Finite Stam Inequality for Symmetric Additive Finite Free Convolution

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Abstract

We study a finite-dimensional Fisher-information–type functional Φ_n defined on the root set of a real-rooted polynomial and its behavior under the symmetric additive finite free convolution $\boxplus_n$. We give a complete discrete integration-by-parts calculus that identifies $\Phi_n(p) = n\,|J_p|^2_{\mathcal{H}_p}$ for an intrinsic score vector J_p on the roots. We then reduce the finite Stam inequality of Problem #4 in *First Proof* (arXiv:2602.05192) to a single score-identification condition for a contraction $\mathbb{E}_{\boxplus} : \mathcal{H}_{p,q} \to \mathcal{H}_r$ on a canonical two-variable Hilbert space. Finally, we show that the most naive candidate obtained by orthogonal projection onto the raw sum-grid subspace $\{f(\lambda_i + \mu_j)\}$ does not satisfy this score-identification in general; thus any proof of the finite Stam inequality by the projection-and-optimize mechanism must employ a different $\boxplus_n$-adapted conditioning map. All proofs are self-contained and deferred to appendices.

Contents

Main results

Our main contribution is a reduction of Problem #4 of *First Proof* (arXiv:2602.05192) to a single score-identification condition for a contraction on a canonical two-variable Hilbert space. In particular, the projection-and-optimize mechanism yields the finite Stam inequality once such a contraction is constructed.

1. **Score representation** of Φ_n as a squared norm (Theorem 3.6; proof in Appendix B).
2. $\boxplus_n$**-sum subspace** $\mathcal{H}_{\boxplus}(p, q) \subset \mathcal{H}_{p,q}$ and the associated contraction $\mathbb{E}_{\boxplus}$ (Appendix D).
3. **Hilbert-space reduction** of the Stam bound to a contraction plus score-identification (Proposition E.4; Appendix E).

4. **Obstruction for the raw sum-grid projection:** the map $\mathsf{E}_{\mathrm{sum}}$ obtained from the raw sum-grid subspace does not satisfy the required score-identification in general (Appendix E.4.1–E.4.2).

Accordingly, constructing a $\boxplus_n$-adapted conditioning map that does satisfy the score-identification remains the remaining step for a complete resolution of Problem #4 via the projection-and-optimize mechanism.

Overview of proofs

The argument follows the classical "projection-and-optimize" route behind Stam's inequality, adapted to the finite free convolution setting.

1. **Root Hilbert space and discrete derivatives.** For a monic polynomial p with simple real roots, we build a finite-dimensional Hilbert space $\mathcal{H}_p$ whose vectors are functions on the root set, with the uniform inner product. A canonical derivative operator on $\mathcal{H}_p$ is defined by differentiating the unique Lagrange interpolant. Its adjoint yields a "score" vector J_p.
2. **Identifying $\Phi_n(p)$ with $|J_p|^2$.** We prove that $\Phi_n(p) = n|J_p|^2_{\mathcal{H}_p}$ where Φ_n is precisely the functional from Problem #4. This is a discrete integration-by-parts identity.
3. **Two-variable space and projection formula.** For p, q we form a two-variable Hilbert space $\mathcal{H}_{p,q}$, a tensor-product root space. We then construct a distinguished n-dimensional "$\boxplus_n$-sum subspace" $\mathcal{H}_{\boxplus}(p,q) \subset \mathcal{H}_{p,q}$ canonically associated to the polynomial $r = p \boxplus_n q$. The key structural statement is the existence of a contraction $\mathbb{E}_{\boxplus} : \mathcal{H}_{p,q} \to \mathcal{H}_r$ that identifies the lifted scores and yields the one-parameter quadratic bound used in the proof of Stam's inequality. Appendix D constructs $\mathbb{E}_{\boxplus}$ from root sensitivities and reduces the remaining verification to a Bessel-type bound for a matching lift.
4. **Optimization.** The contraction property of orthogonal projection gives

$$\Phi_n(r) \le a^2 \Phi_n(p) + (1-a)^2 \Phi_n(q)$$

and optimizing over $a \in \mathbb{R}$ yields the Stam inequality in the reciprocal form.

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1. Introduction

Stam's inequality is a fundamental monotonicity principle for Fisher information under addition of independent variables. Problem #4 in *First Proof* asks for an analogous statement in a finite-dimensional "free" setting, where the operation is the symmetric additive finite free convolution $\boxplus_n$ on degree-n polynomials and the role of Fisher information is played by a root-functional Φ_n. (arXiv)

The operation $\boxplus_n$ is a classical convolution studied since Walsh and Szegő; it is also the symmetric additive convolution appearing in the finite free probability theory developed by Marcus–Spielman–Srivastava, which supplies a rich algebraic calculus (bilinearity, commutation with differentiation/translation, and real-rootedness preservation).

The present paper isolates the precise "Stam mechanism" behind the conjectured inequality and packages it into two structural theorems. While the appendices carry the full proofs, the main text keeps the argument transparent: once the score is identified and the projection identity is established, the inequality is a one-line optimization.

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2. Notation and the finite free convolution $\boxplus_n$

Throughout, $n \geq 2$ is fixed.

2.1 Polynomials and roots

Let $\mathbb{R}_n[x]$ denote the set of real monic polynomials of degree n. For $p \in \mathbb{R}_n[x]$ with all roots real, we write

$$p(x) = \prod_{i=1}^{n}(x - \lambda_i) \qquad \lambda_1 \leq \cdots \leq \lambda_n.$$

When p has repeated roots we will explicitly indicate this (and our functional Φ_n will be $+\infty$ by definition).

2.2 Definition of $\boxplus_n$

Definition 2.1 (Symmetric additive finite free convolution $\boxplus_n$).

Let $p, q \in \mathbb{R}_n[x]$ be written in coefficient form

$$p(x) = \sum_{k=0}^{n} a_k x^{n-k} \qquad q(x) = \sum_{k=0}^{n} b_k x^{n-k} \qquad a_0 = b_0 = 1.$$

Define the polynomial $r = p \boxplus_n q$ by

$$(p \boxplus_n q)(x) = \sum_{k=0}^{n} c_k x^{n-k}$$

where for each $k = 0, 1, \dots, n$

$$c_k = \sum_{i+j=k} \frac{(n-i)!\,(n-j)!}{n!\,(n-k)!}\, a_i b_j.$$

This is exactly the operation denoted $\boxplus_n$ in Problem #4 of *First Proof.* (arXiv)

Remark 2.2 (Equivalence to the standard finite free convolution conventions). If one rewrites $p(x) = \sum_{i=0}^{n} (-1)^i \alpha_i x^{n-i}$ and $q(x) = \sum_{i=0}^{n} (-1)^i \beta_i x^{n-i}$ then Definition 2.1 matches the symmetric additive convolution $+_n$ in Marcus–Spielman–Srivastava. In particular, one obtains the differential-operator identity

$$(p \boxplus_n q)(x) = \frac{1}{n!} \sum_{k=0}^{n} p^{(k)}(x)\, q^{(n-k)}(0)$$

and $\boxplus_n$ commutes with differentiation and translation.

2.3 The functional Φ_n

Definition 2.3 (The functional Φ_n).

Let $p \in \mathbb{R}_n[x]$ have simple real roots $\lambda_1, \dots, \lambda_n$. Define

$$\Phi_n(p) := \sum_{i=1}^{n} \left(\sum_{j=1\ j\neq i}^{n} \frac{1}{\lambda_i - \lambda_j} \right)^2 .$$

If p has a multiple root, set $\Phi_n(p) := +\infty$.

This is exactly the functional in Problem #4 of *First Proof.* (arXiv)

2.4 Statement of the finite Stam inequality

Theorem 2.4 (Conditional finite Stam inequality for $\boxplus_n$).

Let $p, q \in \mathbb{R}_n[x]$ be monic real-rooted polynomials of degree n and let $r = p \boxplus_n q$. Assume Conjecture 4.1 holds whenever p, q have simple real roots. Then

$$\boxed{\frac{1}{\Phi_n(r)} \ \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}} \qquad \text{with the convention } \frac{1}{\infty} = 0.$$

Equivalently, whenever $\Phi_n(p)\Phi_n(q) < \infty$,

$$\boxed{\Phi_n(p \boxplus_n q) \ \leq \frac{\Phi_n(p)\,\Phi_n(q)}{\Phi_n(p) + \Phi_n(q)}}.$$

Assuming Conjecture 4.1, this theorem would answer Problem #4 of *First Proof* in the affirmative. (arXiv)

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3. Root Hilbert spaces and discrete scores

The goal of this section is to reinterpret $\Phi_n(p)$ as a squared norm of an intrinsic score vector on the roots.

3.1 The root-trace inner product

Definition 3.1 (Root-trace Hilbert space $\mathcal{H}_p$).

Assume $p \in \mathbb{R}_n[x]$ has simple real roots $\lambda_1, \dots, \lambda_n$. Let

$$\mathcal{H}_p := \mathbb{R}^n$$

equipped with the uniform inner product

$$\langle u, v \rangle_{\mathcal{H}_p} := \frac{1}{n} \sum_{i=1}^{n} u_i v_i.$$

We interpret $u \in \mathcal{H}_p$ as a function on the root set via $u_i = u(\lambda_i)$.

3.2 Interpolation, discrete differentiation, and the score

Fix p as above. Let

$$\mathrm{Ev}_p : \mathbb{R}_{<n}[x] \to \mathcal{H}_p, \qquad (\mathrm{Ev}_p f)_i := f(\lambda_i)$$

be the evaluation map on the roots, and let

$$\mathrm{Int}_p : \mathcal{H}_p \to \mathbb{R}_{<n}[x]$$

be its inverse (Lagrange interpolation) i.e. the unique polynomial of degree $< n$ matching prescribed values on $\{\lambda_i\}$.

Definition 3.2 (Discrete derivative operator D_p).

Define the operator $D_p : \mathcal{H}_p \to \mathcal{H}_p$ by

$$D_p := \mathrm{Ev}_p \circ \frac{d}{dx} \circ \mathrm{Int}_p.$$

This is the unique way to differentiate functions on a finite set of nodes by differentiating their interpolants.

Definition 3.3 (Score vector J_p).

Let $\mathbf{1} \in \mathcal{H}_p$ be the constant vector with all entries 1. Define

$$J_p := -D_p^* \mathbf{1},$$

where D_p^* is the adjoint with respect to the inner product on $\mathcal{H}_p$.

Lemma 3.4 (Zero-mean property).

If p has simple real roots, then

$$\sum_{i=1}^{n} J_p(\lambda_i) = 0.$$

Proof. Using the explicit formula in Lemma 3.5 below, each unordered pair $\{i, j\}$ contributes $\frac{1}{\lambda_i - \lambda_j} + \frac{1}{\lambda_j - \lambda_i} = 0$.

Lemma 3.5 (Score equals the root interaction field).

For each root λ_i of p

$$J_p(\lambda_i) = \sum_{j=1\ j \neq i}^{n} \frac{1}{\lambda_i - \lambda_j}.$$

A complete proof is given in Appendix B.

Theorem 3.6 (Score representation of Φ_n).

If $p \in \mathbb{R}_n[x]$ has simple real roots, then

$$\boxed{\Phi_n(p) = n \, |J_p|_{\mathcal{H}_p}^2.}$$

Proof. By Lemma 3.5, the i-th entry of J_p is exactly the inner sum defining $\Phi_n(p)$. Therefore,

$$|J_p|_{\mathcal{H}_p}^2 = \frac{1}{n} \sum_{i=1}^{n} J_p(\lambda_i)^2 = \frac{1}{n} \Phi_n(p).$$

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4. Two-variable space and the projection identity

This section states the structural identity that replaces the classical conditional-expectation identity for scores of sums of independent random variables.

4.1 The product root space

Let $p, q \in \mathbb{R}_n[x]$ be monic with simple real roots $\{\lambda_i\}$ and $\{\mu_j\}$. Define the product Hilbert space

$$\mathcal{H}_{p,q} := \mathcal{H}_p \otimes \mathcal{H}_q \cong \mathbb{R}^{n\times n}$$

with inner product

$$\langle U, V\rangle_{\mathcal{H}_{p,q}} := \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} U_{ij}V_{ij}.$$

We lift J_p and J_q to $\mathcal{H}_{p,q}$ by

$$(J_p^{\uparrow})_{ij} := J_p(\lambda_i) \qquad (J_q^{\uparrow})_{ij} := J_q(\mu_j).$$

By Lemma 3.4, these lifts are orthogonal:

$$\langle J_p^{\uparrow}, J_q^{\uparrow}\rangle_{\mathcal{H}_{p,q}} = \left(\frac{1}{n}\sum_{i=1}^{n} J_p(\lambda_i)\right)\left(\frac{1}{n}\sum_{j=1}^{n} J_q(\mu_j)\right) = 0.$$

4.2 The $\boxplus_n$ sum subspace and orthogonal projection

Let $r := p\boxplus_n q$. One may canonically associate an n-dimensional closed subspace

$$\mathcal{H}_{\boxplus}(p, q) \subset \mathcal{H}_{p,q}$$

and an isometric identification

$$\iota_{p,q}^{\boxplus} : \mathcal{H}_r \xrightarrow{\cong} \mathcal{H}_{\boxplus}(p, q)$$

whose construction depends functorially on the convolution structure (Appendix D). The adjoint

$$\mathbb{E}_{\boxplus} := (\iota_{p,q}^{\boxplus})^* : \mathcal{H}_{p,q} \to \mathcal{H}_r$$

acts as a “conditional expectation onto the sum variable” and satisfies

$$|\mathbb{E}_{\boxplus}(U)|_{\mathcal{H}_r} \le |U|_{\mathcal{H}_{p,q}} \qquad \forall U \in \mathcal{H}_{p,q}$$

because $\iota_{p,q}^{\boxplus}$ is an isometry.

Conjecture 4.1 (Score projection identity).

Let $p, q \in \mathbb{R}_n[x]$ be monic with simple real roots, and let $r = p \boxplus_n q$. Then for every $a \in \mathbb{R}$

$$\boxed{J_r = \mathbb{E}_{\boxplus}(a\, J_p^{\uparrow} + (1-a)\, J_q^{\uparrow}).}$$

Appendix E reduces this conjecture to an explicit finite-dimensional verification and shows that it fails for the raw sum-grid conditioning map $\mathsf{E}_{\mathrm{sum}}$ (Appendix E.4.1–E.4.2).

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5. Conditional derivation of the finite Stam inequality

We now derive the finite Stam inequality from Theorem 3.6 and Conjecture 4.1 in the simple-root case, and then explain the limiting extension to multiple roots.

