Smooth shifts of the Φ^4_3 measure are singular: a mollifier-scale proof via total variation

Author. Lior Isthmus
Date. 2026-02-18

Abstract

We study the Φ^4_3 Euclidean field measure μ on $\mathbb{T}^3$ and its behavior under smooth shifts $T_\psi(u) = u + \psi$ with $\psi \in C^\infty(\mathbb{T}^3)$. For nontrivial interaction ($\lambda \neq 0$) we prove that μ and $T_{\psi *}\mu$ are mutually singular for every $\psi \neq 0$. The core mechanism is a deterministic logarithmic mass-renormalization drift (setting-sun) at an ultra-small mollifier scale $\varepsilon_n = e^{-e^n}$, which dominates random fluctuations after normalization. The conclusion is obtained by constructing bounded tests that asymptotically separate the two measures in total variation. This note is a gap-filling companion to Problem 1 of *First Proof* [FP26].

Contents

---

0. Notation and conventions

- $\mathbb{T}^3$ denotes the three-dimensional flat torus.
- $\mathcal{D}'(\mathbb{T}^3)$ is the space of distributions on $\mathbb{T}^3$.
- We use the Fourier basis $e^{2\pi i k\cdot x}$ on $\mathbb{T}^3$. When writing factors like $(m^2 + |k|^2)^{-1}$, we absorb the $(2\pi)^2$ coming from the Laplacian eigenvalues into the notation $|k|^2$; this does not affect any divergence or scaling arguments.
- For f, g integrable, $\langle f, g\rangle := \int_{\mathbb{T}^3} f(x)g(x)\,\mathrm{d}x$.
- We write $A \lesssim B$ if $A \le CB$ for a constant C depending only on fixed parameters (e.g. m, λ, mollifier shape, and the test function ψ), but not on small scales such as ε or large cutoffs such as Λ.
- Throughout, $\lambda \neq 0$ is fixed. The mass parameter is denoted $m > 0$; the precise mass-renormalization convention is immaterial for the separation argument.

---

1. Introduction and main statement

Theorem 1.1 (Singularity of smooth shifts). Let μ be the Φ^4_3 measure on $\mathcal{D}'(\mathbb{T}^3)$ for interaction strength $\lambda \neq 0$. For every $\psi \in C^\infty(\mathbb{T}^3) \setminus \{0\}$, the translated measure $T_{\psi *}\mu$ is mutually singular with μ:

$$\mu \perp T_{\psi *}\mu.$$

Remark 1.2 (Roadmap). The proof will proceed by constructing, for a sequence of scales $\varepsilon_n = e^{-e^n}$, a renormalized cubic observable Y_n such that

1. $Y_n(u) \to 0$ in probability under $u \sim \mu$, while
2. $Y_n(u + \psi) \to \pm\infty$ in probability under $u \sim \mu$, and then separating μ and $T_{\psi *}\mu$ by bounded tests $F_n = \mathbf{1}_{\{|Y_n| \le 1\}}$ in total variation (see §3).

1.1. Positioning and scope

Remark 1.3 (Purpose and provenance). This note is written with a dual purpose. First, it gives a self-contained presentation of the failure of quasi-shift invariance of the interacting Φ^4_3 measure under smooth deterministic translations. Second, it serves as a "gap-filling" document: we isolate the exact fixed-time inputs required by a short small-scale proof sketch (and by AI-generated reproductions of that sketch), and we record where those inputs enter, so that they can be verified and upgraded to full publication-level arguments. This note was written in response to Problem 1 of *First Proof* [FP26].

In this level-A version, the only probabilistic inputs are Proposition 5.1 (fixed-time renormalized cubic decomposition) and Proposition 6.1 (fixed-time tightness of linear/quadratic observables), both of which are standard outputs of the dynamical Φ^4_3 model under stationary quantisation constructions; see, for example, [Hai14, §10], [GIP15], [MW17], [MWX17], [GH21, §4], and [AK20, Thm. 1.1].

Remark 1.4 (Comparison map: Hairer-style vs mollifier-style). There are two common proof skeletons for Theorem 1.1. The first (the "Fourier-cutoff + separating event" route) constructs an almost-sure separating set using projections P_N and a Borel–Cantelli argument. The second (the "mollifier + super-exponential scale" route) uses a renormalized cubic observable at scales $\varepsilon_n = e^{-e^n}$. The present note follows the second route, but replaces the almost-sure separation step by a total-variation separation argument (Lemma 3.1).

For orientation, the correspondences are:

Component	Fourier-cutoff route	Mollifier route (this note)
Ultraviolet regularization	$P_N u$	$u_\varepsilon = \rho_\varepsilon * u$
Log divergence driving singularity	$c_{N,2} \sim \log N$	$C_2(\varepsilon) \sim \kappa \log(\varepsilon^{-1})$ (Proposition 4.5)
Observable	$\langle H_3(P_N u;\, c_N) + 9c_{N,2} P_N u, \psi\rangle$	$Y_n(u)$ in (1)
Separation method	separating set via Borel–Cantelli	TV separation via F_n (Lemma 3.1)
Nontrivial analytic inputs	fixed-time bounds for Wick powers and diagrams	Propositions 5.1 and 6.1 (made explicit)

The advantage of the TV separation step is that it only requires convergence in probability for $Y_n(u)$ and divergence in probability for $Y_n(u + \psi)$, rather than almost-sure convergence along a subsequence.

2. Mollifiers, ultra-small scales, and the observable

Definition 2.1 (Mollifier and smoothing). Fix a nonnegative function $\rho \in C_c^\infty(\mathbb{R}^3)$ with $\int_{\mathbb{R}^3} \rho(x)\,\mathrm{d}x = 1$. Define its $\mathbb{Z}^3$-periodisation

$$\rho^{\mathrm{per}}(x) := \sum_{n\in\mathbb{Z}^3} \rho(x+n), \qquad x \in \mathbb{R}^3,$$

which is a smooth function on $\mathbb{T}^3 \simeq \mathbb{R}^3/\mathbb{Z}^3$. For $\varepsilon \in (0,1)$ set

$$\rho_\varepsilon(x) := \varepsilon^{-3}\rho(x/\varepsilon), \qquad \rho_\varepsilon^{\mathrm{per}}(x) := \sum_{n\in\mathbb{Z}^3} \rho_\varepsilon(x+n),$$

and for $w \in \mathcal{D}'(\mathbb{T}^3)$ define

$$w_\varepsilon := \rho_\varepsilon^{\mathrm{per}} * w,$$

where $*$ denotes convolution on $\mathbb{T}^3$.

Definition 2.2 (Ultra-small scale sequence). Define

$$\varepsilon_n := e^{-e^n}, \qquad \log(\varepsilon_n^{-1}) = e^n.$$

We will repeatedly use that $\log(\varepsilon_n^{-1})$ grows like e^n while ε_n decays super-exponentially.

Lemma 2.3 (Fourier multiplier identity). Let $\hat\rho$ denote the Euclidean Fourier transform of ρ on $\mathbb{R}^3$,

$$\hat\rho(\xi) := \int_{\mathbb{R}^3} e^{-2\pi i\xi\cdot x}\rho(x)\,\mathrm{d}x.$$

Then for every $k \in \mathbb{Z}^3$ one has

$$\widehat{\rho_\varepsilon^{\mathrm{per}}}(k) = \hat\rho(\varepsilon k).$$

Proof. By definition of Fourier coefficients on $\mathbb{T}^3$ and periodicity,

$$\widehat{\rho_\varepsilon^{\mathrm{per}}}(k) = \int_{[0,1)^3} e^{-2\pi i k\cdot x} \sum_{n\in\mathbb{Z}^3} \rho_\varepsilon(x+n)\,\mathrm{d}x = \int_{\mathbb{R}^3} e^{-2\pi i k\cdot x}\rho_\varepsilon(x)\,\mathrm{d}x.$$

Changing variables $x = \varepsilon y$ gives

$$\widehat{\rho_\varepsilon^{\mathrm{per}}}(k) = \int_{\mathbb{R}^3} e^{-2\pi i(\varepsilon k)\cdot y}\rho(y)\,\mathrm{d}y = \hat\rho(\varepsilon k).$$

This choice turns logarithmic divergences in ε into linear growth in e^n while keeping mollification errors super-exponentially small.

Definition 2.4 (Renormalization constants). Let $C_1(\varepsilon)$ and $C_2(\varepsilon)$ denote the tadpole and setting-sun counterterms associated with the mollifier regularization at scale ε. We will use that

- $C_1(\varepsilon)$ diverges like ε^{-1}, and
- $C_2(\varepsilon)$ diverges like $\log(\varepsilon^{-1})$ with a strictly positive coefficient (proved in §4).

Definition 2.5 (Renormalized cubic observable). Fix $\beta \in (1/2, 1)$ and $\psi \in C^\infty(\mathbb{T}^3) \setminus \{0\}$. For $w \in \mathcal{D}'(\mathbb{T}^3)$ set

$$Y_n(w) := e^{-\beta n} \Big\langle w_{\varepsilon_n}^3 - 3C_1(\varepsilon_n) w_{\varepsilon_n} - 9\lambda^2 C_2(\varepsilon_n) w,\ \psi \Big\rangle. \tag{1}$$

Define the bounded test

$$F_n(w) := \mathbf{1}_{\{|Y_n(w)| \le 1\}}. \tag{2}$$

Remark 2.6 (On the coefficient 9 and renormalisation schemes). In the stochastic-quantisation normalisation where the drift is of the form $-\lambda u^3$ (with the corresponding Wick renormalisation), the setting-sun mass counterterm enters with the usual combinatorial coefficient 9; see, e.g., the discussion of counterterms and diagrams in [MWX17, §3] and the invariant-measure formulation in [GH21, §4]. We fix this normalisation throughout.

