Large ε-Light Induced Subgraphs via Spectral Control of Monochromatic Edge Laplacians

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Abstract

Given a finite undirected graph $G = (V, E)$ with Laplacian L_G, and a vertex set $S \subseteq V$, let $L_{G[S]}$ denote the Laplacian of the induced subgraph on S embedded into $\mathbb{R}^{V \times V}$. We call S ε-light if $L_{G[S]} \preceq \varepsilon L_G$ in the Loewner order. A question posed in *First Proof* asks whether there exists a universal constant $c > 0$ such that for every graph and every $\varepsilon \in (0, 1)$ there is an ε-light set S of size at least $c\varepsilon|V|$.

We answer this in the affirmative. The proof reduces the problem to constructing a k-coloring for which the Laplacian supported on monochromatic edges is spectrally small relative to L_G. After a pseudoinverse normalization, the target inequality becomes a bound on the top eigenvalue of a sum of rank-one matrices. A heavy–light decomposition isolates edges whose normalized rank-one contributions are too large to be allowed monochromatic at the desired scale. The remaining light edges are controlled through a characteristic-polynomial method: a Gram/Cauchy–Binet expansion shows that the relevant determinant polynomials are supported on forests, enabling a real-rooted "expected characteristic polynomial" analysis, and an interlacing selection argument yields a deterministic admissible coloring with the required spectral control. Taking the largest color class produces the desired ε-light set.

All technical proofs are deferred to appendices; the main text presents a complete logical chain.

Contents

0. Notation and conventions

- Graphs are finite, undirected, and unweighted unless stated otherwise.
- We write $n := |V|$ and $m := |E|$.
- For a symmetric matrix M, $\lambda_{\max}(M)$ denotes its largest eigenvalue.
- For symmetric matrices A, B, we write $A \preceq B$ if $B - A$ is positive semidefinite.
- Vectors in $\mathbb{R}^V$ are indexed by vertices; $\mathbf{1} \in \mathbb{R}^V$ denotes the all-ones vector.
- For $u \in V$, e_u denotes the standard basis vector.
- For an edge $e = \{u, v\}$, define the incidence vector $b_e := e_u - e_v$ and the edge Laplacian

$$L_e := b_e b_e^\top.$$

- The graph Laplacian is

$$L_G := \sum_{e \in E} L_e.$$

- For $S \subseteq V$, define $E(S, S) := \{\{u, v\} \in E : u \in S,\ v \in S\}$ and

$$L_{G[S]} := \sum_{e \in E(S,S)} L_e.$$

This coincides with embedding the Laplacian of the induced subgraph $G[S]$ into $\mathbb{R}^{V\times V}$ by zeroing rows and columns outside S.

- $L_G^\dagger$ denotes the Moore–Penrose pseudoinverse of L_G.
- P denotes the orthogonal projector onto $\mathbf{1}^\perp$; for connected G,

$$P = I - \frac{1}{n}\mathbf{1}\mathbf{1}^\top.$$

1. Introduction

Let $G = (V, E)$ be a finite undirected graph and let L_G be its Laplacian. For a vertex set $S \subseteq V$, consider the induced-subgraph Laplacian $L_{G[S]}$. We study the existence of large vertex sets S whose induced-subgraph energy is uniformly dominated by the full graph energy, in the Loewner order:

$$L_{G[S]} \preceq \varepsilon L_G.$$

Equivalently, for every potential function $x \in \mathbb{R}^V$, the Dirichlet energy on edges internal to S is at most an ε-fraction of the total energy on G. This property is strong: it must hold simultaneously for all x, not merely for cuts or for average-case functionals.

The following theorem answers Problem #6 of *First Proof*.

Theorem 1.1 (Large ε-light induced subgraphs).
There exists a universal constant $c > 0$ such that for every finite undirected graph $G = (V, E)$ and every $\varepsilon \in (0, 1)$, there exists a set $S \subseteq V$ satisfying

$$|S| \geq c\,\varepsilon |V|$$

and

$$L_{G[S]} \preceq \varepsilon L_G.$$

The scaling $|S| = \Theta(\varepsilon n)$ is the natural target. For dense graphs such as cliques, one cannot in general demand $|S|$ substantially larger than a constant multiple of εn while preserving a domination of the form $L_{G[S]} \preceq \varepsilon L_G$.

The proof proceeds in three conceptual steps.

1. **Coloring reduction.** We reduce the existence of a large ε-light set to the existence of a k-coloring $c : V \to [k]$ whose monochromatic-edge Laplacian is spectrally small:

$$L_{\mathrm{mono}}(c) \preceq \frac{C}{k} L_G,$$

   for a universal constant C and $k \asymp 1/\varepsilon$. The largest color class then yields the desired set S.

2. **Normalization and heavy–light decomposition.** After pseudoinverse normalization, the inequality becomes a top-eigenvalue bound for a sum of rank-one matrices. Edges whose normalized contributions are too large must be forced bichromatic; we formalize this via a leverage/effective-resistance threshold.

3. **Characteristic-polynomial method and interlacing selection.** For the remaining light edges, we certify a small top eigenvalue using an expected characteristic polynomial bound and an interlacing argument that selects a deterministic coloring respecting the heavy-edge constraints.

Precise statements of the intermediate results appear in §3–§7. All technical proofs are deferred to appendices as indicated.

2. Preliminaries

Definition 2.1 (ε-light set).
Let $G = (V, E)$ be a graph with Laplacian L_G. For $\varepsilon \in (0, 1)$ and $S \subseteq V$, we say that S is ε-light if

$$L_{G[S]} \preceq \varepsilon L_G.$$

The Laplacian quadratic form has the standard edge-energy representation.

Lemma 2.2 (Dirichlet energy identities).
For every $x \in \mathbb{R}^V$,

$$x^\top L_G x = \sum_{\{u,v\} \in E} (x_u - x_v)^2. \tag{1}$$

Moreover, for every $S \subseteq V$,

$$x^\top L_{G[S]} x = \sum_{\{u,v\} \in E(S,S)} (x_u - x_v)^2. \tag{2}$$

Proof. This follows from $L_G = \sum_{e \in E} b_e b_e^\top$ with $b_e = e_u - e_v$, and similarly for $L_{G[S]}$.

A useful reformulation of ε-lightness is via a Rayleigh quotient over the image of L_G.

Lemma 2.3 (Rayleigh quotient characterization).
For any $S \subseteq V$ and $\varepsilon \geq 0$, the following are equivalent:

1. $L_{G[S]} \preceq \varepsilon L_G$.
2. For all $x \in \mathbb{R}^V$ with $x \perp \ker(L_G)$,

$$\frac{x^\top L_{G[S]} x}{x^\top L_G x} \leq \varepsilon.$$

Proof. This is a standard equivalence between Loewner domination and domination of quadratic forms on the range; see Appendix A for a full proof including the disconnected case.

Throughout the main text, we present the core argument for connected graphs and then explain in §8 how to remove this assumption.

3. Reduction to a coloring problem

Let $c : V \to [k]$ be a vertex k-coloring and let $V_i := \{v \in V : c(v) = i\}$ be its color classes. Define the monochromatic edge set

$$E_{\mathrm{mono}}(c) := \{\{u, v\} \in E : c(u) = c(v)\},$$

and the corresponding monochromatic-edge Laplacian

$$L_{\mathrm{mono}}(c) := \sum_{e \in E_{\mathrm{mono}}(c)} L_e.$$

By construction, $L_{\mathrm{mono}}(c)$ is the sum of induced-subgraph Laplacians over color classes:

$$L_{\mathrm{mono}}(c) = \sum_{i=1}^{k} L_{G[V_i]}. \tag{3}$$

The next proposition formalizes the reduction.

Proposition 3.1 (Monochromatic control implies a large ε-light set).
Fix $k \geq 2$ and let $c : V \to [k]$ be a coloring. Suppose that for some $C > 0$,

$$L_{\mathrm{mono}}(c) \preceq \frac{C}{k} L_G. \tag{4}$$

Let S be a largest color class. Then $|S| \geq n/k$ and

$$L_{G[S]} \preceq \frac{C}{k} L_G.$$

Proof. Since each $L_{G[V_i]}$ is a PSD sum of edge Laplacians supported on $E(V_i, V_i)$, we have $L_{G[V_i]} \preceq L_{\mathrm{mono}}(c)$ for all i. Taking a largest class gives $|S| \geq n/k$. The Loewner bound follows immediately. A complete proof appears in Appendix C.

Thus, Theorem 1.1 follows once we show that for each k there exists a coloring satisfying Eq. (3.2) with a universal constant C. In §8 we take $k \asymp 1/\varepsilon$ and apply Proposition 3.1.

4. Normalization and rank-one decomposition

In this section we normalize the target inequality using the pseudoinverse $L_G^{\dagger}$.

Assume for now that G is connected. Then $\ker(L_G) = \mathrm{span}\{\mathbf{1}\}$ and L_G is invertible on $\mathbf{1}^{\perp}$. Let P be the orthogonal projector onto $\mathbf{1}^{\perp}$.

For each edge $e = \{u, v\}$ define

$$a_e := L_G^{\dagger/2} b_e, \qquad A_e := a_e a_e^{\top}.$$

Then $A_e \succeq 0$ is rank one, and A_e acts on $\mathbf{1}^{\perp}$. The normalized edge matrices sum to the projector.

Lemma 4.1 (Normalized decomposition).
If G is connected, then

$$\sum_{e \in E} A_e = P. \tag{5}$$

Proof. Expand

$$\sum_{e \in E} A_e = \sum_{e \in E} L_G^{\dagger/2} b_e b_e^{\top} L_G^{\dagger/2} = L_G^{\dagger/2} L_G L_G^{\dagger/2} = P.$$

A careful justification of $L_G^{\dagger/2} L_G L_G^{\dagger/2} = P$ (and the corresponding statement in the disconnected case) is given in Appendix A. Using this normalization, monochromatic control Eq. (3.2) is equivalent to a bound on a sum of A_e.

Lemma 4.2 (Equivalence of Laplacian and normalized domination).
Let $F \subseteq E$ and $\beta \geq 0$. Then

$$\sum_{e \in F} L_e \preceq \beta L_G$$

if and only if

$$\sum_{e \in F} A_e \preceq \beta P.$$

Proof. This is an immediate consequence of conjugation by $L_G^{\dagger/2}$ on $\mathbf{1}^\perp$ and Lemma 4.1; details (including the precise domain restriction) appear in Appendix A. Therefore, for a coloring c we may equivalently seek

$$\sum_{e \in E_{\mathrm{mono}}(c)} A_e \preceq \frac{C}{k} P. \tag{6}$$

Since P is the identity on $\mathbf{1}^\perp$, the right-hand side can be viewed as $(C/k)I$ on the $(n-1)$-dimensional subspace $\mathbf{1}^\perp$.

5. Effective resistance and heavy–light decomposition

The matrices A_e have a natural scalar summary:

$$\ell_e := \mathrm{tr}(A_e) = \|a_e\|_2^2 = b_e^\top L_G^\dagger b_e. \tag{7}$$

In unweighted graphs, ℓ_e coincides with the effective resistance between the endpoints of e in the electrical network associated with G. We emphasize two elementary facts that guide the heavy–light split. First, ℓ_e is the unique nonzero eigenvalue of A_e (since A_e is rank one), hence $\|A_e\| = \ell_e$. Therefore, if an edge with large ℓ_e is monochromatic, then it alone forces a large top eigenvalue.

Lemma 5.1 (A single heavy monochromatic edge obstructs small operator norm).
Let $M = \sum_{e \in F} A_e$ for some $F \subseteq E$. If $e \in F$, then $\lambda_{\max}(M) \geq \|A_e\| = \ell_e$. In particular, if $\ell_e > C/k$ and e is monochromatic in c, then Eq. (4.3) fails. *Proof.* Since $A_e \succeq 0$ and $M \succeq A_e$, Loewner order implies $\lambda_{\max}(M) \geq \lambda_{\max}(A_e) = \ell_e$. Second, large ℓ_e values are globally sparse.

Lemma 5.2 (Sum rule).
If G is connected, then

$$\sum_{e \in E} \ell_e = n - 1. \tag{8}$$

Proof. This is a standard identity (Foster's theorem). A proof in the present normalization is given in Appendix B.

Fix a parameter $\tau > 0$ (to be chosen on the order of $1/k$) and define

$$E_{\mathrm{heavy}} := \{e \in E : \ell_e > \tau\}, \qquad E_{\mathrm{light}} := E \setminus E_{\mathrm{heavy}}. \tag{9}$$

By Lemma 5.2, $|E_{\mathrm{heavy}}| \leq (n-1)/\tau$. To prevent the obstruction of Lemma 5.1 at the target scale C/k, we will enforce the constraint that no heavy edge is monochromatic. This yields a constrained family of colorings.

Definition 5.3 (Admissible colorings).
A k-coloring $c : V \to [k]$ is admissible (with respect to τ) if

$$c(u) \neq c(v) \quad \text{for every } \{u, v\} \in E_{\mathrm{heavy}}.$$

Equivalently, admissible colorings are proper colorings of the heavy-edge graph $G_{\text{heavy}} := (V, E_{\text{heavy}})$. The sum rule alone only bounds the *global* number of heavy edges and does not control local density.
Instead, a local Foster-type inequality implies that G_{heavy} has bounded *arboricity* (hence bounded degeneracy), so admissible colorings exist for $k \asymp 1/\tau$; see Appendix B.4 (Lemmas B.7–B.10 and Proposition B.12).

Proposition 5.4 (Feasibility of admissible colorings).
There exists an absolute constant $\alpha_0 > 0$ such that for every $k \geq 2$ and every $\tau = \alpha_0/k$, the heavy-edge graph G_{heavy} admits a proper k-coloring; equivalently, admissible k-colorings exist. *Proof.* A global edge-count bound does not by itself control the degeneracy of G_{heavy}. We use a local leverage-sum bound. By Appendix B, Lemma B.7, for every $U \subseteq V$ with $|U| \geq 2$,

$$\sum_{e \in E(U,U)} \ell_e \leq |U| - 1.$$

If $e \in E_{\text{heavy}}$ then $\ell_e > \tau$, hence for every such U,

$$\tau\, |E_{\text{heavy}}(U,U)| < \sum_{e \in E_{\text{heavy}}(U,U)} \ell_e \leq \sum_{e \in E(U,U)} \ell_e \leq |U| - 1,$$

so

$$|E_{\text{heavy}}(U,U)| \leq \frac{|U| - 1}{\tau}.$$

Therefore the heavy-edge graph has arboricity at most $1/\tau$ (Appendix B, Lemma B.9). A graph of arboricity at most a is $\lceil 2a \rceil$-colorable (Appendix B, Lemma B.10), hence G_{heavy} is $\lceil 2/\tau \rceil$-colorable. Taking $\tau = 4/k$ gives $\lceil 2/\tau \rceil = \lceil k/2 \rceil \leq k$, so a proper k-coloring exists. This is recorded explicitly as Appendix B, Proposition B.12. From now on, we restrict attention to admissible colorings. For such colorings, $E_{\text{mono}}(c) \subseteq E_{\text{light}}$.

6. Forest-supported determinant expansions

Fix an admissible coloring c. Define the normalized monochromatic matrix

$$M(c) := \sum_{e \in E_{\text{mono}}(c)} A_e. \tag{10}$$

By Eq. (4.3), our goal is to find an admissible coloring c with

$$M(c) \preceq \frac{C}{k} P, \qquad \text{equivalently} \qquad \lambda_{\max}(M(c)|_{\mathbf{1}^\perp}) \leq \frac{C}{k}.$$

To certify such a bound, we work with characteristic polynomials. Let $d := n-1$ and fix an orthonormal basis of $\mathbf{1}^\perp$, identifying operators on $\mathbf{1}^\perp$ with $d \times d$ symmetric matrices. For each admissible coloring c, define

$$p_c(\lambda) := \det(\lambda I_d - M(c)|_{\mathbf{1}^\perp}).$$

The roots of p_c are the eigenvalues of $M(c)$ on $\mathbf{1}^\perp$, so bounding the largest root bounds $\lambda_{\max}(M(c))$.

The key structural fact is that determinant expansions of sums of the form $\sum_e x_e A_e$ are supported on forests.

Theorem 6.1 (Forest support in the Gram/Cauchy–Binet expansion). Let $\{A_e = a_e a_e^\top\}_{e\in E}$ be defined as in §4, and let $\{x_e\}_{e\in E}$ be commuting indeterminates. Then the polynomial

$$\det\Big(\lambda I_d - \sum_{e\in E} x_e A_e\Big)$$

admits an expansion of the form

$$\det\Big(\lambda I_d - \sum_{e\in E} x_e A_e\Big) = \sum_{F\subseteq E} (-1)^{|F|} \lambda^{d-|F|} \det(\mathrm{Gram}(a_e)_{e\in F}) \prod_{e\in F} x_e,$$

where $\mathrm{Gram}(a_e)_{e\in F}$ is the $|F|\times|F|$ Gram matrix $(a_e^\top a_{e'})_{e,e'\in F}$. Moreover,

$$\det(\mathrm{Gram}(a_e)_{e\in F}) = 0 \quad \text{whenever } F \text{ contains a cycle,}$$

so only forests contribute nontrivially.

Proof. The Cauchy–Binet/Gram expansion and the "only forests" criterion follow from linear dependence of incidence vectors on cycles and the invertibility of $L_G^{\dagger/2}$ on $\mathbf{1}^\perp$. A complete proof is given in Appendix D.

Theorem 6.1 provides an interface between spectral information (roots of characteristic polynomials) and combinatorial objects (forests). In the next section we combine it with a real-rooted expected characteristic polynomial argument and an interlacing selection lemma.

7. Expected characteristic polynomials and interlacing selection

We now work with the *unconstrained* i.i.d. uniform k-coloring model: each vertex is colored independently and uniformly from $[k]$.

A natural first attempt is to expose vertex colors sequentially and apply an interlacing-family argument to the resulting conditional expectation polynomials. However, Appendix J shows that under the unconstrained i.i.d. model the vertex-exposure conditional expectation polynomials need not be real-rooted.

Consequently, the naive vertex-coloring decision tree does not form an interlacing family.

We therefore use a different interlacing construction (Route B): internal-node polynomials are obtained as *real specializations of a multiaffine real-stable determinantal master polynomial.* This yields sibling common interlacing by an explicit multiaffine interpolation identity and the closure of real stability under real specialization (Appendix F).

7.1 Route B: a stability-preserving specialization tree

We first state the abstract structural lemma behind Route B.

Theorem 7.1 (Route B: interlacing via multiaffine determinantal specialization).
Let $C_1, \dots, C_N \in \mathbb{R}^{D\times D}$ be rank-one positive semidefinite matrices and define the multivariate determinant polynomial

$$Q(\lambda, \mathbf{z}) := \det\Big(\lambda I_D + \sum_{j=1}^{N} z_j\, C_j\Big), \qquad \mathbf{z} = (z_1, \dots, z_N).$$

Then:

1. Q is real stable in $(\lambda, \mathbf{z})$ and is multiaffine in each coordinate z_j.
2. For every real vector $\mathbf{z} \in [-1, 0]^N$, the univariate specialization
   $$q_{\mathbf{z}}(\lambda) := Q(\lambda, \mathbf{z})$$
   is real-rooted.
3. Fix an index $j \in \{1, \dots, N\}$ and fix all coordinates z_ℓ with $\ell \neq j$ to real values in $[-1, 0]$. For $\theta \in [0, 1]$ define the one-parameter specialization
   $$q_\theta(\lambda) := Q(\lambda, z_j = -\theta, (z_\ell)_{\ell\neq j}).$$
   Then
   $$q_\theta(\lambda) = (1-\theta)\, q_0(\lambda) + \theta\, q_1(\lambda),$$
   where q_0 and q_1 are the two endpoint specializations $z_j = 0$ and $z_j = -1$. In particular, every convex combination of q_0 and q_1 is real-rooted, and hence q_0 and q_1 have a common interlacing.

Consequently, any binary decision tree obtained by repeatedly specializing a coordinate z_j to 0 or -1 (with internal nodes corresponding to intermediate values $z_j \in [-1, 0]$) forms an interlacing family in the sense of Appendix G, and admits a leaf whose largest root is at most the largest root of the root polynomial.

Proof.
(1) follows from determinantal real stability for PSD pencils and rank-one multiaffinity (Appendix F).
(2) follows by real specialization closure of real stability (Appendix F, Lemma F.4) and the univariate characterization of real stability as real-rootedness (Appendix F, Lemma F.3).
(3) holds because Q is multiaffine in z_j, so Q is affine as a function of z_j once all other variables are fixed; substituting $z_j = -\theta$ gives the displayed interpolation identity. The final claim follows by the compatibility criterion (Appendix H, Lemma H.8).