5.1 Simple-root case

Assume p, q have simple real roots and set $r = p \boxplus_n q$. Assume Conjecture 4.1. Fix $a \in \mathbb{R}$. By the conjectured projection identity and the contraction property of $\mathbb{E}_{\boxplus}$

$$|J_r|^2_{\mathcal{H}_r} = \left|\mathbb{E}_{\boxplus}(aJ_p^{\uparrow} + (1-a)J_q^{\uparrow})\right|^2_{\mathcal{H}_r} \le |aJ_p^{\uparrow} + (1-a)J_q^{\uparrow}|^2_{\mathcal{H}_{p,q}}.$$

Using orthogonality of $J_p^{\uparrow}$ and $J_q^{\uparrow}$ noted in §4.1,

$$|aJ_p^{\uparrow} + (1-a)J_q^{\uparrow}|^2_{\mathcal{H}_{p,q}} = a^2|J_p^{\uparrow}|^2_{\mathcal{H}_{p,q}} + (1-a)^2|J_q^{\uparrow}|^2_{\mathcal{H}_{p,q}}.$$

Since $|J_p^{\uparrow}|_{\mathcal{H}_{p,q}} = |J_p|_{\mathcal{H}_p}$ and similarly for q we obtain

$$|J_r|^2_{\mathcal{H}_r} \le a^2|J_p|^2_{\mathcal{H}_p} + (1-a)^2|J_q|^2_{\mathcal{H}_q}.$$

Multiplying by n and applying Theorem 3.6 gives the quadratic bound

$$\boxed{\Phi_n(r) \le a^2\Phi_n(p) + (1-a)^2\Phi_n(q).}$$

Now minimize the right-hand side over $a \in \mathbb{R}$. Writing $A := \Phi_n(p)$ and $B := \Phi_n(q)$ (both > 0 in the simple-root case, $n \ge 2$) the minimum of $a^2A + (1-a)^2B$ is attained at

$$a^* = \frac{B}{A+B}$$

and equals

$$\frac{AB}{A+B}.$$

Therefore,

$$\Phi_n(r) \le \frac{\Phi_n(p)\Phi_n(q)}{\Phi_n(p) + \Phi_n(q)}.$$

Taking reciprocals yields Theorem 2.4 in the simple-root case:

$$\frac{1}{\Phi_n(r)} \ge \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

5.2 Multiple roots

If p has a multiple root, then $\Phi_n(p) = +\infty$ by definition, hence $1/\Phi_n(p) = 0$ and the inequality becomes weaker. The only nontrivial case is when $r = p \boxplus_n q$ may develop multiple roots even if p, q are simple, or conversely when one wants a uniform limiting argument across degenerations. Appendix F provides a perturbation-and-limit argument establishing that the inequality extends to all monic real-rooted polynomials with the convention $1/\infty = 0$.

This completes the conditional derivation of the finite Stam inequality from Conjecture 4.1. The multiple-root extension remains valid once the simple-root case is established.

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6. Equality and low-degree checks

Proposition 6.1 (The case $n = 2$).

For $n = 2$ equality holds in Theorem 2.4 for all monic real-rooted $p, q \in \mathbb{R}_2[x]$ with $\Phi_2(p)\Phi_2(q) < \infty$.

A direct computation is given in Appendix G.

Remark 6.2 (Mechanism of equality). In the proof of §5.1, equality forces simultaneous equality in (i) the projection contraction and (ii) the optimization step. Consequently, equality can be characterized in terms of the lifted scores $J_p^{\uparrow}, J_q^{\uparrow}$ lying in the sum subspace and being proportional in the appropriate sense. A detailed analysis (including classification results in special families) is deferred.

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7. Appendix structure for complete proofs

The complete proofs are organized into the following appendices.

- **Appendix A:** Algebraic properties of $\boxplus_n$ and equivalence of coefficient and differential-operator formulas (including commutation with differentiation and translation).
- **Appendix B:** Discrete interpolation calculus; proof of Lemma 3.5 and Theorem 3.6 (discrete integration by parts and score representation).
- **Appendix C:** Equivalent formulas for Φ_n (pair-interaction representation and auxiliary identities).
- **Appendix D:** Construction of the canonical sum subspace $\mathcal{H}_{\boxplus}(p, q) \subset \mathcal{H}_{p,q}$ and the isometry $\iota_{p,q}^{\boxplus}$.
- **Appendix E:** Reduction of the Stam bound to a single contraction with score identification (Proposition E.4).
- **Appendix F:** Degeneration to multiple roots and stability of the inequality under perturbation.

- **Appendix G:** Equality analysis in low degree (in particular $n = 2$) and additional examples.

References

1. M. Abouzaid, A. J. Blumberg, M. Hairer, J. Kileel, T. G. Kolda, P. D. Nelson, D. Spielman, N. Srivastava, R. Ward, S. Weinberger, L. Williams. *First Proof.* arXiv:2602.05192. (arXiv)
2. A. W. Marcus, D. A. Spielman, N. Srivastava. *Finite free convolutions of polynomials.* arXiv:1504.00350 (v2).

Appendix A. Algebraic properties and equivalent models of $\boxplus_n$

This appendix provides complete proofs of the algebraic identities and structural properties of the symmetric additive finite free convolution $\boxplus_n$ used in the main text, notably in §2.2–§2.4. We also record a stability-based proof that $\boxplus_n$ preserves real-rootedness, ensuring that $\Phi_n(p \boxplus_n q)$ is well-defined whenever p, q are real-rooted.

Throughout, $n \geq 2$ is fixed.

A.1 Bilinear extension and basic identities

For commutation identities with differentiation and translation it is convenient to work with a bilinear extension of $\boxplus_n$ to all polynomials of degree at most n.

Definition A.1 (Bilinear extension of $\boxplus_n$ on $\mathbb{R}_{\leq n}[x]$).

Let $\mathbb{R}_{\leq n}[x]$ be the real vector space of polynomials of degree at most n. For $p \in \mathbb{R}_{\leq n}[x]$ write its *degree-n padded coefficient vector* as

$$p(x) = \sum_{k=0}^{n} a_k x^{n-k} \qquad a_k \in \mathbb{R},$$

where we allow $a_0 = 0$ (this simply corresponds to $\deg p < n$). Define, for $p, q \in \mathbb{R}_{\leq n}[x]$

$$(p \boxplus_n q)(x) := \sum_{k=0}^{n} c_k x^{n-k}$$

with

$$c_k := \sum_{i,j\geq 0 \;\; i+j=k} \frac{(n-i)!\,(n-j)!}{n!\,(n-k)!}\, a_i b_j, \qquad (k = 0, 1, \dots, n)$$

where $q(x) = \sum_{k=0}^{n} b_k x^{n-k}$. {#eq:a-1}

When $p, q \in \mathbb{R}_n[x]$ are monic, we have $a_0 = b_0 = 1$ and $\deg(p \boxplus_n q) = n$.

Proposition A.2 (Bilinearity, commutativity, and degree bound).

The map

$$\boxplus_n : \mathbb{R}_{\leq n}[x] \times \mathbb{R}_{\leq n}[x] \to \mathbb{R}_{\leq n}[x]$$

defined by Definition A.1 is bilinear and commutative. Moreover $\deg(p \boxplus_n q) \leq n$ for all $p, q \in \mathbb{R}_{\leq n}[x]$ and if p, q are monic of degree n then $p \boxplus_n q$ is monic of degree n.

Proof. Bilinearity is immediate from the linearity of the coefficient formula in each input sequence (a_i), (b_j). Commutativity follows from symmetry of the summand under $(i, a_i) \leftrightarrow (j, b_j)$. The output is written explicitly as a linear combination of monomials x^{n-k} for $k = 0, \dots, n$ hence $\deg \leq n$. If p, q are monic of degree n then $a_0 = b_0 = 1$ and

$$c_0 = \frac{(n-0)!(n-0)!}{n!(n-0)!}\, a_0 b_0 = 1,$$

so $p \boxplus_n q$ is monic of degree n.

Proposition A.3 (Identity element).

Let $e_n(x) := x^n$. Then for all $p \in \mathbb{R}_{\leq n}[x]$

$$p \boxplus_n e_n = e_n \boxplus_n p = p.$$

Proof. Write $e_n(x) = \sum_{k=0}^{n} b_k x^{n-k}$ with $b_0 = 1$ and $b_k = 0$ for $k \geq 1$. Then for each k

$$c_k = \sum_{i+j=k} \frac{(n-i)!(n-j)!}{n!(n-k)!}\, a_i b_j = \frac{(n-k)!\, n!}{n!(n-k)!}\, a_k b_0 = a_k.$$

Thus $p \boxplus_n e_n = p$. Commutativity gives $e_n \boxplus_n p = p$.

Proposition A.4 (Alternating-sign convention).

Suppose we write

$$p(x) = \sum_{k=0}^{n} (-1)^k \alpha_k x^{n-k} \qquad q(x) = \sum_{k=0}^{n} (-1)^k \beta_k x^{n-k}.$$

Let $r = p \boxplus_n q$ be defined by Definition A.1 using the padded coefficients $a_k = (-1)^k \alpha_k$ and $b_k = (-1)^k \beta_k$. Then r admits the same alternating form

$$r(x) = \sum_{k=0}^{n} (-1)^k \gamma_k x^{n-k}$$

where

$$\gamma_k = \sum_{i+j=k} \frac{(n-i)!\,(n-j)!}{n!\,(n-k)!}\, \alpha_i \beta_j.$$

Proof. By Definition A.1,

$$c_k = \sum_{i+j=k} \frac{(n-i)!\,(n-j)!}{n!\,(n-k)!}\, a_i b_j = \sum_{i+j=k} \frac{(n-i)!\,(n-j)!}{n!\,(n-k)!}\, (-1)^i \alpha_i\, (-1)^j \beta_j = (-1)^k \gamma_k.$$

Therefore $r(x) = \sum_k c_k x^{n-k} = \sum_k (-1)^k \gamma_k x^{n-k}$.

A.2 Differential-operator formula and commutation rules

The coefficient definition in Definition A.1 admits a particularly useful representation in terms of derivatives at a single point. This formula is standard in the literature on Walsh-type convolutions and finite free convolutions; we prove it here directly by coefficient extraction.

Theorem A.5 (Differential-operator representation).

For all $p, q \in \mathbb{R}_{\le n}[x]$

$$\boxed{(p \boxplus_n q)(x) = \frac{1}{n!} \sum_{k=0}^{n} p^{(k)}(x)\, q^{(n-k)}(0).} \tag{1}$$

Equivalently (by commutativity)

$$\boxed{(p \boxplus_n q)(x) = \frac{1}{n!} \sum_{k=0}^{n} q^{(k)}(x)\, p^{(n-k)}(0).} \tag{2}$$

Proof. Write

$$p(x) = \sum_{i=0}^{n} a_i x^{n-i} \qquad q(x) = \sum_{j=0}^{n} b_j x^{n-j}$$

with padding allowed. Fix $k \in 0, 1, \dots, n$. Since

$$\frac{d^{\,n-k}}{dx^{\,n-k}} x^{n-j} = \begin{cases} (n-k)! & j = k, \\ 0, & j \ne k, \end{cases}$$

we have

$$q^{(n-k)}(0) = (n-k)!\, b_k.$$

Next, for each i and each $k \le n-i$

$$\frac{d^k}{dx^k} x^{n-i} = \frac{(n-i)!}{(n-i-k)!}\, x^{n-i-k}.$$

Therefore,

$$p^{(k)}(x) = \sum_{i=0}^{n-k} a_i \frac{(n-i)!}{(n-i-k)!}\, x^{n-i-k}.$$

Now consider the right-hand side of {#eq:a-2}:

$$\frac{1}{n!} \sum_{k=0}^{n} p^{(k)}(x)\, q^{(n-k)}(0) = \frac{1}{n!} \sum_{k=0}^{n} p^{(k)}(x)\, (n-k)!\, b_k.$$

We extract the coefficient of x^{n-m} for a fixed $m \in \{0, 1, \dots, n\}$. A term x^{n-m} arises from $p^{(k)}(x)$ precisely when $n-i-k = n-m$ i.e. $k = m-i$ with $0 \le i \le m$. For such i, k

$$\operatorname{coeff}_{x^{n-m}} p^{(k)}(x) = a_i \frac{(n-i)!}{(n-m)!}.$$

Multiplying by $\frac{(n-k)!\, b_k}{n!}$ with $k = m-i$ gives a contribution

$$\frac{1}{n!} \cdot a_i \frac{(n-i)!}{(n-m)!} \cdot (n-m+i)!\, b_{m-i}.$$

Summing over $i = 0, \dots, m$ yields the coefficient

$$\operatorname{coeff}_{x^{n-m}} (p \boxplus_n q) = \sum_{i+j=m} \frac{(n-i)!\,(n-j)!}{n!\,(n-m)!}\, a_i b_j,$$

which is exactly c_m from Definition A.1. Since this holds for every m the polynomials agree, proving {#eq:a-2}. Identity {#eq:a-3} follows by commutativity (Proposition A.2).

Corollary A.6 (Operator form).

For each $q \in \mathbb{R}_{\le n}[x]$ define the constant-coefficient differential operator

$$\mathcal{T}_q := \frac{1}{n!} \sum_{k=0}^{n} q^{(n-k)}(0)\, D^k, \qquad D := \frac{d}{dx}. \tag{3}$$

Then for all $p \in \mathbb{R}_{\le n}[x]$

$$p \boxplus_n q = \mathcal{T}_q p.$$

Proof. This is a rephrasing of Theorem A.5.

Proposition A.7 (Commutation with differentiation).

On $\mathbb{R}_{\le n}[x]$ the bilinear map $\boxplus_n$ satisfies

$$D(p \boxplus_n q) = (Dp) \boxplus_n q = p \boxplus_n (Dq).$$

Proof. Differentiate {#eq:a-2} term-by-term:

$$D(p \boxplus_n q)(x) = \frac{1}{n!} \sum_{k=0}^{n} p^{(k+1)}(x)\, q^{(n-k)}(0).$$

Reindexing $m = k + 1$ gives

$$D(p \boxplus_n q)(x) = \frac{1}{n!} \sum_{m=1}^{n+1} p^{(m)}(x)\, q^{(n+1-m)}(0).$$

Since q has degree at most n we have $q^{(n+1-m)}(0) = 0$ for $m = 0$ and $q^{(-1)} \equiv 0$ by convention; thus we may rewrite this as

$$D(p \boxplus_n q)(x) = \frac{1}{n!} \sum_{m=0}^{n} p^{(m)}(x)\, (Dq)^{(n-m)}(0)$$

which is exactly $(p \boxplus_n Dq)(x)$ by Theorem A.5 (applied to p and Dq). This proves $D(p \boxplus_n q) = p \boxplus_n (Dq)$. Commutativity yields $D(p \boxplus_n q) = (Dp) \boxplus_n q$.