We also choose the test function in Y_n to be the same function ψ that defines the shift direction; any fixed test $\varphi \in C^\infty(\mathbb{T}^3)$ with $\langle \psi, \varphi \rangle \neq 0$ would yield the same separation mechanism.

If one works in a different convention in which the linear setting-sun counterterm comes with a nonzero coefficient c_{ss} instead of 9, then all statements and proofs below remain valid after the replacement $9 \mapsto c_{\mathrm{ss}}$ (and the only deterministic requirement is $c_{\mathrm{ss}}\, \kappa \neq 0$ in Proposition 4.5).

Remark 2.7 (Why $\beta \in (1/2, 1)$). The choice $\beta \in (1/2, 1)$ is dictated by the two competing effects in §§5–6: (i) the dominant random fluctuation has variance of order $\log(\varepsilon^{-1})$ and therefore becomes of order e^n at $\varepsilon = \varepsilon_n$, so multiplying by $e^{-\beta n}$ kills it provided $\beta > 1/2$; (ii) the deterministic setting-sun drift grows like $C_2(\varepsilon_n) \simeq \kappa e^n$, so after multiplying by $e^{-\beta n}$ it diverges provided $\beta < 1$.

Remark 2.8 (Using w vs w_{ε_n} in the linear counterterm). In some fixed-time expansions the linear setting-sun term is written with w_ε rather than w. Our definition (1) uses w since the mass counterterm in the stochastic-quantisation equation is linear in the field itself. If one replaces w by w_{ε_n} in (1), the difference is

$$9\lambda^2 e^{-\beta n} C_2(\varepsilon_n)\, \langle w - w_{\varepsilon_n}, \psi \rangle.$$

For $w = u$ with $u \in C^{-1/2-\eta}$, a standard Hölder–Besov smoothing estimate yields that for any $\delta \in (0, 1)$,

$$\|u - u_\varepsilon\|_{C^{-1/2-\eta-\delta}} \lesssim \varepsilon^\delta\, \|u\|_{C^{-1/2-\eta}}.$$

See for instance [BCD11, Ch. 2]. Pairing with the fixed smooth test function ψ and using the duality estimate (Lemma B.1) gives

$$|\langle u - u_{\varepsilon_n}, \psi\rangle| \lesssim \varepsilon_n^\delta \, \|u\|_{C^{-1/2-\eta}}.$$

Since $C_2(\varepsilon_n) \simeq e^n$ and $\varepsilon_n = e^{-e^n}$, we have $e^n \varepsilon_n^\delta \to 0$ for every $\delta > 0$, hence the difference is negligible. Thus either convention leads to the same separation argument.

---

3. Total variation separation

Lemma 3.1 (Separation in total variation). Let $\nu, \tilde{\nu}$ be probability measures on a measurable space. If there exist measurable $F_n : \Omega \to [0,1]$ such that

$$\mathbb{E}_\nu[F_n] \to 1 \qquad \text{and} \qquad \mathbb{E}_{\tilde{\nu}}[F_n] \to 0,$$

then $\|\nu - \tilde{\nu}\|_{\mathrm{TV}} = 1$ and $\nu \perp \tilde{\nu}$.

Proof. By definition of total variation,

$$\|\nu - \tilde{\nu}\|_{\mathrm{TV}} \;=\; \sup_{0 \le F \le 1} \big|\mathbb{E}_\nu[F] - \mathbb{E}_{\tilde{\nu}}[F]\big| \ge \big|\mathbb{E}_\nu[F_n] - \mathbb{E}_{\tilde{\nu}}[F_n]\big| \to 1.$$

Since $\|\nu - \tilde{\nu}\|_{\mathrm{TV}} \le 1$, equality holds, hence the measures are mutually singular.

Remark 3.2 (Application to Theorem 1.1). To prove Theorem 1.1, it suffices to show

$$\mathbb{E}_\mu[F_n] \to 1 \qquad \text{and} \qquad \mathbb{E}_{T_{\psi *}\mu}[F_n] \to 0,$$

with F_n from (2).

---

4. The setting-sun constant has logarithmic divergence

This section proves the key deterministic input:

$$C_2(\varepsilon) = \kappa \log(\varepsilon^{-1}) + O(1), \qquad \kappa > 0.$$

The positivity of κ will be used in §6 to force $Y_n(u + \psi)$ to diverge deterministically.

Definition 4.1 (Setting-sun constant). Let $\widehat{\rho}$ denote the Euclidean Fourier transform of the kernel ρ from Definition 2.1, so that by Lemma 2.3 the torus Fourier coefficient of $\rho_\varepsilon^{\mathrm{per}}$ at mode $k \in \mathbb{Z}^3$ equals $\widehat{\rho}(\varepsilon k)$. Define

$$C_2(\varepsilon) := \sum_{k_1, k_2 \in \mathbb{Z}^3} \frac{|\widehat{\rho}(\varepsilon k_1)|^2 \, |\widehat{\rho}(\varepsilon k_2)|^2 \, |\widehat{\rho}(\varepsilon(k_1 + k_2))|^2}{(m^2 + |k_1|^2)(m^2 + |k_2|^2)(m^2 + |k_1 + k_2|^2)}. \tag{3}$$

Lemma 4.2 (Reduction to sharp cutoff up to $O(1)$). Let $\Lambda = \varepsilon^{-1}$. There exist constants $\delta \in (0,1)$ and $M > 1$ such that

$$c\, S(\delta\Lambda) \;\le\; C_2(\varepsilon) \;\le\; S(M\Lambda) + O(1),$$

where $c > 0$ and

$$S(\Lambda) := \sum_{\substack{k_1, k_2 \in \mathbb{Z}^3 \\ |k_1|, |k_2|, |k_1 + k_2| \le \Lambda}} \frac{1}{(m^2 + |k_1|^2)(m^2 + |k_2|^2)(m^2 + |k_1 + k_2|^2)}. \tag{4}$$

Proof. Since $\widehat{\rho}(0) = 1$ and $\widehat{\rho}$ is continuous, there exist $\delta \in (0,1)$ and $c_0 \in (0,1)$ such that $|\xi| \le \delta$ implies $|\widehat{\rho}(\xi)| \ge c_0$. Hence restricting the sum in (3) to $|k_1|, |k_2|, |k_1 + k_2| \le \delta\Lambda$ yields the lower bound with $c = c_0^6$.

For the upper bound, use rapid decay of $\widehat{\rho}$ to truncate to $|k_1|, |k_2|, |k_1 + k_2| \le M\Lambda$ at the cost of an absolutely summable tail (choose decay order N large). The tail contributes $O(1)$ uniformly in ε.

Lemma 4.3 (Sum–integral comparison). Define

$$I(\Lambda) := \iint_{\substack{|p| \le \Lambda, \\ |q| \le \Lambda, \\ |p+q| \le \Lambda}} \frac{\mathrm{d}p \, \mathrm{d}q}{(m^2 + |p|^2)(m^2 + |q|^2)(m^2 + |p + q|^2)}. \tag{5}$$

Then

$$S(\Lambda) = I(\Lambda) + O(1) \qquad (\Lambda \to \infty).$$

Proof. See Appendix D.1–D.2 for a complete proof. In outline, we introduce a smooth cutoff χ_Λ and define smoothed versions $S_\chi(\Lambda)$ and $I_\chi(\Lambda)$ of $S(\Lambda)$ and $I(\Lambda)$. Poisson summation on $\mathbb{Z}^6$ gives

$$S_\chi(\Lambda) - I_\chi(\Lambda) = \sum_{\ell \in \mathbb{Z}^6 \setminus \{0\}} \widehat{F_\Lambda}(2\pi\ell),$$

where F_Λ is the smoothed kernel. Splitting F_Λ into a local integrable part (near the singular set) and a smooth part (away from it), one obtains uniform (in Λ)

bounds on the nonzero Fourier modes by integration by parts, hence $|S_\chi(\Lambda) - I_\chi(\Lambda)| \lesssim 1$. Finally, $S(\Lambda)$ and $I(\Lambda)$ differ from their smoothed counterparts by $O(1)$, since the discrepancy is supported in a boundary layer where the kernel is $\lesssim \Lambda^{-6}$ and the number of lattice points is $\lesssim \Lambda^6$.

Lemma 4.4 (Riesz convolution identity). For $q \neq 0$,

$$\int_{\mathbb{R}^3} \frac{\mathrm{d}p}{|p|^2|p+q|^2} \;=\; \frac{c}{|q|}$$

for some constant $c > 0$.

Proof. Let $K(x) = |x|^{-2}$. In $\mathbb{R}^3$, $\widehat{K}(\xi) = c_1|\xi|^{-1}$ in the sense of tempered distributions. Hence $\widehat{K * K}(\xi) = c_1^2|\xi|^{-2}$, whose inverse Fourier transform is $c_2|x|^{-1}$. Since $K \geq 0$, the convolution $K * K$ is nonnegative, so the constant in the identity is strictly positive (equivalently, the explicit Riesz potential constant is positive). Evaluating at $x = q$ yields the claim with $c = c_2$.

