu)) = \frac{1}{x_i} \cdot \frac{(1-t)t^{r_i(\mu)-1}}{1-t^{r_i(\mu)}} \;>\; 0.$$

Proof. If $\mu_i > 0$, then $h(\mu) \in C_i(\mu)$ (Definition G.8), hence $r_i(\mu) \geq 1$ and the choice $K = r_i(\mu)$ is allowed. Since $t \in (0,1)$, the probability in eq:app-i-5 is strictly positive, and the Poisson clock at i has rate $1/x_i > 0$.

I.3 Jump-to-hole equals a move-to-front operation on the cyclic word

We now describe the effect of J_i in the (h, w) representation.

Definition I.6 (Move-to-front operator on words). Let A be a finite set and let $w = (w_1, \dots, w_m)$ be a word with distinct letters in A. For $a \in A$ appearing in w, define $\mathsf{M}_a(w)$ to be the word obtained by moving a to the front and preserving the relative order of the other letters:

$$\mathsf{M}_a(w) := (a,\ w_1, \dots, w_{p-1}, w_{p+1}, \dots, w_m), \quad \text{where } w_p = a.$$

Lemma I.7 (Word dynamics under jump-to-hole). Let $\mu \in S_n(\lambda)$, let $h = h(\mu)$, and let $i \in [n]$ with $\mu_i = a > 0$. Then:

1. the hole moves one step counterclockwise:

$$h(\mathsf{J}_i(\mu)) = \sigma^{-1}(h),$$

2. the cyclic word transforms by move-to-front of the moved label:

$$w(\mathsf{J}_i(\mu)) = \mathsf{M}_a(w(\mu)).$$

Proof. Let $h = h(\mu)$. The update $\mathsf{J}_i(\mu)$ is the cyclic block rotation $\mathsf{T}_i(\mu; k)$ with target $j_k = h$ (Definition G.10).

Write the clockwise block from i to h (inclusive) as

$$i = j_0,\ j_1 = \sigma(i),\ \dots,\ j_m = h.$$

By the definition eq:app-g-17 of T_i, the entries on this block are rotated:

$$(\mathsf{J}_i(\mu))_{j_\ell} = \mu_{j_{\ell+1}} \quad (\ell = 0, \dots, m-1), \qquad (\mathsf{J}_i(\mu))_h = \mu_i = a.$$

Since $\mu_h = 0$, the entry at j_{m-1} becomes (0), hence the new hole position is j_{m-1}. But j_{m-1} is the unique predecessor of h on the cycle, i.e. $j_{m-1} = \sigma^{-1}(h)$, proving eq:app-i-7.

For eq:app-i-8, note that the new hole is $\sigma^{-1}(h)$, so the new cyclic word $w(\mathsf{J}_i(\mu))$ begins at site $\sigma(\sigma^{-1}(h)) = h$. By eq:app-i-10, the label at h becomes a, so a becomes the first letter of the new word. The remaining letters of the new word appear in the same clockwise order as before, with the single occurrence of a removed from its old position (because all other sites not equal to (h) either remain unchanged or are shifted forward along the block). This is exactly the move-to-front operation M_a on the old word $w(\mu)$.

Corollary I.8 (Hole shift without changing the word). Let $\mu \in S_n(\lambda)$ and let $w(\mu) = (w_1, \dots, w_{n-1})$. If $i = \sigma(h(\mu))$ is the site immediately clockwise of the hole, then $\mu_i = w_1$ and

$$w(\mathsf{J}_i(\mu)) = w(\mu), \qquad h(\mathsf{J}_i(\mu)) = \sigma^{-1}(h(\mu)).$$

Proof. Apply Lemma I.7 with $a = w_1$. Since $\mathsf{M}_{w_1}(w) = w$, the word is unchanged while the hole shifts by one step.

I.4 Move-to-front generates all words

The following elementary lemma is the combinatorial engine of irreducibility.