Proposition A.8 (Translation covariance).

For $t \in \mathbb{R}$ define the translation operator $(T_t f)(x) := f(x - t)$. Then for all $p, q \in \mathbb{R}_{\le n}[x]$

$$T_t(p \boxplus_n q) = (T_t p) \boxplus_n q = p \boxplus_n (T_t q).$$

Proof. Using Theorem A.5 and the elementary identity $(T_t p)^{(k)}(x) = p^{(k)}(x - t)$

$$((T_t p) \boxplus_n q)(x) = \frac{1}{n!} \sum_{k=0}^{n} (T_t p)^{(k)}(x)\, q^{(n-k)}(0) = \frac{1}{n!} \sum_{k=0}^{n} p^{(k)}(x-t)\, q^{(n-k)}(0) = (p \boxplus_n q)(x-t) = (T_t(p \boxplus_n q))(x).$$

The identity $(p \boxplus_n T_t q) = T_t(p \boxplus_n q)$ follows by commutativity.

A.3 Stability and real-rootedness preservation

We now show that $\boxplus_n$ preserves real-rootedness. We give a short, fully rigorous argument using stability theory and the characterization of stability-preserving linear operators.

Definition A.9 (Stability in one variable).

A complex polynomial $f \in \mathbb{C}[x]$ is called **stable** if $f(z) \neq 0$ for all z with $\operatorname{Im} z > 0$ (i.e. f has no zeros in the open upper half-plane).

Lemma A.10 (Real coefficients: stability $\Leftrightarrow$ real-rootedness).

Let $f \in \mathbb{R}[x]$. Then f is stable if and only if all zeros of f are real.

Proof. If all zeros are real, then certainly f has no zeros in $\{\operatorname{Im} z > 0\}$ hence f is stable. Conversely, assume f is stable but has a non-real zero z. Since f has real coefficients, $\overline{z}$ is also a zero. One of $z, \overline{z}$ lies in the upper half-plane, contradicting stability. Hence all zeros are real.

Theorem A.11 (Borcea–Brändén: stability preservers on $\mathbb{C}_{\leq n}[x]$).

Let $T : \mathbb{C}_{\leq n}[x] \to \mathbb{C}_{\leq n}[x]$ be a complex-linear operator. Define its **algebraic symbol**

$$G_T(z, w) := T\big[(z + w)^n\big] \in \mathbb{C}[z, w].$$

If G_T is stable as a bivariate polynomial (i.e. $G_T(z, w) \neq 0$ whenever $\operatorname{Im} z > 0$ and $\operatorname{Im} w > 0$) then T preserves stability: for every stable $f \in \mathbb{C}_{\leq n}[x]$ the polynomial Tf is stable.

A proof can be found in Borcea–Brändén, *The Lee–Yang and Pólya–Schur programs. I*, Invent. Math. (2009) where a full classification is established.

Proposition A.12 (Symbol computation for $p \mapsto p \boxplus_n q$).

Fix $q \in \mathbb{C}_{\leq n}[x]$ and define the linear operator $T_q : \mathbb{C}_{\leq n}[x] \to \mathbb{C}_{\leq n}[x]$ by

$$T_q(p) := p \boxplus_n q.$$

Then its algebraic symbol equals

$$G_{T_q}(z, w) = q(z + w). \tag{4}$$

Proof. By Theorem A.5,

$$T_q(p)(w) = \frac{1}{n!} \sum_{k=0}^{n} p^{(k)}(w)\, q^{(n-k)}(0).$$

Apply this with $p(w) = (z + w)^n$. Since $\frac{d^k}{dw^k}(z + w)^n = \frac{n!}{(n-k)!}(z + w)^{n-k}$ we obtain

$$G_{T_q}(z, w) = T_q\big[(z+w)^n\big](w) = \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{(n-k)!}(z+w)^{n-k}\, q^{(n-k)}(0) = \sum_{m=0}^{n} \frac{q^{(m)}(0)}{m!}(z+w)^m = q(z+w)$$

where we set $m = n - k$ in the sum and used the Taylor expansion of q at 0.

Theorem A.13 (Real-rootedness preservation).

Let $p, q \in \mathbb{R}_n[x]$ be monic and real-rooted. Then $r := p \boxplus_n q$ is real-rooted (and monic of degree n).

Proof. Since $p, q \in \mathbb{R}[x]$ are real-rooted, Lemma A.10 implies that both are stable. Consider the operator $T_q(p) = p \boxplus_n q$. By Proposition A.12, its algebraic symbol is $G_{T_q}(z, w) = q(z + w)$. If $\operatorname{Im} z > 0$ and $\operatorname{Im} w > 0$ then $\operatorname{Im}(z + w) > 0$ and stability of q gives $q(z+w) \neq 0$. Hence G_{T_q} is bivariate stable. By Theorem A.11, T_q preserves stability, so $r = T_q(p)$ is stable. Finally $r \in \mathbb{R}[x]$ so Lemma A.10 implies that r is real-rooted. Monicity and degree n follow from Proposition A.2.

A.4 Summary for the main text

This appendix establishes the algebraic facts about $\boxplus_n$ used in the main text:

- Definition A.1 and Proposition A.2 justify that $\boxplus_n$ is a well-defined bilinear, commutative operation on degree-$\leq n$ polynomials, and preserves monicity in degree n.
- Theorem A.5 (equations {#eq:a-2}–{#eq:a-3}) supplies the differential-operator representation quoted in Remark 2.2.
- Proposition A.7 and Proposition A.8 prove commutation with differentiation and covariance under translation, respectively.
- Theorem A.13 guarantees that if p, q are real-rooted, then $p \boxplus_n q$ is real-rooted, so $\Phi_n(p \boxplus_n q)$ is well-defined.

A.5 References (Appendix A)

1. J. Borcea and P. Brändén, *The Lee–Yang and Pólya–Schur programs. I. Linear operators preserving stability*, Inventiones Mathematicae (2009).
2. A. W. Marcus, D. A. Spielman, N. Srivastava, *Finite free convolutions of polynomials*, arXiv:1504.00350 (for broader context and additional equivalent formulations of finite free convolutions).

Appendix B. Discrete integration by parts and the score identity

This appendix supplies the complete proofs of Lemma 3.5 and Theorem 3.6 from the main text. The key point is an exact discrete integration-by-parts identity on the root set of a polynomial, which identifies the functional $\Phi_n(p)$ with the squared norm of a canonical “score” vector J_p.

B.0 Clarification on the discrete derivative used for the score

In §3 of the main text we introduced a Hilbert space on the roots and a score defined as an adjoint applied to a constant function. For the proofs, it is most transparent to realize the relevant “derivative” as a **two-point difference quotient** mapping functions on the roots to functions on ordered pairs of distinct roots. This yields an exact adjoint formula and produces precisely the interaction field

$$J_p(\lambda_i) = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}.$$

By Proposition B.6 (equivalently Lemma 3.5), this interaction field coincides with the score $J_p = -D_p^* \mathbf{1}$ from Definition 3.3, so the main text statements are unaffected by using ∂_p in this appendix.

B.1 Root space and pair space

Let $p \in \mathbb{R}_n[x]$ be monic with **simple real roots**

$$p(x) = \prod_{i=1}^{n} (x - \lambda_i) \qquad \lambda_i \in \mathbb{R}, \qquad \lambda_i \neq \lambda_j \quad (i \neq j).$$

Definition B.1 (Root Hilbert space).

Define

$$\mathcal{H}_p := \mathbb{R}^n$$

with the inner product

$$\langle u, v \rangle_{\mathcal{H}_p} := \frac{1}{n} \sum_{i=1}^{n} u_i v_i. \tag{5}$$

We interpret $u \in \mathcal{H}_p$ as a function on the root set via $u_i = u(\lambda_i)$.

Definition B.2 (Pair Hilbert space).

Define

$$\mathcal{H}_p^{(2)} := \{F \in \mathbb{R}^{n \times n} : \ F_{ii} = 0 \text{ for all } i\}$$

with inner product

$$\langle F, G \rangle_{\mathcal{H}_p^{(2)}} := \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} F_{ij} G_{ij}. \tag{6}$$

We also define the constant off-diagonal matrix $\mathbf{1}^{(2)} \in \mathcal{H}_p^{(2)}$ by

$$\mathbf{1}^{(2)}_{ij} := \begin{cases} 1, & i \neq j, \\ 0, & i = j. \end{cases} \tag{7}$$

B.2 A two-point difference quotient and its adjoint

Definition B.3 (Two-point difference quotient operator).

Define the linear map $\partial_p : \mathcal{H}_p \to \mathcal{H}_p^{(2)}$ by

$$(\partial_p u)_{ij} := \begin{cases} \dfrac{u_i - u_j}{\lambda_i - \lambda_j}, & i \neq j, \\ 0, & i = j. \end{cases} \tag{8}$$

Note that $(\partial_p u)_{ij} = (\partial_p u)_{ji}$ for $i \neq j$.

Lemma B.4 (Adjoint of the difference quotient).

Let $V \in \mathcal{H}_p^{(2)}$. The adjoint $\partial_p^* : \mathcal{H}_p^{(2)} \to \mathcal{H}_p$ with respect to the inner products {#eq:b-1}–{#eq:b-2} is given by

$$(\partial_p^* V)_i = \frac{1}{n} \sum_{j=1\ j\neq i}^{n} \frac{V_{ij} + V_{ji}}{\lambda_i - \lambda_j} \qquad i = 1, \dots, n. \tag{9}$$

Proof. Fix $u \in \mathcal{H}_p$ and $V \in \mathcal{H}_p^{(2)}$. By definition,

$$\langle \partial_p u, V \rangle_{\mathcal{H}_p^{(2)}} = \frac{1}{n^2} \sum_{i \neq j} \frac{u_i - u_j}{\lambda_i - \lambda_j} V_{ij}.$$

Expand the numerator:

$$\sum_{i\neq j} \frac{u_i - u_j}{\lambda_i - \lambda_j} V_{ij} = \sum_{i\neq j} \frac{u_i}{\lambda_i - \lambda_j} V_{ij} - \sum_{i\neq j} \frac{u_j}{\lambda_i - \lambda_j} V_{ij}.$$

In the second term, swap the dummy indices $i \leftrightarrow j$:

$$\sum_{i\neq j} \frac{u_j}{\lambda_i - \lambda_j} V_{ij} = \sum_{i\neq j} \frac{u_i}{\lambda_j - \lambda_i} V_{ji} = -\sum_{i\neq j} \frac{u_i}{\lambda_i - \lambda_j} V_{ji}.$$

Hence

$$\sum_{i\neq j} \frac{u_i - u_j}{\lambda_i - \lambda_j} V_{ij} = \sum_{i\neq j} \frac{u_i}{\lambda_i - \lambda_j} (V_{ij} + V_{ji}) = \sum_{i=1}^{n} u_i \sum_{j\neq i} \frac{V_{ij} + V_{ji}}{\lambda_i - \lambda_j}.$$

Therefore

$$\langle \partial_p u, V\rangle_{\mathcal{H}_p^{(2)}} = \frac{1}{n^2}\sum_{i=1}^{n} u_i \sum_{j\neq i} \frac{V_{ij}+V_{ji}}{\lambda_i-\lambda_j} = \left\langle u,\ \Big(\frac{1}{n}\sum_{j\neq i}\frac{V_{ij}+V_{ji}}{\lambda_i-\lambda_j}\Big)_{i=1}^{n}\right\rangle_{\mathcal{H}_p}.$$

By the defining property of the adjoint, this proves {#eq:b-5}.

B.3 The score vector and discrete integration by parts

Definition B.5 (Score vector).

Define the score $J_p \in \mathcal{H}_p$ (agreeing with Definition 3.3) by

$$J_p := \frac{n}{2}\,\partial_p^* \mathbf{1}^{(2)}. \tag{10}$$

Proposition B.6 (Explicit score formula).

For each $i = 1, \dots, n$

$$\boxed{J_p(\lambda_i) = \sum_{j=1\ j\neq i}^{n} \frac{1}{\lambda_i - \lambda_j}.} \tag{11}$$

Proof. Apply Lemma B.4 with $V = \mathbf{1}^{(2)}$. Then for each i

$$(\partial_p^* \mathbf{1}^{(2)})_i = \frac{1}{n}\sum_{j\neq i}\frac{\mathbf{1}_{ij}^{(2)}+\mathbf{1}_{ji}^{(2)}}{\lambda_i-\lambda_j} = \frac{1}{n}\sum_{j\neq i}\frac{1+1}{\lambda_i-\lambda_j} = \frac{2}{n}\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j}.$$

Multiplying by $n/2$ as in {#eq:b-6} yields {#eq:b-7}.

Corollary B.7 (Zero-mean property).

$$\sum_{i=1}^{n} J_p(\lambda_i) = 0. \tag{12}$$

Proof. Sum {#eq:b-7} over i. Each unordered pair $\{i, j\}$ contributes $\frac{1}{\lambda_i-\lambda_j} + \frac{1}{\lambda_j-\lambda_i} = 0$.

Proposition B.8 (Discrete integration by parts).

For every $u \in \mathcal{H}_p$

$$\boxed{\langle u, J_p\rangle_{\mathcal{H}_p} = \frac{n}{2}\,\langle \partial_p u,\ \mathbf{1}^{(2)}\rangle_{\mathcal{H}_p^{(2)}}.} \tag{13}$$

Proof. By Definition B.5 and the definition of adjoint,

$$\langle u, J_p \rangle_{\mathcal{H}_p} = \left\langle u,\ \frac{n}{2}\partial_p^* \mathbf{1}^{(2)} \right\rangle_{\mathcal{H}_p} = \frac{n}{2}\left\langle \partial_p u,\ \mathbf{1}^{(2)} \right\rangle_{\mathcal{H}_p^{(2)}}.$$

B.4 Proofs of Lemma 3.5 and Theorem 3.6

We now show that the score defined in §3 (and realized concretely by Definition B.5) matches the interaction field in the definition of Φ_n and that Φ_n is its squared norm.

Proof of Lemma 3.5. Lemma 3.5 asserts that for each root λ_i of p

$$J_p(\lambda_i) = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}.$$

This is exactly Proposition B.6 (equation {#eq:b-7}).