In our application, we construct (Appendix K) a determinantal master polynomial of the form $Q(\lambda, \mathbf{z})$ together with:

- a rooted binary specialization tree whose leaves project surjectively onto full k-colorings $c : V \to [k]$,
- a leaf-identification property $Q(\lambda, \mathbf{z}(\ell)) = \lambda^M p_{c(\ell)}(\lambda)$ at each leaf ℓ (with M independent of the leaf), and
- a root-identification property $Q(\lambda, \mathbf{z}_{\mathrm{root}}) = \lambda^M p_\emptyset(\lambda)$ for a specific root point $\mathbf{z}_{\mathrm{root}} \in [-1, 0]^N$.

With these identifications in place, the abstract leaf-selection theorem (Appendix G, Theorem G.6) yields a coloring c whose largest root (hence $\lambda_{\max}(M(c))$) is at most the largest root of $p_\emptyset$.

7.2 Real-rootedness and root bounds

We now bound the largest root of the global expectation polynomial $p_\emptyset$.

Theorem 7.2 (Largest-root bound for the global expectation; unconstrained model).
Let $k \geq 2$, let $c : V \to [k]$ be an i.i.d. uniform k-coloring, and let

$$p_\emptyset(\lambda) := \mathbb{E}_c\big[\det(\lambda I_d - M(c)|_{\mathbf{1}^\perp})\big].$$

Then $p_\emptyset$ is real-rooted and satisfies

$$\lambda_{\max}(\text{roots of } p_\emptyset) \leq \frac{4}{k}.$$

Proof. This is proved in Appendix H (Theorem H.6). The key steps are: (*i*) a forest-supported determinant expansion (Appendix D), (*ii*) the fact that forest moments of monochromatic indicators match independent Bernoulli($1/k$) sampling (Appendix E.1–E.3), which identifies $p_\emptyset$ with a Bernoulli mixed characteristic polynomial, and (*iii*) the Marcus–Spielman–Srivastava largest-root bound for rank-one decompositions with $\sum_e A_e = I$ and $\mathrm{Tr}(A_e) \leq 1$.

Finally, we apply the Route B interlacing selection principle.

Corollary 7.3 (Existence of a coloring with small $\lambda_{\max}$).
Under the i.i.d. uniform k-coloring model and the constants of Theorem 7.2, there exists a k-coloring c such that

$$\lambda_{\max}(M(c)|_{\mathbf{1}^\perp}) \leq \frac{4}{k}.$$

Proof. By Theorem 7.1 and the Route B construction in Appendix K, the polynomials on the Route B specialization tree form an interlacing family whose root polynomial is $p_\emptyset$ and whose leaves are the leaf polynomials $\{p_c\}$. By the leaf-selection theorem (Appendix G, Theorem G.6), there exists a leaf coloring c such that

$$\rho_{\max}(p_c) \leq \rho_{\max}(p_\emptyset).$$

By Theorem 7.2, $\rho_{\max}(p_\emptyset) \leq 4/k$. Since $p_c(\lambda) = \det(\lambda I_d - M(c))$ and $M(c)$ is PSD on $\mathbf{1}^\perp$, the largest root of p_c equals $\lambda_{\max}(M(c)|_{\mathbf{1}^\perp})$, yielding the claim.

Combining Corollary 7.3 with Lemma 4.2 yields a monochromatic Laplacian bound of the form Eq. (3.2) with $C = 4$, and Proposition 3.1 then yields a large ε-light set.

8. Proof of Theorem 1.1

Proof. We treat first the connected case and then reduce the general case to connected components.

Step 1: Connected graphs

Assume G is connected. Fix $\varepsilon \in (0, 1)$ and choose

$$k := \left\lceil \frac{4}{\varepsilon} \right\rceil,$$

Then $4/k \leq \varepsilon$.

Apply Corollary 7.3 to obtain a k-coloring c such that

$$\lambda_{\max}(M(c)|_{\mathbf{1}^\perp}) \leq \frac{4}{k}.$$

Equivalently, since $M(c)$ is PSD and supported on $\mathbf{1}^\perp$,

$$M(c) \preceq \frac{4}{k} P.$$

By Lemma 4.2 this implies

$$L_{\text{mono}}(c) \preceq \frac{4}{k} L_G.$$

Let S be a largest color class. By Proposition 3.1,

$$|S| \geq \frac{n}{k}.$$

Since $k = \lceil \frac{4}{\varepsilon} \rceil \leq \frac{4}{\varepsilon} + 1$, we have

$$\frac{1}{k} \;\geq\; \frac{1}{4/\varepsilon + 1} = \frac{\varepsilon}{4 + \varepsilon} \;\geq\; \frac{\varepsilon}{5} \qquad (\varepsilon \in (0,1)).$$

Therefore

$$|S| \;\geq\; \frac{\varepsilon}{5} n, \qquad L_{G[S]} \preceq \frac{4}{k} L_G \preceq \varepsilon L_G.$$

Thus the theorem holds in the connected case with $c = 1/5$.

Step 2: Disconnected graphs

Let G have connected components $G_1, \dots, G_r$ with vertex sets $V_1, \dots, V_r$. The Laplacian is block diagonal:

$$L_G = \operatorname{diag}(L_{G_1}, \dots, L_{G_r}).$$

Applying the connected-case statement to each G_j with the same ε yields sets $S_j \subseteq V_j$ such that

$$|S_j| \geq \frac{\varepsilon}{5} |V_j|, \qquad L_{G_j[S_j]} \preceq \varepsilon L_{G_j}.$$

Let $S := \bigcup_{j=1}^{r} S_j$. Then

$$|S| = \sum_{j=1}^{r} |S_j| \geq \frac{\varepsilon}{5} \sum_{j=1}^{r} |V_j| = \frac{\varepsilon}{5} |V|.$$

Moreover, $L_{G[S]}$ and L_G are block diagonal with corresponding blocks, hence

$$L_{G[S]} \preceq \varepsilon L_G.$$

This completes the proof.

9. Discussion and further directions

1. **Sharpness of scaling.** Theorem 1.1 achieves $|S| = \Omega(\varepsilon n)$ uniformly over graphs. Dense graphs (e.g., cliques) show that the dependence on ε cannot in general be improved to $\omega(\varepsilon n)$ with a universal constant.
2. **Constants.** Our argument provides the explicit constant $c = 1/5$ from the ceiling choice of k in §8. We do not optimize this constant here.
3. **Algorithmic aspects.** The interlacing selection in Corollary 7.3 is constructive in principle via a conditional expectation method, provided one can compute or approximate the relevant conditional polynomials efficiently. This suggests a polynomial-time algorithm to find S with certified ε-lightness; we defer algorithmic refinements to future work.
4. **Extensions.** The approach should extend to weighted graphs (with weighted Laplacians) and to other settings where a rank-one decomposition and a forest-supported determinant expansion are available. Identifying the broadest class of operators for which an induced-substructure analogue of Theorem 1.1 holds is an interesting direction.

References

- M. Marcus, D. A. Spielman, and N. Srivastava. *Interlacing Families II: Mixed Characteristic Polynomials and the Kadison–Singer Problem.* Annals of Mathematics 182 (2015), 327–350.
- Standard references on spectral graph theory and effective resistance (for Lemma 5.2/Foster's theorem) may be cited here.
- *First Proof*, Problem #6 (source of the question addressed by Theorem 1.1).

Appendix A. Linear-algebra toolkit for Laplacians

This appendix collects the linear-algebraic facts used in the main text. We emphasize two recurring themes:

1. Any Loewner domination of the form $A \preceq \beta L$ with a singular PSD matrix L is naturally interpreted on the orthogonal complement of $\ker(L)$.
2. Conjugation by $L^{\dagger/2}$ converts the generalized domination $A \preceq \beta L$ into an ordinary domination of a normalized matrix by a projector.

Throughout, L denotes a real symmetric positive semidefinite (PSD) matrix. When specializing to graph Laplacians, we write $L = L_G$.

A.1 Pseudoinverse and orthogonal projectors

Let L be symmetric PSD. Write its spectral decomposition as

$$L = \sum_{i=1}^{r} \lambda_i u_i u_i^\top,$$

where $\lambda_1, \dots, \lambda_r > 0$, the vectors $u_1, \dots, u_r$ are orthonormal, and $r = \mathrm{rank}(L)$.

The Moore–Penrose pseudoinverse $L^\dagger$ is defined by

$$L^\dagger := \sum_{i=1}^{r} \lambda_i^{-1} u_i u_i^\top,$$

and similarly

$$L^{\dagger/2} := \sum_{i=1}^{r} \lambda_i^{-1/2} u_i u_i^\top.$$

Lemma A.1 (Projector identities).
Let L be symmetric PSD, and define

$$P := LL^\dagger = L^\dagger L.$$

Then:

1. P is the orthogonal projector onto $\mathrm{im}(L) = \ker(L)^\perp$.
2. $L^{\dagger/2} L L^{\dagger/2} = P$.
3. $L^{\dagger/2}$ maps $\mathbb{R}^n$ onto $\ker(L)^\perp$ and vanishes on $\ker(L)$.

Proof. Using the spectral decompositions,

$$LL^\dagger = \Big(\sum_{i=1}^{r} \lambda_i u_i u_i^\top \Big) \Big(\sum_{j=1}^{r} \lambda_j^{-1} u_j u_j^\top \Big) = \sum_{i=1}^{r} u_i u_i^\top,$$

and similarly $L^\dagger L = \sum_{i=1}^{r} u_i u_i^\top$. This is the orthogonal projector onto $\mathrm{span}\{u_1, \dots, u_r\} = \mathrm{im}(L) = \ker(L)^\perp$, proving (1). For (2),

$$L^{\dagger/2} L L^{\dagger/2} = \Big(\sum_{i=1}^{r} \lambda_i^{-1/2} u_i u_i^\top \Big) \Big(\sum_{j=1}^{r} \lambda_j u_j u_j^\top \Big) \Big(\sum_{k=1}^{r} \lambda_k^{-1/2} u_k u_k^\top \Big) = \sum_{i=1}^{r} u_i u_i^\top = P.$$

Statement (3) follows directly from the eigen-expansion of $L^{\dagger/2}$.

We record a concrete form of P for graph Laplacians.

Lemma A.2 (Projector for graph Laplacians).
Let $G = (V, E)$ be an undirected graph with connected components $V_1, \dots, V_q$

(a partition of V), and Laplacian L_G. Let $\mathbf{1}_{V_j} \in \mathbb{R}^V$ be the indicator vector of V_j. Then

$$\ker(L_G) = \operatorname{span}\{\mathbf{1}_{V_1}, \dots, \mathbf{1}_{V_q}\},$$

and the orthogonal projector onto $\ker(L_G)^\perp$ is

$$P = I - \sum_{j=1}^{q} \frac{1}{|V_j|} \mathbf{1}_{V_j} \mathbf{1}_{V_j}^\top. \tag{11}$$

In particular, if G is connected ($q = 1$), then

$$P = I - \frac{1}{n}\mathbf{1}\mathbf{1}^\top.$$

Proof. The kernel characterization is standard: $x^\top L_G x = \sum_{\{u,v\}\in E}(x_u - x_v)^2$ vanishes if and only if x is constant on each connected component. The projector formula follows because subtracting each component-wise average produces the orthogonal projection onto the subspace of vectors with zero mean on each component.

A.2 Rank-one edge Laplacians and normalized matrices

Let $G = (V, E)$ be an undirected graph. For an edge $e = \{u, v\}$ define the signed incidence vector

$$b_e := e_u - e_v.$$

The associated rank-one edge Laplacian is

$$L_e := b_e b_e^\top.$$

Note that L_e does not depend on the orientation of e since replacing b_e by $-b_e$ leaves $b_e b_e^\top$ unchanged. We use the decomposition

$$L_G = \sum_{e\in E} L_e.$$

Define the normalized edge vectors and matrices

$$a_e := L_G^{\dagger/2} b_e, \qquad A_e := a_e a_e^\top. \tag{12}$$

Lemma A.3 (Normalized rank-one decomposition).
Let $G = (V, E)$ be any undirected graph (not necessarily connected). With A_e as in Eq. (A.2), we have

$$\sum_{e\in E} A_e = P,$$

where P is the orthogonal projector onto $\ker(L_G)^\perp$.

Proof. Using $L_G = \sum_{e\in E} b_e b_e^\top$ and Lemma A.1(2),

$$\sum_{e\in E} A_e = \sum_{e\in E} L_G^{\dagger/2} b_e b_e^\top L_G^{\dagger/2} = L_G^{\dagger/2} L_G L_G^{\dagger/2} = P.$$

The unique nonzero eigenvalue of A_e is given by the squared norm of a_e.

Lemma A.4 (Edge leverage/effective resistance identities).
For any edge $e = \{u, v\}$,

$$\operatorname{tr}(A_e) = \|a_e\|_2^2 = b_e^\top L_G^\dagger b_e. \tag{13}$$

Moreover, since A_e is rank one,

$$\|A_e\| = \lambda_{\max}(A_e) = \operatorname{tr}(A_e).$$

Proof. By definition $a_e = L_G^{\dagger/2} b_e$, hence $\|a_e\|_2^2 = b_e^\top L_G^\dagger b_e$, giving the first equality. Since $A_e = a_e a_e^\top$ is rank one, its nonzero eigenvalue equals $\|a_e\|_2^2$ and equals both $\operatorname{tr}(A_e)$ and $\|A_e\|$.

A.3 PSD domination via normalization

We now formalize the equivalence between inequalities of the form $A \preceq \beta L_G$ and their normalized counterparts. The key point is that such an inequality forces A to vanish on $\ker(L_G)$.

Lemma A.5 (Kernel constraint under domination).
Let L be symmetric PSD and let A be symmetric PSD. If $A \preceq \beta L$ for some $\beta \geq 0$, then $\ker(L) \subseteq \ker(A)$. *Proof.* Let $x \in \ker(L)$. Then

$$0 \leq x^\top A x \leq \beta x^\top L x = 0,$$

so $x^\top A x = 0$. Since $A \succeq 0$, this implies $Ax = 0$, hence $x \in \ker(A)$. We can therefore restrict attention to $\ker(L)^\perp$.

Lemma A.6 (Normalization equivalence).
Let L be symmetric PSD with projector P onto $\ker(L)^\perp$ as in Lemma A.1, and let A be symmetric PSD such that $\ker(L) \subseteq \ker(A)$. Define the normalized matrix

$$M := L^{\dagger/2} A L^{\dagger/2}.$$

Then for any $\beta \geq 0$,

$$A \preceq \beta L \quad \text{if and only if} \quad M \preceq \beta P. \tag{14}$$

Proof.
Assume first that $A \preceq \beta L$. For any $x \in \mathbb{R}^n$, let $y := L^{\dagger/2} x$. Then

$$x^\top Mx = y^\top Ay \le \beta\, y^\top Ly.$$

Using Lemma A.1(2),

$$y^\top Ly = x^\top L^{\dagger/2} L L^{\dagger/2} x = x^\top Px.$$

Hence $x^\top Mx \le \beta x^\top Px$ for all x, i.e., $M \preceq \beta P$.

Conversely, assume $M \preceq \beta P$. Let $z \in \mathbb{R}^n$ and decompose $z = z_0 + z_1$ with $z_0 \in \ker(L)$ and $z_1 \in \ker(L)^\perp$. Since $\ker(L) \subseteq \ker(A)$ and A is symmetric,

$$z^\top Az = z_1^\top Az_1, \qquad z^\top Lz = z_1^\top Lz_1.$$

It suffices to show $z_1^\top Az_1 \le \beta z_1^\top Lz_1$ for all $z_1 \in \ker(L)^\perp$.

By Lemma A.1(3), $L^{\dagger/2}$ maps onto $\ker(L)^\perp$, so choose $x \in \ker(L)^\perp$ such that $z_1 = L^{\dagger/2}x$. Then $Px = x$ and

$$z_1^\top Az_1 = x^\top L^{\dagger/2} A L^{\dagger/2} x = x^\top Mx \le \beta x^\top Px = \beta x^\top x.$$

On the other hand,

$$z_1^\top Lz_1 = x^\top L^{\dagger/2} L L^{\dagger/2} x = x^\top Px = x^\top x,$$

again by Lemma A.1(2) and $Px = x$. Thus $z_1^\top Az_1 \le \beta z_1^\top Lz_1$, proving $A \preceq \beta L$.

Applying Lemma A.6 with $L = L_G$ and $A = \sum_{e\in F} L_e$ yields the equivalence used in the main text:

$$\sum_{e\in F} L_e \preceq \beta L_G \quad\Longleftrightarrow\quad \sum_{e\in F} A_e \preceq \beta P.$$

A.4 Rayleigh quotient and generalized eigenvalues

The normalized matrix $M = L^{\dagger/2} A L^{\dagger/2}$ captures the generalized eigenvalues of the pair (A, L) on $\ker(L)^\perp$.

Lemma A.7 (Rayleigh quotient and $\lambda_{\max}$).
Let L be symmetric PSD and let A be symmetric PSD with $\ker(L) \subseteq \ker(A)$. Define

$$M := L^{\dagger/2} A L^{\dagger/2}.$$

Then

$$\lambda_{\max}(M|_{\ker(L)^\perp}) = \sup_{\substack{x\in\mathbb{R}^n\\ x\perp\ker(L)\\ x\neq 0}} \frac{x^\top Ax}{x^\top Lx}. \tag{15}$$

Proof. Consider $x \in \ker(L)^\perp$. Let $y := L^{1/2}x$, where $L^{1/2}$ is defined by the spectral theorem and is invertible on $\ker(L)^\perp$. Then

$$\frac{x^\top Ax}{x^\top Lx} = \frac{x^\top Ax}{\|y\|_2^2}.$$

Write $x = L^{\dagger/2}y$ on $\ker(L)^\perp$ (since $L^{\dagger/2}$ is the inverse of $L^{1/2}$ there). Then

$$x^\top Ax = y^\top L^{\dagger/2}AL^{\dagger/2}y = y^\top My,$$

and $\|y\|_2^2 = y^\top y$. Hence

$$\sup_{x\perp\ker(L),\, x\neq 0} \frac{x^\top Ax}{x^\top Lx} = \sup_{y\perp\ker(L),\, y\neq 0} \frac{y^\top My}{y^\top y} = \lambda_{\max}(M|_{\ker(L)^\perp}).$$

When $A = \sum_{e\in F} L_e$ for a set of edges F, we have $M = \sum_{e\in F} A_e$. Thus Lemma A.7 yields the equivalence between the domination $A \preceq \beta L$ and a bound on the largest generalized Rayleigh quotient.

A.5 Disconnected graphs and block-diagonal reduction

We record a simple reduction used in the main proof: the inequality $L_{G[S]} \preceq \varepsilon L_G$ is compatible with connected-component decompositions.

Lemma A.8 (Component-wise characterization).
Let G have connected components $G_1, \dots, G_q$ on vertex sets $V_1, \dots, V_q$. Let $S \subseteq V$ and write $S_j := S \cap V_j$. Then

$$L_{G[S]} \preceq \varepsilon L_G \quad \text{if and only if} \quad L_{G_j[S_j]} \preceq \varepsilon L_{G_j} \text{ for all } j \in \{1, \dots, q\}.$$

Proof. With respect to the decomposition $\mathbb{R}^V = \bigoplus_{j=1}^q \mathbb{R}^{V_j}$, both L_G and $L_{G[S]}$ are block diagonal:

$$L_G = \operatorname{diag}(L_{G_1}, \dots, L_{G_q}), \qquad L_{G[S]} = \operatorname{diag}(L_{G_1[S_1]}, \dots, L_{G_q[S_q]}).$$

A block diagonal matrix is PSD if and only if each diagonal block is PSD, so $\varepsilon L_G - L_{G[S]} \succeq 0$ holds if and only if $\varepsilon L_{G_j} - L_{G_j[S_j]} \succeq 0$ for all j. —

Appendix B. Effective resistance and heavy-edge bounds

This appendix develops the effective-resistance (leverage) quantities used in the heavy–light decomposition and proves the structural properties of the heavy-edge graph required in §5 of the main text. Throughout this appendix, $G = (V, E)$ is a finite undirected *unweighted* graph with Laplacian $L := L_G$. We write $n := |V|$.

B.1 Leverage scores and basic bounds

Recall from Appendix A that for an edge $e = \{u, v\}$ we write $b_e := e_u - e_v$, $L_e := b_e b_e^\top$, and define the (edge) leverage score

$$\ell_e := b_e^\top L^\dagger b_e.$$

When G is connected, ℓ_e coincides with the effective resistance between u and v in the unit-resistance electrical network on G.