Proposition 4.5 (Logarithmic divergence with positive coefficient). There exists $\kappa > 0$ such that

$$C_2(\varepsilon) \;=\; \kappa \log(\varepsilon^{-1}) + O(1) \qquad (\varepsilon \downarrow 0). \tag{6}$$

In particular,

$$C_2(\varepsilon_n) = \kappa e^n + O(1). \tag{7}$$

Proof. By Lemma 4.2 it suffices to prove

$$S(\Lambda) = \kappa \log \Lambda + O(1).$$

By Lemma 4.3 it suffices to prove

$$I(\Lambda) = \kappa \log \Lambda + O(1).$$

Write

$$I(\Lambda) = \int_{|q|\leq\Lambda} \frac{\mathrm{d}q}{m^2+|q|^2}\, J_\Lambda(q), \qquad J_\Lambda(q) := \int_{\substack{|p|\leq\Lambda,\\ |p+q|\leq\Lambda}} \frac{\mathrm{d}p}{(m^2+|p|^2)(m^2+|p+q|^2)}.$$

Define

$$J(q) := \int_{\mathbb{R}^3} \frac{\mathrm{d}p}{|p|^2|p+q|^2}.$$

By Lemma D.6 and Corollary D.7 (Appendix D), the contributions to $I(\Lambda)$ coming from (i) the truncation constraints $|p| \le \Lambda$, $|p+q| \le \Lambda$, (ii) the region $|q| \le 1$, and (iii) replacing $(m^2 + |\cdot|^2)^{-1}$ by $|\cdot|^{-2}$ on the region $1 \le |q| \le \Lambda/2$, are all $O(1)$ as $\Lambda \to \infty$. Consequently,

$$I(\Lambda) = \int_{1 \le |q| \le \Lambda/2} \frac{\mathrm{d}q}{|q|^2} J(q) + O(1).$$

By Lemma 4.4, $J(q) = c|q|^{-1}$ with $c > 0$, hence

$$I(\Lambda) = c \int_{1 \le |q| \le \Lambda/2} \frac{\mathrm{d}q}{|q|^3} + O(1) = 4\pi c \int_1^{\Lambda/2} \frac{\mathrm{d}r}{r} + O(1) = (4\pi c) \log \Lambda + O(1).$$

Thus $\kappa = 4\pi c > 0$ and (6) follows by Lemmas 4.2–4.3. The specialization (7) uses $\log(\varepsilon_n^{-1}) = e^n$.

---

5. Fluctuation bound at ultra-small scales

This section provides the probabilistic input needed to show that, under $u \sim \mu$, the observable $Y_n(u)$ defined in (1) converges to 0 in probability. The key estimate is that the dominant centered fluctuation has variance $\lesssim \log(\varepsilon^{-1})$, which becomes $\lesssim e^n$ at $\varepsilon = \varepsilon_n$.

Proposition 5.1 (Fixed-time renormalized cubic decomposition). Let μ be the Φ^4_3 measure on $\mathbb{T}^3$ at $\lambda \neq 0$, realised as the time-marginal of a stationary solution to the renormalised stochastic quantisation equation. Then for each $\varepsilon \in (0,1)$ there exist real-valued random variables R_ε and Θ_ε such that

$$\left\langle u_\varepsilon^3 - 3C_1(\varepsilon)u_\varepsilon - 9\lambda^2 C_2(\varepsilon)u,\ \psi \right\rangle = R_\varepsilon + \Theta_\varepsilon, \tag{8}$$

with the following properties:

1. (Tight remainder) $\{R_\varepsilon\}_{\varepsilon \in (0,1)}$ is tight.
2. (Centered fluctuation) $\mathbb{E}[\Theta_\varepsilon] = 0$ for all ε.
3. (Variance bound) There exists $C_\psi < \infty$ such that

$$\mathrm{Var}(\Theta_\varepsilon) \le C_\psi\, C_2(\varepsilon) \qquad \text{for all } \varepsilon \in (0,1). \tag{9}$$

Proof. This is a repackaging of standard fixed-time estimates for the dynamical Φ^4_3 model. Uniform covariance bounds for the mollified stochastic diagrams (including the unique diagram of homogeneity $-3/2$ corresponding to the setting-sun subdiagram) are proved for arbitrary mollifiers in [MWX17, §3]. Identification of the nonlinear (stationary) solution with a finite sum of these diagrams plus a regular remainder is established in the regularity-structure framework

[Hai14, §10] and in the paracontrolled framework [MW17, §2–§3]; see also the invariant-measure formulation in [GH21, §4]. Existence of the Φ^4_3 invariant measure as the law of a stationary solution (hence uniform-in-time moment estimates) is proved in [AK20, Thm. 1.1] and [GH21, §4.3]. Combining these results yields the stated decomposition and variance bound.

Remark 5.2. Proposition 5.1 isolates the only probabilistic input needed for Proposition 5.5: a fixed-time renormalized cubic pairing can be decomposed into a tight remainder plus a centered fluctuation whose variance grows at most logarithmically in the ultraviolet scale. Lemma 5.3 below shows that the variance control is naturally expressed in terms of the same triple-propagator kernel that defines $C_2(\varepsilon)$.

Lemma 5.3 (A canonical chaos bound implies (9)**).** Let $\xi(k)_{k\in\mathbb{Z}^3}$ be centered i.i.d. Gaussian modes with Wick products $:\xi(k_1)\xi(k_2):$. Let $\sigma_\varepsilon(k)$ be the Fourier multiplier

$$\sigma_\varepsilon(k) := \frac{\widehat{\rho}(\varepsilon k)}{\sqrt{m^2+|k|^2}}. \tag{10}$$

Define

$$\Theta_\varepsilon := \sum_{k_1,k_2\in\mathbb{Z}^3} \overline{\widehat{\psi}(k_1+k_2)}\,\sigma_\varepsilon(k_1)\sigma_\varepsilon(k_2)\sigma_\varepsilon(k_1+k_2)\ :\xi(k_1)\xi(k_2):. \tag{11}$$

Then $\mathbb{E}[\Theta_\varepsilon]=0$ and

$$\operatorname{Var}(\Theta_\varepsilon) \le 2\|\psi\|_{L^2}^2 \sum_{k_1,k_2\in\mathbb{Z}^3} |\sigma_\varepsilon(k_1)|^2\,|\sigma_\varepsilon(k_2)|^2\,|\sigma_\varepsilon(k_1+k_2)|^2. \tag{12}$$

In particular, using (3),

$$\operatorname{Var}(\Theta_\varepsilon) \le 2\|\psi\|_{L^2}^2 C_2(\varepsilon). \tag{13}$$

Proof. Since $:\xi(k_1)\xi(k_2):$ is centered, $\mathbb{E}[\Theta_\varepsilon]=0$. Write $\Theta_\varepsilon = \sum_{k_1,k_2} a_{k_1,k_2} :\xi(k_1)\xi(k_2):$ with

$$a_{k_1,k_2} = \overline{\widehat{\psi}(k_1+k_2)}\,\sigma_\varepsilon(k_1)\sigma_\varepsilon(k_2)\sigma_\varepsilon(k_1+k_2).$$

Wiener chaos isometry (for second chaos) yields

$$\operatorname{Var}(\Theta_\varepsilon) = 2\sum_{k_1,k_2} |a_{k_1,k_2}|^2.$$

Thus

$$\mathrm{Var}(\Theta_\varepsilon) = 2 \sum_{k_1,k_2} |\widehat{\psi}(k_1+k_2)|^2 |\sigma_\varepsilon(k_1)|^2 |\sigma_\varepsilon(k_2)|^2 |\sigma_\varepsilon(k_1+k_2)|^2.$$

Re-index by $k = k_1 + k_2$ and apply Parseval,

$$\sum_k |\widehat{\psi}(k)|^2 = \|\psi\|_{L^2}^2,$$

to obtain (12), and then (13) by the definition of $C_2(\varepsilon)$.

Lemma 5.4 (Tightness times a vanishing scalar). Let $Z_{n_{n\geq 1}}$ be a tight family of real random variables and let $a_n \to 0$ be deterministic. Then $a_n Z_n \to 0$ in probability.

Proof. Fix $\delta > 0$. Tightness implies that for every $\eta > 0$ there exists $M < \infty$ such that $\sup_n \mathbb{P}(|Z_n| > M) \leq \eta$. Then

$$\mathbb{P}(|a_n Z_n| > \delta) \leq \mathbb{P}(|Z_n| > M) + \mathbb{P}(|a_n| M > \delta) \leq \eta + \mathbf{1}_{\{|a_n| > \delta/M\}}.$$

Let $n \to \infty$, so the indicator vanishes, then $\eta \downarrow 0$.

Proposition 5.5 ($Y_n(u) \to 0$ under μ). Assume Proposition 5.1. Then for $u \sim \mu$,

$$Y_n(u) \to 0 \qquad \text{in probability as } n \to \infty.$$

Proof. By definition (1) and decomposition (8) with $\varepsilon = \varepsilon_n$,

$$Y_n(u) = e^{-\beta n}(R_{\varepsilon_n} + \Theta_{\varepsilon_n}).$$

Since R_ε is tight, Lemma 5.4 gives $e^{-\beta n} R_{\varepsilon_n} \to 0$ in probability.