Proof of Theorem 3.6. By Definition 2.3 of Φ_n and Lemma 3.5,

$$\Phi_n(p) = \sum_{i=1}^{n} \left(J_p(\lambda_i) \right)^2.$$

On the other hand, by the definition of the norm in $\mathcal{H}_p$ (equation {#eq:b-1})

$$|J_p|_{\mathcal{H}_p}^2 = \langle J_p, J_p \rangle_{\mathcal{H}_p} = \frac{1}{n}\sum_{i=1}^{n} \left(J_p(\lambda_i) \right)^2.$$

Therefore,

$$\boxed{\Phi_n(p) = n\,|J_p|_{\mathcal{H}_p}^2,}$$

which is Theorem 3.6.

B.5 Remarks for later use

1. The operator ∂_p is a finite-dimensional analogue of the free difference quotient; Proposition B.8 is the corresponding integration-by-parts identity.
2. The normalization in Definition B.5 is chosen so that the score agrees **exactly** with the interaction field in Problem #4 and Theorem 3.6 takes the clean form $\Phi_n(p) = n|J_p|^2$.
3. Corollary B.7 (zero mean) is the only property of J_p needed to deduce the orthogonality statement in §4.1 of the main text.

Appendix C. Equivalent formulas for Φ_n

This appendix collects several equivalent representations of the functional

$$\Phi_n(p) = \sum_{i=1}^{n} \left(\sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right)^2$$

for a monic degree-n polynomial p with real zeros $\lambda_1, \dots, \lambda_n$. These identities are useful for (i) structural understanding, (ii) degeneracy analysis (multiple roots) and (iii) alternative computations.

Throughout this appendix we assume:

- $p \in \mathbb{R}_n[x]$ is monic.
- p has **simple real roots** $\lambda_1 < \dots < \lambda_n$.

Under this assumption, all expressions below are finite and unambiguous.

C.1 Logarithmic-derivative identities on the roots

We first record standard identities relating root sums to values of p'/p and p''/p'.

Lemma C.1 (Logarithmic derivative).

Let $p(x) = \prod_{i=1}^{n}(x - \lambda_i)$ with pairwise distinct roots $\lambda_i \in \mathbb{C}$. Then for all $x \notin \{\lambda_1, \dots, \lambda_n\}$

$$\frac{p'(x)}{p(x)} = \sum_{i=1}^{n} \frac{1}{x - \lambda_i}. \tag{14}$$

Proof. Differentiate $\log p(x) = \sum_i \log(x - \lambda_i)$ on the domain where $p(x) \neq 0$ or compute directly:

$$p'(x) = \sum_{i=1}^{n} \prod_{j \neq i} (x - \lambda_j) = p(x) \sum_{i=1}^{n} \frac{1}{x - \lambda_i}.$$

Divide by $p(x)$.

Lemma C.2 (Root evaluation of p''/p').

Assume p has simple roots $\lambda_1, \dots, \lambda_n$. Then for each i

$$\frac{p''(\lambda_i)}{p'(\lambda_i)} = 2 \sum_{j=1}^{n}\sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}. \tag{15}$$

Proof. Write $p(x) = (x - \lambda_i) g_i(x)$ where $g_i(x) = \prod_{j \neq i}(x - \lambda_j)$ and $g_i(\lambda_i) = p'(\lambda_i) \neq 0$. Then

$$p'(x) = g_i(x) + (x - \lambda_i) g_i'(x) \qquad p''(x) = 2g_i'(x) + (x - \lambda_i) g_i''(x).$$

Evaluating at $x = \lambda_i$ gives

$$p'(\lambda_i) = g_i(\lambda_i) \qquad p''(\lambda_i) = 2g_i'(\lambda_i).$$

Now

$$\frac{g_i'(x)}{g_i(x)} = \sum_{j\neq i} \frac{1}{x - \lambda_j}$$

by Lemma C.1 applied to g_i. Evaluating at $x = \lambda_i$ yields

$$\frac{g_i'(\lambda_i)}{g_i(\lambda_i)} = \sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j}.$$

Combine with $p''(\lambda_i) = 2g_i'(\lambda_i)$ and $p'(\lambda_i) = g_i(\lambda_i)$ to obtain {#eq:c-2}.

Corollary C.3 (Derivative form of Φ_n).

If p has simple real roots, then

$$\Phi_n(p) = \frac{1}{4}\sum_{i=1}^{n} \left(\frac{p''(\lambda_i)}{p'(\lambda_i)}\right)^2. \tag{16}$$

Proof. By Lemma C.2, the inner sum defining $\Phi_n(p)$ equals $\frac{1}{2}\, p''(\lambda_i)/p'(\lambda_i)$. Square and sum over i.

C.2 Pair-interaction representation

The next identity is the key "two-body" representation: the square of the interaction field collapses to a pure sum of inverse squared gaps.

Proposition C.4 (Pair-interaction formula).

Let $p \in \mathbb{R}_n[x]$ have simple real roots $\lambda_1, \dots, \lambda_n$. Then

$$\boxed{\Phi_n(p) = \sum_{i,j=1\ i\neq j}^{n} \frac{1}{(\lambda_i - \lambda_j)^2} = 2\sum_{1\le i<j\le n} \frac{1}{(\lambda_i - \lambda_j)^2}.} \tag{17}$$

Proof. Set

$$S_i := \sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j}.$$

Then $\Phi_n(p) = \sum_i S_i^2$. Expanding,

$$\sum_{i=1}^{n} S_i^2 = \sum_{i=1}^{n}\sum_{j\neq i} \frac{1}{(\lambda_i - \lambda_j)^2} + \sum_{i=1}^{n}\ \sum_{j\neq i}\ \sum_{k\neq i\ \ k\neq j} \frac{1}{(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)}. \tag{18}$$

The first term on the right-hand side of {#eq:c-5} is exactly $\sum_{i \neq j}(\lambda_i - \lambda_j)^{-2}$. We show that the triple sum vanishes.

Fix three distinct indices a, b, c. Consider the cyclic sum

$$T(a,b,c) := \frac{1}{(\lambda_a - \lambda_b)(\lambda_a - \lambda_c)} + \frac{1}{(\lambda_b - \lambda_a)(\lambda_b - \lambda_c)} + \frac{1}{(\lambda_c - \lambda_a)(\lambda_c - \lambda_b)}.$$

With common denominator $(\lambda_a - \lambda_b)(\lambda_b - \lambda_c)(\lambda_c - \lambda_a)$ the numerator becomes

$$(\lambda_b - \lambda_c) + (\lambda_c - \lambda_a) + (\lambda_a - \lambda_b) = 0,$$

so $T(a,b,c) = 0$. The triple sum in {#eq:c-5} is precisely the sum of $T(a,b,c)$ over all unordered triples $\{a, b, c\}$ hence it is 0.

Therefore $\Phi_n(p) = \sum_{i \neq j}(\lambda_i - \lambda_j)^{-2}$. The second equality in {#eq:c-4} is immediate since the summand is symmetric in (i, j).

Corollary C.5 (Blow-up at collisions).

Let $p \in \mathbb{R}_n[x]$ have simple real roots $\lambda_1 < \cdots < \lambda_n$. Then

$$\Phi_n(p) \;\geq\; \frac{2}{(\lambda_{k+1} - \lambda_k)^2} \qquad (k = 1, \dots, n-1). \tag{19}$$

In particular, if a sequence of polynomials p_m has two roots whose gap tends to 0 then $\Phi_n(p_m) \to +\infty$ and hence $1/\Phi_n(p_m) \to 0$.

Proof. By Proposition C.4,

$$\Phi_n(p) = 2 \sum_{i<j} \frac{1}{(\lambda_i - \lambda_j)^2} \geq 2 \cdot \frac{1}{(\lambda_{k+1} - \lambda_k)^2}.$$

The limit statement follows.

C.3 Critical-point representation

We now express $\Phi_n(p)$ as a sum over the critical points of p, i.e. the zeros of p'.

Lemma C.6 (Interlacing and simplicity of critical points).

Let $p \in \mathbb{R}_n[x]$ have simple real roots $\lambda_1 < \cdots < \lambda_n$. Then p' has exactly $n - 1$ simple real roots $\alpha_1 < \cdots < \alpha_{n-1}$ and they strictly interlace the roots of p:

$$\lambda_1 < \alpha_1 < \lambda_2 < \alpha_2 < \cdots < \alpha_{n-1} < \lambda_n. \tag{20}$$

Moreover, $p(\alpha_k) \neq 0$ for all k.

Proof. By Rolle's theorem, between each consecutive pair $\lambda_i < \lambda_{i+1}$ there exists at least one root of p'. Since $\deg p' = n-1$ there are exactly $n-1$ roots of p' counting multiplicity, hence there is exactly one root in each interval $(\lambda_i, \lambda_{i+1})$. These roots are necessarily distinct, thus p' has $n-1$ distinct real roots, and hence they are simple. The interlacing {#eq:c-7} follows. Finally each α_k lies strictly between two roots of p, so $p(\alpha_k) \neq 0$.

Theorem C.7 (Critical-point formula via residues).

Let $p \in \mathbb{R}_n[x]$ be monic with simple real roots. Let $\alpha_1, \dots, \alpha_{n-1}$ be the roots of p'. Then

$$\boxed{\Phi_n(p) = -\frac{1}{4} \sum_{k=1}^{n-1} \frac{p''(\alpha_k)}{p(\alpha_k)}.} \tag{21}$$

Proof. Consider the rational function

$$F(z) := \frac{p''(z)^2}{p'(z)\, p(z)}. \tag{22}$$

Its poles occur precisely at the zeros of p and of p'. By Lemma C.6, these are all simple, and they are disjoint.

Step 1: Residues at zeros of p. Fix i. Since λ_i is a simple root, we have

$$p(z) = (z-\lambda_i)p'(\lambda_i) + O((z-\lambda_i)^2) \qquad p'(z) = p'(\lambda_i) + O(z-\lambda_i) \qquad p''(z) = p''(\lambda_i) + O(z-\lambda_i).$$

Therefore

$$F(z) = \frac{p''(\lambda_i)^2 + O(z-\lambda_i)}{(p'(\lambda_i) + O(z-\lambda_i))((z-\lambda_i)p'(\lambda_i) + O((z-\lambda_i)^2))} = \frac{p''(\lambda_i)^2}{p'(\lambda_i)^2} \cdot \frac{1}{z-\lambda_i} + O(1)$$

so

$$\mathrm{Res}(F; \lambda_i) = \frac{p''(\lambda_i)^2}{p'(\lambda_i)^2}. \tag{23}$$

By Lemma C.2,

$$\frac{p''(\lambda_i)}{p'(\lambda_i)} = 2 \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}$$

so

$$\mathrm{Res}(F; \lambda_i) = 4 \left(\sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right)^2.$$

Summing over i yields

$$\sum_{i=1}^{n} \mathrm{Res}(F; \lambda_i) = 4\Phi_n(p). \tag{24}$$

Step 2: Residues at zeros of p'. Fix k. Since α_k is a simple zero of p' we have

$$p'(z) = (z-\alpha_k)p''(\alpha_k)+O((z-\alpha_k)^2) \qquad p(z) = p(\alpha_k)+O(z-\alpha_k) \qquad p''(z) = p''(\alpha_k)+O(z-\alpha_k).$$

Thus

$$F(z) = \frac{p''(\alpha_k)^2 + O(z-\alpha_k)}{((z-\alpha_k)p''(\alpha_k) + O((z-\alpha_k)^2))(p(\alpha_k) + O(z-\alpha_k))} = \frac{p''(\alpha_k)}{p(\alpha_k)} \cdot \frac{1}{z-\alpha_k} + O(1)$$

and hence

$$\operatorname{Res}(F; \alpha_k) = \frac{p''(\alpha_k)}{p(\alpha_k)}. \tag{25}$$

Step 3: Residue at infinity. Since p is monic of degree n as $z \to \infty$

$$p(z) \sim z^n, \quad p'(z) \sim nz^{n-1} \quad p''(z) \sim n(n-1)z^{n-2}.$$

Therefore

$$F(z) \sim \frac{n^2(n-1)^2 z^{2n-4}}{(nz^{n-1}) \cdot z^n} = \frac{n(n-1)^2}{z^3}$$

so the Laurent expansion of $F(z)$ at infinity has no $1/z$ term and

$$\operatorname{Res}(F; \infty) = 0. \tag{26}$$

Step 4: Apply the residue theorem. The sum of residues at all finite poles plus the residue at infinity is zero:

$$0 = \sum_{i=1}^{n} \operatorname{Res}(F; \lambda_i) + \sum_{k=1}^{n-1} \operatorname{Res}(F; \alpha_k) + \operatorname{Res}(F; \infty).$$

Combine {#eq:c-11} {#eq:c-12} and {#eq:c-13} to obtain

$$0 = 4\Phi_n(p) + \sum_{k=1}^{n-1} \frac{p''(\alpha_k)}{p(\alpha_k)}.$$

Rearranging gives {#eq:c-8}.

C.4 Positive "curvature" form at critical points

The critical-point formula can be rewritten as a manifestly positive double sum.

Proposition C.8 (Positive double-sum representation).

Let $p \in \mathbb{R}_n[x]$ be monic with simple real roots $\lambda_1, \dots, \lambda_n$ and let $\alpha_1, \dots, \alpha_{n-1}$ be the roots of p'. Then for each k

$$\frac{p''(\alpha_k)}{p(\alpha_k)} = -\sum_{i=1}^{n} \frac{1}{(\alpha_k - \lambda_i)^2} \tag{27}$$

and consequently

$$\boxed{\Phi_n(p) = \frac{1}{4} \sum_{k=1}^{n-1} \sum_{i=1}^{n} \frac{1}{(\alpha_k - \lambda_i)^2}.} \tag{28}$$

Proof. By Lemma C.1,

$$\frac{p'(x)}{p(x)} = \sum_{i=1}^{n} \frac{1}{x - \lambda_i}.$$

Differentiate:

$$\left(\frac{p'(x)}{p(x)} \right)' - \sum_{i=1}^{n} \frac{1}{(x - \lambda_i)^2}. \tag{29}$$

On the other hand,

$$\left(\frac{p'}{p} \right)' = \frac{p''}{p} - \left(\frac{p'}{p} \right)^2.$$

At a critical point α_k we have $p'(\alpha_k) = 0$, hence $(p'/p)(\alpha_k) = 0$ and so

$$\left(\frac{p'}{p} \right)' (\alpha_k) = \frac{p''(\alpha_k)}{p(\alpha_k)}.$$

Comparing with {#eq:c-16} at $x = \alpha_k$ proves {#eq:c-14}. Substituting {#eq:c-14} into Theorem C.7 (equation {#eq:c-8}) yields {#eq:c-15}.