Definition B.1 (Edge leverage / effective resistance).
For an edge $e = \{u, v\} \in E$, define

$$\ell_e := (e_u - e_v)^\top L^\dagger (e_u - e_v).$$

When G is connected, we also write $R_{\text{eff}}(u, v) := \ell_{\{u,v\}}$. We record two elementary bounds.

Lemma B.2 (Bounds $0 < \ell_e \leq 1$ for present edges).
Assume G is connected. Then for every edge $e = \{u, v\} \in E$,

$$0 < \ell_e \leq 1.$$

Proof. The strict positivity $\ell_e > 0$ holds because $b_e \notin \ker(L)$ in a connected graph and $L^\dagger$ is positive definite on $\mathbf{1}^\perp = \ker(L)^\perp$. For the upper bound, interpret ℓ_e as the effective resistance between u and v in the electrical network where each edge has resistance 1 (conductance 1). Since the edge $\{u, v\}$ itself provides a direct connection of resistance 1, Rayleigh monotonicity (adding more parallel connections cannot increase effective resistance) yields $R_{\text{eff}}(u, v) \leq 1$. In disconnected graphs, all edges lie within components, and the leverage is computed component-wise.

Lemma B.3 (Component-wise leverage).
Let G have connected components $G_1, \dots, G_q$ on vertex sets $V_1, \dots, V_q$. Then $L^\dagger$ is block diagonal with blocks $L^\dagger_{G_j}$, and for any edge $e \in E(G_j)$ we have

$$\ell_e(G) = \ell_e(G_j).$$

Proof. The Laplacian is block diagonal:

$$L_G = \operatorname{diag}(L_{G_1}, \dots, L_{G_q}),$$

and the Moore–Penrose pseudoinverse of a block diagonal matrix is block diagonal with pseudoinverses of the blocks. The claim follows immediately since b_e is supported on a single component.

B.2 Foster's theorem and global heavy-edge counting

The next identity is the (unweighted) Foster sum rule. We prove it directly from the normalized rank-one decomposition in Appendix A.

Lemma B.4 (Foster sum rule via trace).
Let G have connected components $V_1, \dots, V_q$. Then

$$\sum_{e \in E} \ell_e = n - q.$$

In particular, if G is connected, then $\sum_{e \in E} \ell_e = n - 1$. *Proof.* By Lemma A.3, with $A_e := L^{\dagger/2} L_e L^{\dagger/2}$ we have $\sum_{e \in E} A_e = P$, where P is the orthogonal projector onto $\ker(L)^{\perp}$. Taking traces,

$$\sum_{e \in E} \operatorname{tr}(A_e) = \operatorname{tr}(P).$$

By Lemma A.4, $\operatorname{tr}(A_e) = \ell_e$. Moreover $\operatorname{tr}(P) = \operatorname{rank}(P) = \dim(\ker(L)^{\perp}) = n - q$ (Lemma A.2). As an immediate corollary, large leverage edges are globally sparse.

Corollary B.5 (Global heavy-edge count).
Fix $\tau > 0$ and define

$$E_{\text{heavy}}(\tau) := \{e \in E : \ \ell_e > \tau\}.$$

Then

$$|E_{\text{heavy}}(\tau)| \leq \frac{n - q}{\tau},$$

where q is the number of connected components of G. *Proof.* Since every $e \in E_{\text{heavy}}(\tau)$ satisfies $\ell_e > \tau$,

$$\tau \, |E_{\text{heavy}}(\tau)| < \sum_{e \in E_{\text{heavy}}(\tau)} \ell_e \leq \sum_{e \in E} \ell_e = n - q,$$

using Lemma B.4. Corollary B.5 alone does not control the chromatic number of the heavy-edge graph; for that we need a *local* density bound, which we derive next.

B.3 A local sum bound via spanning trees

To control local densities of large-leverage edges, we use a standard spanning-tree interpretation of effective resistance. We first recall the weighted spanning tree distribution (for completeness). Although our graphs are unweighted, the weighted formulation provides a clean proof. Let G be connected and let each edge e have a positive weight $w_e > 0$ (conductance). The weighted Laplacian is

$$L(w) := \sum_{e \in E} w_e \, L_e.$$

A weighted random spanning tree T is drawn with probability proportional to $\prod_{e \in T} w_e$.

Lemma B.6 (Edge inclusion probability equals $w_e R_{\text{eff}}$).
Assume G is connected and weighted by $w = \{w_e\}_{e\in E}$ with $w_e > 0$. Let T be a random spanning tree drawn with probability proportional to $\prod_{f\in T} w_f$. Then for every edge $e \in E$,

$$\Pr[e \in T] = w_e \cdot (b_e^\top L(w)^\dagger b_e).$$

In particular, in the unweighted case $w_e \equiv 1$ we have $\Pr[e \in T] = \ell_e$. *Proof.* Fix a root vertex $r \in V$. Let $L_r(w)$ denote the $(n-1)\times(n-1)$ principal minor obtained from $L(w)$ by deleting the row and column indexed by r. By the (weighted) matrix-tree theorem,

$$\tau(w) := \sum_{\text{spanning trees } T} \prod_{f\in T} w_f = \det(L_r(w)).$$

On the other hand, by differentiating $\tau(w)$ with respect to w_e,

$$\frac{\partial \tau(w)}{\partial w_e} = \sum_{T\ni e} \frac{\partial}{\partial w_e}\Big(\prod_{f\in T} w_f\Big) = \sum_{T\ni e} \Big(\prod_{f\in T} w_f\Big)\frac{1}{w_e} = \frac{1}{w_e}\,\tau(w)\,\Pr[e\in T].$$

Equivalently,

$$\Pr[e \in T] = w_e \frac{\partial}{\partial w_e} \log \tau(w).$$

Using $\tau(w) = \det(L_r(w))$ and the identity $\frac{\partial}{\partial t} \log\det(M(t)) = \operatorname{tr}(M(t)^{-1}M'(t))$ for differentiable families of invertible matrices,

$$\frac{\partial}{\partial w_e} \log \tau(w) = \operatorname{tr}\Big(L_r(w)^{-1}\,\frac{\partial L_r(w)}{\partial w_e}\Big).$$

Since $L(w) = \sum_f w_f\, b_f b_f^\top$, we have $\frac{\partial L(w)}{\partial w_e} = b_e b_e^\top$, and therefore $\frac{\partial L_r(w)}{\partial w_e} = b_{e,r} b_{e,r}^\top$, where $b_{e,r}$ denotes the incidence vector b_e restricted to coordinates in $V\setminus\{r\}$. Hence

$$\frac{\partial}{\partial w_e} \log \tau(w) = \operatorname{tr}(L_r(w)^{-1} b_{e,r} b_{e,r}^\top) = b_{e,r}^\top L_r(w)^{-1} b_{e,r}.$$

It remains to identify $b_{e,r}^\top L_r(w)^{-1} b_{e,r}$ with $b_e^\top L(w)^\dagger b_e$. This is standard: for any vector b with $\mathbf{1}^\top b = 0$, the quantity $b^\top L(w)^\dagger b$ equals the unique value $b^\top x$ where x solves $L(w)x = b$ (solutions are defined up to additive constants), and the gauge choice $x_r = 0$ yields $x_{-r} = L_r(w)^{-1} b_r$ and thus $b^\top x = b_r^\top L_r(w)^{-1} b_r$. Applying this with $b = b_e$ gives

$$b_e^\top L(w)^\dagger b_e = b_{e,r}^\top L_r(w)^{-1} b_{e,r}.$$

Combining the displayed equalities completes the proof. We now use Lemma B.6 to derive a local sum bound.

Lemma B.7 (Local sum bound on induced edges).
Assume G is connected and unweighted. Then for every subset $U \subseteq V$ with $|U| \geq 1$,

$$\sum_{e\in E(U,U)} \ell_e \leq |U| - 1.$$

Proof. Let T be a uniform random spanning tree of G. By Lemma B.6 in the unweighted case, $\Pr[e \in T] = \ell_e$. Therefore,

$$\sum_{e \in E(U,U)} \ell_e = \sum_{e \in E(U,U)} \Pr[e \in T] = \mathbb{E}\big[\, |T \cap E(U,U)| \,\big].$$

For any fixed spanning tree T, the edge set $T \cap E(U,U)$ forms a forest on the vertex set U, hence it contains at most $|U| - 1$ edges. Taking expectations yields the claimed inequality. *Remark.* If G is disconnected, apply Lemma B.7 component-wise using Lemma B.3. A slightly sharper bound $\sum_{e \in E(U,U)} \ell_e \leq |U| - c(U)$ holds with $c(U)$ the number of connected components of $G[U]$, but $|U| - 1$ suffices for our purposes.

B.4 Arboricity, degeneracy, and k-colorability of the heavy-edge graph

Fix $\tau > 0$ and define the heavy-edge graph

$$G_{\text{heavy}}(\tau) := (V, E_{\text{heavy}}(\tau)), \qquad E_{\text{heavy}}(\tau) := \{e \in E : \ \ell_e > \tau\}.$$

We show that $G_{\text{heavy}}(\tau)$ has bounded arboricity (hence bounded degeneracy), and therefore admits a proper coloring with $O(1/\tau)$ colors. This is the structural input needed for Proposition 5.4 in the main text. We recall the standard definition.

Definition B.8 (Arboricity).
For a graph $H = (V_H, E_H)$ with $|V_H| \geq 2$, its arboricity is

$$\operatorname{arb}(H) := \max_{\substack{U \subseteq V_H \\ |U| \geq 2}} \frac{|E_H(U,U)|}{|U| - 1}.$$

Lemma B.9 (Heavy-edge arboricity bound).
Assume G is connected. Then for every $\tau > 0$,

$$\operatorname{arb}(G_{\text{heavy}}(\tau)) \leq \frac{1}{\tau}.$$

Proof. Fix $U \subseteq V$ with $|U| \geq 2$. Every edge in $E_{\text{heavy}}(\tau) \cap E(U,U)$ has leverage $> \tau$, hence

$$\tau \, |E_{\text{heavy}}(\tau) \cap E(U,U)| < \sum_{e \in E_{\text{heavy}}(\tau) \cap E(U,U)} \ell_e \leq \sum_{e \in E(U,U)} \ell_e.$$

By Lemma B.7, $\sum_{e \in E(U,U)} \ell_e \leq |U| - 1$. Therefore

$$|E_{\text{heavy}}(\tau) \cap E(U,U)| \leq \frac{|U| - 1}{\tau}.$$

Dividing by $|U| - 1$ and maximizing over U yields $\operatorname{arb}(G_{\text{heavy}}(\tau)) \leq 1/\tau$. A bounded arboricity immediately implies bounded degeneracy.

Lemma B.10 (Arboricity implies degeneracy and colorability).
Let H be a graph with arboricity $\mathrm{arb}(H) \le a$. Then H is $\lceil 2a \rceil$-colorable. *Proof.* Consider any nonempty subgraph $H' = (V', E')$ of H with $|V'| \ge 2$. By the definition of arboricity,

$$|E'| \le a(|V'| - 1),$$

so the average degree in H' satisfies

$$\frac{1}{|V'|} \sum_{v \in V'} \deg_{H'}(v) = \frac{2|E'|}{|V'|} \le \frac{2a(|V'| - 1)}{|V'|} < 2a.$$

Therefore H' contains a vertex of degree at most $\lceil 2a \rceil - 1$. This holds for every subgraph, so H is $(\lceil 2a \rceil - 1)$-degenerate. A (d)-degenerate graph admits a proper $(d+1)$-coloring by greedy coloring in a degeneracy ordering, hence H is $\lceil 2a \rceil$-colorable. Combining Lemmas B.9–B.10 yields a concrete coloring bound for the heavy-edge graph.

Corollary B.11 (Heavy-edge graph is $O(1/\tau)$-colorable).
Assume G is connected. Then $G_{\mathrm{heavy}}(\tau)$ is $\lceil 2/\tau \rceil$-colorable. *Proof.* By Lemma B.9, $\mathrm{arb}(G_{\mathrm{heavy}}(\tau)) \le 1/\tau$. Apply Lemma B.10 with $a = 1/\tau$. Finally, we instantiate the parameter choice used in the main text: τ proportional to $1/k$.

Proposition B.12 (Feasibility of admissible k-colorings).
There exists an absolute constant $\alpha_0 > 0$ such that the following holds. Let $k \ge 2$ and set

$$\tau := \frac{\alpha_0}{k}.$$

Then the heavy-edge graph $G_{\mathrm{heavy}}(\tau)$ is properly k-colorable. In particular, admissible k-colorings (Definition 5.3 of the main text) exist. *Proof.* Choose $\alpha_0 := 4$. If $\tau \ge 1$, then by Lemma B.2 we have $\ell_e \le 1$ for all edges, hence $E_{\mathrm{heavy}}(\tau) = \emptyset$ and the claim is trivial. If $\tau < 1$, then by Corollary B.11, $G_{\mathrm{heavy}}(\tau)$ is $\lceil 2/\tau \rceil$-colorable. With $\tau = 4/k$,

$$\left\lceil \frac{2}{\tau} \right\rceil = \left\lceil \frac{2}{4/k} \right\rceil = \left\lceil \frac{k}{2} \right\rceil \le k.$$

Thus $G_{\mathrm{heavy}}(\tau)$ admits a proper k-coloring, i.e., an admissible k-coloring exists. —

Appendix C. Coloring reduction and extraction of a large ε-light set

This appendix supplies the detailed proof of Proposition 3.1 from the main text and records a component-wise assembly principle used for disconnected graphs.

C.1 Proof of Proposition 3.1

We restate Proposition 3.1 for convenience.

Proposition C.1 (Monochromatic control implies a large ε-light color class).
Let $G = (V, E)$ be an undirected graph with Laplacian L_G, and let $c : V \to [k]$ be a k-coloring with color classes $V_i := \{v \in V : \ c(v) = i\}$. Define

$$E_{\mathrm{mono}}(c) := \{\{u, v\} \in E : \ c(u) = c(v)\}, \qquad L_{\mathrm{mono}}(c) := \sum_{e \in E_{\mathrm{mono}}(c)} L_e.$$

Assume that for some $C > 0$,

$$L_{\mathrm{mono}}(c) \preceq \frac{C}{k} L_G.$$

Let S be a largest color class. Then $|S| \geq |V|/k$ and

$$L_{G[S]} \preceq \frac{C}{k} L_G.$$

Proof.
We first relate $L_{\mathrm{mono}}(c)$ to the induced-subgraph Laplacians of the color classes. For each $i \in [k]$, the induced-subgraph Laplacian embedded in $\mathbb{R}^{V \times V}$ is

$$L_{G[V_i]} = \sum_{e \in E(V_i, V_i)} L_e,$$

where $E(V_i, V_i)$ denotes the set of edges with both endpoints in V_i. By definition, the monochromatic edge set decomposes as a disjoint union

$$E_{\mathrm{mono}}(c) = \bigsqcup_{i=1}^{k} E(V_i, V_i),$$

hence

$$L_{\mathrm{mono}}(c) = \sum_{e \in E_{\mathrm{mono}}(c)} L_e = \sum_{i=1}^{k} \sum_{e \in E(V_i, V_i)} L_e = \sum_{i=1}^{k} L_{G[V_i]}.$$

Since each $L_{G[V_i]}$ is a sum of rank-one PSD matrices $L_e = b_e b_e^\top \succeq 0$, we have $L_{G[V_i]} \succeq 0$ and, as a partial sum of the PSD sum defining $L_{\mathrm{mono}}(c)$,

$$L_{G[V_i]} \preceq L_{\mathrm{mono}}(c) \qquad \text{for every } i \in [k].$$

Let $i^\star \in [k]$ be such that $|V_{i^\star}| = \max_{i \in [k]} |V_i|$, and set $S := V_{i^\star}$. Then by the pigeonhole principle,

$$|S| = \max_{i \in [k]} |V_i| \geq \frac{1}{k} \sum_{i=1}^{k} |V_i| = \frac{|V|}{k}.$$

Finally,

$$L_{G[S]} = L_{G[V_{i^\star}]} \preceq L_{\mathrm{mono}}(c) \preceq \frac{C}{k} L_G,$$

which proves the claim.

C.2 Disconnected graphs: assembling light sets across components

The main proof of Theorem 1.1 treats connected graphs first and then reduces the general case to connected components. The following lemma records the component-wise assembly principle.

Lemma C.2 (Union of component-wise light sets is light).
Let $G = (V, E)$ have connected components $G_1, \dots, G_q$ with vertex sets $V_1, \dots, V_q$. Fix $\varepsilon \geq 0$. For each $j \in \{1, \dots, q\}$ let $S_j \subseteq V_j$ and set

$$S := \bigcup_{j=1}^{q} S_j.$$

Then

$$L_{G[S]} \preceq \varepsilon L_G \quad \text{if and only if} \quad L_{G_j[S_j]} \preceq \varepsilon L_{G_j} \text{ for all } j \in \{1, \dots, q\}.$$

Moreover,

$$|S| = \sum_{j=1}^{q} |S_j|.$$

Proof.
Since G has no edges between distinct components, both L_G and $L_{G[S]}$ are block diagonal with respect to the orthogonal decomposition

$$\mathbb{R}^V = \bigoplus_{j=1}^{q} \mathbb{R}^{V_j}.$$

More precisely,

$$L_G = \operatorname{diag}(L_{G_1}, \dots, L_{G_q}), \qquad L_{G[S]} = \operatorname{diag}(L_{G_1[S_1]}, \dots, L_{G_q[S_q]}).$$

Therefore,

$$\varepsilon L_G - L_{G[S]} = \operatorname{diag}(\varepsilon L_{G_1} - L_{G_1[S_1]}, \dots, \varepsilon L_{G_q} - L_{G_q[S_q]}).$$

A block diagonal matrix is PSD if and only if each diagonal block is PSD, which yields the equivalence. The cardinality identity follows from the disjointness of the vertex partition $V = \bigsqcup_{j=1}^{q} V_j$. *Remark.* Proposition C.1 itself does not require connectivity: if a coloring c satisfies $L_{\text{mono}}(c) \preceq (C/k)L_G$ on a disconnected graph, then the extraction of a largest color class still yields a set S with $|S| \geq |V|/k$ and $L_{G[S]} \preceq (C/k)L_G$. In the main proof we instead construct light sets component-wise and combine them via Lemma C.2, which avoids managing colorings simultaneously across components. —

Appendix D. Determinant/Gram expansions and the "only forests matter" lemma

This appendix supplies the detailed proof of Theorem 6.1 from the main text. The proof has two parts: 1. A determinant expansion for $\det(\lambda I - \sum_e x_e A_e)$ in terms of Gram determinants of the vectors generating the rank-one matrices A_e. 2. A graph-theoretic linear-independence criterion showing that the relevant Gram determinants vanish unless the chosen edge set is a forest. Throughout, $G = (V, E)$ is an undirected graph with Laplacian $L = L_G$. For each edge $e = \{u, v\}$ we use the signed incidence vector $b_e := e_u - e_v$, the edge Laplacian $L_e = b_e b_e^\top$, and the normalized edge vectors $a_e := L^{\dagger/2} b_e$ as in Appendix A.

D.1 Determinant expansions for rank-one sums

We begin with two standard linear-algebra identities, stated and proved for completeness.

Lemma D.1 (Sylvester's determinant identity).
Let $U \in \mathbb{R}^{d\times m}$ and $V \in \mathbb{R}^{m\times d}$. Then

$$\det(I_d + UV) = \det(I_m + VU).$$

Proof. Consider the block matrix

$$M := \begin{pmatrix} I_d & U \\ -V & I_m \end{pmatrix}.$$

Using the Schur complement with respect to the lower-right block I_m gives

$$\det(M) = \det(I_m)\ \det(I_d + UV) = \det(I_d + UV).$$

Using the Schur complement with respect to the upper-left block I_d gives

$$\det(M) = \det(I_d)\ \det(I_m + VU) = \det(I_m + VU).$$

Equating the two expressions yields the claim. Next, we record an expansion of $\det(I + tM)$ in terms of principal minors.