Lemma I.9 (Transitivity of move-to-front). Let A be a finite set and let $\mathcal{W}(A)$ be the set of all words listing each element of A exactly once. Fix any target word

$$v = (v_1, \dots, v_m) \in \mathcal{W}(A).$$

Then for every initial word $w \in \mathcal{W}(A)$ there exists a finite sequence of move-to-front operations that transforms w into v.

In particular, the explicit sequence

$$w^{(0)} := w, \qquad w^{(r)} := \mathsf{M}_{v_{m-r+1}}(w^{(r-1)}) \quad (r = 1, \dots, m)$$

satisfies $w^{(m)} = v$.

Proof. We argue by induction on r. For $r = 1$, $w^{(1)}$ begins with v_m. Suppose $w^{(r-1)}$ begins with the block $(v_{m-r+2}, \dots, v_m)$ in this order. Applying $\mathsf{M}_{v_{m-r+1}}$ moves v_{m-r+1} to the front while preserving the relative order of all other letters, hence the front block becomes $(v_{m-r+1}, v_{m-r+2}, \cdots, v_m)$. After m steps we obtain v.

I.5 Irreducibility of the bottom chain

We now combine the previous facts to construct, for any pair $\mu, \nu \in S_n(\lambda)$, a directed path $\mu \to \nu$ using only jump-to-hole transitions, each of which occurs with positive rate (Lemma I.5).

Theorem I.10 (Irreducibility of X^{SP} on $S_n(\lambda)$). Assume λ is restricted strict, $t \in (0,1)$, and $x \in (0,\infty)^n$. Then the continuous-time bottom chain X^{SP} of Appendix G is **irreducible** on $S_n(\lambda)$: for every $\mu, \nu \in S_n(\lambda)$, the state ν is reachable from μ with positive probability in finite time.

Proof. Fix $\mu, \nu \in S_n(\lambda)$. Let

$$(h_0, w_0) := (h(\mu), w(\mu)), \qquad (h_\star, w_\star) := (h(\nu), w(\nu)).$$

Step 1: Match the cyclic word. By Lemma I.9 (with $A = \lambda \setminus 0$), there exists a finite sequence of move-to-front operations on words that transforms w_0 into $w_\star$. Concretely, write $w_\star = (v_1, \dots, v_{n-1})$ and apply $\mathsf{M}_{v_{n-1}}, \mathsf{M}_{v_{n-2}}, \dots, \mathsf{M}_{v_1}$ in that order.

By Lemma I.7, each move-to-front M_a on the current word is realized by a jump-to-hole transition $\mathsf{J}_{i(a)}$ at the unique site (i(a)) currently carrying label a. Hence there exists a finite sequence of bottom states

$$\mu = \mu^{(0)} \;\to\; \mu^{(1)} \;\to\; \cdots \;\to\; \mu^{(m)}$$

such that

$$w(\mu^{(m)}) = w_\star.$$

Each arrow in eq:app-i-15 is a jump-to-hole transition, hence has positive rate by Lemma I.5.

Step 2: Adjust the hole position without changing the word. At this point the word agrees with ν, but the hole position may differ. By Corollary I.8, if we repeatedly apply the jump-to-hole update at the site immediately clockwise of the hole, then the cyclic word remains unchanged while the hole shifts by one step counterclockwise each time.

Since the hole position evolves on a cycle, by applying this word-preserving hole shift sufficiently many times we can move the hole from $h(\mu^{(m)})$ to $h_\star$ while keeping the word equal to $w_\star$. Thus there exists $\ell \geq 0$ such that

$$\mu^{(m)} \;\to\; \mu^{(m+1)} \;\to\; \cdots \;\to\; \mu^{(m+\ell)}$$

and

$$h(\mu^{(m+\ell)}) = h_\star, \qquad w(\mu^{(m+\ell)}) = w_\star.$$

By Remark I.3, $(h_\star, w_\star)$ uniquely specifies ν, hence $\mu^{(m+\ell)} = \nu$.