C.5 Notes for later appendices

1. **Degenerations.** Corollary C.5 is the basic input for treating multiple-root limits (Appendix F): whenever a pair of roots collides, Φ_n diverges, and the reciprocal $1/\Phi_n$ vanishes.
2. **Interlacing structure.** Lemma C.6 provides the strict interlacing needed in Theorem C.7 to ensure all poles are simple and disjoint.
3. **Alternative computations.** The positive form {#eq:c-15} can be numerically stable in regimes where gaps $(\lambda_i - \lambda_j)$ are extremely small, because it distributes the blow-up across distances to critical points.

Appendix D. Construction of the two-variable space and the $\boxplus_n$ sum projection

This appendix constructs the two-variable Hilbert space $\mathcal{H}_{p,q}$, the distinguished n-dimensional sum subspace $\mathcal{H}_{\boxplus}(p,q) \subset \mathcal{H}_{p,q}$, and the associated orthogonal projection $P_{\boxplus}$. We also construct an **isometric identification**

$$\iota^{\boxplus}_{p,q} : \mathcal{H}_r \xrightarrow{\cong} \mathcal{H}_{\boxplus}(p,q) \qquad r := p \boxplus_n q,$$

and define the contraction

$$\mathbb{E}_{\boxplus} := (\iota_{p,q}^{\boxplus})^* : \mathcal{H}_{p,q} \to \mathcal{H}_r$$

used in §4–§5 of the main text.

In addition, Appendix D' (Sections D'.1–D'.3) records a permutation-average model for $\boxplus_n$ and two derivative identities (a tilt representation and a collision-safe decomposition) that are convenient when translating identities about $r = p \boxplus_n q$ into identities for the matched-sum family $\{h_\sigma\}_{\sigma \in S_n}$.

D.1 Standing assumptions and root Hilbert spaces

Fix $n \geq 2$. In this appendix we assume:

- $p, q \in \mathbb{R}_n[x]$ are monic and have **simple real roots**

$$p(x) = \prod_{i=1}^{n}(x - \lambda_i) \qquad \lambda_1 < \cdots < \lambda_n,$$

$$q(x) = \prod_{j=1}^{n}(x - \mu_j) \qquad \mu_1 < \cdots < \mu_n.$$

- $r := p \boxplus_n q$ has **simple real roots**

$$r(x) = \prod_{k=1}^{n}(x - \rho_k) \qquad \rho_1 < \cdots < \rho_n.$$

The multiple-root case is handled by perturbation and limiting arguments in Appendix F.

Definition D.1 (Root Hilbert spaces and product space).

Define

$$\mathcal{H}_p := \mathbb{R}^n, \qquad \langle u, v \rangle_{\mathcal{H}_p} := \frac{1}{n}\sum_{i=1}^{n} u_i v_i, \tag{30}$$

$$\mathcal{H}_q := \mathbb{R}^n, \qquad \langle u, v \rangle_{\mathcal{H}_q} := \frac{1}{n}\sum_{j=1}^{n} u_j v_j, \tag{31}$$

$$\mathcal{H}_r := \mathbb{R}^n, \qquad \langle u, v \rangle_{\mathcal{H}_r} := \frac{1}{n}\sum_{k=1}^{n} u_k v_k, \tag{32}$$

and the two-variable (product) Hilbert space

$$\mathcal{H}_{p,q} := \mathcal{H}_p \otimes \mathcal{H}_q \cong \mathbb{R}^{n\times n} \qquad \langle U, V\rangle_{\mathcal{H}_{p,q}} := \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} U_{ij}V_{ij}. \tag{33}$$

We interpret $U \in \mathcal{H}_{p,q}$ as a function on the root grid $\{(\lambda_i, \mu_j)\}_{i,j}$.

D'.1 A permutation model for $\boxplus_n$

This section records an explicit permutation average representation of the symmetric additive finite free convolution $\boxplus_n$. It will be used as an auxiliary model when translating identities about $r := p \boxplus_n q$ into identities for a finite family of "matched-sum" polynomials.

Fix $n \geq 2$, and let $p, q \in \mathbb{R}_n[x]$ be monic. Write

$$p(x) = \prod_{i=1}^{n}(x - \lambda_i), \qquad q(x) = \prod_{i=1}^{n}(x - \mu_i),$$

where the roots are listed with multiplicity (no ordering is needed for this section).

For each permutation $\sigma \in S_n$ define the matched-sum polynomial

$$h_\sigma(x) := \prod_{i=1}^{n}\big(x - (\lambda_i + \mu_{\sigma(i)})\big). \tag{34}$$

Lemma D'.1 (Permutation-average model for $\boxplus_n$).

For all monic $p, q \in \mathbb{R}_n[x]$,

$$\boxed{(p \boxplus_n q)(x) = \frac{1}{n!}\sum_{\sigma\in S_n} h_\sigma(x).} \tag{35}$$

Proof. Write the elementary symmetric expansions

$$p(x) = \sum_{i=0}^{n}(-1)^i e_i(\lambda)\, x^{n-i}, \qquad q(x) = \sum_{j=0}^{n}(-1)^j e_j(\mu)\, x^{n-j},$$

where $e_i(\lambda)$ is the i-th elementary symmetric polynomial in $\lambda_1, \dots, \lambda_n$, and similarly for μ.

Fix $k \in \{0, 1, \dots, n\}$. Expanding h_σ and extracting the coefficient of x^{n-k} gives

$$[x^{n-k}]\, h_\sigma(x) = (-1)^k e_k(\lambda_1 + \mu_{\sigma(1)}, \dots, \lambda_n + \mu_{\sigma(n)}).$$

Expanding the elementary symmetric polynomial of the matched sums,

$$e_k(\lambda_i + \mu_{\sigma(i)}) = \sum_{\substack{A\subset[n]\\ |A|=k}} \prod_{i\in A} (\lambda_i + \mu_{\sigma(i)}),$$

and then expanding each product by choosing which factors contribute λ and which contribute μ, yields

$$e_k(\lambda_i + \mu_{\sigma(i)}) = \sum_{i=0}^{k} \sum_{\substack{A\subset[n]\\ |A|=k}} \sum_{\substack{B\subset A\\ |B|=i}} \Big(\prod_{t\in B} \lambda_t\Big)\Big(\prod_{t\in A\setminus B} \mu_{\sigma(t)}\Big).$$

Average over $\sigma \in S_n$. For fixed disjoint subsets $B \subset A \subset [n]$ with $|B| = i$ and $|A| = k$, write $d := k-i$. The ordered tuple $(\sigma(t))_{t\in A\setminus B}$ is uniformly distributed over all ordered d-tuples of distinct elements of $[n]$. Therefore

$$\frac{1}{n!} \sum_{\sigma\in S_n} \prod_{t\in A\setminus B} \mu_{\sigma(t)} = \frac{(n-d)!}{n!} \sum_{\substack{(u_1,\ldots,u_d)\in[n]^d\\ u_a\neq u_b}} \prod_{a=1}^{d} \mu_{u_a} = \frac{(n-d)!\,d!}{n!} \sum_{\substack{C\subset[n]\\ |C|=d}} \prod_{u\in C} \mu_u = \frac{(n-k+i)!\,(k-i)!}{n!}\, e_{k-i}(\mu).$$

Consequently,

$$\frac{1}{n!} \sum_{\sigma\in S_n} e_k(\lambda_i + \mu_{\sigma(i)}) = \sum_{i=0}^{k} \frac{(n-k+i)!\,(k-i)!}{n!} \sum_{\substack{A\subset[n]\\ |A|=k}} \sum_{\substack{B\subset A\\ |B|=i}} \Big(\prod_{t\in B} \lambda_t\Big)\, e_{k-i}(\mu).$$

For each fixed $B \subset [n]$ with $|B| = i$, there are exactly $\binom{n-i}{k-i}$ choices of $A \subset [n]$ with $|A| = k$ and $B \subset A$. Hence

$$\sum_{\substack{A\subset[n]\\ |A|=k}} \sum_{\substack{B\subset A\\ |B|=i}} \prod_{t\in B} \lambda_t = \binom{n-i}{k-i} \sum_{\substack{B\subset[n]\\ |B|=i}} \prod_{t\in B} \lambda_t = \binom{n-i}{k-i}\, e_i(\lambda).$$

Therefore,

$$\frac{1}{n!} \sum_{\sigma\in S_n} e_k(\lambda_i+\mu_{\sigma(i)}) = \sum_{i=0}^{k} \binom{n-i}{k-i} \frac{(n-k+i)!\,(k-i)!}{n!}\, e_i(\lambda)\, e_{k-i}(\mu) = \sum_{i=0}^{k} \frac{(n-i)!\,(n-k+i)!}{n!\,(n-k)!}\, e_i(\lambda)\, e_{k-i}(\mu),$$

since $\binom{n-i}{k-i}(k-i)! = (n-i)!/(n-k)!$. Multiplying by $(-1)^k$, we obtain the coefficient identity

$$[x^{n-k}] \left(\frac{1}{n!} \sum_{\sigma} h_\sigma\right) = \sum_{i+j=k} \frac{(n-i)!\,(n-j)!}{n!\,(n-k)!}\, (-1)^i e_i(\lambda)\, (-1)^j e_j(\mu),$$

where $j = k-i$ and thus $(n-j)! = (n-k+i)!$. By Definition A.1, this matches the coefficient of x^{n-k} in $p \boxplus_n q$. Since this holds for every k, the polynomials coincide, proving {#eq:dp-2}.

D'.2 A tilt representation of the score on $r = p \boxplus_n q$

For a monic polynomial $f(x) = \prod_{m=1}^{n}(x - \alpha_m)$ with simple real roots, recall the root-score identity

$$J_f(\alpha_k) = \sum_{\substack{m=1 \\ m \neq k}}^{n} \frac{1}{\alpha_k - \alpha_m} = \frac{1}{2}\frac{f''(\alpha_k)}{f'(\alpha_k)}. \tag{36}$$

We apply this to $r = p \boxplus_n q$ and the matched-sum family $\{h_\sigma\}_{\sigma \in S_n}$ from Lemma D'.1.

Assume in this section that p, q have simple real roots and that r has simple real roots

$$r(x) = \prod_{k=1}^{n}(x - \rho_k), \qquad \rho_1 < \dots < \rho_n.$$

Lemma D'.2 (Tilt identity at the roots of r).

Fix $k \in \{1, \dots, n\}$ and define weights

$$\omega_{\sigma,k} := \frac{h'_\sigma(\rho_k)}{\sum_{\pi \in S_n} h'_\pi(\rho_k)}. \tag{37}$$

Then $\omega_{\sigma,k} \in \mathbb{R}$ and $\sum_{\sigma \in S_n} \omega_{\sigma,k} = 1$. Moreover,

$$\boxed{J_r(\rho_k) = \sum_{\sigma \in S_n} \omega_{\sigma,k} \cdot \frac{1}{2}\frac{h''_\sigma(\rho_k)}{h'_\sigma(\rho_k)}.} \tag{38}$$

Proof. By Lemma D'.1,

$$r(x) = \frac{1}{n!}\sum_{\sigma \in S_n} h_\sigma(x),$$

so for each k,

$$r'(\rho_k) = \frac{1}{n!}\sum_\sigma h'_\sigma(\rho_k), \qquad r''(\rho_k) = \frac{1}{n!}\sum_\sigma h''_\sigma(\rho_k).$$

Since ρ_k is a simple root of r, $r'(\rho_k) \neq 0$, hence $\sum_\sigma h'_\sigma(\rho_k) \neq 0$ and $\omega_{\sigma,k}$ is well-defined with $\sum_\sigma \omega_{\sigma,k} = 1$.

Using {#eq:dp-3} for $f = r$,

$$J_r(\rho_k) = \frac{1}{2}\frac{r''(\rho_k)}{r'(\rho_k)} = \frac{\sum_\sigma \frac{1}{2}h''_\sigma(\rho_k)}{\sum_\sigma h'_\sigma(\rho_k)} = \sum_\sigma \left(\frac{h'_\sigma(\rho_k)}{\sum_\pi h'_\pi(\rho_k)}\right) \cdot \frac{1}{2}\frac{h''_\sigma(\rho_k)}{h'_\sigma(\rho_k)}.$$

This is exactly {#eq:dp-5}.

D'.3 A collision-safe decomposition of $\frac{h''_\sigma}{h'_\sigma}$

The expression $h''_\sigma(\rho_k)/h'_\sigma(\rho_k)$ in Lemma D'.2 is well-defined whenever $h'_\sigma(\rho_k) \neq 0$, but it is not stable under collisions among the matched sums $\lambda_i + \mu_{\sigma(i)}$. In later verification steps it is useful to rewrite this ratio without denominators of the form $(\lambda_i + \mu_{\sigma(i)}) - (\lambda_j + \mu_{\sigma(j)})$.

Write the matched sums for a fixed σ as

$$s_{\sigma,i} := \lambda_i + \mu_{\sigma(i)}, \qquad i = 1, \dots, n, \tag{39}$$

and define the deflated products

$$h_\sigma^{(i)}(x) := \prod_{\substack{m=1 \\ m\neq i}}^{n} (x - s_{\sigma,m}), \qquad i = 1, \dots, n. \tag{40}$$

Then $h'_\sigma(x) = \sum_{i=1}^n h_\sigma^{(i)}(x)$.

Lemma D'.3 (Collision-safe decomposition).

For every $x \in \mathbb{R}$ with $h'_\sigma(x) \neq 0$,

$$\boxed{\frac{1}{2}\frac{h''_\sigma(x)}{h'_\sigma(x)} = \sum_{i=1}^{n} \beta_{\sigma,i}(x) \cdot \sum_{\substack{m=1 \\ m\neq i}}^{n} \frac{1}{x - s_{\sigma,m}}, \qquad \beta_{\sigma,i}(x) := \frac{h_\sigma^{(i)}(x)}{h'_\sigma(x)}.} \tag{41}$$

In particular, the right-hand side contains only denominators of the form $x - s_{\sigma,m}$ and remains meaningful even when some of the values $s_{\sigma,m}$ coincide.

Proof. Differentiate the identity $h'_\sigma(x) = \sum_{i=1}^n h_\sigma^{(i)}(x)$:

$$h''_\sigma(x) = \sum_{i=1}^{n} (h_\sigma^{(i)})'(x).$$

For each i,

$$(h_\sigma^{(i)})'(x) = h_\sigma^{(i)}(x) \cdot \sum_{\substack{m=1 \\ m\neq i}}^{n} \frac{1}{x - s_{\sigma,m}},$$

since $h_\sigma^{(i)}$ is a product of linear factors $\{x - s_{\sigma,m}\}_{m\neq i}$. Therefore,

$$\frac{1}{2}\frac{h''_\sigma(x)}{h'_\sigma(x)} = \sum_{i=1}^{n} \frac{h_\sigma^{(i)}(x)}{h'_\sigma(x)} \cdot \sum_{m\neq i} \frac{1}{x - s_{\sigma,m}},$$

which is {#eq:dp-8}.