For the fluctuation, by (9) and Proposition 4.5,

$$\mathrm{Var}(e^{-\beta n}\Theta_{\varepsilon_n}) = e^{-2\beta n}\mathrm{Var}(\Theta_{\varepsilon_n}) \leq e^{-2\beta n} C_\psi C_2(\varepsilon_n) = O\big(e^{-2\beta n} e^n\big) = O\big(e^{(1-2\beta)n}\big).$$

Since $\beta > 1/2$, the right-hand side tends to 0, hence $e^{-\beta n}\Theta_{\varepsilon_n} \to 0$ in L^2, and therefore in probability. Summing the two contributions yields $Y_n(u) \to 0$ in probability.

Corollary 5.6 (Acceptance probability under μ). Assume Proposition 5.1. Then

$$\mathbb{E}_\mu[F_n] = \mathbb{P}_\mu(|Y_n(u)| \le 1) \to 1.$$

Proof. By Proposition 5.5, $Y_n(u) \to 0$ in probability under $u \sim \mu$. Therefore

$$\mathbb{P}_\mu(|Y_n(u)| > 1) \to 0,$$

and hence $\mathbb{P}_\mu(|Y_n(u)| \le 1) \to 1$.

---

6. Shift expansion and deterministic divergence

This section expands $Y_n(u + \psi)$ and shows that the logarithmically divergent linear counterterm produces a deterministic drift of size $e^{(1-\beta)n}$, while all remaining random terms vanish after multiplying by $e^{-\beta n}$.

Proposition 6.1 (Negative H"older bounds, Wick-square convergence, and variable-test tightness). Let μ be the Φ^4_3 measure on $\mathbb{T}^3$ at $\lambda \neq 0$, realised as the time-marginal of a stationary solution to the renormalised stochastic quantisation equation. Fix $\eta > 0$.

1. (Fixed-time regularity and Wick square.) There exists a random distribution $:u^2: \in C^{-1-\eta}(\mathbb{T}^3)$ such that for every $p < \infty$,

   $$\mathbb{E}\,\|u\|^p_{C^{-1/2-\eta}} < \infty, \qquad \mathbb{E}\,\|:u^2:\|^p_{C^{-1-\eta}} < \infty,$$

   and moreover

   $$u_\varepsilon^2 - C_1(\varepsilon) \to :u^2: \quad \text{in probability in } C^{-1-\eta} \text{ as } \varepsilon \downarrow 0.$$

2. (Uniform pairing with varying tests.) Let $(\varphi_\varepsilon)_{\varepsilon\in(0,1)} \subset C^\infty(\mathbb{T}^3)$ be a deterministic family such that

   $$\sup_{\varepsilon\in(0,1)} \|\varphi_\varepsilon\|_{C^{1+\eta}} < \infty.$$

   Then the families of real random variables

   $$\{\langle u, \varphi_\varepsilon\rangle\}_{\varepsilon\in(0,1)} \quad \text{and} \quad \{\langle u_\varepsilon^2 - C_1(\varepsilon), \varphi_\varepsilon\rangle\}_{\varepsilon\in(0,1)}$$

   are tight. In particular, for any fixed $\psi \in C^\infty(\mathbb{T}^3)$, the choices

   $$\varphi_\varepsilon = \psi\,\psi_\varepsilon, \qquad \varphi_\varepsilon = \psi\,\psi_\varepsilon^2$$

   satisfy the uniform bound above, hence produce tight families.

Proof. Item 1 is a standard fixed-time output of the dynamical Φ^4_3 model (regularity structures / paracontrolled calculus) together with the identification of μ as the law of a stationary solution; see, for example, [Hai14, §10] for the construction of Wick powers and fixed-time bounds, [MW17, §§2–3] and [MWX17, §3] for the diagrammatic estimates in the dynamical model, and [GH21, §4] (together with [AK20, Thm. 1.1]) for the invariant-measure formulation at fixed time.

For item 2, apply the duality estimate of Lemma B.1: for any $f \in C^{-1-\eta}$ and any $\varphi \in C^{1+\eta}$,

$$|\langle f, \varphi\rangle| \lesssim \|f\|_{C^{-1-\eta}} \|\varphi\|_{C^{1+\eta}}.$$

By item 1, the family $\{u_\varepsilon^2 - C_1(\varepsilon)\}_\varepsilon$ is tight in $C^{-1-\eta}$, and the uniform bound on $\|\varphi_\varepsilon\|_{C^{1+\eta}}$ implies tightness of the pairings. The same argument applies to $\langle u, \varphi_\varepsilon\rangle$ using $u \in C^{-1/2-\eta}$ and the embedding $C^{1+\eta} \hookrightarrow C^{1/2+\eta}$. Finally, if $\psi \in C^\infty$, then convolution with the L^1-normalised kernel $\rho_\varepsilon^{\mathrm{per}}$ is bounded on each C^k norm, so $\sup_{\varepsilon\in(0,1)} \|\psi_\varepsilon\|_{C^k} < \infty$ for all k, and therefore $\sup_\varepsilon \|\psi\,\psi_\varepsilon\|_{C^{1+\eta}} < \infty$ and $\sup_\varepsilon \|\psi\,\psi_\varepsilon^2\|_{C^{1+\eta}} < \infty$.

Remark 6.2. Proposition 6.1 is the fixed-time regularity input used to discard the non-divergent terms in the shift expansion (14). The point is not merely that pairings against a *fixed* smooth test function are tight, but that the pairings remain tight for *varying* tests φ_ε with uniformly bounded $C^{1+\eta}$ norms (item 2). This is exactly what is needed for the n-dependent tests $\psi\,\psi_{\varepsilon_n}$ and $\psi\,\psi_{\varepsilon_n}^2$ in Proposition 6.4.

For readers who prefer to separate analytic from probabilistic inputs: item 2 is a deterministic "sufficient condition" consequence of negative H"older tightness via Lemma B.1 (and is essentially Proposition B.2 specialised to families of tests with uniform $C^{1+\eta}$ bounds).

Lemma 6.3 (Shift identity for Y_n). Let $u \in \mathcal{D}'(\mathbb{T}^3)$ and $\psi \in C^\infty(\mathbb{T}^3)$. Set $v = u + \psi$. Then

$$\begin{aligned} Y_n(v) = Y_n(u) &+ e^{-\beta n}\Big\langle 3\psi_{\varepsilon_n}(u_{\varepsilon_n}^2 - C_1(\varepsilon_n)), \psi\Big\rangle \\ &+ e^{-\beta n}\Big\langle 3\psi_{\varepsilon_n}^2 u_{\varepsilon_n} + \psi_{\varepsilon_n}^3, \psi\Big\rangle - 9\lambda^2 e^{-\beta n} C_2(\varepsilon_n)\langle\psi,\psi\rangle. \end{aligned} \tag{14}$$

Proof. Since $v_{\varepsilon_n} = u_{\varepsilon_n} + \psi_{\varepsilon_n}$, expand

$$v_{\varepsilon_n}^3 = u_{\varepsilon_n}^3 + 3\psi_{\varepsilon_n} u_{\varepsilon_n}^2 + 3\psi_{\varepsilon_n}^2 u_{\varepsilon_n} + \psi_{\varepsilon_n}^3.$$

Insert this into the definition of Y_n in (1), subtract $Y_n(u)$, and collect terms.

Proposition 6.4 ($Y_n(u+\psi) \to \pm\infty$ under μ). Assume Propositions 5.1 and 6.1. Then for $u \sim \mu$ and nonzero $\psi \in C^\infty(\mathbb{T}^3)$,

$$|Y_n(u+\psi)| \to \infty \qquad \text{in probability.}$$

Proof. Using Lemma 6.3 with $v = u + \psi$ and Proposition 5.5,

$$Y_n(u+\psi) = Y_n(u) + T_n^{(2)} + T_n^{(3)} + D_n,$$

where

$$T_n^{(2)} := e^{-\beta n}\Big\langle 3\psi_{\varepsilon_n}(u_{\varepsilon_n}^2 - C_1(\varepsilon_n)), \psi\Big\rangle,$$

$$T_n^{(3)} := e^{-\beta n}\Big\langle 3\psi_{\varepsilon_n}^2 u_{\varepsilon_n} + \psi_{\varepsilon_n}^3, \psi\Big\rangle,$$

and

$$D_n := -9\lambda^2 e^{-\beta n} C_2(\varepsilon_n)\langle \psi, \psi\rangle.$$

Step 1: $T_n^{(2)} \to 0$ in probability. Fix $\eta > 0$ as in Proposition 6.1. Set $\varphi_n := \psi\,\psi_{\varepsilon_n}$. Since $\psi \in C^\infty$ and convolution with the L^1-normalised kernel $\rho_{\varepsilon_n}^{\mathrm{per}}$ is bounded on each C^k norm, we have $\sup_n \|\varphi_n\|_{C^{1+\eta}} < \infty$. Proposition 6.1(2) then yields that the family

$$\Big\{\langle u_{\varepsilon_n}^2 - C_1(\varepsilon_n), \varphi_n\rangle\Big\}_{n\geq 1}$$

is tight. Lemma 5.4 with $a_n = e^{-\beta n}$ yields $T_n^{(2)} \to 0$ in probability.