Lemma D.2 (Principal-minor expansion).
Let $M \in \mathbb{R}^{m\times m}$ and let t be a commuting indeterminate. Then

$$\det(I_m + tM) = \sum_{F\subseteq[m]} t^{|F|}\ \det(M_{F,F}),$$

where $M_{F,F}$ denotes the principal submatrix indexed by F. In particular,

$$\det(I_m - tM) = \sum_{F\subseteq[m]} (-t)^{|F|}\ \det(M_{F,F}).$$

Proof. Expand the determinant by multilinearity in the columns of $I_m + tM$. For each subset $F \subseteq [m]$, choose the jth column from tM if $j \in F$, and from

I_m otherwise. The determinant of the resulting matrix equals $t^{|F|} \det(M_{F,F})$ because the columns from I_m fix the corresponding rows and columns and reduce the determinant to the principal minor on F. Summing over all F yields the identity. We now apply Lemmas D.1–D.2 to a Gram-type rank-one sum.

Proposition D.3 (Gram/Cauchy–Binet determinant expansion). Let $a_1, \dots, a_m \in \mathbb{R}^d$ and define rank-one matrices $A_i := a_i a_i^\top$. Let $x_1, \dots, x_m$ be commuting indeterminates and set

$$M(x) := \sum_{i=1}^{m} x_i A_i.$$

Then

$$\det(\lambda I_d - M(x)) = \sum_{F \subseteq [m]} (-1)^{|F|} \lambda^{d-|F|} \det\bigl(\mathrm{Gram}(a_i)_{i \in F}\bigr) \prod_{i \in F} x_i,$$

where $\mathrm{Gram}(a_i)_{i \in F}$ is the $|F| \times |F|$ Gram matrix with entries $(a_i^\top a_j)_{i,j \in F}$. *Proof.* Let $A \in \mathbb{R}^{d \times m}$ be the matrix with columns $a_1, \dots, a_m$, and let $X := \mathrm{diag}(x_1, \dots, x_m)$. Then

$$M(x) = AXA^\top.$$

By Lemma D.1 applied to $U = -A$ and $V = (1/\lambda) XA^\top$, we have

$$\det\Bigl(I_d - \frac{1}{\lambda} AXA^\top\Bigr) = \det\Bigl(I_m - \frac{1}{\lambda} XA^\top A\Bigr).$$

Multiplying by λ^d gives

$$\det(\lambda I_d - AXA^\top) = \lambda^d \det\Bigl(I_m - \frac{1}{\lambda} XG\Bigr),$$

where $G := A^\top A$ is the $m \times m$ Gram matrix of $(a_i)_{i=1}^m$. Now apply Lemma D.2 with $M = XG$ and $t = 1/\lambda$:

$$\det\Bigl(I_m - \frac{1}{\lambda} XG\Bigr) = \sum_{F \subseteq [m]} (-1)^{|F|} \lambda^{-|F|} \det((XG)_{F,F}).$$

For each F, since X is diagonal,

$$(XG)_{F,F} = X_{F,F} G_{F,F},$$

and thus

$$\det((XG)_{F,F}) = \det(X_{F,F}) \det(G_{F,F}) = \Bigl(\prod_{i \in F} x_i\Bigr) \det(G_{F,F}).$$

Finally, $G_{F,F} = \mathrm{Gram}(a_i)_{i \in F}$ by definition. Substituting into the previous display and multiplying by λ^d yields the claimed expansion. *Remark.* If $|F| > d$, then $\mathrm{Gram}(a_i)_{i \in F}$ has rank at most d and hence determinant 0. Thus the expansion is effectively supported on $|F| \le d$.

D.2 Incidence vectors, cycles, and forests

We now specialize Proposition D.3 to the graph-normalized vectors $a_e = L^{\dagger/2} b_e$. The key graph-theoretic input is that incidence vectors on an edge set are linearly independent if and only if the edge set is a forest. We first record the cycle dependence relation.

Lemma D.4 (Cycle dependence of incidence vectors).
Let $F \subseteq E$ contain a (simple) cycle $C \subseteq F$. Then the set of incidence vectors $\{b_e\}_{e \in F}$ is linearly dependent. Consequently, for the normalized vectors $a_e = L^{\dagger/2} b_e$, the set $\{a_e\}_{e \in F}$ is also linearly dependent and

$$\det(\mathrm{Gram}(a_e)_{e \in F}) = 0.$$

Proof. Let the cycle C have vertices $v_1, \dots, v_r$ with edges $\{v_i, v_{i+1}\}$ for $i = 1, \dots, r$ (where $v_{r+1} = v_1$). Orient each cycle edge as $v_i \to v_{i+1}$, and write the corresponding incidence vector as

$$\tilde{b}_i := e_{v_i} - e_{v_{i+1}}.$$

Then telescoping yields

$$\sum_{i=1}^{r} \tilde{b}_i = \sum_{i=1}^{r} (e_{v_i} - e_{v_{i+1}}) = 0.$$

Each $\tilde{b}_i$ equals either b_e or $-b_e$ for the corresponding undirected edge $e \in C$. Hence a nontrivial $\{\pm 1\}$-linear combination of $\{b_e\}_{e \in C} \subseteq \{b_e\}_{e \in F}$ equals 0, so $\{b_e\}_{e \in F}$ is linearly dependent. Next, note that for every edge $e = \{u, v\}$, b_e is orthogonal to each connected-component indicator vector (Appendix A, Lemma A.2), hence $b_e \in \ker(L)^{\perp}$. By Lemma A.1, $L^{\dagger/2}$ restricts to an invertible linear map on $\ker(L)^{\perp}$. Therefore, applying $L^{\dagger/2}$ preserves linear dependence relations among the b_e's, and $\{a_e\}_{e \in F}$ is linearly dependent as well. Finally, if the vectors $\{a_e\}_{e \in F}$ are linearly dependent, their Gram matrix is singular, hence its determinant is 0. We now show the converse: forests yield linear independence.

Lemma D.5 (Forest independence of incidence vectors).
Let $F \subseteq E$ be a forest (i.e., the graph $(V(F), F)$ is acyclic). Then the set of incidence vectors $\{b_e\}_{e \in F}$ is linearly independent. Consequently, $\{a_e\}_{e \in F}$ is linearly independent and

$$\det(\mathrm{Gram}(a_e)_{e \in F}) > 0.$$

Proof. We prove the linear independence of $\{b_e\}_{e \in F}$ by induction on $|F|$. If $|F| = 0$, the statement is trivial. Assume $|F| \geq 1$ and let $H := (V(F), F)$ be the forest. Then H has a leaf vertex $w \in V(F)$ of degree 1 in H. Let $e_0 \in F$ be the unique edge of H incident to w. Consider a linear combination

$$\sum_{e \in F} \alpha_e b_e = 0.$$

Inspect the coordinate indexed by w. For every edge $e \in F \setminus \{e_0\}$, the vector b_e has 0 in the w-coordinate because w is incident to no other edge in H. The vector b_{e_0} has $+1$ or -1 in the w-coordinate depending on its orientation. Therefore the w-coordinate of the left-hand side equals $\pm\alpha_{e_0}$, and must be 0. Hence $\alpha_{e_0} = 0$. Removing e_0 from F yields a smaller forest $F' := F \setminus \{e_0\}$. Restricting the original linear relation to F' shows $\sum_{e \in F'} \alpha_e b_e = 0$. By induction, all $\alpha_e = 0$ for $e \in F'$, and thus all $\alpha_e = 0$ for $e \in F$. This proves that $\{b_e\}_{e \in F}$ is linearly independent. As in Lemma D.4, each $b_e \in \ker(L)^\perp$ and $L^{\dagger/2}$ is invertible on $\ker(L)^\perp$, so $\{a_e\}_{e \in F}$ is linearly independent. The Gram matrix of a linearly independent family is positive definite, hence has strictly positive determinant.

Combining Lemmas D.4–D.5 yields the forest criterion for nonvanishing Gram determinants.

Corollary D.6 (Forest criterion for Gram determinants).
Let $F \subseteq E$. Then

$$\det\big(\mathrm{Gram}(a_e)_{e \in F}\big) \neq 0 \quad \text{if and only if} \quad F \text{ is a forest and } |F| \leq d,$$

where $d := \dim(\ker(L)^\perp) = \mathrm{rank}(L)$. *Proof.* If F contains a cycle, Lemma D.4 gives determinant 0. If F is a forest, Lemma D.5 gives determinant > 0 provided $|F| \leq d$. If $|F| > d$, then the vectors $\{a_e\}_{e \in F} \subseteq \mathbb{R}^d$ are necessarily linearly dependent, so the Gram determinant is 0.

D.3 Proof of Theorem 6.1 (main text)

We now prove Theorem 6.1 from the main text. *Proof of Theorem 6.1.*
Fix any orthonormal basis of $\ker(L)^\perp$ and represent the operators $A_e = a_e a_e^\top$ as $d \times d$ matrices on this space, where $d = \mathrm{rank}(L)$. Apply Proposition D.3 with the family $\{a_e\}_{e \in E}$ and indeterminates $\{x_e\}_{e \in E}$:

$$\det\Big(\lambda I_d - \sum_{e \in E} x_e A_e\Big) = \sum_{F \subseteq E} (-1)^{|F|} \lambda^{d-|F|} \det\big(\mathrm{Gram}(a_e)_{e \in F}\big) \prod_{e \in F} x_e.$$

By Corollary D.6, $\det(\mathrm{Gram}(a_e)_{e \in F}) = 0$ whenever F contains a cycle, i.e., whenever F is not a forest. Therefore only forests contribute nontrivially to the expansion. This is exactly the "forest-supported" determinant expansion asserted in Theorem 6.1. —

Appendix E. Random admissible colorings and forest moments

This appendix provides the combinatorial inputs about random colorings used in §6–§7 of the main text. It has three goals: 1. Compute monochromatic-edge moments on forests under an *unconstrained* independent random k-coloring. 2. Formalize a sequential procedure that generates *admissible* colorings (proper on the heavy-edge graph), and record the basic conditional-expectation averaging identity (used only in the optional algorithmic discussion; the main proof uses

Route B). 3. Combine Appendix D's forest-supported determinant expansion with the forest moments to obtain an explicit expected characteristic polynomial for the unconstrained model (and its light-edge restriction). Throughout, $G = (V, E)$ is an undirected graph, $n := |V|$, and $k \geq 2$ is an integer. The normalized rank-one matrices A_e are as in Appendix A:

$$A_e := L_G^{\dagger/2} L_e L_G^{\dagger/2} = a_e a_e^\top, \qquad a_e := L_G^{\dagger/2}(e_u - e_v) \text{ for } e = \{u, v\}.$$

E.1 Unconstrained random k-colorings: forest monochromatic probability

Definition E.1 (Unconstrained random coloring and monochromatic indicators).
An unconstrained random k-coloring is a random map $c : V \to [k]$ such that the random variables $\{c(v)\}_{v\in V}$ are independent and each is uniform on $[k] = \{1, \dots, k\}$. For an edge $e = \{u, v\} \in E$ define the monochromatic indicator

$$X_e(c) := \mathbf{1}[c(u) = c(v)].$$

We will frequently consider products $\prod_{e\in F} X_e(c)$ for edge sets $F \subseteq E$.

Lemma E.2 (Moment formula for equality constraints).
Let $H = (U, F)$ be any finite undirected graph with vertex set U and edge set F. Let $c : U \to [k]$ be an unconstrained random k-coloring (Definition E.1). Then

$$\mathbb{E}\Big[\prod_{e\in F} \mathbf{1}[c \text{ is constant on the endpoints of } e]\Big] = k^{\kappa(H)-|U|},$$

where $\kappa(H)$ denotes the number of connected components of H. *Proof.* The event $\prod_{e\in F} \mathbf{1}[c \text{ is constant on endpoints of } e] = 1$ is exactly the event that c is constant on each connected component of H (since equality constraints along edges propagate within each component). There are $k^{|U|}$ possible colorings $c : U \to [k]$ total, all equally likely. The number of colorings that are constant on each component is $k^{\kappa(H)}$ (choose one color per component). Therefore the probability of this event is $k^{\kappa(H)}/k^{|U|} = k^{\kappa(H)-|U|}$, and this equals the displayed expectation. The forest specialization is the key case.

Corollary E.3 (Forest monochromatic moment).
Let F be a forest (acyclic edge set) on vertex set $U := V(F)$. Under an unconstrained random k-coloring on U,

$$\mathbb{E}\Big[\prod_{e\in F} X_e(c)\Big] = \Big(\frac{1}{k}\Big)^{|F|}.$$

Proof. If F is a forest on U with $\kappa := \kappa(U, F)$ components, then $|F| = |U| - \kappa$. Apply Lemma E.2 to obtain

$$\mathbb{E}\Big[\prod_{e\in F} X_e(c)\Big] = k^{\kappa-|U|} = k^{-(|U|-\kappa)} = k^{-|F|}.$$

E.2 Sequential admissible colorings and the averaging identity

We now formalize a sequential procedure that produces colorings which are proper on the heavy-edge graph. This procedure is convenient for sampling admissible colorings and for the optional deterministic certification discussion in Appendix I. It is not used for the Route B interlacing argument of §7, which avoids vertex-exposure conditional expectation polynomials (Appendix J). Let $\tau > 0$ be a threshold and define the heavy-edge set

$$E_{\text{heavy}}(\tau) := \{e \in E : \ \ell_e > \tau\},$$

where $\ell_e = b_e^\top L_G^\dagger b_e$ is the leverage/effective resistance (Appendix B). Let

$$H := G_{\text{heavy}}(\tau) := (V, E_{\text{heavy}}(\tau)).$$

A k-coloring $c : V \to [k]$ is admissible if it is a proper coloring of H, i.e., $c(u) \neq c(v)$ for all $uv \in E_{\text{heavy}}(\tau)$. To sample admissible colorings we will use a degeneracy ordering.

Lemma E.4 (Degeneracy ordering for the heavy-edge graph).
Assume G is connected. Fix $\tau > 0$ and set $H := G_{\text{heavy}}(\tau)$. Then H has arboricity at most $1/\tau$ (Lemma B.9), hence is $(\lceil 2/\tau \rceil - 1)$-degenerate (Lemma B.10). In particular, there exists an ordering $v_1, \dots, v_n$ of V such that for every $t \in \{1, \dots, n\}$, the vertex v_t has at most $\lceil 2/\tau \rceil - 1$ neighbors in $\{v_1, \dots, v_{t-1}\}$ within H. *Proof.* Lemma B.9 gives $\operatorname{arb}(H) \le 1/\tau$. Lemma B.10 shows that every subgraph of H has a vertex of degree at most $\lceil 2/\tau \rceil - 1$, so H is $(\lceil 2/\tau \rceil - 1)$-degenerate. The standard degeneracy-order construction (iteratively remove a minimum-degree vertex and reverse the removal order) yields an ordering with the claimed "bounded number of earlier neighbors" property. We can now define the sequential admissible coloring procedure.

Definition E.5 (Sequential admissible k-coloring distribution).
Fix an ordering $v_1, \dots, v_n$ as in Lemma E.4 and an integer k such that

$$k > \lceil 2/\tau \rceil - 1.$$

We define a random coloring $c : V \to [k]$ sequentially as follows. For $t = 1, 2, \dots, n$, having already colored $v_1, \dots, v_{t-1}$, define the *available color set*

$$A_t := [k] \setminus \{\, c(u) : \ u \in N_H(v_t) \cap \{v_1, \dots, v_{t-1}\} \,\},$$

where $N_H(v)$ denotes the neighbor set of v in the heavy-edge graph H. Choose $c(v_t)$ uniformly at random from A_t, independently of any further randomness. By Lemma E.4 and the inequality $k > \lceil 2/\tau \rceil - 1$, we have $|A_t| \ge 1$ for all t. The next lemma confirms admissibility.

Lemma E.6 (The sequential procedure outputs an admissible coloring).
The random coloring produced by Definition E.5 is a proper k-coloring of H,

i.e., $c(u) \neq c(v)$ for every heavy edge $uv \in E_{\mathrm{heavy}}(\tau)$. *Proof.* Fix any heavy edge $uv \in E_{\mathrm{heavy}}(\tau)$. Without loss of generality, suppose $u = v_s$ and $v = v_t$ with $s < t$. When v_t is colored, the vertex $u = v_s$ is already colored and is a heavy neighbor of v_t, hence $c(u)$ is excluded from A_t by definition. Therefore $c(v) \neq c(u)$. Finally, we record the conditional-expectation averaging identity used in §7. Let σ denote a partial admissible assignment: a map from a subset $\mathrm{dom}(\sigma) \subseteq V$ to $[k]$ that is proper on $H[\mathrm{dom}(\sigma)]$. Given such σ and an uncolored vertex $v \notin \mathrm{dom}(\sigma)$, define the allowed color set under σ as

$$A_\sigma(v) := [k] \setminus \{\sigma(u) : \; u \in N_H(v) \cap \mathrm{dom}(\sigma)\}.$$

Let $p_c(\lambda)$ denote the characteristic polynomial used in the main text (restricted to $\ker(L_G)^\perp$), and for a partial assignment σ define

$$p_\sigma(\lambda) := \mathbb{E}\big[p_c(\lambda) \mid \sigma\big],$$

where the expectation is over the continuation of the sequential procedure from σ.

Lemma E.7 (Conditional expectation is an average over allowed colors).
Let σ be any partial admissible assignment and let v be the next vertex to be colored in the sequential procedure. Then

$$p_\sigma(\lambda) = \frac{1}{|A_\sigma(v)|} \sum_{i \in A_\sigma(v)} p_{\sigma \cup (v \mapsto i)}(\lambda).$$

Proof. By Definition E.5, conditioned on σ, the next color choice assigns $c(v) = i$ uniformly over $i \in A_\sigma(v)$. Therefore, for any random variable $Z(c)$ depending on the completed coloring,

$$\mathbb{E}[Z(c) \mid \sigma] = \frac{1}{|A_\sigma(v)|} \sum_{i \in A_\sigma(v)} \mathbb{E}[Z(c) \mid \sigma \cup (v \mapsto i)].$$

Apply this with $Z(c) = p_c(\lambda)$ to obtain the identity.

E.3 Expected characteristic polynomials: the unconstrained model

This section combines Appendix D (forest-supported determinant expansions) with Corollary E.3 (forest moments) to obtain an explicit expected characteristic polynomial for the *unconstrained* random coloring model. This polynomial serves as a baseline object for the root bounds proved later (Appendix H). Let $d := \mathrm{rank}(L_G) = n - q$, where q is the number of connected components of G. Let $\ker(L_G)^\perp$ be identified with $\mathbb{R}^d$ via any orthonormal basis, so that each A_e is a $d \times d$ PSD matrix on that subspace. For an unconstrained random k-coloring $c : V \to [k]$ define

$$M_{\mathrm{mono}}(c) := \sum_{e \in E} X_e(c)\, A_e, \qquad p_c(\lambda) := \det(\lambda I_d - M_{\mathrm{mono}}(c)).$$

Proposition E.8 (Expected characteristic polynomial equals Bernoulli edge sampling).
Let $c : V \to [k]$ be an unconstrained random coloring and let $Y = \{Y_e\}_{e\in E}$ be independent Bernoulli random variables with

$$\Pr[Y_e = 1] = \frac{1}{k} \quad \text{for all } e \in E.$$

Then

$$\mathbb{E}_c[\det(\lambda I_d - \sum_{e\in E} X_e(c)A_e)] = \mathbb{E}_Y[\det(\lambda I_d - \sum_{e\in E} Y_e A_e)].$$

Moreover, both expectations admit the explicit forest expansion

$$\mathbb{E}[p(\lambda)] = \sum_{\substack{F\subseteq E \\ F \text{ is a forest}}} (-1)^{|F|}\lambda^{d-|F|} \det(\mathrm{Gram}(a_e)_{e\in F})\Big(\frac{1}{k}\Big)^{|F|}. \tag{16}$$

Proof. Start from Proposition D.3 specialized to the family $\{a_e\}_{e\in E}$:

$$\det\Big(\lambda I_d - \sum_{e\in E} x_e A_e\Big) = \sum_{F\subseteq E} (-1)^{|F|}\lambda^{d-|F|} \det(\mathrm{Gram}(a_e)_{e\in F}) \prod_{e\in F} x_e.$$

Substitute $x_e = X_e(c)$ and take expectation over c:

$$\mathbb{E}_c[p_c(\lambda)] = \sum_{F\subseteq E} (-1)^{|F|}\lambda^{d-|F|} \det(\mathrm{Gram}(a_e)_{e\in F})\mathbb{E}_c\Big[\prod_{e\in F} X_e(c)\Big].$$

By Corollary D.6, the Gram determinant vanishes unless F is a forest with $|F| \le d$, so the sum restricts to forests. For each forest F, Corollary E.3 yields

$$\mathbb{E}_c\Big[\prod_{e\in F} X_e(c)\Big] = (1/k)^{|F|}.$$

This proves the forest expansion Eq. (E.1).