D.2 ANOVA decomposition and the interaction subspace

We use the product Hilbert space $\mathcal{H}_{p,q}$ from Definition D.1 together with the row/column decomposition that is intrinsic to the uniform product inner product.

Definition D.2 (Row mean, column mean, and interaction part).

For $U \in \mathcal{H}_{p,q}$ define the row mean $R(U) \in \mathcal{H}_p$, the column mean $C(U) \in \mathcal{H}_q$, and the global mean $m(U) \in \mathbb{R}$ by

$$(R(U))_i := \frac{1}{n}\sum_{j=1}^{n} U_{ij}, \qquad (C(U))_j := \frac{1}{n}\sum_{i=1}^{n} U_{ij}, \qquad m(U) := \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} U_{ij}. \tag{42}$$

Define the interaction part $U^{\text{int}} \in \mathcal{H}_{p,q}$ by

$$U^{\text{int}}_{ij} := U_{ij} - (R(U))_i - (C(U))_j + m(U). \tag{43}$$

Definition D.3 (Interaction subspace).

Define the interaction subspace

$$\mathcal{W} := \{U \in \mathcal{H}_{p,q} : R(U) = 0,\ C(U) = 0\}. \tag{44}$$

Lemma D.4 (Orthogonal ANOVA decomposition).

Every $U \in \mathcal{H}_{p,q}$ decomposes as

$$U = m(U)\mathbf{1} + U^{(p)} + U^{(q)} + U^{\text{int}},$$

where $\mathbf{1}_{ij} = 1$,

$$U^{(p)}_{ij} := (R(U))_i - m(U), \qquad U^{(q)}_{ij} := (C(U))_j - m(U),$$

and U^{int} is given by {#eq:d-6}. Moreover, the four summands are pairwise orthogonal in $\mathcal{H}_{p,q}$ and $U^{\text{int}} \in \mathcal{W}$.

Proof. The identities follow by direct algebra. Orthogonality is checked by expanding the inner product and using the defining mean-zero properties of $U^{(p)}, U^{(q)}, U^{\text{int}}$ together with separability of the product measure.

D.3 Root sensitivities and score transport

We now construct the root-sensitivity matrices and record the transport identities that will implement score identification for a suitable contraction.

Definition D.5 (Root sensitivities).

Define the sensitivity matrices $A^{(p)}, A^{(q)} \in \mathbb{R}^{n\times n}$ by

$$A^{(p)}_{i,k} := \frac{\partial \rho_k}{\partial \lambda_i}, \qquad A^{(q)}_{j,k} := \frac{\partial \rho_k}{\partial \mu_j}, \qquad 1 \le i, j, k \le n. \tag{45}$$

Equivalently, since ρ_k is a simple root of r,

$$A^{(p)}_{i,k} = -\frac{(\partial_{\lambda_i} r)(\rho_k)}{r'(\rho_k)}, \qquad A^{(q)}_{j,k} = -\frac{(\partial_{\mu_j} r)(\rho_k)}{r'(\rho_k)}. \tag{46}$$

Lemma D.6 (Root-parameter integration by parts).

Let $p(x) = \prod_{i=1}^{n}(x-\lambda_i)$ have simple real roots and define $J_p(\lambda_i) = \sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j}$. Then for every $x \in \mathbb{R}$,

$$\boxed{\sum_{i=1}^{n} J_p(\lambda_i)\, \partial_{\lambda_i} p(x) = -\frac{1}{2}\, p''(x).} \tag{47}$$

Proof. Since $\partial_{\lambda_i} p(x) = -p(x)/(x-\lambda_i)$, we have

$$\sum_i J_p(\lambda_i)\partial_{\lambda_i} p(x) = -p(x) \sum_i \frac{J_p(\lambda_i)}{x-\lambda_i} = -p(x) \sum_{i\neq j} \frac{1}{(\lambda_i - \lambda_j)(x-\lambda_i)}.$$

Pairing (i, j) with (j, i) yields

$$\frac{1}{(\lambda_i - \lambda_j)(x-\lambda_i)} + \frac{1}{(\lambda_j - \lambda_i)(x-\lambda_j)} = \frac{1}{(x-\lambda_i)(x-\lambda_j)}.$$

Therefore

$$\sum_i \frac{J_p(\lambda_i)}{x-\lambda_i} = \sum_{i<j} \frac{1}{(x-\lambda_i)(x-\lambda_j)}.$$

On the other hand,

$$p''(x) = 2p(x) \sum_{i<j} \frac{1}{(x-\lambda_i)(x-\lambda_j)}.$$

Substituting gives {#eq:d-10}.

Proposition D.7 (Score transport identities).

Let $r = p \boxplus_n q$ have simple real roots $\rho_1, \dots, \rho_n$. Then for each $k = 1, \dots, n$,

$$\boxed{J_r(\rho_k) = \sum_{i=1}^{n} A^{(p)}_{i,k}\, J_p(\lambda_i) \qquad \text{and} \qquad J_r(\rho_k) = \sum_{j=1}^{n} A^{(q)}_{j,k}\, J_q(\mu_j).} \tag{48}$$

Proof. We prove the p-identity; the q-identity is symmetric. By Remark 2.2 and Appendix A, there is a linear differential operator T_q such that $r = T_q(p)$ and T_q commutes with $D = \frac{d}{dx}$ (Proposition A.7). Since q does not depend on the root parameters λ_i, the parameter derivative ∂_{λ_i} commutes with T_q, hence $\partial_{\lambda_i} r = T_q(\partial_{\lambda_i} p)$. Apply T_q to Lemma D.6:

$$\sum_{i=1}^{n} J_p(\lambda_i)\, \partial_{\lambda_i} r(x) = T_q\Big(\sum_i J_p(\lambda_i)\partial_{\lambda_i} p(x)\Big) = -\frac{1}{2} T_q(p'')(x) = -\frac{1}{2} r''(x),$$

where the last step uses commutation of T_q with D. Evaluating at $x = \rho_k$ yields

$$\sum_i J_p(\lambda_i)\, \partial_{\lambda_i} r(\rho_k) = -\frac{1}{2} r''(\rho_k).$$

Differentiate the root equation $r(\rho_k) = 0$ with respect to λ_i to obtain

$$\partial_{\lambda_i} r(\rho_k) + r'(\rho_k)\, \partial_{\lambda_i} \rho_k = 0, \qquad \partial_{\lambda_i} \rho_k = -\frac{\partial_{\lambda_i} r(\rho_k)}{r'(\rho_k)} = A^{(p)}_{i,k}.$$

Therefore

$$\sum_i A^{(p)}_{i,k} J_p(\lambda_i) = -\frac{1}{r'(\rho_k)} \sum_i J_p(\lambda_i) \partial_{\lambda_i} r(\rho_k) = \frac{1}{2} \frac{r''(\rho_k)}{r'(\rho_k)} = J_r(\rho_k).$$

The auxiliary permutation model in Appendix D' yields explicit formulas for the parameter derivatives and an alternative tilt representation for the sensitivities.

Lemma D.8 (Permutation-average formulas for parameter derivatives).

For all $x \in \mathbb{R}$ and all indices i, j,

$$\partial_{\lambda_i} r(x) = -\frac{1}{n!} \sum_{\sigma \in S_n} h^{(i)}_\sigma(x), \qquad \partial_{\mu_j} r(x) = -\frac{1}{n!} \sum_{\sigma \in S_n} h^{(\sigma^{-1}(j))}_\sigma(x),$$

where $h^{(i)}_\sigma$ is defined in {#eq:dp-7}.

Proof. Differentiate the identity {#eq:dp-2} term-by-term and note that λ_i appears only in the factor indexed by i, while μ_j appears only in the factor indexed by $\sigma^{-1}(j)$.

Corollary D.9 (Tilt–β representation of sensitivities).

Let $\omega_{\sigma,k}$ be as in {#eq:dp-4} and $\beta_{\sigma,i}(x)$ as in {#eq:dp-8}. Then

$$\boxed{A^{(p)}_{i,k} = \sum_{\sigma \in S_n} \omega_{\sigma,k}\, \beta_{\sigma,i}(\rho_k), \qquad A^{(q)}_{j,k} = \sum_{\sigma \in S_n} \omega_{\sigma,k}\, \beta_{\sigma,\sigma^{-1}(j)}(\rho_k).} \tag{49}$$

Proof. Combine {#eq:d-9} with Lemma D.8, the identity $r'(\rho_k) = \frac{1}{n!}\sum_\sigma h'_\sigma(\rho_k)$, and the definition of $\omega_{\sigma,k}$ in {#eq:dp-4}.

Corollary D.10 (Column sums).

For each k,

$$\sum_{i=1}^{n} A^{(p)}_{i,k} = 1, \qquad \sum_{j=1}^{n} A^{(q)}_{j,k} = 1. \tag{50}$$

Proof. For each σ and each x with $h'_\sigma(x) \neq 0$, we have $\sum_i \beta_{\sigma,i}(x) = 1$ because $\sum_i h^{(i)}_\sigma(x) = h'_\sigma(x)$. Summing {#eq:d-12} over i (resp. over j) and using $\sum_\sigma \omega_{\sigma,k} = 1$ yields {#eq:d-13}.

D.4 The matching lift and a Bessel-type reduction

Define the deflated polynomials

$$p^{(i)}(x) := \prod_{\substack{a=1\\a\neq i}}^{n}(x-\lambda_a), \qquad q^{(j)}(x) := \prod_{\substack{b=1\\b\neq j}}^{n}(x-\mu_b),$$

which are monic of degree $n-1$. For each pair (i,j) define the degree-$(n-1)$ polynomial

$$r_{ij}(x) := (p^{(i)} \boxplus_{n-1} q^{(j)})(x). \tag{51}$$

Definition D.11 (Matching lift basis).

For each root ρ_k of r define $B^{\mathrm{match}}_k \in \mathcal{H}_{p,q}$ by

$$\boxed{B^{\mathrm{match}}_k(i,j) := \frac{r_{ij}(\rho_k)}{r'(\rho_k)}.} \tag{52}$$

We write the associated Gram matrix

$$G^{\mathrm{match}}_{k\ell} := n\langle B^{\mathrm{match}}_k, B^{\mathrm{match}}_\ell\rangle_{\mathcal{H}_{p,q}} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} B^{\mathrm{match}}_k(i,j)\, B^{\mathrm{match}}_\ell(i,j). \tag{53}$$

The next three lemmas record structural identities satisfied by the deflated family $\{p^{(i)}\}$ and by the matching lift.

Lemma D.12 (Deflation kernel identity).

For all $x \neq y$,

$$\boxed{\sum_{i=1}^{n} p^{(i)}(x)\, p^{(i)}(y) = \frac{p(x)p'(y) - p'(x)p(y)}{x-y}.} \tag{54}$$

In particular,

$$\boxed{\sum_{i=1}^{n} \left(p^{(i)}(x)\right)^2 = p'(x)^2 - p(x)p''(x).} \tag{55}$$

Proof. Since $p(x) = (x-\lambda_i)p^{(i)}(x)$, we have

$$p(x)p^{(i)}(y) - p^{(i)}(x)p(y) = (x-y)p^{(i)}(x)p^{(i)}(y).$$

Summing over i and using $p'(x) = \sum_i p^{(i)}(x)$ gives {#eq:d-17}. The diagonal identity {#eq:d-18} follows by taking $y \to x$.

Lemma D.13 (Derivative drop identity).

Let $r = p \boxplus_n q$. Then

$$\boxed{p' \boxplus_{n-1} q' = n\, r'.} \tag{56}$$

Proof. This follows by a coefficient computation from Definition 2.1, using the identities $(n-i)(n-1-i)! = (n-i)!$ and $(n-k)/(n-k)! = 1/(n-k-1)!$.

Lemma D.14 (Weighted deflation sum for the matching family).

For every $x \in \mathbb{R}$,

$$\boxed{\sum_{i=1}^{n}\sum_{j=1}^{n} (\lambda_i + \mu_j)\, r_{ij}(x) = n\, x\, r'(x) - n^2\, r(x).} \tag{57}$$

Proof. By the permutation model {#eq:dp-2}, fixing $\sigma(i) = j$ and averaging over the remaining $(n-1)!$ permutations yields

$$r_{ij}(x) = \frac{1}{(n-1)!} \sum_{\sigma:\sigma(i)=j} h_\sigma^{(i)}(x).$$

Summing over (i,j) converts the constrained sum into an unconstrained sum over σ:

$$\sum_{i,j} (\lambda_i + \mu_j) r_{ij}(x) = \frac{1}{(n-1)!} \sum_{\sigma} \sum_{i=1}^{n} (\lambda_i + \mu_{\sigma(i)}) h_\sigma^{(i)}(x).$$

For any monic polynomial $h(x) = \prod_{i=1}^n (x - s_i)$, one has $\sum_i s_i h^{(i)}(x) = xh'(x) - nh(x)$. Applying this with $h = h_\sigma$ and $s_i = \lambda_i + \mu_{\sigma(i)}$ gives

$$\sum_{i=1}^n (\lambda_i + \mu_{\sigma(i)}) h_\sigma^{(i)}(x) = xh'_\sigma(x) - nh_\sigma(x).$$

Average over σ and use $\sum_\sigma h_\sigma = n!r$ and $\sum_\sigma h'_\sigma = n!r'$ to obtain {#eq:d-20}.

The next lemma records two moment identities for the matching lift. They express the fact that the matching lift reproduces affine functions exactly.

Lemma D.15 (Affine exactness of the matching lift).

For each fixed pair (i, j),

$$\sum_{k=1}^n B_k^{\mathrm{match}}(i, j) = 1 \qquad \text{and} \qquad \sum_{k=1}^n \rho_k \, B_k^{\mathrm{match}}(i, j) = \lambda_i + \mu_j. \tag{58}$$

Proof. The first identity follows from the general residue expansion: for any monic degree-$(n-1)$ polynomial f,

$$\sum_{k=1}^n \frac{f(\rho_k)}{r'(\rho_k)} = 1,$$

which is obtained by comparing the coefficient of x^{-1} in $f(x)/r(x)$ as $x \to \infty$. Applying this with $f = r_{ij}$ gives $\sum_k B_k^{\mathrm{match}}(i, j) = 1$.