Step 2: $T_n^{(3)} \to 0$ in probability. Set $\widetilde{\varphi}_n := \psi\,\psi_{\varepsilon_n}^2$. As above, $\sup_n \|\widetilde{\varphi}_n\|_{C^{1+\eta}} < \infty$, so Proposition 6.1(2) implies that the family $\{\langle u, \widetilde{\varphi}_n\rangle\}_n$ is tight, and therefore $e^{-\beta n}\langle u, \widetilde{\varphi}_n\rangle \to 0$ in probability. To justify replacing u by u_{ε_n} in this pairing, note that Proposition 6.1(1) gives $u \in C^{-1/2-\eta}$ with finite moments. By the standard smoothing estimate (see [BCD11, Ch. 2]) one has $\|u - u_{\varepsilon_n}\|_{C^{-1/2-\eta-\delta}} \lesssim \varepsilon_n^\delta \|u\|_{C^{-1/2-\eta}}$ for any $\delta \in (0,1)$, and since $\sup_n \|\widetilde{\varphi}_n\|_{C^{1/2+\eta+\delta}} < \infty$, Lemma B.1 implies $|\langle u - u_{\varepsilon_n}, \widetilde{\varphi}_n\rangle| \to 0$ in probability. The purely deterministic term satisfies

$$e^{-\beta n}\,|\langle \psi_{\varepsilon_n}^3, \psi\rangle| \leq e^{-\beta n}\,\|\psi_{\varepsilon_n}^3\|_{L^2}\,\|\psi\|_{L^2} \lesssim e^{-\beta n} \to 0,$$

so $T_n^{(3)} \to 0$ in probability.

Step 3: D_n diverges deterministically. By Proposition 4.5, $C_2(\varepsilon_n) = \kappa e^n + O(1)$ with $\kappa > 0$, hence

$$D_n = -(9\lambda^2\kappa)e^{(1-\beta)n}\|\psi\|_{L^2}^2 + o(e^{(1-\beta)n}).$$

Since $\beta < 1$ and $\psi \neq 0$, the right-hand side diverges with magnitude $e^{(1-\beta)n}$.

Combining $Y_n(u) \to 0$ and $T_n^{(2)}, T_n^{(3)} \to 0$ in probability with $|D_n| \to \infty$ yields $|Y_n(u+\psi)| \to \infty$ in probability.

Corollary 6.5 (Rejection probability under the shifted law). Assume Propositions 5.1 and 6.1. Then

$$\mathbb{E}_{T_{\psi *}\mu}[F_n] = \mathbb{E}_\mu[F_n(u+\psi)] = \mathbb{P}_\mu(|Y_n(u+\psi)| \le 1) \to 0.$$

Proof. This follows from Proposition 6.4.

---

7. Proof of Theorem 1.1

Proof. Let F_n be defined by (2). Under Proposition 5.1, Corollary 5.6 yields

$$\mathbb{E}_\mu[F_n] \to 1.$$

Under Propositions 5.1 and 6.1, Corollary 6.5 yields

$$\mathbb{E}_{T_{\psi *}\mu}[F_n] \to 0.$$

Applying Lemma 3.1 to $\nu = \mu$ and $\tilde{\nu} = T_{\psi *}\mu$ concludes that $\mu \perp T_{\psi *}\mu$.

Appendices

Appendix A. Fixed-time cubic decomposition and the second-chaos fluctuation

This appendix explains a concrete route to Proposition 5.1. The main point is that the dominant *centered* fluctuation in the renormalized cubic pairing can be represented as a second Wiener chaos with kernel controlled by the setting-sun constant $C_2(\varepsilon)$.

A.1. Gaussian base field and Wick products

Definition A.1 (Gaussian Fourier modes and multipliers). Let $\xi(k)_{k\in\mathbb{Z}^3}$ be centered complex Gaussian variables such that

- $\xi(-k) = \overline{\xi(k)}$ (so that the resulting field is real-valued),
- $\mathbb{E}[\xi(k)\overline{\xi(\ell)}] = \delta_{k\ell}$.

Define the Fourier multiplier

$$\sigma_\varepsilon(k) := \frac{\widehat{\rho}(\varepsilon k)}{\sqrt{m^2 + |k|^2}}. \tag{15}$$

Define a Gaussian random distribution X^ε by its Fourier coefficients

$$\widehat{X^\varepsilon}(k) := \sigma_\varepsilon(k)\xi(k). \tag{16}$$

Then X^ε is centered and satisfies

$$\mathbb{E}\big[\widehat{X^\varepsilon}(k)\overline{\widehat{X^\varepsilon}(\ell)}\big] = \delta_{k\ell}\,\frac{|\widehat{\rho}(\varepsilon k)|^2}{m^2 + |k|^2}. \tag{17}$$

Definition A.2 (Wick product on the Fourier modes). For $k_1, k_2 \in \mathbb{Z}^3$ define the Wick product

$$:\xi(k_1)\xi(k_2): := \xi(k_1)\xi(k_2) - \mathbb{E}[\xi(k_1)\xi(k_2)]. \tag{18}$$

This is centered and belongs to the second Wiener chaos of ξ.

A.2. A canonical second-chaos fluctuation with setting-sun kernel

Definition A.3 (Second-chaos fluctuation). Let $\psi \in C^\infty(\mathbb{T}^3)$. Define

$$\Theta_\varepsilon := \sum_{k_1,k_2\in\mathbb{Z}^3} \overline{\widehat{\psi}(k_1+k_2)}\,\sigma_\varepsilon(k_1)\sigma_\varepsilon(k_2)\sigma_\varepsilon(k_1+k_2)\ :\xi(k_1)\xi(k_2):. \tag{19}$$

Lemma A.4 (Second-chaos isometry and C_2 control). Let Θ_ε be defined by (19). Then $\mathbb{E}[\Theta_\varepsilon] = 0$ and

$$\operatorname{Var}(\Theta_\varepsilon) \le 2\|\psi\|_{L^2}^2 \sum_{k_1,k_2\in\mathbb{Z}^3} |\sigma_\varepsilon(k_1)|^2\,|\sigma_\varepsilon(k_2)|^2\,|\sigma_\varepsilon(k_1+k_2)|^2. \tag{20}$$

In particular, with $C_2(\varepsilon)$ from (3),

$$\operatorname{Var}(\Theta_\varepsilon) \le 2\|\psi\|_{L^2}^2 C_2(\varepsilon). \tag{21}$$

Proof. Write $\Theta_\varepsilon = \sum_{k_1,k_2} a_{k_1,k_2} :\xi(k_1)\xi(k_2):$ with

$$a_{k_1,k_2} = \overline{\widehat{\psi}(k_1+k_2)}\,\sigma_\varepsilon(k_1)\sigma_\varepsilon(k_2)\sigma_\varepsilon(k_1+k_2).$$

Second-chaos isometry gives $\mathrm{Var}(\Theta_\varepsilon) = 2\sum_{k_1,k_2} |a_{k_1,k_2}|^2$, hence

$$\mathrm{Var}(\Theta_\varepsilon) = 2\sum_{k_1,k_2} |\widehat{\psi}(k_1+k_2)|^2 |\sigma_\varepsilon(k_1)|^2 |\sigma_\varepsilon(k_2)|^2 |\sigma_\varepsilon(k_1+k_2)|^2.$$

Re-index by $k = k_1 + k_2$ and apply Parseval $\sum_k |\widehat{\psi}(k)|^2 = \|\psi\|_{L^2}^2$ to obtain (20). The bound (21) follows by the definition of $C_2(\varepsilon)$ and (15).

A.3. From renormalized cubic pairings to Proposition 5.1

In the body (Proposition 5.1), we require a fixed-time decomposition of the renormalized cubic pairing into a tight remainder plus a centered fluctuation with variance $\lesssim C_2(\varepsilon)$. We record a convenient "reduction lemma" explaining how Lemma A.4 supplies the needed variance control once a second-chaos representation is known.

Lemma A.5 (Reduction to a second-chaos representation). Suppose that for each $\varepsilon \in (0,1)$, one can write

$$\left\langle u_\varepsilon^3 - 3C_1(\varepsilon)u_\varepsilon - 9\lambda^2 C_2(\varepsilon)u,\ \psi \right\rangle = R_\varepsilon + \widetilde{\Theta}_\varepsilon, \tag{22}$$

where:

1. R_ε is tight;
2. $\widetilde{\Theta}_\varepsilon$ is centered and belongs to the second Wiener chaos of an underlying Gaussian family;
3. $\widetilde{\Theta}_\varepsilon$ admits a kernel representation of the form (19), up to a finite sum of terms whose variances are uniformly bounded in ε.

Then Proposition 5.1 holds, i.e. $\mathrm{Var}(\widetilde{\Theta}_\varepsilon) \lesssim C_2(\varepsilon)$.

Proof. By Lemma A.4, the principal second-chaos term has variance bounded by a constant multiple of $C_2(\varepsilon)$. Any additional second-chaos contributions with uniformly bounded variance can be absorbed into R_ε (tightness is stable under summation), and the same variance bound continues to hold up to a modified constant C_ψ.

Remark A.6 (Where the representation comes from). In stochastic quantization constructions of Φ^4_3, the fixed-time renormalized cubic observable is expressed as a finite sum of diagrammatic terms (Wiener chaos components) plus remainders with better regularity. The only part that grows like $\log(\varepsilon^{-1})$ at the level of fluctuations is the setting-sun-shaped contribution, which can be identified with a second-chaos term whose kernel is controlled by the same triple-propagator sum defining $C_2(\varepsilon)$. Lemma A.5 packages precisely what is needed from that identification.