Combining Steps 1–2 yields a directed path $\mu \to \nu$ along transitions of strictly positive rate. In a finite-state continuous-time Markov chain, any such path is followed with positive probability in finite time. Therefore X^{SP} is irreducible.

I.6 Uniqueness of the stationary distribution

For completeness, we record the standard consequence used implicitly in the main text.

Corollary I.11 (Uniqueness of π). Under the assumptions of Theorem I.10, the bottom chain X^{SP} has a unique stationary distribution π on $S_n(\lambda)$. In particular, the stationary distribution identified in Appendix H (Corollary H.13) is the unique stationary distribution.

Proof. The chain X^{SP} is a continuous-time Markov chain on a finite state space. By Theorem I.10 it is irreducible. Every finite irreducible continuous-time Markov chain is positive recurrent and admits a unique stationary distribution.

Remark I.12 (Aperiodicity). No separate aperiodicity argument is needed: continuous-time irreducible chains on finite state spaces converge to stationarity without periodicity obstructions. (If one considers an embedded discrete-time jump chain, it may have a nontrivial period; the continuous-time holding times remove this obstruction.)

End of Appendix I

This completes the proof package needed for the uniqueness part of the stationary distribution statement in the main text.

Appendix J. Worked Examples and Computational Checks

This appendix provides worked examples of the bottom chain X^{SP} (Appendix G) and outlines a reproducible computational protocol to verify stationarity on small instances. The results here are supplementary: the rigorous stationarity proof is given in Appendix H, and irreducibility is given in Appendix I.

J.1 What is being checked

There are two complementary checks one can perform for small n.

1. **Markov-chain check (bottom chain only).** Build the generator $\mathcal{L}$ explicitly from the intrinsic update rule (Appendix G / Definition G.14), solve the linear system $\pi\mathcal{L} = 0$ with $\sum_\mu \pi(\mu) = 1$, and confirm that π is a probability vector. This check is independent of the queue model.
2. **Partition-function check (queue model).** Enumerate the positive queue space $\Omega^+(\lambda)$ (Appendix E) and compute bottom weights

$$W^+(\mu) = \sum_{Q^\circ:\ \Phi^+(Q^\circ)=\mu} \mathrm{wt}^+(Q^\circ), \qquad Z^+ = \sum_\mu W^+(\mu),$$

then compare $W^+(\mu)/Z^+$ to the stationary vector obtained from $\mathcal{L}$. Theorems H.11–H.13 imply these agree.

In what follows we focus on the explicit bottom chain computation and give a concrete numeric example.

J.2 The case $n = 2$: explicit generator and stationary distribution

Let $n = 2$ and $\lambda = (a, 0)$ with $a \geq 2$ (restricted strict). Then

$$S_2(\lambda) = (a, 0), (0, a).$$

The bottom chain X^{SP} has only deterministic jumps: whenever the clock at the site carrying the nonzero label rings, the label jumps into the hole (there are no intermediate candidates when $n = 2$).

Proposition J.1 (Two-state generator and stationary distribution). Let $x_1, x_2 > 0$. In the ordered basis $((a, 0), (0, a))$, the generator is

$$\mathcal{L} = \begin{pmatrix} -\frac{1}{x_1} & \frac{1}{x_1} \\ \frac{1}{x_2} & -\frac{1}{x_2} \end{pmatrix}.$$

The unique stationary distribution is

$$\pi(a, 0) = \frac{x_1}{x_1 + x_2}, \qquad \pi(0, a) = \frac{x_2}{x_1 + x_2}.$$

Proof. From the definition of the bottom chain, $(a, 0) \to (0, a)$ occurs exactly when clock (1) rings, with rate $1/x_1$, and $(0, a) \to (a, 0)$ occurs exactly when clock (2) rings, with rate $1/x_2$. This yields eq:app-j-2-gen. The stationary balance equation $\pi(a, 0) \cdot (1/x_1) = \pi(0, a) \cdot (1/x_2)$ and normalization $\pi(a, 0) + \pi(0, a) = 1$ imply eq:app-j-2-pi.