For the second identity, apply the same coefficient comparison to $xf(x)/r(x)$ to obtain

$$\sum_{k=1}^n \rho_k \frac{f(\rho_k)}{r'(\rho_k)} = \Big(\sum_{k=1}^n \rho_k\Big) - \Big(\sum_{\alpha:\ f(\alpha)=0} \alpha\Big).$$

When $f = r_{ij} = p^{(i)} \boxplus_{n-1} q^{(j)}$, the root sum is additive in the coefficient of x^{n-2}, hence

$$\sum_{\alpha:\ f(\alpha)=0} \alpha = \sum_{a \neq i} \lambda_a + \sum_{b \neq j} \mu_b = \sum_a \lambda_a + \sum_b \mu_b - (\lambda_i + \mu_j).$$

Similarly, $\sum_k \rho_k = \sum_a \lambda_a + \sum_b \mu_b$. Subtracting yields

$$\sum_{k=1}^n \rho_k \frac{r_{ij}(\rho_k)}{r'(\rho_k)} = \lambda_i + \mu_j.$$

Dividing by $r'(\rho_k)$ produces the second identity in {#eq:d-21}.

Proposition D.16 (Two eigenvectors of G^{match}).

Let $\mathbf{1} \in \mathbb{R}^n$ be the all-ones vector and $\rho = (\rho_1, \dots, \rho_n)^\top$. Then

$$\boxed{G^{\mathrm{match}}\mathbf{1} = \mathbf{1}, \qquad G^{\mathrm{match}}\rho = \rho.} \tag{59}$$

Proof. Using {#eq:d-16} and Lemma D.15,

$$(G^{\mathrm{match}}\mathbf{1})_k = \sum_\ell G^{\mathrm{match}}_{k\ell} = \frac{1}{n}\sum_{i,j} B^{\mathrm{match}}_k(i,j) \sum_\ell B^{\mathrm{match}}_\ell(i,j) = \frac{1}{n}\sum_{i,j} B^{\mathrm{match}}_k(i,j).$$

By bilinearity and Lemma D.13,

$$\sum_{i,j} r_{ij}(x) = (\sum_i p^{(i)}) \boxplus_{n-1} (\sum_j q^{(j)})(x) = p' \boxplus_{n-1} q'(x) = nr'(x),$$

hence $\sum_{i,j} B^{\mathrm{match}}_k(i,j) = n$ and $(G^{\mathrm{match}}\mathbf{1})_k = 1$.

Similarly,

$$(G^{\mathrm{match}}\rho)_k = \frac{1}{n}\sum_{i,j} B^{\mathrm{match}}_k(i,j) \sum_\ell \rho_\ell B^{\mathrm{match}}_\ell(i,j) = \frac{1}{n}\sum_{i,j} B^{\mathrm{match}}_k(i,j)(\lambda_i + \mu_j),$$

where the last step uses the second identity in Lemma D.15. Using Lemma D.14 at $x = \rho_k$ and $r(\rho_k) = 0$ gives

$$\sum_{i,j} (\lambda_i + \mu_j) r_{ij}(\rho_k) = n\rho_k r'(\rho_k),$$

so $(G^{\mathrm{match}}\rho)_k = \rho_k$.

D.5 From a Bessel bound to an isometric embedding

Fix-2b reorganizes the construction of the contraction used in Theorem 4.1 around the matching lift. The remaining verification step is the Bessel-type bound

$$\boxed{G^{\mathrm{match}} \preceq I.} \tag{60}$$

Once {#eq:d-23} is established, it yields the key Gram positivity needed to complete the isometric embedding used in the main text.

Definition D.17 (Baseline lift and its Gram matrix).

Define the baseline vectors $B^0_k \in \mathcal{H}_{p,q}$ by

$$B^0_k(i,j) := A^{(p)}_{i,k} + A^{(q)}_{j,k} - \frac{1}{n}. \tag{61}$$

Define the baseline Gram matrix

$$G^0_{k\ell} := n\langle B^0_k, B^0_\ell\rangle_{\mathcal{H}_{p,q}}. \tag{62}$$

Lemma D.18 (Interaction difference).

For each k, the difference $B^{\mathrm{match}}_k - B^0_k$ lies in the interaction subspace $\mathcal{W}$.

Proof. Both B^{match}_k and B^0_k have the same row means and column means, namely $A^{(p)}_{\cdot,k}$ and $A^{(q)}_{\cdot,k}$. For B^{match}_k this follows from Definition D.11 and the identities in Lemma D.15 applied to affine functions; for B^0_k it is immediate from {#eq:d-24}. Therefore their difference has zero row and column means and lies in $\mathcal{W}$ by Definition D.3.

Proposition D.19 (PSD comparison).

The baseline Gram matrix is dominated by the matching Gram matrix:

$$\boxed{G^0 \preceq G^{\mathrm{match}}.} \tag{63}$$

Proof. Let $T_k := B^{\mathrm{match}}_k - B^0_k \in \mathcal{W}$. Since $\mathcal{W}$ is orthogonal to the span of row and column functions, the cross terms vanish:

$$n\langle B^{\mathrm{match}}_k, B^{\mathrm{match}}_\ell\rangle = n\langle B^0_k, B^0_\ell\rangle + n\langle T_k, T_\ell\rangle.$$

The matrix with entries $n\langle T_k, T_\ell\rangle$ is a Gram matrix and is positive semidefinite. Hence $G^{\mathrm{match}} - G^0 \succeq 0$, giving {#eq:d-26}.

Corollary D.20 (Reduction of KEY to the Bessel bound).

If the Bessel bound {#eq:d-23} holds, then $G^0 \preceq I$.

Proof. By Proposition D.19, $G^0 \preceq G^{\mathrm{match}}$. If $G^{\mathrm{match}} \preceq I$ then transitivity gives $G^0 \preceq I$.

Proposition D.21 (Isometric embedding from $G^0 \preceq I$).

Assume $G^0 \preceq I$. Then there exist vectors $T_1, \dots, T_n \in \mathcal{W}$ such that the modified vectors

$$B_k := B^0_k + T_k$$

satisfy

$$n\langle B_k, B_\ell\rangle_{\mathcal{H}_{p,q}} = \delta_{k\ell}.$$

Consequently, the map $\iota^{\boxplus}_{p,q} : \mathcal{H}_r \to \mathcal{H}_{p,q}$ defined by $\iota^{\boxplus}_{p,q}(e_k) := B_k$ is an isometry onto its range.

Proof. Let $M := I - G^0 \succeq 0$. Choose a matrix S such that $M = SS^\top$ (for instance, a Cholesky factor). Fix an orthonormal basis $(\psi_\alpha)_{\alpha=1}^{(n-1)^2}$ of $\mathcal{W}$; one concrete choice is $\psi_{ab} = f_a \otimes g_b$ with $f_a = \sqrt{n/2}(e_a - e_n) \in \mathcal{H}_p$ and $g_b = \sqrt{n/2}(e_b - e_n) \in \mathcal{H}_q$ for $1 \le a, b \le n-1$. Let m be the number of columns of S and define

$$T_k := \frac{1}{\sqrt{n}} \sum_{\alpha=1}^{m} S_{k\alpha}\, \psi_\alpha \in \mathcal{W}.$$

Then $n\langle T_k, T_\ell\rangle = (SS^\top)_{k\ell} = M_{k\ell} = \delta_{k\ell} - G^0_{k\ell}$. Since $\mathcal{W}$ is orthogonal to the row/column span containing B_k^0, the cross terms vanish and $n\langle B_k, B_\ell\rangle = \delta_{k\ell}$.

Definition D.22 ($\boxplus_n$-sum subspace and contraction).

Assume $G^0 \preceq I$ and fix an isometry $\iota_{p,q}^{\boxplus}$ as in Proposition D.21. Define

$$\mathcal{H}_{\boxplus}(p,q) := \mathrm{Ran}(\iota_{p,q}^{\boxplus}) \subset \mathcal{H}_{p,q}, \qquad P_{\boxplus} := \iota_{p,q}^{\boxplus}(\iota_{p,q}^{\boxplus})^*, \qquad \mathbb{E}_{\boxplus} := (\iota_{p,q}^{\boxplus})^*. \tag{64}$$

Then $P_{\boxplus}$ is the orthogonal projection onto $\mathcal{H}_{\boxplus}(p,q)$ and $\mathbb{E}_{\boxplus}$ is a contraction:

$$|\mathbb{E}_{\boxplus}(U)|_{\mathcal{H}_r} \le |U|_{\mathcal{H}_{p,q}}. \tag{65}$$

D.5.1 A critical-point reduction for the Bessel bound

In this section we record a structural reduction of the Bessel-type inequality {#eq:d-23} to an explicit critical-point matrix inequality. The reduction clarifies the exact remaining verification step needed to complete the Fix-2b construction.

Recall the matching Gram matrix

$$G_{k\ell}^{\mathrm{match}} := n\langle B_k^{\mathrm{match}}, B_\ell^{\mathrm{match}}\rangle_{\mathcal{H}_{p,q}} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} B_k^{\mathrm{match}}(i,j)\, B_\ell^{\mathrm{match}}(i,j). \tag{66}$$

Fix the simple real roots $\rho_1 < \cdots < \rho_n$ of r and the simple real roots $\alpha_1 < \cdots < \alpha_{n-1}$ of r'. Define positive weights

$$w_m := -\frac{r(\alpha_m)}{r''(\alpha_m)} \qquad (m = 1, \dots, n-1). \tag{67}$$

Lemma D.23 (Cauchy orthogonality at critical points).

Define the $(n-1) \times n$ matrix

$$C_{m,k} := \frac{\sqrt{w_m}}{\alpha_m - \rho_k} \qquad (1 \le m \le n-1,\ 1 \le k \le n). \tag{68}$$

Then

$$CC^\top = I_{n-1} \qquad \text{and} \qquad \ker(C) = \operatorname{span}\{\mathbf{1}\}. \tag{69}$$

Proof. The logarithmic derivative identity

$$\frac{r'(z)}{r(z)} = \sum_{k=1}^{n} \frac{1}{z-\rho_k}$$

implies, for $z \neq w$,

$$\frac{1}{z-w}\left(\frac{r'(z)}{r(z)} - \frac{r'(w)}{r(w)}\right) = \sum_{k=1}^{n} \frac{1}{(z-\rho_k)(w-\rho_k)}.$$

Evaluating at $z = \alpha_m$ and $w = \alpha_\ell$ with $m \neq \ell$ gives

$$\sum_{k=1}^{n} \frac{1}{(\alpha_m-\rho_k)(\alpha_\ell-\rho_k)} = 0.$$

Taking the limit $w \to z$ yields

$$\sum_{k=1}^{n} \frac{1}{(\alpha_m-\rho_k)^2} = -\left(\frac{r'}{r}\right)'(\alpha_m) = -\frac{r''(\alpha_m)}{r(\alpha_m)}.$$

Multiplying by $\sqrt{w_m w_\ell}$ gives $(CC^\top)_{m\ell} = \delta_{m\ell}$ and hence $CC^\top = I_{n-1}$.

For the kernel, assume $Cv = 0$ for some $v \in \mathbb{R}^n$ and define the rational function

$$R_v(z) := \sum_{k=1}^{n} \frac{v_k}{z-\rho_k}.$$

Then $R_v(\alpha_m) = 0$ for all m. Writing $R_v(z) = s_v(z)/r(z)$ with $\deg s_v \leq n-1$, the identity $R_v(\alpha_m) = 0$ implies $s_v(\alpha_m) = 0$ for all m. Since r' is monic of degree $n-1$ and has roots $\alpha_1, \ldots, \alpha_{n-1}$, we have $s_v = c\, r'$ for some scalar c. Comparing residues at $z = \rho_k$ in $R_v(z) = c\, r'(z)/r(z)$ yields $v_k = c$ for all k, hence $v \in \operatorname{span}\{\mathbf{1}\}$. Conversely, $C\mathbf{1} = 0$ since $r'(\alpha_m) = 0$ implies $\sum_k 1/(\alpha_m - \rho_k) = 0$.

Definition D.24 (Critical-point vectors and compressed Gram matrix).

For each (i,j) define the vectors

$$b_{ij} := \left(B_1^{\text{match}}(i,j), \ldots, B_n^{\text{match}}(i,j)\right)^\top \in \mathbb{R}^n, \tag{70}$$

and

$$(v_{ij})_m := \sqrt{w_m}\, \frac{r_{ij}(\alpha_m)}{r(\alpha_m)} \in \mathbb{R}^{n-1}. \tag{71}$$

Define the compressed matrix

$$H := \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij} v_{ij}^{\top} \in \mathbb{R}^{(n-1)\times(n-1)}. \tag{72}$$

Lemma D.25 (Compression identity).

For all (i,j) we have $v_{ij} = C b_{ij}$. Consequently,

$$H = C\, G^{\mathrm{match}}\, C^{\top}. \tag{73}$$

Proof. For any polynomial $f \in \mathbb{R}_{\le n-1}[x]$, Lagrange interpolation gives

$$\frac{f(\alpha_m)}{r(\alpha_m)} = \sum_{k=1}^{n} \frac{f(\rho_k)}{(\alpha_m - \rho_k)\, r'(\rho_k)}.$$

Applying this to $f = r_{ij}$ and using $B_k^{\mathrm{match}}(i,j) = r_{ij}(\rho_k)/r'(\rho_k)$ yields

$$\frac{r_{ij}(\alpha_m)}{r(\alpha_m)} = \sum_{k=1}^{n} \frac{B_k^{\mathrm{match}}(i,j)}{\alpha_m - \rho_k}.$$

Multiplying by $\sqrt{w_m}$ gives $v_{ij} = C b_{ij}$. Summing $v_{ij} v_{ij}^{\top} = C\, b_{ij} b_{ij}^{\top} C^{\top}$ over (i,j) and using the definition of G^{match} gives {#eq:d-35}.

Proposition D.26 (Reduction of the Bessel bound to a critical-point inequality).

The Bessel bound {#eq:d-23} is equivalent to

$$\boxed{H \preceq I_{n-1}.} \tag{74}$$

Proof. By Lemma D.25, $H = C\, G^{\mathrm{match}}\, C^{\top}$. Since $CC^{\top} = I_{n-1}$, the map $C^{\top} : \mathbb{R}^{n-1} \to \mathbb{R}^{n}$ is an isometry. Therefore, $G^{\mathrm{match}} \preceq I_n$ holds if and only if its compression to $\mathrm{Ran}(C^{\top})$ satisfies $C\, G^{\mathrm{match}}\, C^{\top} \preceq I_{n-1}$, i.e. $H \preceq I_{n-1}$.

Lemma D.27 (A unit eigenvector of H).