Appendix B. Tightness of linear and quadratic observables

This appendix addresses Proposition 6.1. We first show that Proposition 6.1 follows from uniform bounds in negative Hölder norms for u and for the renormalized square $:u^2:$. We then give a detailed finite-dimensional Lyapunov–Itô argument (Galerkin level) to support stationary moment bounds, which are a standard input into the fixed-time regularity theory.

B.1. Tightness from negative-regularity bounds

Lemma B.1 (Duality estimate for smooth pairings). Let $\alpha > 0$. For any distribution $f \in C^{-\alpha}(\mathbb{T}^3)$ and any $\varphi \in C^\infty(\mathbb{T}^3)$,

$$|\langle f, \varphi \rangle| \lesssim \|f\|_{C^{-\alpha}} \|\varphi\|_{C^\alpha}. \tag{23}$$

Proof. This is the standard Hölder–Besov duality estimate: $C^{-\alpha}$ is the dual of C^α (up to standard identifications), and smooth φ belongs to all C^α.

Proposition B.2 (A sufficient condition for Proposition 6.1). Let u be a random distribution on $\mathbb{T}^3$. Suppose that for some $\eta_0 > 0$ and all $p < \infty$:

1. $\sup_{\varepsilon \in (0,1)} \mathbb{E}\|u_\varepsilon\|^p_{C^{-1/2-\eta_0}} < \infty$;
2. there exists a random distribution $:u^2: \in C^{-1-\eta_0}$ such that $u_\varepsilon^2 - C_1(\varepsilon) \to :u^2:$ in probability in $C^{-1-\eta_0}$, and $\sup_{\varepsilon \in (0,1)} \mathbb{E}\|u_\varepsilon^2 - C_1(\varepsilon)\|^p_{C^{-1-\eta_0}} < \infty$.

Then Proposition 6.1 holds.

Proof. Fix $\varphi \in C^\infty$. Apply Lemma B.1 with $\alpha = 1/2 + \eta_0$ to obtain

$$|\langle u_\varepsilon, \varphi \rangle| \lesssim \|u_\varepsilon\|_{C^{-1/2-\eta_0}} \|\varphi\|_{C^{1/2+\eta_0}},$$

so uniform L^p bounds on $\|u_\varepsilon\|_{C^{-1/2-\eta_0}}$ imply uniform L^p bounds on $\langle u_\varepsilon, \varphi \rangle$, hence tightness.

Similarly, apply Lemma B.1 with $\alpha = 1 + \eta_0$:

$$|\langle u_\varepsilon^2 - C_1(\varepsilon), \varphi \rangle| \lesssim \|u_\varepsilon^2 - C_1(\varepsilon)\|_{C^{-1-\eta_0}} \|\varphi\|_{C^{1+\eta_0}}.$$

Uniform L^p bounds imply tightness of $\langle u_\varepsilon^2 - C_1(\varepsilon), \varphi \rangle$.

Remark B.3 (What remains to fully prove Proposition 6.1). By Proposition B.2, it suffices to establish uniform moment bounds for u in $C^{-1/2-\kappa}$ and for the renormalized square in $C^{-1-\kappa}$, along with convergence $u_\varepsilon^2 - C_1(\varepsilon) \to :u^2:$. These are standard outputs of Φ^4_3 constructions. The remainder of this appendix provides a detailed stationary Lyapunov estimate at the Galerkin level, which is a conventional ingredient in the invariant-measure approach.

B.2. Galerkin-level stationary energy estimates (Lyapunov–Itô)

This section writes out, in finite dimension, the Itô computation leading to stationary energy moment bounds.

Definition B.4 (Galerkin projection and finite-dimensional noise). Let $H_N := \mathrm{span}\{e_k : |k| \le N\} \subset L^2(\mathbb{T}^3)$ be the Fourier Galerkin space and P_N the orthogonal projection. Let W_t^N be an H_N-valued Brownian motion. For a spatial mollifier ρ_ε, set $W_t^{\varepsilon,N} := \rho_\varepsilon W_t^N$ and denote its covariance operator by

$$Q^{\varepsilon,N} := \rho_\varepsilon P_N \rho_\varepsilon^*. \tag{24}$$

Definition B.5 (Finite-dimensional renormalized dynamics). Fix a renormalized mass parameter $m_{\mathrm{ren}}^2 > 0$ and consider the finite-dimensional SDE on H_N:

$$du_t^{\varepsilon,N} = P_N(\Delta u_t^{\varepsilon,N} - m_{\mathrm{ren}}^2 u_t^{\varepsilon,N} - \lambda (u_t^{\varepsilon,N})^3)\, dt + dW_t^{\varepsilon,N}. \tag{25}$$

Definition B.6 (Energy functional). Define $\mathcal{E} : H_N \to \mathbb{R}$ by

$$\mathcal{E}(u) := \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{m_{\mathrm{ren}}^2}{2}\|u\|_{L^2}^2 + \frac{\lambda}{4}|u|_{L^4}^4. \tag{26}$$

Lemma B.7 (Itô formula for $\mathcal{E}(u_t^{\varepsilon,N})$). Let $u_t = u_t^{\varepsilon,N}$ solve (25). Then

$$d\mathcal{E}(u_t) = -\|D\mathcal{E}(u_t)\|_{L^2}^2\, dt + dM_t + \frac{1}{2}\,\mathrm{Tr}(Q^{\varepsilon,N}(-\Delta + m_{\mathrm{ren}}^2))\, dt + \frac{3\lambda}{2}\,\mathrm{Tr}(Q^{\varepsilon,N} M_{u_t^2})\, dt, \tag{27}$$

where M_t is a real-valued martingale and M_{u^2} denotes multiplication by u^2 on H_N.

Proof. This is the finite-dimensional Itô formula applied to $\mathcal{E}$, using

$$D\mathcal{E}(u) = -\Delta u + m_{\mathrm{ren}}^2 u + \lambda u^3, \qquad D^2\mathcal{E}(u)[h] = (-\Delta + m_{\mathrm{ren}}^2)h + 3\lambda u^2 h,$$

and the drift in (25).

Lemma B.8 (Trace bound for the quadratic term). There exists a constant $C_N < \infty$ such that for all $\varepsilon \in (0,1)$ and all $u \in H_N$,

$$\mathrm{Tr}(Q^{\varepsilon,N} M_{u^2}) \le \mathrm{Tr}(Q^{\varepsilon,N})\|u\|_{L^2}^2 \le C_N \|u\|_{L^2}^2. \tag{28}$$

Proof. Since $Q^{\varepsilon,N}$ is positive, $\mathrm{Tr}(Q^{\varepsilon,N} M_{u^2}) \le |M_{u^2}|_{\mathrm{op}} \mathrm{Tr}(Q^{\varepsilon,N}) = |u|_2^2 \mathrm{Tr}(Q^{\varepsilon,N})$. Moreover, $\mathrm{Tr}(Q^{\varepsilon,N}) \le \dim(H_N)$ and $\dim(H_N) \sim N^3$.

Proposition B.9 (Lyapunov inequality for $\mathcal{E}$). There exist constants $A_N, B_N < \infty$ such that for all $\varepsilon \in (0,1)$,

$$\frac{d}{dt}\mathbb{E}[\mathcal{E}(u_t)] \le A_N + B_N\, \mathbb{E}[\mathcal{E}(u_t)]. \tag{29}$$

Proof. Take expectations in (27); the martingale term vanishes. Use Lemma B.8 and the coercive bound $\|u\|_{L^2}^2 \le \frac{2}{m_{\mathrm{ren}}^2}\mathcal{E}(u)$ from (26) to absorb $|u|_2^2$ into $\mathcal{E}(u)$.

Proposition B.10 (Existence of an invariant measure and energy moments). For each fixed (ε, N), the Markov process defined by (25) admits an invariant probability measure $\mu^{\varepsilon,N}$. Moreover, for each $p \ge 1$,

$$\mathbb{E}_{\mu^{\varepsilon,N}}[\mathcal{E}(u)^p] < \infty. \tag{30}$$

Proof. Tightness of time-averaged laws follows from Proposition B.9 since $\mathcal{E}$ has compact sublevel sets in finite dimension. Krylov–Bogoliubov yields existence of an invariant measure. For moments, apply Itô to $\mathcal{E}^p$ (a polynomial), bound the quadratic variation term using $|Q^{\varepsilon,N}|_{\mathrm{op}} \le 1$, and conclude finiteness at stationarity.

Remark B.11 (Role of Proposition B.10). Proposition B.10 supplies stationary integrability of energy in a rigorous finite-dimensional setting. To reach Proposition 6.1 for the limiting Φ_3^4 measure, one must combine such stationary energy control with the fixed-time distributional regularity theory (enhanced noise + reconstruction). This is the place where paracontrolled calculus or regularity structures enter.

B.3. From stationary energy control to Proposition 6.1: what remains

The remaining step is to upgrade energy-type moments to distributional norms in negative Hölder regularity, and to construct the renormalized square $:u^2:$ at fixed time. This is exactly the content of the standard Φ_3^4 regularity theory.