Next, substitute $x_e = Y_e$ and take expectation over independent Bernoulli variables. Since the Y_e are independent,

$$\mathbb{E}_Y\Big[\prod_{e\in F} Y_e\Big] = (1/k)^{|F|}.$$

The same determinant expansion therefore produces exactly the same forest expansion, and hence the two expected characteristic polynomials coincide.

We will also use a light-edge specialization, obtained by setting heavy-edge variables to zero.

Corollary E.9 (Light-edge restriction by specialization).
Let $E_{\text{light}} \subseteq E$ be any subset (e.g., the light-edge set from a heavy–light decomposition). Define

$$M_{\text{light}}(c) := \sum_{e \in E_{\text{light}}} X_e(c)\, A_e, \qquad p_c^{\text{light}}(\lambda) := \det(\lambda I_d - M_{\text{light}}(c)).$$

Then under the unconstrained random coloring,

$$\mathbb{E}_c[p_c^{\text{light}}(\lambda)] = \sum_{\substack{F \subseteq E_{\text{light}} \\ F \text{ is a forest}}} (-1)^{|F|} \lambda^{d-|F|} \det\bigl(\mathrm{Gram}(a_e)_{e \in F}\bigr) \Bigl(\frac{1}{k}\Bigr)^{|F|}.$$

Equivalently,

$$\mathbb{E}_c\bigl[\det(\lambda I_d - \sum_{e \in E_{\text{light}}} X_e(c) A_e)\bigr] = \mathbb{E}_Y\bigl[\det(\lambda I_d - \sum_{e \in E_{\text{light}}} Y_e A_e)\bigr],$$

where $\{Y_e\}_{e \in E_{\text{light}}}$ are independent Bernoulli$(1/k)$ variables.

Proof. Apply Proposition E.8 to the family of vectors indexed only by E_{light}, or equivalently specialize $x_e = 0$ for $e \notin E_{\text{light}}$ in Proposition D.3 before taking expectations.

Remark. In the main text, admissibility enforces that heavy edges are never monochromatic, i.e., $X_e(c) = 0$ for $e \in E_{\text{heavy}}(\tau)$. The sequential admissible coloring procedure (Definition E.5) is designed to support interlacing-based selection through Lemma E.7. The real-rootedness and root bounds for the corresponding expected polynomials are developed in Appendices F–H.

Appendix F. Real stability and mixed characteristic polynomials (toolkit)

This appendix collects the polynomial-analytic tools used later to prove real-rootedness statements for expected characteristic polynomials. The emphasis is on a minimal, self-contained toolkit tailored to the present paper's setting, where the matrices of interest are PSD and (in our application) rank one.

We work with complex polynomials in several variables. We write

$$\mathbb{H} := \{z \in \mathbb{C} : \ \mathrm{Im}(z) > 0\}$$

for the open upper half-plane.

F.1 Stable and real-stable polynomials

Definition F.1 (Stable polynomial).
Let $p \in \mathbb{C}[z_1, \dots, z_m]$ be a nonzero polynomial. We say that p is *stable* if

$$p(z_1, \dots, z_m) \neq 0 \quad \text{whenever} \quad z_1, \dots, z_m \in \mathbb{H}.$$

Definition F.2 (Real-stable polynomial).
A polynomial $p \in \mathbb{R}[z_1, \dots, z_m]$ is *real stable* if it is stable in the sense of Definition F.1.

In one variable, stability coincides with real-rootedness.

Lemma F.3 (Univariate real stability equals real-rootedness).
Let $p \in \mathbb{R}[z]$ be a nonzero univariate polynomial. Then p is real stable if and only if all zeros of p are real.

Proof. If p has a zero ζ with $\mathrm{Im}(\zeta) > 0$, then p is not stable by definition. Conversely, if all zeros are real, then $p(z) \neq 0$ whenever $\mathrm{Im}(z) > 0$, so p is stable.

The next closure property is the one we use most: specializing some variables to real values preserves real stability (or yields the zero polynomial).

Lemma F.4 (Specialization to real values).
Let $p \in \mathbb{R}[z_1, \dots, z_m]$ be real stable, and fix $r \in \mathbb{R}$. Define

$$q(z_1, \dots, z_{m-1}) := p(z_1, \dots, z_{m-1}, r).$$

Then either $q \equiv 0$ or q is real stable.

Proof. For $t > 0$ define

$$q_t(z_1, \dots, z_{m-1}) := p(z_1, \dots, z_{m-1}, r + it).$$

Since $r + it \in \mathbb{H}$ and p is stable, each q_t is stable (as a polynomial in $z_1, \dots, z_{m-1}$). As $t \downarrow 0$, the coefficients of q_t converge to those of q. By Hurwitz's theorem applied to any univariate restriction (fix $z_1, \dots, z_{m-2} \in \mathbb{H}$ and view the remaining variable as univariate), the limit is either stable or identically zero. Since stability is defined by nonvanishing on $\mathbb{H}^{m-1}$, we obtain the claim.

We will also use stability under real shifts and under (univariate) differentiation.

Lemma F.5 (Real shifts preserve stability).
Let $p \in \mathbb{C}[z_1, \dots, z_m]$ be stable and let $t \in \mathbb{R}$. Define

$$q(z_1, \dots, z_m) := p(z_1, \dots, z_{j-1}, z_j - t, z_{j+1}, \dots, z_m).$$

Then q is stable.

Proof. If $z_1, \dots, z_m \in \mathbb{H}$, then $z_j - t \in \mathbb{H}$ as well (subtracting a real number does not change the imaginary part). Hence $q(z) \neq 0$ by stability of p.

Lemma F.6 (Partial derivatives preserve stability).
Let $p \in \mathbb{C}[z_1, \dots, z_m]$ be stable. Then for each $j \in \{1, \dots, m\}$, the partial derivative $\partial_{z_j} p$ is either identically zero or stable.

Proof. Fix $z_i \in \mathbb{H}$ for all $i \neq j$, and consider the univariate polynomial

$$\varphi(w) := p(z_1, \dots, z_{j-1}, w, z_{j+1}, \dots, z_m) \in \mathbb{C}[w].$$

By stability of p, $\varphi(w) \neq 0$ for all $w \in \mathbb{H}$, hence all zeros of φ lie in $\mathbb{C} \setminus \mathbb{H}$, which is a closed half-plane. By the Gauss–Lucas theorem, all zeros of $\varphi'(w)$ lie in the convex hull of the zeros of φ, hence also in $\mathbb{C} \setminus \mathbb{H}$. Therefore $\varphi'(w) \neq 0$ for all $w \in \mathbb{H}$.

But $\varphi'(w) = (\partial_{z_j} p)(z_1, \dots, z_{j-1}, w, z_{j+1}, \dots, z_m)$. Since the choice of $z_i \in \mathbb{H}$ for $i \neq j$ was arbitrary, we conclude that $\partial_{z_j} p$ is stable unless it is identically zero.

Finally, we note a simple identity for multiaffine polynomials.

Lemma F.7 (A differential operator equals a shift for multiaffine polynomials).
Let $p \in \mathbb{C}[z_1, \dots, z_m]$ be *multiaffine* in the variable z_j, i.e., of the form

$$p(z) = p_0(z_{\neq j}) + z_j\, p_1(z_{\neq j})$$

with p_0, p_1 independent of z_j. Then for every $t \in \mathbb{C}$,

$$(1 - t\,\partial_{z_j})p(z) = p(z_1, \dots, z_{j-1}, z_j - t, z_{j+1}, \dots, z_m).$$

Proof. Since $\partial_{z_j} p = p_1$, we have

$$(1 - t\,\partial_{z_j})p = p_0 + z_j p_1 - t p_1 = p_0 + (z_j - t)p_1,$$

which is exactly p with z_j replaced by $z_j - t$.

Combining Lemma F.7 with Lemma F.5 yields that for multiaffine stable polynomials, each operator $(1 - t\,\partial_{z_j})$ with $t \in \mathbb{R}$ preserves stability.

F.2 Determinantal real stability

We now record the determinantal stability statement used throughout the paper.

Theorem F.8 (Determinantal polynomials are real stable).
Let $A_0, A_1, \dots, A_m$ be real symmetric $d \times d$ matrices such that A_0 is positive definite and A_i is positive semidefinite for $i = 1, \dots, m$. Define

$$p(z_0, z_1, \dots, z_m) := \det\Big(\sum_{i=0}^{m} z_i A_i\Big).$$

Then p is real stable.

Proof. The coefficients of p are real since all A_i are real. To prove stability, take $z_0, \dots, z_m \in \mathbb{H}$ and consider the complex matrix

$$M := \sum_{i=0}^{m} z_i A_i.$$

Write $M = H + iK$ where H and K are real symmetric matrices:

$$H := \sum_{i=0}^{m} \mathrm{Re}(z_i)\, A_i, \qquad K := \sum_{i=0}^{m} \mathrm{Im}(z_i)\, A_i.$$

Since $\mathrm{Im}(z_i) > 0$ for all i, and $A_0 \succ 0$, we have

$$K \succeq \mathrm{Im}(z_0)\, A_0 \succ 0,$$

so K is positive definite.

We claim that M is invertible. Indeed, if $Mv = 0$ for some $v \in \mathbb{C}^d$, then

$$0 = v^* M v = v^* H v + i\, v^* K v.$$

Taking imaginary parts yields $v^* K v = 0$. Since $K \succ 0$, this implies $v = 0$. Hence $\ker(M) = \{0\}$ and M is invertible, so $\det(M) \neq 0$. This holds for all $(z_0, \dots, z_m) \in \mathbb{H}^{m+1}$, proving stability.

We will use the following special case repeatedly.

Corollary F.9 (Stability of $xI + \sum z_i A_i$).
Let $A_1, \dots, A_m$ be real symmetric PSD $d \times d$ matrices. Define

$$q(x, z_1, \dots, z_m) := \det\Big(x I_d + \sum_{i=1}^{m} z_i A_i\Big).$$

Then q is real stable in the variables $(x, z_1, \ldots, z_m)$.

Proof. Apply Theorem F.8 with $A_0 = I_d$ and $z_0 = x$.

F.3 Mixed characteristic polynomials for rank-one families

In our application, the matrices A_e are rank one: $A_e = a_e a_e^\top$. In this setting, the determinant polynomial in Corollary F.9 is multiaffine in the variables z_i, and hence the operators in Lemma F.7 apply directly.

Lemma F.10 (Multiaffinity for rank-one inputs).
Let $A_1, \ldots, A_m$ be rank-one PSD matrices in $\mathbb{R}^{d\times d}$. Then the polynomial

$$q(x, z) := \det\Big(xI_d + \sum_{i=1}^{m} z_i A_i\Big)$$

is multiaffine in the variables $z_1, \ldots, z_m$.

Proof. Apply the Gram/Cauchy–Binet expansion (Appendix D, Proposition D.3) to the rank-one decomposition $A_i = a_i a_i^\top$. Every monomial in the expansion has the form $\prod_{i\in F} z_i$ with each index appearing at most once.

We now define a p-parameter mixed characteristic polynomial tailored to independent Bernoulli selection.

Definition F.11 (p-mixed characteristic polynomial, rank-one case).
Let $A_1, \ldots, A_m$ be rank-one PSD matrices in $\mathbb{R}^{d\times d}$ and let $p \in [0, 1]$. Define

$$\mu_p[A_1, \ldots, A_m](x) := \left(\prod_{i=1}^{m}(1 - p\,\partial_{z_i})\right) \det\Big(xI_d + \sum_{i=1}^{m} z_i A_i\Big)\Bigg|_{z_1=\cdots=z_m=0}. \tag{17}$$

Because of multiaffinity (Lemma F.10), the operator in Definition F.11 has a simple evaluation form.

Lemma F.12 (Shift/evaluation form).
Under the assumptions of Definition F.11,

$$\mu_p[A_1, \ldots, A_m](x) = \det\Big(xI_d + \sum_{i=1}^{m}(-p)\,A_i\Big)\Big|_{\text{expanded multiaffinely in the } z_i} = q(x, -p, \ldots, -p),$$

where $q(x, z) = \det(xI_d + \sum_{i=1}^{m} z_i A_i)$ is viewed as a multiaffine polynomial in z and then evaluated at $z_i = -p$. *Proof.* Since q is multiaffine in each z_i, Lemma F.7 yields

$$(1 - p\,\partial_{z_i})q(x, z) = q(x, z_1, \ldots, z_{i-1}, z_i - p, z_{i+1}, \ldots, z_m).$$

Applying this consecutively for $i = 1, \ldots, m$ shifts each z_i by $-p$. Evaluating at $z = 0$ yields $q(x, -p, \ldots, -p)$. The next proposition identifies μ_p with an expected characteristic polynomial under independent Bernoulli sampling.

Proposition F.13 (Bernoulli representation).
Let $A_1, \dots, A_m$ be rank-one PSD matrices in $\mathbb{R}^{d\times d}$. Let $Y_1, \dots, Y_m$ be independent Bernoulli random variables with

$$\Pr[Y_i = 1] = p, \qquad \Pr[Y_i = 0] = 1 - p.$$

Then

$$\mathbb{E}\left[\det\Big(xI_d - \sum_{i=1}^m Y_i A_i\Big)\right] = \mu_p[A_1, \dots, A_m](x). \tag{18}$$

Proof. Let

$$q(x, z) := \det\Big(xI_d + \sum_{i=1}^m z_i A_i\Big),$$

which is multiaffine in z by Lemma F.10. Observe that

$$\det\Big(xI_d - \sum_{i=1}^m Y_i A_i\Big) = q(x, -Y_1, \dots, -Y_m).$$

Expand q in its multiaffine form:

$$q(x, z) = \sum_{F\subseteq[m]} c_F(x) \prod_{i\in F} z_i,$$

for certain coefficients $c_F(x) \in \mathbb{R}[x]$. Then

$$q(x, -Y_1, \dots, -Y_m) = \sum_{F\subseteq[m]} c_F(x)\, (-1)^{|F|} \prod_{i\in F} Y_i.$$

Taking expectations and using independence,

$$\mathbb{E}\left[\prod_{i\in F} Y_i\right] = p^{|F|}.$$

Hence

$$\mathbb{E}\left[q(x, -Y)\right] = \sum_{F\subseteq[m]} c_F(x)\, (-1)^{|F|} p^{|F|} = q(x, -p, \dots, -p).$$

By Lemma F.12, $q(x, -p, \dots, -p) = \mu_p[A_1, \dots, A_m](x)$, proving the claim.

Finally, we record the real-rootedness consequence that will be used later.

Corollary F.14 (Real-rootedness of μ_p).
Under the assumptions of Proposition F.13, the polynomial $\mu_p[A_1, \dots, A_m](x) \in \mathbb{R}[x]$ is real-rooted.

Proof. Consider the determinantal polynomial $q(x, z)$ from Corollary F.9. It is real stable in (x, z), hence by Lemma F.4 its specialization $x \mapsto q(x, -p, \dots, -p)$ is either real stable (as a univariate polynomial in x) or identically zero. In our setting it is not identically zero because $q(x, 0, \dots, 0) = \det(xI_d) = x^d$, and stability-preserving shifts do not annihilate the leading term.

Therefore $q(x, -p, \dots, -p)$ is real stable in one variable x, and by Lemma F.3 it is real-rooted. By Lemma F.12, this polynomial equals $\mu_p[A_1, \dots, A_m](x)$.

Remark. The definition in Eq. (F.1) matches the usual "mixed characteristic polynomial" formalism (up to the parameter p), but here we rely on rank-one multiaffinity (Lemma F.10) to reduce the stability-preserving operator $(1-p\partial_{z_i})$ to a simple shift. For general (higher-rank) PSD matrices, one can recover analogous real-rootedness statements via polarization and stability-preserving linear operators, as in the interlacing families literature.

Appendix G. Interlacing families with admissibility constraints

This appendix formalizes the interlacing-family selection step used in Corollary 7.3 of the main text. The role of this appendix is purely *structural*: once we know that at each branching step the candidate polynomials have a common interlacing, a deterministic choice achieving the largest-root bound follows by a simple greedy argument.

We emphasize that admissibility constraints (proper coloring on the heavy-edge graph) merely remove some branches from the decision tree; the interlacing-family selection works verbatim on the remaining (allowed) children.

Throughout this appendix, a univariate real polynomial is said to be *real-rooted* if all its zeros are real. For a real-rooted polynomial f, we write

$$\rho_{\max}(f)$$

for its largest real root (counted with multiplicity).

G.1 Interlacing and common interlacing

Definition G.1 (Interlacing).
Let $f, g \in \mathbb{R}[x]$ be real-rooted polynomials with $\deg(f) = d$ and $\deg(g) = d - 1$, and assume both have positive leading coefficients. Write the roots of f as

$$\alpha_1 \le \alpha_2 \le \cdots \le \alpha_d$$

and the roots of g as

$$\beta_1 \le \beta_2 \le \cdots \le \beta_{d-1}.$$

We say that g *interlaces* f if

$$\alpha_1 \le \beta_1 \le \alpha_2 \le \cdots \le \beta_{d-1} \le \alpha_d.$$

Definition G.2 (Common interlacing).
Let $f_1, \dots, f_r \in \mathbb{R}[x]$ be real-rooted polynomials of the same degree d, each with positive leading coefficient. We say that $\{f_i\}_{i=1}^r$ has a *common interlacing* if there exists a real-rooted polynomial $g \in \mathbb{R}[x]$ of degree $d-1$ such that g interlaces f_i for every $i \in \{1, \dots, r\}$.

The next lemma records the basic consequence of common interlacing that we use in the selection argument.

Lemma G.3 (Convex combinations under a common interlacing).
Let $f_1, \dots, f_r \in \mathbb{R}[x]$ be real-rooted polynomials of degree d with positive leading coefficients, and assume they have a common interlacing. Let $\alpha_1, \dots, \alpha_r \ge 0$ with $\sum_{i=1}^r \alpha_i = 1$, and define

$$f(x) := \sum_{i=1}^{r} \alpha_i f_i(x).$$

Then:

1. f is real-rooted of degree d.
2. The family $\{f_i\}_{i=1}^r$ and f share a common interlacing.
3. The largest root satisfies

$$\min_i \rho_{\max}(f_i) \;\le\; \rho_{\max}(f) \;\le\; \max_i \rho_{\max}(f_i).$$

Proof. Let g be a common interlacing polynomial of degree $d-1$ (Definition G.2). Let its roots be

$$\beta_1 \le \cdots \le \beta_{d-1},$$

and set $\beta_0 := -\infty$ and $\beta_d := +\infty$.

Since g interlaces each f_i and every f_i has positive leading coefficient, all f_i have the same sign pattern on the intervals

$$(\beta_0, \beta_1),\ (\beta_1, \beta_2),\ \dots,\ (\beta_{d-1}, \beta_d),$$

namely the sign alternates between consecutive intervals and is positive on (β_{d-1}, β_d).

Because f is a convex combination of the f_i, the polynomial f has the same sign as all f_i on each interval (it cannot change sign if all summands have the same sign). Therefore f must have a real root in each of the d intervals, and hence is real-rooted of degree d. Moreover, the roots of f lie one-per-interval between successive roots of g, so g interlaces f as well.

Finally, the largest root of each f_i lies in (β_{d-1}, β_d), and the same holds for f. Since all roots in that final interval are ordered, the largest root of f must lie between the minimum and maximum of the largest roots of the f_i's.

We extract a one-step "good child" rule.

Lemma G.4 (Selecting a child with no larger largest root).
Under the hypotheses of Lemma G.3, there exists an index $i^\star \in \{1, \dots, r\}$ such that

$$\rho_{\max}(f_{i^\star}) \le \rho_{\max}(f).$$

Proof. By Lemma G.3(3),

$$\min_i \rho_{\max}(f_i) \ \le\ \rho_{\max}(f).$$

Choose $i^\star$ attaining the minimum.

G.2 Interlacing families on rooted trees (with forbidden branches)

We now state the tree-structured notion used in interlacing-family arguments.

Definition G.5 (Interlacing family on a rooted tree).
Let $\mathcal{T}$ be a rooted finite tree. Each node $u \in \mathcal{T}$ is assigned a real-rooted polynomial $p_u \in \mathbb{R}[x]$ of a fixed degree d, with positive leading coefficient. We say that $\{p_u\}_{u \in \mathcal{T}}$ is an *interlacing family* if the following hold for every internal node u:

1. Let $\mathrm{Child}(u)$ denote the set of children of u. The polynomials $\{p_v\}_{v \in \mathrm{Child}(u)}$ have a common interlacing.
2. The parent polynomial is a convex combination of its children:

$$p_u(x) = \sum_{v \in \mathrm{Child}(u)} w_{u \to v}\, p_v(x),$$

where $w_{u\to v} \geq 0$ and $\sum_{v\in\mathrm{Child}(u)} w_{u\to v} = 1$.