Remark J.2 (Independence of t when $n = 2$). The truncated-geometric choice parameter t does not enter when $n = 2$, because each eligible candidate set has size 1.

J.3 The case $n = 3$: transitions from a canonical state

Let $n = 3$ and $\lambda = (a, b, 0)$ with $a > b \geq 2$. Then $|S_3(\lambda)| = 6$. In this case, every eligible candidate set has size $r \in 1, 2$, so the truncated-geometric probabilities simplify:

- if $r = 1$, the move is deterministic;
- if $r = 2$, then

$$\mathbb{P}(K = 1) = \frac{1 - t}{1 - t^2} = \frac{1}{1 + t}, \qquad \mathbb{P}(K = 2) = \frac{(1 - t)t}{1 - t^2} = \frac{t}{1 + t}.$$

We illustrate the updates out of the state $\mu = (a, b, 0)$, which has hole at site (3).

Example J.3 (Outgoing transitions from $(a, b, 0)$). Let $\mu = (a, b, 0)$ and $h(\mu) = 3$.

- If clock (2) rings, the site (2) carries label b and the interval $[2 \to h]$ contains only the hole; hence $r_2(\mu) = 1$ and the update is deterministic:

$$(a, b, 0) \xrightarrow{\frac{1}{x_2}} (a, 0, b).$$

- If clock (1) rings, the site (1) carries label a and there are two eligible targets in $[1 \to h]$: site (2) (label $b < a$) and the hole (label $0 < a$). Hence $r_1(\mu) = 2$. The update chooses:
  - target site (2) with probability $1/(1+t)$, producing the adjacent swap:

$$(a, b, 0) \ \to \ (b, a, 0),$$

  - target site (3) (the hole) with probability $t/(1 + t)$, producing the cyclic rotation

$$(a, b, 0) \ \to \ (b, 0, a).$$

Thus the total jump rates out of $(a, b, 0)$ are

$$(a, b, 0) \xrightarrow{\frac{1}{x_2}} (a, 0, b), \qquad (a, b, 0) \xrightarrow{\frac{1}{x_1} \cdot \frac{1}{1+t}} (b, a, 0), \qquad (a, b, 0) \xrightarrow{\frac{1}{x_1} \cdot \frac{t}{1+t}} (b, 0, a).$$

Remark J.4 (General rule for $n = 3$). The preceding computation is representative: the ring geometry and the single hole ensure that from any state and site i, the ordered eligible list consists of the hole plus possibly one smaller positive label in between, so $r_i(\mu) \leq 2$.

J.4 Numerical example: explicit generator and stationarity residual

We now present a concrete numerical instance of the $n = 3$ chain and compute its stationary distribution by linear algebra.

Example J.5 (A concrete 6×6 generator). Take

$$\lambda = (3, 2, 0), \qquad t = \frac{1}{2}, \qquad x = (x_1, x_2, x_3) = (4, 5, 6).$$

List the six states in the order

$$\mu^{(1)} = (3, 2, 0), \mu^{(2)} = (0, 2, 3), \mu^{(3)} = (3, 0, 2), \mu^{(4)} = (2, 0, 3), \mu^{(5)} = (0, 3, 2), \mu^{(6)} = (2, 3, 0).$$

Using the explicit rate formula of the bottom chain (Appendix G / Proposition G.15), one obtains the generator matrix $\mathcal{L}$ (rows sum to (0)):

$$\mathcal{L} \approx \begin{pmatrix} -0.45 & 0 & 0.20 & 0.083333 & 0 & 0.166667 \\ 0.166667 & -0.366667 & 0 & 0 & 0 & 0.20 \\ 0 & 0.166667 & -0.416667 & 0 & 0.25 & 0 \\ 0 & 0.25 & 0.111111 & -0.416667 & 0.055556 & 0 \\ 0.066667 & 0.133333 & 0 & 0 & -0.366667 & 0.166667 \\ 0 & 0 & 0.25 & 0.20 & 0 & -0.45 \end{pmatrix}.$$

Solving $\pi\mathcal{L} = 0$ with $\sum_{k=1}^{6} \pi(\mu^{(k)}) = 1$ yields the stationary distribution

$$\pi \approx (0.106572,\ 0.226098,\ 0.200283,\ 0.115893,\ 0.154116,\ 0.197039).$$

A direct residual check gives

$$|\pi\mathcal{L}|_\infty \leq 2 \times 10^{-16},$$

confirming stationarity numerically.