Proposition B.12 (Standard fixed-time outputs for Φ_3^4; quoted). Let μ be the Φ_3^4 measure on $\mathbb{T}^3$ at $\lambda \neq 0$, realised as the time-marginal of a stationary solution to the renormalised stochastic quantisation equation. Then for every $\eta > 0$ and every $p < \infty$:

1. (Field regularity) $u \in C^{-1/2-\eta}(\mathbb{T}^3)$ almost surely and $\mathbb{E}\|u\|_{C^{-1/2-\eta}}^p < \infty$.

2. (Renormalized square) There exists a random distribution $:u^2: \in C^{-1-\eta}(\mathbb{T}^3)$ such that $\mathbb{E}\| :u^2: \|^p_{C^{-1-\eta}} < \infty$ and

$$u_\varepsilon^2 - C_1(\varepsilon) \to :u^2: \quad \text{in probability in } C^{-1-\eta}(\mathbb{T}^3) \quad (\varepsilon \downarrow 0).$$

These statements are standard outputs of the construction of the dynamical Φ^4_3 model and its invariant measure; see for example [Hai14, §10], [GIP15], [MW17, §2–§3], and the invariant-measure formulations in [AK20, Thm. 1.1] and [GH21, §4].

Proof. This proposition is quoted from the standard Φ^4_3 construction and invariant-measure theory; see the cited references.

Remark B.13 (Self-contained roadmap). To remove Proposition B.12, one would proceed by:

- constructing the enhanced noise $(X, :X^2:, :X^3:, \mathcal{I}(:X^2:), \mathcal{I}(:X^3:))$ at the level of the mollified equation,
- establishing uniform (in ε) moment bounds in parabolic Hölder spaces for these objects,
- solving the remainder equation via a paracontrolled fixed-point argument,
- proving tightness and convergence of $(u^\varepsilon, (u^\varepsilon)^2 - C_1(\varepsilon))$ in suitable negative Hölder spaces,
- passing to the limit and identifying the invariant measure as μ.

This program is substantial and typically occupies a full technical appendix. For the main result of this paper (Theorem 1.1), it suffices to use the outputs summarized in Proposition B.12.

Appendix C. Summary of key inputs and their minimal uses

This appendix provides a dependency map between key inputs and conclusions in the main text.

C.1. Proposition 5.1 (cubic decomposition)

Remark C.1. Proposition 5.1 is used only in §5 to prove Proposition 5.5 and Corollary 5.6. Its two essential parts are:

- tightness of the remainder R_ε (so $e^{-\beta n} R_{\varepsilon_n} \to 0$),
- variance bound $\mathrm{Var}(\Theta_\varepsilon) \lesssim C_2(\varepsilon)$ (so $e^{-\beta n}\Theta_{\varepsilon_n} \to 0$ in L^2).

Lemma A.4 and Lemma A.5 show that once the dominant fluctuation admits a second-chaos representation, the required variance bound reduces to the definition of $C_2(\varepsilon)$.

C.2. Proposition 6.1 (tightness of linear and quadratic observables)

Remark C.2. Proposition 6.1 is used only in §6 to show that the two error terms $T_n^{(2)}$ and $T_n^{(3)}$ in the shift expansion vanish in probability. Proposition B.2 provides a sufficient condition for Proposition 6.1 in terms of negative Hölder moment bounds for u and the renormalized square $: u^2 :$.

C.3. Deterministic input from §4

Remark C.3. Proposition 4.5 (log divergence with $\kappa > 0$) is the only deterministic renormalization input needed to force the drift term in §6 to dominate:

$$-9\lambda^2 e^{-\beta n} C_2(\varepsilon_n)\langle \psi, \psi \rangle \sim -(9\lambda^2\kappa)e^{(1-\beta)n}\|\psi\|_{L^2}^2.$$

This term is responsible for the failure of quasi-invariance under smooth translations.

Appendix D. Deterministic estimates for §4

This appendix supplies complete proofs of the two deterministic statements used in §4:

1. the sum–integral comparison in Lemma 4.3, and

2. the $O(1)$ control of boundary and mass errors implicit in the proof of Proposition 4.5.

Throughout, constants may depend on m and on the choice of smooth cutoff, but are uniform in the large parameter $\Lambda \geq 2$.

D.1. Sum–integral comparison via Poisson summation

The sharp-cutoff quantities in §4 are

- the lattice sum $S(\Lambda)$ in (4), and

- the continuum integral $I(\Lambda)$ in (5).

We first replace sharp constraints by smooth cutoffs and compare the resulting sum and integral using Poisson summation on $\mathbb{Z}^6$.

Definition D.1 (Smooth cutoff and smoothed kernels). Fix $\chi \in C_c^\infty(\mathbb{R}^3)$ such that $\chi(\xi) = 1$ for $|\xi| \leq 1$ and $\chi(\xi) = 0$ for $|\xi| \geq 2$. For $\Lambda \geq 1$ set $\chi_\Lambda(\xi) := \chi(\xi/\Lambda)$.

Define the smoothed lattice sum and integral

$$S_\chi(\Lambda) := \sum_{k_1,k_2\in\mathbb{Z}^3} \chi_\Lambda(k_1)\chi_\Lambda(k_2)\chi_\Lambda(k_1+k_2)\,\frac{1}{(m^2+|k_1|^2)(m^2+|k_2|^2)(m^2+|k_1+k_2|^2)}, \tag{31}$$

and

$$I_\chi(\Lambda) := \iint_{\mathbb{R}^3\times\mathbb{R}^3} \chi_\Lambda(p)\chi_\Lambda(q)\chi_\Lambda(p+q)\,\frac{\mathrm{d}p\,\mathrm{d}q}{(m^2+|p|^2)(m^2+|q|^2)(m^2+|p+q|^2)}. \tag{32}$$

Let

$$F_\Lambda(p,q) := \chi_\Lambda(p)\chi_\Lambda(q)\chi_\Lambda(p+q)\,\frac{1}{(m^2+|p|^2)(m^2+|q|^2)(m^2+|p+q|^2)}. \tag{33}$$

Then $S_\chi(\Lambda) = \sum_{n\in\mathbb{Z}^6} F_\Lambda(n)$ and $I_\chi(\Lambda) = \int_{\mathbb{R}^6} F_\Lambda$.

Lemma D.2 (Smoothed sum–integral comparison).
There exists $C<\infty$ such that for all $\Lambda \geq 2$,

$$|S_\chi(\Lambda) - I_\chi(\Lambda)| \leq C.$$

Proof. Poisson summation in $\mathbb{R}^6$ gives

$$S_\chi(\Lambda) = \sum_{\ell\in\mathbb{Z}^6} \widehat{F}_\Lambda(2\pi\ell), \qquad I_\chi(\Lambda) = \widehat{F}_\Lambda(0),$$

hence

$$S_\chi(\Lambda) - I_\chi(\Lambda) = \sum_{\ell\in\mathbb{Z}^6\setminus\{0\}} \widehat{F}_\Lambda(2\pi\ell).$$

It therefore suffices to show that $\widehat{F}_\Lambda(\xi)$ decays faster than any power of $|\xi|$, uniformly in Λ, away from $\xi = 0$.

To this end, split $F_\Lambda = F_\Lambda^{\mathrm{loc}} + F_\Lambda^{\mathrm{reg}}$ using a smooth partition of unity subordinate to the sets

$$U := \{(p,q) : |p|\leq 1 \text{ or } |q|\leq 1 \text{ or } |p+q|\leq 1\}, \qquad U^c.$$

On U, the singular factors are locally integrable in $\mathbb{R}^6$ (indeed, $|p|^{-2}$ and $|q|^{-2}$ are integrable in three dimensions, and products remain integrable in six dimensions on bounded sets), and the cutoff is bounded. Hence $F_\Lambda^{\mathrm{loc}} \in L^1(\mathbb{R}^6)$ with $\|F_\Lambda^{\mathrm{loc}}\|_{L^1} \lesssim 1$ uniformly in Λ, so $|\widehat{F}_\Lambda^{\mathrm{loc}}(\xi)| \leq \|F_\Lambda^{\mathrm{loc}}\|_{L^1} \lesssim 1$.

On U^c the function F_Λ^{reg} is smooth, and for every multiindex α one has

$$\sup_{\Lambda\geq 2} \|\partial^\alpha F_\Lambda^{\mathrm{reg}}\|_{L^1(\mathbb{R}^6)} < \infty.$$

This follows because all denominators are bounded below on U^c, while the derivatives of χ_Λ contribute factors of $\Lambda^{-|\alpha|}$ supported in a boundary layer of volume $O(\Lambda^{6-|\alpha|})$, yielding L^1 bounds uniform in Λ.

Integrating by parts in the Fourier transform gives, for any $N \geq 0$,

$$|\widehat{F}_\Lambda^{\mathrm{reg}}(\xi)| \leq C_N(1+|\xi|)^{-N}, \qquad \text{uniformly in } \Lambda \geq 2.$$

Combining the L^1 bound on $\widehat{F}_\Lambda^{\mathrm{loc}}$ with the rapid decay of $\widehat{F}_\Lambda^{\mathrm{reg}}$, we obtain

$$\sum_{\ell \in \mathbb{Z}^6 \setminus \{0\}} |\widehat{F}_\Lambda(2\pi\ell)| \leq \sum_{\ell \neq 0} \big(|\widehat{F}_\Lambda^{\mathrm{loc}}(2\pi\ell)| + |\widehat{F}_\Lambda^{\mathrm{reg}}(2\pi\ell)|\big) \lesssim \sum_{\ell \neq 0} (1+|\ell|)^{-N} < \infty$$

for N large, with a bound independent of Λ. This proves the claim.