A subset of children may be forbidden by constraints; in that case $\mathrm{Child}(u)$ is understood to be the *allowed* set of children. Definition G.5 depends only on the remaining children and their convex weights.

The following theorem is the standard "leaf selection" lemma.

Theorem G.6 (Leaf selection by largest-root monotonicity).
Let $\{p_u\}_{u\in\mathcal{T}}$ be an interlacing family (Definition G.5). Then there exists a leaf ℓ of $\mathcal{T}$ such that

$$\rho_{\max}(p_\ell) \leq \rho_{\max}(p_{\mathrm{root}}).$$

Proof. Starting at the root node u_0, define a path inductively as follows. If u_t is a leaf, stop. Otherwise, by Definition G.5(1) the children $\{p_v\}_{v\in\mathrm{Child}(u_t)}$ have a common interlacing, and by Definition G.5(2) the parent p_{u_t} is a convex combination of the child polynomials. Lemma G.4 therefore yields a child $v_t \in \mathrm{Child}(u_t)$ such that

$$\rho_{\max}(p_{v_t}) \leq \rho_{\max}(p_{u_t}).$$

Set $u_{t+1} := v_t$ and continue.

Since $\mathcal{T}$ is finite, the process terminates at some leaf ℓ. Along the selected path the largest root is nonincreasing, hence

$$\rho_{\max}(p_\ell) \leq \rho_{\max}(p_{\mathrm{root}}).$$

Remark. The proof is constructive: once the polynomials are explicitly available (or evaluable), the choice rule in Lemma G.4 yields a deterministic selection.

G.3 Application: the Route B specialization tree

Appendix J shows that conditional expectation polynomials along the naive vertex-exposure coloring tree need not be real-rooted. The main text therefore uses Route B: a binary specialization tree built from a multiaffine real-stable determinantal polynomial (Theorem 7.1).

Fix rank-one PSD matrices $C_1, \ldots, C_N$ and the master polynomial

$$Q(\lambda, \mathbf{z}) = \det\Big(\lambda I_D + \sum_{j=1}^{N} z_j\, C_j\Big).$$

Assume we are given a rooted binary tree $\mathcal{T}$ whose nodes specify a vector $\mathbf{z}(u) \in [-1,0]^N$ with the following properties:

- The root node u_{root} is labeled by $\mathbf{z}(u_{\mathrm{root}}) = \mathbf{z}_{\mathrm{root}}$.
- Each internal node u chooses an index $j(u) \in \{1, \dots, N\}$ and a parameter $\theta(u) \in [0,1]$ such that $z_{j(u)}(u) = -\theta(u)$.
- The two children $u0$ and $u1$ of u are obtained by replacing $z_{j(u)}(u)$ by the endpoint values 0 and -1, respectively, leaving all other coordinates unchanged:

$$\mathbf{z}(u0) := \mathbf{z}(u)\big|_{z_{j(u)}=0}, \qquad \mathbf{z}(u1) := \mathbf{z}(u)\big|_{z_{j(u)}=-1}.$$

- Leaves have all coordinates in $\{0, -1\}$.

To each node $u \in \mathcal{T}$ assign the univariate polynomial

$$p_u(\lambda) := Q(\lambda, \mathbf{z}(u)).$$

By Theorem 7.1(2), every p_u is real-rooted. Moreover, by Theorem 7.1(3) we have the explicit one-step interpolation identity

$$p_u(\lambda) = (1 - \theta(u))\, p_{u0}(\lambda) + \theta(u)\, p_{u1}(\lambda),$$

so the parent is a convex combination of its children.

The remaining interlacing-family requirement is sibling common interlacing.

Lemma G.7 (Sibling common interlacing in Route B).
At every internal node u of the Route B specialization tree, the two children polynomials $\{p_{u0}, p_{u1}\}$ have a common interlacing.

Proof.
Fix an internal node u and abbreviate $q_0 := p_{u0}$ and $q_1 := p_{u1}$. For any $\theta \in [0,1]$, multiaffinity gives

$$(1-\theta)q_0(\lambda) + \theta q_1(\lambda) = Q(\lambda, \mathbf{z}(u)\big|_{z_{j(u)}=-\theta}).$$

The right-hand side is a specialization of the real-stable polynomial Q at real values in $[-1,0]$, hence is real-rooted (Theorem 7.1(2)). Therefore every convex combination of q_0 and q_1 is real-rooted. By the compatibility criterion (Appendix H, Lemma H.8), q_0 and q_1 have a common interlacing.

Thus $\{p_u\}_{u\in\mathcal{T}}$ is an interlacing family (Definition G.5), and we can select a good leaf.

Corollary G.8 (Route B leaf selection).
There exists a leaf ℓ of $\mathcal{T}$ such that

$$\rho_{\max}(p_\ell) \le \rho_{\max}(p_{u_{\mathrm{root}}}).$$

Proof. Apply Theorem G.6 to the interlacing family $\{p_u\}_{u\in\mathcal{T}}$.

In the main text, Appendix K constructs a specific Route B specialization tree $\mathcal{T}$ in which:

- leaves correspond bijectively to full k-colorings c of V,
- the leaf polynomial is $p_\ell(\lambda) = p_c(\lambda)$, and
- the root polynomial is $p_{u_{\mathrm{root}}}(\lambda) = p_\emptyset(\lambda)$.

With these identifications, Corollary G.8 is exactly the selection step used in Corollary 7.3.

Appendix H. Largest-root bounds and verification of sibling interlacing

H.1 Objectives and how this appendix closes the proof

This appendix supplies the two missing technical inputs deferred earlier:

1. A largest-root bound for the global expectation polynomial $p_\emptyset(\lambda)$ (Theorem 7.2).
2. A verification of the sibling common-interlacing property required by the Route B interlacing selection lemma (Lemma G.7), thereby justifying the recursive "choose a child whose largest root does not increase" step used in Appendix G.

Throughout, we keep the notation from the main text and Appendices D–G. In particular:

- $G = (V, E)$ is a connected undirected graph with Laplacian L.
- P denotes the orthogonal projection onto $\mathbf{1}^\perp$.
- For each edge e, we define the rank-one positive semidefinite matrix

$$A_e := a_e a_e^\top, \qquad a_e := L^{\dagger/2} b_e, \qquad b_e := \mathbf{e}_u - \mathbf{e}_v \quad (e = \{u, v\}).$$

- The leverage score is $\ell_e := \mathrm{Tr}(A_e) = \|a_e\|_2^2$.
- For a k-coloring $c : V \to [k]$, the normalized monochromatic Laplacian is

$$M(c) := L^{\dagger/2} L_{\text{mono}}(c)\, L^{\dagger/2} = \sum_{e \in E_{\text{mono}}(c)} A_e,$$

and the leaf polynomial is

$$p_c(\lambda) := \det(\lambda I - M(c)).$$

The internal-node polynomials on the Route B tree are obtained by specializing the determinantal master polynomial $Q(\lambda, \mathbf{z})$ at real points $\mathbf{z} \in [-1, 0]^N$ (Theorem 7.1 and Appendix G).

H.2 A largest-root bound for Bernoulli mixed characteristic polynomials

We begin with a dimension-free largest-root estimate for the expected characteristic polynomial of a Bernoulli sum of rank-one PSD matrices. This will be the engine behind Theorem 7.2.

Definition H.1 (Bernoulli mixed characteristic polynomial).
Let $A_1, \dots, A_m$ be rank-one PSD matrices in $\mathbb{R}^{d\times d}$ and let $Y_1, \dots, Y_m$ be independent Bernoulli random variables with parameter $p \in (0, 1)$. Define

$$\mu_p(\lambda; A_1, \dots, A_m) := \mathbb{E}\Big[\det\Big(\lambda I - \sum_{i=1}^{m} Y_i A_i\Big)\Big].$$

By Appendix F, μ_p is real-rooted.

Theorem H.2 (Largest-root bound for Bernoulli sums).
Assume:

- $A_i = a_i a_i^\top$ are rank-one PSD,
- $\sum_{i=1}^{m} A_i = I_d$,
- $\operatorname{Tr}(A_i) = \|a_i\|_2^2 \le \tau$ for all i.

Then the largest real root of $\mu_p(\cdot; A_1, \dots, A_m)$ is at most

$$\rho_{\max}(\mu_p(\cdot; A_1, \dots, A_m)) \;\le\; p\,(1 + \sqrt{\tau})^2.$$

Proof.
Define independent random vectors v_i by

$$v_i := \begin{cases} a_i/\sqrt{p}, & \text{with probability } p, \\ 0, & \text{with probability } 1 - p. \end{cases}$$

Then $v_i v_i^\top = (Y_i/p) A_i$, and therefore

$$\sum_{i=1}^{m} v_i v_i^\top = \frac{1}{p} \sum_{i=1}^{m} Y_i A_i.$$

Moreover,

$$\mathbb{E}[v_i v_i^\top] = A_i, \qquad \mathbb{E}\|v_i\|_2^2 = \|a_i\|_2^2 = \mathrm{Tr}(A_i) \le \tau.$$

We now invoke the Marcus–Spielman–Srivastava largest-root theorem for mixed characteristic polynomials (equivalently, for sums of independent random rank-one matrices with bounded second moments). Concretely, under the conditions $\sum_i \mathbb{E}[v_i v_i^\top] = I_d$ and $\mathbb{E}\|v_i\|_2^2 \le \tau$, the expected characteristic polynomial

$$\mathbb{E}\Big[\det\Big(xI - \sum_{i=1}^{m} v_i v_i^\top\Big)\Big]$$

has all real roots and satisfies

$$\rho_{\max}\Big(\mathbb{E}\Big[\det\Big(xI - \sum_{i=1}^{m} v_i v_i^\top\Big)\Big]\Big) \;\le\; (1+\sqrt{\tau})^2.$$

Using the identity

$$\det\Big(xI - \frac{1}{p}\sum_{i=1}^{m} Y_i A_i\Big) = p^{-d}\,\det\Big(pxI - \sum_{i=1}^{m} Y_i A_i\Big),$$

we obtain

$$\mathbb{E}\Big[\det\Big(xI - \frac{1}{p}\sum_{i=1}^{m} Y_i A_i\Big)\Big] = p^{-d}\,\mu_p(px; A_1, \dots, A_m).$$

Scaling by the positive factor p^{-d} does not change roots, so the largest root of $\mu_p(px; A_1, \dots, A_m)$ is at most $(1+\sqrt{\tau})^2$. Equivalently, the largest root of $\mu_p(\lambda; A_1, \dots, A_m)$ is at most $p(1+\sqrt{\tau})^2$.

Corollary H.3 (Uniform constant bound).
Under the hypotheses of Theorem H.2, if in addition $\tau \le 1$ then

$$\rho_{\max}(\mu_p(\cdot; A_1, \dots, A_m)) \le 4p.$$

Proof.
Since $\tau \le 1$, we have $(1+\sqrt{\tau})^2 \le (1+1)^2 = 4$. Apply Theorem H.2.

H.3 Proof of Theorem 7.2 (largest root of $p_\emptyset$)

We now connect Theorem H.2 to the global expectation polynomial in our coloring framework.

Recall the global expectation polynomial

$$p_\emptyset(\lambda) := \mathbb{E}_c\big[\det(\lambda I - M(c))\big],$$

where c is drawn from the coloring distribution specified in Appendix E.

Lemma H.4 (Normalization of $\{A_e\}$).
On the subspace $\mathbf{1}^\perp$, one has

$$\sum_{e\in E} A_e = I_{\mathbf{1}^\perp}.$$

Proof.
This is the operator identity

$$\sum_{e\in E} A_e = L^{\dagger/2}\Big(\sum_{e\in E} b_e b_e^\top\Big)L^{\dagger/2} = L^{\dagger/2} L\, L^{\dagger/2} = P,$$

and P restricts to the identity on $\mathbf{1}^\perp$.

Lemma H.5 (Identification of $p_\emptyset$ with a Bernoulli mixed characteristic polynomial).
For the unconstrained i.i.d. uniform k-coloring model, one has the exact identity

$$p_\emptyset(\lambda) = \mu_{1/k}(\lambda; \{A_e\}_{e\in E}),$$

where the Bernoulli model samples each edge independently with probability $1/k$.

Proof.
This is precisely the forest-expansion identification established in Appendix E: the determinant expansion of p_c is supported on forests (Appendix D), and the joint monochromatic indicators have the same moments as independent Bernoulli($1/k$) indicators on forests (Appendix E). Therefore the expected characteristic polynomials agree.

Theorem H.6 (Theorem 7.2 with explicit constants).
Assume the unconstrained i.i.d. uniform k-coloring model. Then

$$\rho_{\max}(p_\emptyset) \le \frac{4}{k}.$$

In particular, one may take $C_0 = 4$ in Theorem 7.2.

Proof.
By Lemma H.4, $\{A_e\}$ satisfies $\sum_{e \in E} A_e = I_{\mathbf{1}^\perp}$. Moreover, for every edge e, A_e is rank-one PSD with

$$\mathrm{Tr}(A_e) = \ell_e \leq 1$$

(the leverage score/effective resistance bound). Applying Corollary H.3 with $p = 1/k$ yields

$$\rho_{\max}(\mu_{1/k}(\cdot; \{A_e\})) \leq \frac{4}{k}.$$

By Lemma H.5, $p_\emptyset = \mu_{1/k}(\cdot; \{A_e\})$, so the same bound holds for $p_\emptyset$.

Remark H.7 (Why "heavy-edge admissibility" can be enforced implicitly).
If the main text defines a "heavy" threshold $\tau = \alpha_0/k$ and forbids monochromatic heavy edges, then any coloring c satisfying $\lambda_{\max}(M(c)) \leq C_0/k$ automatically has no monochromatic edges with $\ell_e > C_0/k$. Thus, choosing $\alpha_0 > C_0$ ensures that the selected coloring is automatically "admissible" in the heavy-edge sense, even if the interlacing selection is carried out in the unconstrained model.

This observation is only used to reconcile the "admissibility" bookkeeping with the fact that the cleanest largest-root estimate (Theorem H.6) is stated for the unconstrained model.

H.4 Verification of Lemma G.7 (Route B sibling common interlacing)

We now justify Lemma G.7 for the Route B specialization tree introduced in §7.1 and Appendix G.

The proof relies on a standard compatibility criterion for real-rooted polynomials (Obreschkoff/Dedieu-type): a finite family of real-rooted polynomials $\{q_i\}$ with positive leading coefficients has a common interlacing if and only if every convex combination $\sum_i \alpha_i q_i$ (with $\alpha_i \geq 0$, $\sum_i \alpha_i = 1$) is real-rooted.

Lemma H.8 (Compatibility criterion).
Let $q_1, \dots, q_r$ be real-rooted degree-d polynomials with positive leading coefficients. Then the following are equivalent:

1. $\{q_i\}$ has a common interlacing.
2. For every choice of nonnegative weights $\alpha_1, \dots, \alpha_r$ with $\sum_i \alpha_i = 1$, the convex combination $\sum_i \alpha_i q_i$ is real-rooted.

Proof.
We show (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Let $r(x)$ be a common interlacer with (real) roots $\beta_1 \leq \cdots \leq \beta_{d-1}$. For convenience set $\beta_0 := -\infty$ and $\beta_d := +\infty$. For each i and each $j \in \{0, 1, \ldots, d-1\}$, the interlacing hypothesis implies that q_i has exactly one root (counting multiplicity) in every open interval (β_j, β_{j+1}). In particular, because each q_i has positive leading coefficient, the values $q_i(\beta_j)$ are all nonzero and share the same sign, and this sign alternates with j. (If some β_j coincides with a root of some q_i, perturb β_j slightly and take a limit; real-rootedness is closed under coefficient-wise limits.)

Now fix nonnegative weights $\alpha_1, \ldots, \alpha_r$ with $\sum_i \alpha_i = 1$ and let $q = \sum_i \alpha_i q_i$. For each j, [q(*j)*= {i=1}^r _i q_i(_j)] is a convex combination of numbers with the same sign, hence $q(\beta_j) \neq 0$ and has that same sign. Since these signs alternate with j, by the intermediate value theorem q has at least one real root in each interval (β_j, β_{j+1}), for $j = 0, 1, \ldots, d-1$. There are d such intervals, so q has at least d real roots counting multiplicity. Because $\deg q = d$, all its roots are real.

(2) $\Rightarrow$ (1). By restricting to convex combinations supported on two indices, (2) implies that for every pair $i \neq j$, every convex combination $tq_i + (1-t)q_j$ is real-rooted. For degree-d real-rooted polynomials with positive leading coefficients, this pairwise compatibility is equivalent to the existence of a common interlacing for $\{q_i, q_j\}$ (see, e.g., Marcus–Spielman–Srivastava (2015, Lemma 3.4)).
Write the roots of q_i as $\alpha_{i,1} \leq \cdots \leq \alpha_{i,d}$. Pairwise common interlacing implies that for all i, j and all $\ell = 1, \ldots, d-1$, [*{i, }* {j, +1}.] Define $L_\ell := \max_i \alpha_{i,\ell}$ and $R_\ell := \min_i \alpha_{i,\ell+1}$. The inequalities above give $L_\ell \leq R_\ell$, so choose any $\beta_\ell \in [L_\ell, R_\ell]$. Then for each i we have $\alpha_{i,\ell} \leq \beta_\ell \leq \alpha_{i,\ell+1}$, which means that the degree-$(d-1)$ polynomial [r(x):= *{ =1}^{d-1} (x-)*] interlaces every q_i. Hence $\{q_i\}$ has a common interlacing.

$\square$

We apply Lemma H.8 to the two children polynomials at a Route B split.

Fix the Route B master polynomial $Q(\lambda, \mathbf{z})$ of Theorem 7.1 and a node u with splitting coordinate $j = j(u)$. Let $u0$ and $u1$ denote the two children, so that $z_j(u0) = 0$ and $z_j(u1) = -1$ and all other coordinates agree with u.

Proposition H.9 (Convex mixtures of Route B children remain real-rooted).
For any $\theta \in [0, 1]$, the convex combination

$$(1-\theta)\, p_{u0}(\lambda) + \theta\, p_{u1}(\lambda)$$

is real-rooted.

Proof.
By multiaffinity of Q in z_j (Theorem 7.1(1)),

$$(1-\theta)\,p_{u0}(\lambda)+\theta\,p_{u1}(\lambda)=Q(\lambda,\mathbf{z}(u)\big|_{z_j=-\theta}).$$

The right-hand side is a specialization of the real-stable polynomial Q at real values in $[-1,0]$, hence is real-rooted (Theorem 7.1(2)).

Corollary H.10 (Route B sibling common interlacing).
At every internal node u of the Route B specialization tree, the two children polynomials $\{p_{u0},p_{u1}\}$ have a common interlacing.

Proof.
The child polynomials p_{u0} and p_{u1} are real-rooted (Theorem 7.1(2)). By Proposition H.9, every convex combination of the children is real-rooted. Applying Lemma H.8 yields a common interlacing.

H.5 Consequences for the main theorem

Combining:

- Corollary H.10 (Route B sibling common interlacing),
- The Route B interlacing selection argument of Appendix G (Corollary G.8),
- The largest-root bound of Theorem H.6,

we conclude that the Route B recursion can select a deterministic coloring $\hat{c}$ such that

$$\lambda_{\max}(M(\hat{c}))\le\rho_{\max}(p_\emptyset)\le\frac{4}{k}.$$

This completes the proof of Corollary 7.3 and hence the main theorem.

Appendix I. Deterministic selection and certification (optional)

This appendix records an explicit deterministic selection procedure corresponding to the interlacing-family argument in Appendix G, and a practical certificate for verifying that an output set S is ε-light.

This appendix is logically optional: the existence proof of Theorem 1.1 is complete without it. Its purpose is to make the "there exists a good leaf" step operational.

I.1 Deterministic interlacing selection on the Route B specialization tree

Appendix G reduces the "there exists a good leaf" step to an interlacing-family selection problem on a rooted tree. In the corrected proof, the relevant tree is not the naive vertex-exposure coloring tree (Appendix J), but the Route B binary specialization tree of §7.1.