Remark J.6 (Interpretation). This computation is strictly a bottom-chain check: it does not use the queue model. By Definition H.2 and Theorem H.10, the same π is obtained by pushing forward the lifted stationary measure on $\Omega^+(\lambda)$.

J.5 A reproducible protocol for comparing π with queue fiber weights

This section records a fully deterministic protocol for verifying the identity

$$\pi(\mu) = \frac{W^+(\mu)}{Z^+}$$

on small instances by brute force.

Algorithm J.7 (Brute-force comparison of bottom stationarity and queue weights).

Input: n, restricted strict λ, parameters $t \in (0,1)$, $x \in (0,\infty)^n$. **Output:** a diagnostic report verifying $\pi_{\mathrm{MC}} = \pi_{\mathrm{queue}}$ on $S_n(\lambda)$.

1. **Enumerate $S_n(\lambda)$.** List all permutations μ of λ.
2. **Build the bottom generator $\mathcal{L}$.** For each $\mu \in S_n(\lambda)$ and each site $i \in [n]$:
   - compute the eligible set $C_i(\mu)$ and its ordered list;
   - for each $k = 1, \dots, |C_i(\mu)|$, compute $\nu = \mathsf{T}_i(\mu; k)$ and add rate

     $$\frac{1}{x_i} \cdot \frac{(1-t)t^{k-1}}{1-t^{|C_i(\mu)|}}$$

     to $\mathcal{L}(\mu, \nu)$;
   - set diagonal entries so rows sum to (0).
3. **Solve for π_{MC}.** Solve $\pi_{\mathrm{MC}}\mathcal{L} = 0$ with $\sum_\mu \pi_{\mathrm{MC}}(\mu) = 1$.
4. **Enumerate $\Omega^+(\lambda)$ and compute wt^+.** Enumerate all semi-signed MLQs $Q^\circ \in \Omega^+(\lambda)$ (Appendix E) and compute $\mathrm{wt}^+(Q^\circ)$ (Appendix E / Definition E.9, Proposition E.10), using:
   - the $q = 1$ classic-layer factors from Appendix D, and
   - the neutral factors $(x_i - \kappa)$ with $\kappa = t^{-(n-1)}$.
5. **Compute queue bottom weights.** For each $\mu \in S_n(\lambda)$, compute

   $$W^+(\mu) = \sum_{Q^\circ:\ \Phi^+(Q^\circ)=\mu} \mathrm{wt}^+(Q^\circ), \qquad Z^+ = \sum_\mu W^+(\mu),$$

   and set $\pi_{\mathrm{queue}}(\mu) = W^+(\mu)/Z^+$.
6. **Compare.** Report the maximum deviation

   $$\max_{\mu \in S_n(\lambda)} |\pi_{\mathrm{MC}}(\mu) - \pi_{\mathrm{queue}}(\mu)|.$$

By Theorem H.10 and Corollary I.11, this should vanish exactly (up to arithmetic error).

Remark J.8 (Complexity and feasibility). For $n \leq 4$, the state space $|S_n(\lambda)| = n!$ is small, and the bottom generator is trivial to compute. The limiting factor is the enumeration of $\Omega^+(\lambda)$, whose size grows rapidly with n and λ_1. For worked checks, one typically chooses λ_1 small *e.g.*$(L \leq 3)$ and $n \leq 4$.

End of Appendix J

This concludes the example and computation appendix.