Lemma D.3 (Sharp vs smooth cutoff differs by $O(1)$).
Let $S(\Lambda)$ and $I(\Lambda)$ be as in §4. Then for $\Lambda \geq 2$,

$$S(\Lambda) = S_\chi(\Lambda) + O(1), \qquad I(\Lambda) = I_\chi(\Lambda) + O(1).$$

Proof. The difference between sharp constraints and the smooth weights is supported in the boundary layer where at least one of $|p|, |q|, |p+q|$ lies in $[\Lambda, 2\Lambda]$. In this region the kernel satisfies

$$\frac{1}{(m^2+|p|^2)(m^2+|q|^2)(m^2+|p+q|^2)} \lesssim \Lambda^{-6},$$

while the boundary layer has volume $O(\Lambda^6)$ in $\mathbb{R}^6$ and contains $O(\Lambda^6)$ lattice points in $\mathbb{Z}^6$. Therefore both the lattice and continuum discrepancies are bounded by $O(\Lambda^6 \cdot \Lambda^{-6}) = O(1)$.

Corollary D.4 (Lemma 4.3).
For $\Lambda \to \infty$,

$$S(\Lambda) = I(\Lambda) + O(1).$$

Proof. Combine Lemma D.2 and Lemma D.3.

D.2. Boundary and mass errors in the inner integral of Proposition 4.5

Recall the inner integral from the proof of Proposition 4.5:

$$J_\Lambda(q) := \int_{\substack{|p|\leq\Lambda,\\ |p+q|\leq\Lambda}} \frac{\mathrm{d}p}{(m^2+|p|^2)(m^2+|p+q|^2)}. \tag{34}$$

Introduce the untruncated massive and massless variants

$$J_m(q) := \int_{\mathbb{R}^3} \frac{\mathrm{d}p}{(m^2+|p|^2)(m^2+|p+q|^2)}, \qquad J(q) := \int_{\mathbb{R}^3} \frac{\mathrm{d}p}{|p|^2|p+q|^2}. \tag{35}$$

The next lemma quantifies the replacements $J_\Lambda \mapsto J_m \mapsto J$ and the approximation $(m^2 + |q|^2)^{-1} \mapsto |q|^{-2}$ on the region $1 \le |q| \le \Lambda/2$.

Lemma D.6 (Integrated $O(1)$ control of boundary and mass errors). There exists $C < \infty$ such that for all $\Lambda \ge 2$:

1. (Truncation error)

$$\int_{|q|\le\Lambda} \frac{\mathrm{d}q}{m^2 + |q|^2} \left|J_\Lambda(q) - J_m(q)\right| \le C.$$

2. (Mass error in the inner integral)

$$\int_{1\le|q|\le\Lambda/2} \frac{\mathrm{d}q}{m^2 + |q|^2} \left|J_m(q) - J(q)\right| \le C.$$

3. (Mass error in the outer factor)

$$\int_{1\le|q|\le\Lambda/2} \left|\frac{1}{m^2 + |q|^2} - \frac{1}{|q|^2}\right| J(q)\,\mathrm{d}q \le C.$$

Proof.
(1) Note that $J_\Lambda(q)$ differs from $J_m(q)$ only by integrating over the complement of the truncated region, i.e. points p for which either $|p| > \Lambda$ or $|p+q| > \Lambda$. For $|q| \le \Lambda$, the set $\{|p| > \Lambda\} \cup \{|p+q| > \Lambda\}$ is contained in $\{|p| > \Lambda/2\}$ up to a fixed constant factor. Hence, using $(m^2 + |p|^2)^{-1} \lesssim |p|^{-2}$,

$$|J_\Lambda(q) - J_m(q)| \le \int_{|p|>\Lambda/2} \frac{\mathrm{d}p}{(m^2 + |p|^2)(m^2 + |p+q|^2)} \lesssim \int_{|p|>\Lambda/2} \frac{\mathrm{d}p}{|p|^4} \lesssim \Lambda^{-1}.$$

Therefore

$$\int_{|q|\le\Lambda} \frac{\mathrm{d}q}{m^2 + |q|^2} |J_\Lambda(q) - J_m(q)| \lesssim \Lambda^{-1} \int_{|q|\le\Lambda} \frac{\mathrm{d}q}{m^2 + |q|^2} \lesssim \Lambda^{-1} \cdot \Lambda \lesssim 1.$$

(2) For $|q| \ge 1$, write

$$\frac{1}{m^2 + |p|^2} - \frac{1}{|p|^2} = -\frac{m^2}{|p|^2(m^2 + |p|^2)}.$$

Expanding $J_m(q) - J(q)$ by replacing each factor once and using symmetry gives

$$|J_m(q) - J(q)| \lesssim \int_{\mathbb{R}^3} \frac{m^2\,\mathrm{d}p}{|p|^2(m^2 + |p|^2)} \cdot \frac{1}{m^2 + |p+q|^2} + \int_{\mathbb{R}^3} \frac{\mathrm{d}p}{|p|^2} \cdot \frac{m^2}{|p+q|^2(m^2 + |p+q|^2)}.$$

Both integrals are bounded by $C|q|^{-2}$ for $|q| \ge 1$ by splitting the p-integral into $|p| \le |q|/2$ and $|p| > |q|/2$ and using $|p+q| \simeq |q|$ on the first region and decay on the second. Consequently,

$$\int_{1\le|q|\le\Lambda/2} \frac{\mathrm{d}q}{m^2 + |q|^2} |J_m(q) - J(q)| \lesssim \int_{1\le|q|\le\Lambda/2} \frac{\mathrm{d}q}{|q|^2} \cdot \frac{1}{|q|^2} \lesssim \int_1^\infty r^{-2}\,\mathrm{d}r < \infty.$$

(3) For $|q| \geq 1$,

$$\left| \frac{1}{m^2 + |q|^2} - \frac{1}{|q|^2} \right| = \frac{m^2}{|q|^2(m^2 + |q|^2)} \lesssim |q|^{-4}.$$

By Lemma 4.4 (Riesz identity), $J(q) = c|q|^{-1}$ for $q \neq 0$. Hence

$$\int_{1 \leq |q| \leq \Lambda/2} |q|^{-4} \, J(q) \, \mathrm{d}q \lesssim \int_{1 \leq |q| \leq \Lambda/2} |q|^{-5} \, \mathrm{d}q \lesssim \int_1^\infty r^{-3} \, \mathrm{d}r < \infty,$$

uniformly in Λ. This proves the claim.

Corollary D.7 (Error reduction used in Proposition 4.5).
With $I(\Lambda)$ as in (5), one has

$$I(\Lambda) = \int_{1 \leq |q| \leq \Lambda/2} \frac{\mathrm{d}q}{|q|^2} \, J(q) + O(1).$$

In particular, the informal "boundary effects" and "mass effects" in the proof of Proposition 4.5 contribute only $O(1)$.

Proof. Combine the three items in Lemma D.6, together with the trivial bound that the contribution of the region $|q| \leq 1$ to $I(\Lambda)$ is $O(1)$ (since $(m^2+|q|^2)^{-1} \lesssim 1$ and $J_m(q)$ is locally integrable).

References

[FP26] M. Abouzaid, A. J. Blumberg, M. Hairer, J. Kileel, T. G. Kolda, P. D. Nelson, D. Spielman, N. Srivastava, R. Ward, S. Weinberger, and L. Williams. *First Proof.* arXiv:2602.05192 (2026). doi:10.48550/arXiv.2602.05192.

[AK20] S. Albeverio and S. Kusuoka. *The invariant measure and the flow associated to the Φ^4_3-quantum field model.* Annali della Scuola Normale Superiore di Pisa, Classe di Scienze (5) **20** (2020), no. 4, 1359–1427. arXiv:1711.07108.

[BCD11] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier Analysis and Nonlinear Partial Differential Equations.* Grundlehren der mathematischen Wissenschaften, vol. 343, Springer, 2011.

[GH21] M. Gubinelli and M. Hofmanová. *A PDE construction of the Euclidean Φ^4_3 quantum field theory.* Communications in Mathematical Physics **384** (2021), no. 1, 1–75. doi:10.1007/s00220-021-04022-0.

[GIP15] M. Gubinelli, P. Imkeller, and N. Perkowski. *Paracontrolled distributions and singular PDEs.* Forum of Mathematics, Pi **3** (2015), e6.

[Hai14] M. Hairer. *A theory of regularity structures.* Inventiones Mathematicae **198** (2014), no. 2, 269–504. arXiv:1303.5113.

[MW17] J.-C. Mourrat and H. Weber. *The dynamic Φ^4_3 model comes down from infinity.* Communications in Mathematical Physics **356** (2017), no. 3, 673–753. arXiv:1601.01234.

[MWX17] J.-C. Mourrat, H. Weber, and W. Xu. *Construction of Φ^4_3 diagrams for pedestrians.* In *From particle systems to partial differential equations*, Springer Proc. Math. Stat. 209, Springer, 2017, pp. 1–46. Also available as arXiv:1610.08897 (2016).