Fix the Route B master polynomial

$$Q(\lambda, \mathbf{z}) = \det\Big(\lambda I_D + \sum_{j=1}^{N} z_j\, C_j\Big)$$

and the associated Route B specialization tree $\mathcal{T}$ from Appendix G / Appendix K. Each node $u \in \mathcal{T}$ is labeled by a vector $\mathbf{z}(u) \in [-1, 0]^N$ and carries the real-rooted polynomial

$$p_u(\lambda) := Q(\lambda, \mathbf{z}(u)).$$

If u is an internal node with splitting coordinate $j = j(u)$ and parameter $\theta = \theta(u)$ (so $z_j(u) = -\theta$), then the two children $u0, u1$ satisfy

$$p_u(\lambda) = (1 - \theta)\, p_{u0}(\lambda) + \theta\, p_{u1}(\lambda),$$

and the children have a common interlacing (Lemma G.7). Therefore Lemma G.4 implies the existence of a child whose largest root does not increase.

Algorithm I.1 (Greedy Route B interlacing selection).
Input: an evaluation oracle for $Q(\lambda, \mathbf{z})$ on real $\mathbf{z} \in [-1, 0]^N$ (or directly for the node polynomials p_u), and a representation of the Route B tree $\mathcal{T}$.
Output: a leaf ℓ of $\mathcal{T}$, hence (in the application) a full k-coloring $\hat{c}$.

1. Initialize $u \leftarrow u_{\text{root}}$.
2. While u is not a leaf:
    - Let $u0, u1$ be the two children of u.
    - Compute (or approximate) the largest roots $\rho_{\max}(p_{u0})$ and $\rho_{\max}(p_{u1})$.
    - Set $u \leftarrow u0$ if $\rho_{\max}(p_{u0}) \le \rho_{\max}(p_{u1})$, and otherwise set $u \leftarrow u1$.
3. Output $\ell := u$.

By construction, the largest root along the selected path is nonincreasing, so $\rho_{\max}(p_\ell) \le \rho_{\max}(p_{u_{\text{root}}})$. In particular, when $p_{u_{\text{root}}} = p_\emptyset$ and $p_\ell = p_{\hat{c}}$ as in Appendix K, this produces a coloring $\hat{c}$ satisfying Corollary 7.3.

Remark. This appendix is optional: the existence proof uses only the non-constructive selection theorem (Appendix G, Theorem G.6). The algorithmic content here is meant as a template for making the selection step operational once an explicit representation of the Route B master polynomial Q is available.

I.2 Certification of ε-lightness for a candidate set S

Although Theorem 1.1 provides a theoretical guarantee, it is often useful to certify ε-lightness of a given candidate set S numerically or symbolically.

We record an equivalent spectral certificate.

Proposition I.5 (Spectral certificate for $L_{G[S]} \preceq \varepsilon L_G$).
Let $G = (V, E)$ be any undirected graph (possibly disconnected) with Laplacian L_G, and let $S \subseteq V$. Define the generalized Rayleigh quotient

$$R_S(x) := \frac{x^\top L_{G[S]} x}{x^\top L_G x}, \qquad x \in \ker(L_G)^\perp,\ x \neq 0.$$

Then the following are equivalent:

1. $L_{G[S]} \preceq \varepsilon L_G$.
2. $\sup_{x \in \ker(L_G)^\perp,\ x \neq 0} R_S(x) \le \varepsilon$.
3. The largest eigenvalue of the normalized matrix

$$M_S := L_G^{\dagger/2} L_{G[S]} L_G^{\dagger/2}$$

satisfies

$$\lambda_{\max}(M_S|_{\ker(L_G)^\perp}) \le \varepsilon.$$

Proof. (1) $\Leftrightarrow$ (2) is Lemma 2.3 in the main text (with full proof in Appendix A). The equality

$$\sup_{x \perp \ker(L_G), x \neq 0} \frac{x^\top L_{G[S]} x}{x^\top L_G x} = \lambda_{\max}(L_G^{\dagger/2} L_{G[S]} L_G^{\dagger/2}|_{\ker(L_G)^\perp})$$

is exactly Lemma A.7, giving (2) $\Leftrightarrow$ (3).

Remark. In practice, one may compute $\lambda_{\max}(M_S)$ by solving the generalized eigenvalue problem

$$L_{G[S]} x = \lambda L_G x$$

on $\ker(L_G)^\perp$ (or component-wise on each connected component), and checking whether the largest generalized eigenvalue is at most ε.

I.3 Computational remarks

Theorem I.3 is constructive in the following formal sense: it provides an explicit deterministic decision rule (Algorithm I.1) that produces a valid coloring whenever the required polynomial objects are available.

At the same time, we emphasize that the *computational* cost of implementing Algorithm I.1 depends on how one represents and evaluates the conditional expectation polynomials p_σ and their largest roots. In full generality:

- Computing p_σ exactly by enumerating completions of σ is exponential in n.
- Even for the unconstrained i.i.d. coloring model, the forest-supported expansion in Appendix E involves a sum over all forests, which is exponential in general.

Therefore, the present paper does not claim a fully efficient (polynomial-time) implementation of Algorithm I.1 for arbitrary graphs. Instead, Appendix I should be read as an explicit *derandomization blueprint* consistent with the interlacing-family existence proof.

Two directions for further algorithmic refinement are:

1. **Barrier-function implementations.** In many interlacing arguments, one can avoid explicit manipulation of characteristic polynomials by maintaining a barrier potential (e.g., a resolvent trace) and applying a pessimistic estimator argument. Developing a barrier-based implementation adapted to the monochromatic-edge Laplacian setting is a promising direction.

2. **Special graph classes.** For restricted families of graphs (bounded treewidth, planar graphs, or graphs with additional algebraic structure), one may hope to evaluate the relevant conditional polynomials or barrier functions efficiently.

We leave a detailed algorithmic complexity analysis to future work.

Appendix K. Route B construction of the determinantal specialization tree

This appendix records the concrete Route B objects invoked in §7.1 and Appendix G.

The role of Route B is to replace the naive vertex-exposure conditioning (which fails to preserve real-rootedness; Appendix J) by a stability-preserving specialization tree whose internal nodes are obtained as real specializations of a multiaffine real-stable determinantal polynomial.

Throughout this appendix we work in the unconstrained i.i.d. uniform k-coloring model.

K.1 Goal: a master polynomial whose leaves are the coloring polynomials

Recall the leaf polynomials

$$p_c(\lambda) = \det(\lambda I_d - M(c)), \qquad M(c) = \sum_{e \in E} A_e \, X_e(c),$$

where $X_e(c) \in \{0, 1\}$ indicates whether e is monochromatic under c.

Route B requires a multivariate determinantal polynomial

$$Q(\lambda, \mathbf{z}) = \det\Big(\lambda I_D + \sum_{j=1}^{N} z_j \, C_j\Big)$$

with rank-one PSD coefficients C_j.

For any selection vector $\delta \in \{0, 1\}^N$, specializing $z_j = -\delta_j \in \{0, -1\}$ yields

$$Q(\lambda, -\delta) = \det\Big(\lambda I_D - \sum_{j=1}^{N} \delta_j \, C_j\Big).$$

Moreover, since each C_j has rank one, we may write $C_j = v_j v_j^\top$ and assemble $V = [v_1 \ \cdots \ v_N]$, so that

$$\sum_{j=1}^{N} \delta_j \, C_j = V \, D_\delta \, V^\top, \qquad D_\delta := \operatorname{diag}(\delta_1, \dots, \delta_N).$$

Finally, multiaffinity implies that for any $\theta \in [0, 1]$ and any fixed values of the other coordinates,

$$Q(\lambda, z_j = -\theta) = (1 - \theta) \, Q(\lambda, z_j = 0) + \theta \, Q(\lambda, z_j = -1).$$

Route B uses this identity to realize convex mixtures as real specializations along the specialization tree.

Route B further requires:

- a rooted binary specialization tree $\mathcal{T}$ in which each node u carries a real vector $\mathbf{z}(u) \in [-1, 0]^N$,
- a *leaf identification* map $\ell \mapsto c(\ell)$ from leaves ℓ to k-colorings such that

$$p_\ell(\lambda) = Q(\lambda, \mathbf{z}(\ell)) = p_{c(\ell)}(\lambda),$$

- a *root identification* such that

$$p_{u_{\mathrm{root}}}(\lambda) = Q(\lambda, \mathbf{z}(u_{\mathrm{root}})) = p_{\emptyset}(\lambda).$$

Given these objects, Theorem 7.1 and Appendix G imply the existence of a leaf ℓ with $\rho_{\max}(p_{c(\ell)}) \le \rho_{\max}(p_{\emptyset})$, which is exactly Corollary 7.3.

The remaining task is therefore to explain one explicit choice of $(Q, \mathcal{T}, \mathbf{z}(\cdot))$ satisfying the two identifications above.

K.2 Construction: two-stage (double-cut) atomization and binary refinement

We sketch a concrete construction based on two standard ingredients:

1. **Rank-one atomization.** Each edge Laplacian is rank one: $A_e = a_e a_e^{\top}$. Determinantal polynomials with rank-one PSD coefficients are real stable and multiaffine (Appendix F).
2. **Double-cut refinement.** A k-coloring can be encoded by a collection of discrete parameters, and any finite k-ary choice can be refined into a sequence of binary choices. Route B performs interlacing selection along this refined binary tree, using multiaffinity to realize convex mixtures as real specializations.

We now describe one such encoding, sufficient for the interlacing argument.

Step 1 (Encoding colorings by tree differences).
Fix a spanning tree T of G rooted at an arbitrary vertex v_0. Every coloring $c : V \to [k]$ induces edge-differences $\xi \in (\mathbb{Z}_k)^T$ defined by

$$\xi_{uv} := c(v) - c(u) \pmod{k} \qquad (uv \in T,\ u \text{ parent of } v).$$

Conversely, given $(c(v_0), \xi)$, the coloring c is recovered uniquely by propagation along the tree. Under the i.i.d. uniform coloring model, the root color $c(v_0)$ is uniform in $[k]$ and the differences ξ_{uv} are independent and uniform in $\mathbb{Z}_k$.

Step 2 (Binary refinement of a uniform $\mathbb{Z}_k$ variable via subset splitting).
Fix once and for all a rooted full binary tree $\mathcal{B}$ whose leaves are labeled by the elements of $\mathbb{Z}_k$. Each node $u \in \mathcal{B}$ carries a nonempty subset $S(u) \subseteq \mathbb{Z}_k$ consisting of the labels of the leaves below u. Thus $S(u_{\mathrm{root}}) = \mathbb{Z}_k$, and if u has children u_L, u_R then $S(u) = S(u_L) \dot\cup S(u_R)$.

Starting at the root, if u is internal we move to u_L with probability $|S(u_L)|/|S(u)|$ and to u_R with probability $|S(u_R)|/|S(u)|$. We stop at a leaf ℓ and output its label. Equivalently, each internal node u carries a Bernoulli decision with parameter

$$\theta(u) := \frac{|S(u_R)|}{|S(u)|} \in (0,1).$$

Lemma K.1 (Correctness of subset splitting).
The leaf label produced by this refinement is exactly uniform on $\mathbb{Z}_k$. Moreover, conditioning on reaching a node u, the output is uniform on $S(u)$.

Proof.
Fix a leaf ℓ with root-to-leaf path $u_0 \to u_1 \to \cdots \to u_T = \ell$. Then

$$\mathbb{P}[\ell] = \prod_{t=0}^{T-1} \frac{|S(u_{t+1})|}{|S(u_t)|} = \frac{|S(\ell)|}{|S(u_0)|} = \frac{1}{k},$$

since $|S(\ell)| = 1$ and $|S(u_0)| = k$. The conditional statement follows by applying the same telescoping argument inside the subtree rooted at u.

Apply this refinement independently for each tree edge difference ξ_f (and, if desired, for the root color $c(v_0)$). This yields a global rooted binary tree whose leaves correspond to full assignments of all differences $\xi_f \in \mathbb{Z}_k$ and hence to full colorings (after choosing the root color).

Lemma K.2 (Tree-difference conditioning can break real-rootedness).
Even if a uniform coloring is exposed through spanning-tree differences, the naive conditional-expectation polynomials under partial exposure need not be real-rooted.

Proof.
Let $G = C_4$ be the cycle on four vertices and let $k = 2$. Use the Foster-normalized rank-one matrices $A_e = a_e a_e^\top$ from Appendix J, so that $\langle a_e, a_e \rangle = 3/4$ for all e and $\langle a_e, a_f \rangle = -1/4$ for all distinct $e \neq f$ (hence $\sum_{e \in E} A_e = P$ on $\mathbf{1}^\perp$).

Fix the spanning tree T consisting of three consecutive edges of the cycle, and encode a coloring by tree differences $\xi = (\xi_1, \xi_2, \xi_3) \in (\mathbb{Z}_2)^3$ together with the root color. Condition on the event $\xi_1 = 0$ and average over the remaining independent differences $(\xi_2, \xi_3) \in (\mathbb{Z}_2)^2$.

If $(\xi_2, \xi_3) = (0,0)$ then all four edges are monochromatic, so $M = P$ and therefore

$$p(\lambda) = \det(\lambda I_3 - P) = (\lambda - 1)^3$$

on $\mathbf{1}^\perp$. In each of the other three cases, exactly two edges are monochromatic and they form a forest $\{e, f\}$, so $M = A_e + A_f$. The nonzero eigenvalues of $A_e + A_f$ are the eigenvalues of the Gram matrix $\begin{pmatrix} \langle a_e, a_e \rangle & \langle a_e, a_f \rangle \\ \langle a_f, a_e \rangle & \langle a_f, a_f \rangle \end{pmatrix} = \begin{pmatrix} 3/4 & -1/4 \\ -1/4 & 3/4 \end{pmatrix}$, namely 1 and 1/2. Hence

$$p(\lambda) = \det(\lambda I_3 - (A_e + A_f)) = \lambda(\lambda - 1)(\lambda - 1/2).$$

Averaging these four cases yields

$$\begin{aligned} p_{\xi_1=0}(\lambda) &= \frac{1}{4}(\lambda-1)^3 + \frac{3}{4}\,\lambda(\lambda-1)(\lambda-1/2) \\ &= \frac{1}{8}(\lambda-1)(8\lambda^2 - 7\lambda + 2). \end{aligned}$$

The discriminant of the quadratic factor equals $(-7)^2 - 4 \cdot 8 \cdot 2 = -15$, so $p_{\xi_1=0}$ has non-real roots.

Lemma K.3 (Obstruction to marginal-probability labeling in correlated one-hot models).
Let $Q(\lambda, z_1, z_2)$ be multiaffine in (z_1, z_2) and suppose it contains a nonzero mixed term $z_1 z_2$. Then, in general,

$$Q(\lambda, -\tfrac{1}{2}, -\tfrac{1}{2}) \neq \tfrac{1}{2}Q(\lambda, -1, 0) + \tfrac{1}{2}Q(\lambda, 0, -1).$$

Consequently, labeling internal nodes by coordinatewise marginals $z_j = -\Pr[\text{choice } j]$ does not, in general, realize the leaf average via Theorem 7.1(3) unless all mixed terms vanish.

Proof.
Write $Q = \alpha + \beta z_1 + \gamma z_2 + \delta z_1 z_2$ with $\delta \neq 0$. Then

$$Q(\lambda, -\tfrac{1}{2}, -\tfrac{1}{2}) = \alpha - \tfrac{1}{2}(\beta + \gamma) + \tfrac{1}{4}\delta$$

while

$$\tfrac{1}{2}Q(\lambda, -1, 0) + \tfrac{1}{2}Q(\lambda, 0, -1) = \alpha - \tfrac{1}{2}(\beta + \gamma).$$

The difference is $\delta/4 \neq 0$.

Step 3 (Atomized determinantal master polynomial).
The required auxiliary space and the rank-one coefficients $C_{e,\eta}$, as well as the leaf/root identification, are developed in Appendix K.4 and implemented in §§K.5–K.6.

K.4 A physical–internal (multi-witness) bridge for Step 3

In view of Lemma K.2 and Lemma K.3, a naive specialization tree that simply exposes a uniform coloring (or its spanning-tree differences) and identifies internal nodes with conditional expectation polynomials cannot be used to form an interlacing family. Step 3 therefore requires an auxiliary parameter space whose additional degrees of freedom ("witness variables") allow one to connect the root polynomial $p_\emptyset$ to the leaf polynomials p_c through a Route B specialization tree.

The purpose of this section is to formalize the Step 3 construction target in a self-contained manner.

K.4.1 Physical variables: spanning-tree differences Fix a rooted spanning tree $T \subseteq E$ and orient its edges away from the root v_0. For a coloring $c : V \to \mathbb{Z}_k$, define the tree differences

$$\xi_f := c(\mathrm{head}(f)) - c(\mathrm{tail}(f)) \in \mathbb{Z}_k, \qquad f \in T.$$

Together with the root color $c(v_0)$, these differences determine c uniquely. Moreover, under the uniform coloring model, the root color and the differences $(\xi_f)_{f\in T}$ are independent and uniform.

We will generate each $\xi_f \sim \mathrm{Unif}(\mathbb{Z}_k)$ via the subset-splitting refinement of Lemma K.1.

K.4.2 Internal variables: partial-sum witnesses for non-tree edges For each non-tree edge $e = \{u, v\} \in E \setminus T$, let $P_T(u, v) = f_{e,1}, \dots, f_{e,\ell(e)}$ be the unique simple path in T from u to v. Let $s_{e,i} \in \{+1, -1\}$ be the sign induced by the comparison between the tree orientation of $f_{e,i}$ and the path orientation.

Introduce witness variables

$$\sigma_{e,0}, \sigma_{e,1}, \dots, \sigma_{e,\ell(e)} \in \mathbb{Z}_k$$

subject to the constraints

$$\sigma_{e,0} = 0, \qquad \sigma_{e,i} = \sigma_{e,i-1} + s_{e,i}\, \xi_{f_{e,i}} \pmod{k} \quad (i = 1, \dots, \ell(e)). \tag{19}$$

Then, for the coloring c determined by $(c(v_0), \xi)$,

$$c(u) - c(v) \equiv \sigma_{e,\ell(e)} \pmod{k}, \qquad X_e(c) = \mathbf{1}[\sigma_{e,\ell(e)} = 0]. \tag{20}$$

Thus the monochromaticity of each non-tree edge is reduced to a local end-test on a single witness value.

K.4.3 Step 3 target: a Route B specialization tree with witnesses Let $\mathcal{L}$ denote the set of all complete assignments to the physical variables $(c(v_0), \xi)$ together with witness variables $(\sigma_{e,i})$ satisfying the constraints in (19). Each leaf $\ell \in \mathcal{L}$ projects to a well-defined coloring $c(\ell)$.

Step 3 asks for an explicit determinantal master polynomial and a specialization tree with the following properties.

Proposition K.4 (Reduction of Step 3 to a witness-compatible determinantal master polynomial).
Suppose there exist:

1. a multiaffine real-stable determinantal polynomial

$$Q(\lambda, \mathbf{z}) = \det\Big(\lambda I_D + \sum_{j=1}^{N} z_j \, C_j\Big),$$

   whose coefficient matrices C_j are rank-one PSD;

2. a rooted binary specialization tree $\mathcal{T}$ that labels each node u with a specialization vector $\mathbf{z}(u) \in [-1, 0]^N$ such that each internal node interpolates its children as in Theorem 7.1(3), i.e. the two children differ in exactly one coordinate $z_{j(u)} \in \{0, -1\}$ and the parent sets that coordinate to $z_{j(u)} = -\theta(u)$ for some $\theta(u) \in [0, 1]$;

3. a surjective map from leaves of $\mathcal{T}$ onto k-colorings of V such that every leaf specialization equals the corresponding leaf polynomial, up to a coloring-independent power of λ:

$$Q(\lambda, \mathbf{z}(\ell)) = \lambda^M \, p_{c(\ell)}(\lambda) \qquad (\ell \text{ a leaf of } \mathcal{T});$$

4. a root identification

$$Q(\lambda, \mathbf{z}(u_{\mathrm{root}})) = \lambda^M \, p_{\emptyset}(\lambda).$$

Then Corollary 7.3 follows by the interlacing selection argument of Appendix G.

Proof.
By (1), Q is real stable and multiaffine in each z_j. By (2), Theorem 7.1 and Appendix G imply the existence of a leaf ℓ with

$$\rho_{\max}(Q(\cdot, \mathbf{z}(\ell))) \leq \rho_{\max}(Q(\cdot, \mathbf{z}(u_{\mathrm{root}}))).$$

Using (3)–(4), and noting that multiplying by λ^M does not change the largest root, we obtain

$$\rho_{\max}(p_{c(\ell)}) \leq \rho_{\max}(p_{\emptyset}),$$

which is exactly Corollary 7.3.

We carry out this construction in §§K.5–K.6.

K.5 Construction of a witness-compatible determinantal master polynomial

We construct a multiaffine real-stable determinantal polynomial whose variables correspond to the Route B refinement decisions. The construction is deliberately high-dimensional but completely explicit.

K.5.1 The extended assignment space Fix $k \geq 2$ and a rooted spanning tree $T \subseteq E$ oriented away from the root v_0.

We consider two families of discrete variables:

- **Physical variables**: the tree differences $\xi = (\xi_f)_{f \in T} \in (\mathbb{Z}_k)^T$.
- **Internal (witness) variables**: for every non-tree edge $e = \{u, v\} \in E \setminus T$, introduce a terminal witness

$$y_e \in \mathbb{Z}_k.$$

  Intuitively, y_e will ultimately be forced to equal the tree-path sum $c(u) - c(v)$, but at the root it will be independent.

Define the extended assignment space

$$\Omega := (\mathbb{Z}_k)^T \times (\mathbb{Z}_k)^{E \setminus T}, \qquad \omega = (\xi, y).$$

Given $\omega = (\xi, y)$ we define the *extended monochromaticity indicators*

$$\widetilde{X}_f(\omega) := \mathbf{1}[\xi_f = 0] \quad (f \in T), \qquad \widetilde{X}_e(\omega) := \mathbf{1}[y_e = 0] \quad (e \in E \setminus T).$$

These indicators are fully local on Ω.

Later, in K.6, we will force y_e to agree with the true tree-path sum, so that $\widetilde{X}_e(\omega)$ becomes the genuine monochromaticity indicator $X_e(c)$.

K.5.2 The lifted space and rank-one atomization Let $\mathsf{H} := \mathbb{R}^\Omega$ with the standard orthonormal basis $\{|\omega\rangle : \omega \in \Omega\}$. For each edge $e \in E$, define the diagonal projector

$$Q_e := \sum_{\omega \in \Omega} \widetilde{X}_e(\omega)\ |\omega\rangle\langle\omega|.$$

Let $A_e = a_e a_e^\top$ be the rank-one PSD matrices on $\mathbf{1}^\perp$ defined in §4.

Define the lifted matrix acting on $\mathbf{1}^\perp \otimes \mathsf{H}$:

$$\mathcal{M} := \sum_{e \in E} A_e \otimes Q_e.$$

This matrix is PSD.

Now atomize the diagonal operators into rank-one pieces: for each $\omega \in \Omega$, set $B_\omega := |\omega\rangle\langle\omega|$ and define

$$C_{e,\omega} := \widetilde{X}_e(\omega)\ (A_e \otimes B_\omega).$$

Then $C_{e,\omega} \succeq 0$ and $\mathrm{rank}(C_{e,\omega}) \le 1$, and

$$\sum_{\omega \in \Omega} C_{e,\omega} = A_e \otimes Q_e.$$

K.5.3 The determinantal master polynomial Introduce variables $\{z_{e,\omega}\}_{e \in E,\ \omega \in \Omega}$ and define

$$Q(\lambda, \mathbf{z}) := \det\Big(\lambda I_{d|\Omega|} + \sum_{e \in E} \sum_{\omega \in \Omega} z_{e,\omega}\, C_{e,\omega}\Big).$$

Lemma K.5 (Real stability and multiaffinity).
The polynomial $Q(\lambda, \mathbf{z})$ is real stable in $(\lambda, \mathbf{z})$ and multiaffine in each coordinate $z_{e,\omega}$.

Proof.
Each coefficient matrix $C_{e,\omega}$ is PSD and rank at most one. The determinantal stability criterion (Appendix F) implies real stability. Rank-one coefficients imply that the degree in each variable $z_{e,\omega}$ is at most one, hence multiaffinity.

K.6 Construction of the specialization tree and identification of leaves/roots

We now construct a rooted binary specialization tree $\mathcal{T}$ satisfying Proposition K.4 for the polynomial $Q(\lambda, \mathbf{z})$ from K.5.

K.6.1 A Route B specialization tree over consistent assignments Let

$$\Omega_{\mathrm{cons}} := \{\hat{\omega} = (\xi, \hat{y}) \in \Omega :\ \hat{y}_e = \Delta_e(\xi)\ \forall e \in E \setminus T\}.$$

For each $\hat{\omega} \in \Omega_{\mathrm{cons}}$ we will define a leaf specialization vector $\mathbf{z}(\hat{\omega})$ as in K.6.3. In this section we build an explicit rooted binary specialization tree $\mathcal{T}$ whose leaves are exactly these specializations, and such that the induced leaf weights are uniform on Ω_{cons}.

Fix an arbitrary edge $e_* \in E$. For each $\hat{\omega} \in \Omega_{\mathrm{cons}}$, consider the coordinate $z_{e_*,\hat{\omega}}$. By definition of the leaf specializations in K.6.3, the coordinate $z_{e_*,\hat{\omega}}$ takes value -1 at the leaf corresponding to $\hat{\omega}$ and takes value 0 at every other leaf.

We order the elements of Ω_{cons} arbitrarily as $\hat{\omega}^{(1)}, \dots, \hat{\omega}^{(m)}$, where $m := |\Omega_{\mathrm{cons}}|$. Starting from the root, we perform a sequential selection procedure: at stage t we split on the single coordinate $z_{e_*,\hat{\omega}^{(t)}}$. Let $m_t := m - t + 1$ be the number of remaining candidates at stage t, and set

$$\theta_t := \frac{1}{m_t}.$$

The two children set $z_{e_*,\hat{\omega}^{(t)}}$ to 0 (reject) and -1 (accept), respectively, and the parent sets it to $-\theta_t$. If the accept child is taken, all remaining coordinates are then fixed deterministically, one at a time, to match the leaf vector $\mathbf{z}(\hat{\omega}^{(t)})$ (using degenerate splits with $\theta \in \{0, 1\}$). If the reject child is taken, we proceed to stage $t + 1$.

This produces a rooted binary specialization tree in which each internal node differs from its children in exactly one coordinate, and the induced leaf weights are uniform: every $\hat{\omega} \in \Omega_{\mathrm{cons}}$ is selected with probability $1/m$.

K.6.2 (Reserved) We do not require any additional refinement beyond the specialization tree described in K.6.1.

K.6.3 Leaf specializations For a refined leaf $\hat{\omega} \in \Omega$, define the specialization vector $\mathbf{z}(\hat{\omega})$ by

$$z_{e,\omega}(\hat{\omega}) := \begin{cases} -1, & \omega = \hat{\omega}, \\ 0, & \omega \neq \hat{\omega}. \end{cases}$$

Then the matrix inside the determinant in K.5.3 is block diagonal across $\omega \in \Omega$, and only the $\hat{\omega}$-block contributes nontrivially. Therefore

$$Q(\lambda, \mathbf{z}(\hat{\omega})) = \lambda^{d(|\Omega|-1)} \det\Big(\lambda I_d - \sum_{e \in E} \widetilde{X}_e(\hat{\omega})\, A_e\Big) = \lambda^{d(|\Omega|-1)}\, p_{c_\xi}(\lambda),$$

where the last equality uses $\widetilde{X}_e(\hat{\omega}) = X_e(c_\xi)$.

This proves the leaf identification (Proposition K.4(3)) with $M = d(|\Omega| - 1)$.

Lemma K.7 (Route B averaging identity).
Let $Q(\lambda, \mathbf{z})$ be multiaffine in every coordinate z_j. Suppose a rooted binary tree $\mathcal{T}$ labels each node u by $\mathbf{z}(u)$ such that:

- each internal node u has children $u0, u1$ differing only in one coordinate $j(u)$,

- at the children, $z_{j(u)}(u0) = 0$ and $z_{j(u)}(u1) = -1$ (all other coordinates equal),
- at the parent, $z_{j(u)}(u) = -\theta(u)$ with $\theta(u) \in [0, 1]$.

Then for each internal node u,

$$Q(\lambda, \mathbf{z}(u)) = (1 - \theta(u))\, Q(\lambda, \mathbf{z}(u0)) + \theta(u)\, Q(\lambda, \mathbf{z}(u1)).$$

Consequently, for the root r,

$$Q(\lambda, \mathbf{z}(r)) = \sum_{\ell \text{ leaf}} w(\ell)\, Q(\lambda, \mathbf{z}(\ell)),$$

where $w(\ell)$ is the product of the split weights along the root-to-leaf path.

Proof.
Fix an internal node u and write $Q(\lambda, \mathbf{z}) = A(\lambda) + B(\lambda)\, z_{j(u)}$ after freezing all other coordinates; this is possible since Q is multiaffine. Then

$$Q(\lambda, \mathbf{z}(u0)) = A(\lambda), \quad Q(\lambda, \mathbf{z}(u1)) = A(\lambda) - B(\lambda), \quad Q(\lambda, \mathbf{z}(u)) = A(\lambda) - \theta(u) B(\lambda),$$

hence $Q(u) = (1 - \theta) Q(u0) + \theta Q(u1)$. Iterating down the tree yields the root identity.

K.6.4 Root identification By Lemma K.7, the root specialization polynomial equals the weighted average of the leaf polynomials:

$$Q(\lambda, \mathbf{z}(u_{\text{root}})) = \sum_{\ell \text{ leaf}} w(\ell)\, Q(\lambda, \mathbf{z}(\ell)).$$

By the construction in K.6.1, the leaf weights $w(\ell)$ are uniform on Ω_{cons}. By K.6.3, for every $\hat{\omega} \in \Omega_{\text{cons}}$,

$$Q(\lambda, \mathbf{z}(\hat{\omega})) = \lambda^{d(|\Omega|-1)}\, p_{c_\xi}(\lambda),$$

where c_ξ is the coloring determined by the tree differences ξ.

Therefore

$$\lambda^{-d(|\Omega|-1)}\, Q(\lambda, \mathbf{z}(u_{\text{root}})) = \mathbb{E}_{\widehat{\omega} \sim \text{Unif}(\Omega_{\text{cons}})} \big[p_{c_\xi}(\lambda) \big].$$

Finally, the map $\xi \mapsto c_\xi$ is a bijection between $(\mathbb{Z}_k)^T$ and the set of colorings modulo global shifts. Since $M(c)$ (and hence p_c) is invariant under global shifts of c, the above expectation equals the uniform-coloring expectation defining $p_{\emptyset}(\lambda)$.

Thus

$$Q(\lambda, \mathbf{z}(u_{\mathrm{root}})) = \lambda^{d(|\Omega|-1)}\, p_{\emptyset}(\lambda),$$

which is Proposition K.4(4).

K.7 Decision-indexed refinement coordinates

Remark. The main proof of Corollary 7.3 uses only the existence of the specialization tree from K.6.1 together with Lemma K.7. The decision-indexed discussion below provides additional perspective but is not required for the argument.

The construction in K.5–K.6 is written using variables indexed by full assignments $\omega \in \Omega$. To make the "one-coordinate refinement" hypothesis of Theorem 7.1(3) explicit, it is convenient to re-index the auxiliary coordinates by the internal refinement decisions rather than by full assignments.

Fix the subset-splitting tree $\mathcal{B}$ for $\mathbb{Z}_k$ from Lemma K.1 and let $\mathcal{I}$ denote its internal nodes. For each refined coordinate X (either a tree difference ξ_f or a witness coordinate y_e) and each $u \in \mathcal{I}$, let $b_{X,u} \in \{0,1\}$ record whether the refinement path takes the right child at u. We introduce the corresponding Route B variables $z_{X,u} \in \{0,-1\}$ via $z_{X,u} = -b_{X,u}$, and internal-node values $z_{X,u} = -\theta(u)$ as in Lemma K.1.

The following selector lemma shows that, after expanding the auxiliary space if necessary, one can realize leaf projectors and their internal-node interpolants using only the decision-indexed variables. This is the point at which the "one-coordinate refinement" hypothesis becomes manifest.

Lemma K.8 (Decision-indexed leaf selectors for a fixed refinement tree).
Let $\mathcal{B}$ be a fixed rooted binary refinement tree for $\mathbb{Z}_k$, and let $\mathrm{Leaves}(\mathcal{B})$ denote its leaves. There exists a real inner-product space H_X with orthonormal basis $\{e_\ell\}_{\ell \in \mathrm{Leaves}(\mathcal{B})}$ and, for each internal node u of $\mathcal{B}$, orthogonal projectors $P_{u,\mathrm{L}}, P_{u,\mathrm{R}} \succeq 0$ on H_X such that:

1. $P_{u,\mathrm{L}}$ is the orthogonal projector onto $\mathrm{span}\{e_\ell : \ell \in \mathrm{Leaves}(u_\mathrm{L})\}$ and $P_{u,\mathrm{R}}$ is the orthogonal projector onto $\mathrm{span}\{e_\ell : \ell \in \mathrm{Leaves}(u_\mathrm{R})\}$.
2. For every internal node u, $P_{u,\mathrm{L}} + P_{u,\mathrm{R}}$ is the identity on $\mathrm{span}\{e_\ell : \ell \in \mathrm{Leaves}(u)\}$.

For a target leaf $\ell^\star \in \mathrm{Leaves}(\mathcal{B})$, define decision-indexed specializations $(z_{u,\mathrm{L}}, z_{u,\mathrm{R}}) \in \{0,-1\}^2$ by

- $z_{u,\mathrm{L}}(\ell^\star) = 0$ and $z_{u,\mathrm{R}}(\ell^\star) = -1$ if $\ell^\star \in \mathrm{Leaves}(u_\mathrm{L})$,

- $z_{u,\mathrm{L}}(\ell^\star) = -1$ and $z_{u,\mathrm{R}}(\ell^\star) = 0$ if $\ell^\star \in \mathrm{Leaves}(u_\mathrm{R})$.

Then the matrix product

$$\Pi_X(\ell^\star) := \prod_{u \in \mathrm{Int}(\mathcal{B})} (I_{\mathsf{H}_X} + z_{u,\mathrm{L}}(\ell^\star)\, P_{u,\mathrm{L}} + z_{u,\mathrm{R}}(\ell^\star)\, P_{u,\mathrm{R}})$$

equals the rank-one projector $e_{\ell^\star} e_{\ell^\star}^\top$. Moreover, for each internal node u, setting $(z_{u,\mathrm{L}}, z_{u,\mathrm{R}}) = (-(1-\theta(u)), -\theta(u))$ yields the convex interpolation

$$\Pi_X(u) = (1-\theta(u))\, \Pi_X(u_\mathrm{L}) + \theta(u)\, \Pi_X(u_\mathrm{R}),$$

where $\theta(u) := |\mathrm{Leaves}(u_\mathrm{R})|/|\mathrm{Leaves}(u)|$.

Proof.
The projectors $P_{u,\mathrm{L}}, P_{u,\mathrm{R}}$ split the subspace spanned by leaves under u into the disjoint left and right subspaces. For the specialization associated to $\ell^\star$, the factor at u equals $I - P_{u,\mathrm{R}}$ if $\ell^\star$ lies in the left subtree and equals $I - P_{u,\mathrm{L}}$ if $\ell^\star$ lies in the right subtree. Thus each factor removes exactly the leaf subspace not containing $\ell^\star$. Intersecting these constraints over all internal nodes leaves precisely the one-dimensional space $\mathrm{span}(e_{\ell^\star})$, hence the product equals $e_{\ell^\star} e_{\ell^\star}^\top$. The interpolation identity follows from multiaffinity in the pair $(z_{u,\mathrm{L}}, z_{u,\mathrm{R}})$ and the definition of $\theta(u)$.

Applying Lemma K.8 coordinatewise and taking tensor products over all refined coordinates produces mutually orthogonal rank-one leaf projectors indexed by full assignments, but generated by decision-indexed variables. In particular, leaf selection may be expressed using only the refinement decisions.

K.3 Summary of what is used in the main proof

Only the following facts about the Route B construction are used downstream:

1. There exists a multiaffine real-stable determinantal polynomial $Q(\lambda, \mathbf{z})$ with rank-one PSD coefficients.
2. There exists a rooted binary specialization tree $\mathcal{T}$ labeling each node with $\mathbf{z}(u) \in [-1, 0]^N$ such that internal nodes interpolate their children as in Theorem 7.1(3).
3. The root polynomial equals $p_\emptyset$ and the leaves equal the leaf polynomials p_c for full k-colorings.

Given (1)–(3), the interlacing argument is completely self-contained (Theorem 7.1 and Appendix G), and Corollary 7.3 follows from the root bound of Appendix H.

Appendix J. A counterexample to conditional real-rootedness under vertex exposure

This appendix records a minimal counterexample showing that conditional expectation polynomials under the unconstrained i.i.d. vertex-exposure filtration need not be real-rooted.

Proposition J.1 (A minimal counterexample).
Let $G = C_4$ be the cycle on four vertices and let $k = 2$. Let $c : V \to [2]$ be an unconstrained i.i.d. uniform 2-coloring, and let σ be the partial assignment fixing two adjacent vertices to color 1. Then the conditional expectation polynomial $p_\sigma(\lambda) = \mathbb{E}[p_c(\lambda) \mid \sigma]$ has non-real roots. In fact,

$$p_\sigma(\lambda) = \frac{1}{8}\lambda(\lambda - 1)(8\lambda^2 - 7\lambda + 2),$$

and the quadratic factor has negative discriminant -15.

Proof.
Label the vertices of C_4 as $1, 2, 3, 4$ in cyclic order and let the edges be $e_{12}, e_{23}, e_{34}, e_{41}$. Let L be the Laplacian of C_4 and $L^\dagger$ its Moore–Penrose pseudoinverse. One checks that

$$L^\dagger = \frac{1}{16}\begin{pmatrix} 5 & -1 & -3 & -1 \\ -1 & 5 & -1 & -3 \\ -3 & -1 & 5 & -1 \\ -1 & -3 & -1 & 5 \end{pmatrix}.$$

For an edge $e = \{u, v\}$, write $b_e := \mathbf{e}_u - \mathbf{e}_v$ and $A_e := L^{\dagger/2} b_e b_e^\top L^{\dagger/2} = a_e a_e^\top$ with $a_e := L^{\dagger/2} b_e$. The Gram inner products satisfy

$$\langle a_e, a_f \rangle = a_e^\top a_f = b_e^\top L^\dagger b_f.$$

A direct evaluation using the displayed $L^\dagger$ shows that for every edge e,

$$\langle a_e, a_e \rangle = \frac{3}{4},$$

and for every distinct pair of edges $e \neq f$,

$$\langle a_e, a_f \rangle = -\frac{1}{4}.$$

Now fix the partial assignment σ given by $c(1) = c(2) = 1$. There are four completions of σ to a full coloring, corresponding to the choices of $(c(3), c(4)) \in \{1, 2\}^2$.

- If $(c(3), c(4)) = (1, 1)$, then all four edges are monochromatic, so $M(c) = \sum_{e \in E} A_e$. By Foster's theorem in the normalized form Eq. (5.2), $\sum_{e \in E} A_e = P$, the projection onto $\mathbf{1}^{\perp}$. Thus $M(c)$ has eigenvalues $0, 1, 1, 1$, and hence $p_c(\lambda) = \lambda(\lambda - 1)^3$.
- In each of the other three completions, exactly two edges are monochromatic. In every case the monochromatic edge set is a forest of size two, say $\{e, f\}$. Then $M(c) = A_e + A_f = a_e a_e^{\top} + a_f a_f^{\top}$. The nonzero eigenvalues of $M(c)$ agree with the eigenvalues of the 2×2 Gram matrix

  $$G_{\{e,f\}} = \begin{pmatrix} \langle a_e, a_e \rangle & \langle a_e, a_f \rangle \\ \langle a_f, a_e \rangle & \langle a_f, a_f \rangle \end{pmatrix} = \begin{pmatrix} 3/4 & -1/4 \\ -1/4 & 3/4 \end{pmatrix}.$$

  The eigenvalues of $G_{\{e,f\}}$ are 1 and $1/2$. Therefore $M(c)$ has eigenvalues $0, 0, 1, 1/2$, and hence $p_c(\lambda) = \lambda^2(\lambda - 1)(\lambda - 1/2)$.

Averaging over the four completions (each of conditional probability $1/4$), we obtain

$$p_\sigma(\lambda) = \frac{1}{4}\lambda(\lambda - 1)^3 + \frac{3}{4}\lambda^2(\lambda - 1)(\lambda - 1/2) = \frac{1}{8}\lambda(\lambda - 1)(8\lambda^2 - 7\lambda + 2).$$

The discriminant of the quadratic factor is $(-7)^2 - 4 \cdot 8 \cdot 2 = -15 < 0$, so p_σ has non-real roots.

Remark J.2.
This rules out a direct application of an interlacing-family argument along the unconstrained vertex-coloring tree. In the main proof we instead use Route B: a binary specialization tree based on multiaffine determinantal specializations, which is compatible with real-stability preservers (Appendices G, H, K).