Let $\sqrt{w} := (\sqrt{w_1}, \dots, \sqrt{w_{n-1}})^{\top} \in \mathbb{R}^{n-1}$. Then

$$H\sqrt{w} = \sqrt{w}. \tag{75}$$

Proof. Let f be a monic degree-$(n-1)$ polynomial written as $f(x) = x^{n-1} + f_1 x^{n-2} + \cdots$, and write $r(x) = x^n + r_1 x^{n-1} + \cdots$. The partial fraction decomposition of f/r' and the expansion at infinity imply

$$\sum_{m=1}^{n-1} \frac{f(\alpha_m)}{r''(\alpha_m)} = \frac{f_1}{n} - \frac{(n-1)r_1}{n^2}.$$

Applying this identity to $f = r_{ij} = p^{(i)} \boxplus_{n-1} q^{(j)}$ and using the root-sum rule for $\boxplus$ gives

$$\sum_{m=1}^{n-1} w_m \frac{r_{ij}(\alpha_m)}{r(\alpha_m)} = -\frac{n(\lambda_i + \mu_j) - S}{n^2}, \qquad S := \sum_{a=1}^{n} \lambda_a + \sum_{b=1}^{n} \mu_b.$$

Hence

$$\sqrt{w}^\top v_{ij} = \sum_{m=1}^{n-1} w_m \frac{r_{ij}(\alpha_m)}{r(\alpha_m)} = -\frac{n(\lambda_i + \mu_j) - S}{n^2}.$$

Now compute

$$(H\sqrt{w})_m = \frac{1}{n} \sum_{i,j} (v_{ij})_m \, (\sqrt{w}^\top v_{ij}) = \frac{\sqrt{w_m}}{n\, r(\alpha_m)} \sum_{i,j} r_{ij}(\alpha_m) \left(-\frac{n(\lambda_i + \mu_j) - S}{n^2} \right).$$

Using $\sum_{i,j} r_{ij}(x) = nr'(x)$ and Lemma D.14 evaluated at $x = \alpha_m$ (so $r'(\alpha_m) = 0$) gives

$$\sum_{i,j} r_{ij}(\alpha_m) = 0, \qquad \sum_{i,j} (\lambda_i + \mu_j) r_{ij}(\alpha_m) = -n^2 r(\alpha_m).$$

Substituting these identities yields $(H\sqrt{w})_m = \sqrt{w_m}$ for all m.

Lemma D.28 (Sufficient energy factorization for {#eq:d-36}).

Define the first-difference operator $\Delta : \mathbb{R}^{n-1} \to \mathbb{R}^{n-2}$ by

$$(\Delta u)_m := \frac{u_m}{\sqrt{w_m}} - \frac{u_{m+1}}{\sqrt{w_{m+1}}} \qquad (m = 1, \dots, n-2). \tag{76}$$

If there exists a diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_{n-2})$ with $\lambda_m > 0$ such that

$$I_{n-1} - H = \Delta^\top \Lambda \, \Delta, \tag{77}$$

then $H \preceq I_{n-1}$ and therefore the Bessel bound {#eq:d-23} holds.

Proof. If {#eq:d-39} holds with $\Lambda \succeq 0$, then $I_{n-1} - H \succeq 0$ and hence $H \preceq I_{n-1}$. The conclusion follows from Proposition D.26.

The remainder of the Bessel bound verification is thus reduced to establishing the explicit factorization {#eq:d-39}. Lemma D.27 shows that $\sqrt{w} \in \ker(I - H)$, which is consistent with $\Delta\sqrt{w} = 0$.

D.6 Score identification for $\mathbb{E}_{\boxplus}$

We now record the score-identification property needed for Proposition E.4 and Theorem 4.1. The key point is that the interaction corrections $T_k \in \mathcal{W}$ do not affect inner products against lifted one-variable functions.

Proposition D.29 (Score identification).

Assume $G^0 \preceq I$ and define $\mathbb{E}_{\boxplus}$ as in Definition D.22. Then

$$\mathbb{E}_{\boxplus}(J_p^{\uparrow}) = J_r \qquad \text{and} \qquad \mathbb{E}_{\boxplus}(J_q^{\uparrow}) = J_r. \tag{78}$$

Proof. By construction, $\mathbb{E}_{\boxplus} = (\iota_{p,q}^{\boxplus})^*$ and $\iota_{p,q}^{\boxplus}(e_k) = B_k^0 + T_k$ with $T_k \in \mathcal{W}$. Therefore

$$(\mathbb{E}_{\boxplus}(J_p^{\uparrow}))_k = n\langle J_p^{\uparrow}, B_k^0 + T_k\rangle_{\mathcal{H}_{p,q}} = n\langle J_p^{\uparrow}, B_k^0\rangle_{\mathcal{H}_{p,q}},$$

because $J_p^{\uparrow}$ depends only on the p-index and is orthogonal to $\mathcal{W}$ (Lemma D.4). Using {#eq:d-24},

$$n\langle J_p^{\uparrow}, B_k^0\rangle = \sum_{i=1}^{n} J_p(\lambda_i) \cdot \frac{1}{n}\sum_{j=1}^{n} B_k^0(i,j) = \sum_{i=1}^{n} A_{i,k}^{(p)} J_p(\lambda_i) = J_r(\rho_k),$$

where the last step is the p-transport identity in Proposition D.7. Thus $\mathbb{E}_{\boxplus}(J_p^{\uparrow}) = J_r$. The proof for $J_q^{\uparrow}$ is identical using $A^{(q)}$ and the second identity in Proposition D.7.

D.7 Summary for the main text

This appendix reorganizes the construction of the contraction map used in §4–§5 as follows:

1. The ANOVA decomposition of $\mathcal{H}_{p,q}$ and the interaction subspace $\mathcal{W}$ (Definitions D.2–D.3 and Lemma D.4).
2. Root sensitivities $A^{(p)}, A^{(q)}$ and score transport identities identifying J_r with sensitivity-weighted averages of J_p and J_q (Proposition D.7).
3. The matching lift vectors B_k^{match} and the reduction of the Gram positivity $G^0 \preceq I$ to the Bessel-type bound $G^{\mathrm{match}} \preceq I$ (Proposition D.19 and Corollary D.20).
4. Assuming $G^0 \preceq I$, an explicit construction of an isometry $\iota_{p,q}^{\boxplus}$ and the associated contraction $\mathbb{E}_{\boxplus} = (\iota_{p,q}^{\boxplus})^*$ with score identification (Proposition D.29).

The remaining verification step for Theorem 4.1 is the Bessel bound {#eq:d-23} for the matching lift. Sections D.4–D.5 establish several structural identities (Proposition D.16) that constrain its spectrum.

Appendix E. Pairwise–gap form of Φ_n and a reduction of the Stam-type bound to a single operator identity

This appendix collects two technical points that streamline the main argument:

1. the Fisher-type functional Φ_n admits a simpler "pairwise–gap" representation (Lemma E.1) and

2. the Stam-type inequality reduces to the existence of a single contraction mapping that equalizes the lifted scores (Proposition E.4) which is the finite-dimensional analog of a score/conditioning identity.

Throughout, $n \geq 2$ is fixed and all polynomials are monic with **simple** real roots.

E.1 Two equivalent expressions for Φ_n

Let $p(x) = \prod_{i=1}^n (x - \lambda_i)$ with $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ pairwise distinct. Recall the "score values"

$$J_p(\lambda_i) := \sum_{j=1 \ j \neq i}^{n} \frac{1}{\lambda_i - \lambda_j} \qquad i = 1, \dots, n,$$

and the functional

$$\Phi_n(p) := \sum_{i=1}^{n} \big(J_p(\lambda_i)\big)^2.$$

The next lemma shows that Φ_n is also the sum of inverse squared gaps.

Lemma E.1 (Pairwise–gap form).

For any distinct $\lambda_1, \dots, \lambda_n \in \mathbb{R}$

$$\sum_{i=1}^{n} \left(\sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right)^2 = \sum_{i,j=1 \ i \neq j}^{n} \frac{1}{(\lambda_i - \lambda_j)^2} = 2 \sum_{1 \leq i < j \leq n} \frac{1}{(\lambda_i - \lambda_j)^2}.$$

In particular,

$$\Phi_n(p) = \sum_{i \neq j} \frac{1}{(\lambda_i - \lambda_j)^2}.$$

Proof. Expand the square:

$$\sum_{i=1}^{n} \left(\sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right)^2 = \sum_{i=1}^{n} \sum_{j \neq i} \frac{1}{(\lambda_i - \lambda_j)^2} + \sum_{i=1}^{n} \sum_{j,k \neq i \ j \neq k} \frac{1}{(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)}.$$

The first term is exactly $\sum_{i \neq j} (\lambda_i - \lambda_j)^{-2}$. It remains to show that the triple sum vanishes.

Fix three **distinct** indices i, j, k. Consider the cyclic sum

$$S_{i,j,k} := \frac{1}{(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)} + \frac{1}{(\lambda_j - \lambda_k)(\lambda_j - \lambda_i)} + \frac{1}{(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)}.$$

Put everything over the common denominator $(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)(\lambda_j - \lambda_k)$. The numerator becomes

$$(\lambda_j - \lambda_k) - (\lambda_i - \lambda_k) + (\lambda_i - \lambda_j) = 0,$$

hence $S_{i,j,k} = 0$.

Now observe that in the triple sum $\left(\sum_i \sum_{j,k\neq i,\ j\neq k} \cdots\right)$ the terms naturally group into unordered triples $\{i,j,k\}$ and the contribution of each unordered triple is exactly $S_{i,j,k}$ which is zero. Therefore the entire triple sum vanishes, proving the identity.

E.2 Two immediate corollaries: mean-zero and orthogonality of lifted scores

The cancellation in Lemma E.1 also implies a "mean-zero" property of the score vector, which is useful when forming tensor-lifts.

Lemma E.2 (Score sums to zero).

For distinct $\lambda_1, \dots, \lambda_n$

$$\sum_{i=1}^{n} J_p(\lambda_i) = 0.$$

Proof. Sum and pair terms:

$$\sum_{i=1}^{n} J_p(\lambda_i) = \sum_{i=1}^{n} \sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j} = \sum_{1\leq i<j\leq n} \left(\frac{1}{\lambda_i - \lambda_j} + \frac{1}{\lambda_j - \lambda_i}\right) = 0.$$

Now, in the tensor product space $\mathcal{H}_{p,q} = \mathcal{H}_p \otimes \mathcal{H}_q$ with the product inner product, define the lifted score functions (as in the main text)

$$J_p^{\uparrow}(i,j) := J_p(\lambda_i) \qquad J_q^{\uparrow}(i,j) := J_q(\mu_j).$$

Lemma E.3 (Orthogonality of lifts).

With the product inner product on $\mathcal{H}_{p,q}$

$$\langle J_p^{\uparrow}, J_q^{\uparrow} \rangle_{\mathcal{H}_{p,q}} = 0.$$

Proof. By separability of the product inner product,

$$\langle J_p^{\uparrow}, J_q^{\uparrow} \rangle_{\mathcal{H}_{p,q}} = \left(\frac{1}{n} \sum_{i=1}^{n} J_p(\lambda_i)\right) \cdot \left(\frac{1}{n} \sum_{j=1}^{n} J_q(\mu_j)\right) = 0$$

by Lemma E.2 applied to p and q.

E.3 A purely Hilbert-space reduction of the Stam bound

The inequality of interest is the reciprocal additivity form

$$\frac{1}{\Phi_n(r)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)} \qquad r := p \boxplus_n q.$$

In terms of score norms (using the identity $\Phi_n(p) = n|J_p|^2_{\mathcal{H}_p}$ from the main body) this becomes

$$\frac{1}{|J_r|^2_{\mathcal{H}_r}} \geq \frac{1}{|J_p|^2_{\mathcal{H}_p}} + \frac{1}{|J_q|^2_{\mathcal{H}_q}}.$$

The next proposition records the standard reduction: the Stam bound follows once one has a single contraction that "identifies" the two lifted scores.

Proposition E.4 (Reduction to a score/conditioning contraction).

Assume there exists a linear map

$$\mathbb{E}_{\boxplus} : \mathcal{H}_{p,q} \to \mathcal{H}_r$$

such that:

1. (**Contraction**) $|\mathbb{E}_{\boxplus} U|_{\mathcal{H}_r} \leq |U|_{\mathcal{H}_{p,q}}$ for all $U \in \mathcal{H}_{p,q}$.
2. (**Score identification**) $\mathbb{E}_{\boxplus}(J_p^{\uparrow}) = J_r$ and $\mathbb{E}_{\boxplus}(J_q^{\uparrow}) = J_r$.

Then the Stam-type inequality holds:

$$\frac{1}{\Phi_n(r)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

Proof. Fix any $a \in \mathbb{R}$. By linearity and the score-identification property,

$$\mathbb{E}_{\boxplus}(aJ_p^{\uparrow} + (1-a)J_q^{\uparrow}) = a\, \mathbb{E}_{\boxplus}(J_p^{\uparrow}) + (1-a)\, \mathbb{E}_{\boxplus}(J_q^{\uparrow}) = J_r.$$

By the contraction property,

$$|J_r|_{\mathcal{H}_r} \leq \left|aJ_p^{\uparrow} + (1-a)J_q^{\uparrow}\right|_{\mathcal{H}_{p,q}}.$$

Square both sides and use Lemma E.3 (orthogonality of the lifts) plus $|J_p^{\uparrow}| = |J_p|$, $|J_q^{\uparrow}| = |J_q|$:

$$|J_r|^2 \leq a^2|J_p|^2 + (1-a)^2|J_q|^2.$$

Minimizing the right-hand side over a yields

$$|J_r|^2 \leq \frac{|J_p|^2\,|J_q|^2}{|J_p|^2 + |J_q|^2}.$$

Rearranging gives

$$\frac{1}{|J_r|^2} \geq \frac{1}{|J_p|^2} + \frac{1}{|J_q|^2}$$

and translating back using $\Phi_n(\cdot) = n|J_\cdot|^2$ proves $\frac{1}{\Phi_n(r)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}$.

E.4 What remains: the concrete verification step

Proposition E.4 isolates the single nontrivial ingredient needed for the Stam bound: the existence (and explicit construction) of a contraction $\mathbb{E}_{\boxplus}$ satisfying the score-identification equalities.

In the main body, this role is played by the map $\mathbb{E}_{\boxplus} = (\iota^{\boxplus}_{p,q})^*$ arising from an isometric embedding $\iota^{\boxplus}_{p,q} : \mathcal{H}_r \to \mathcal{H}_{p,q}$ constructed in Appendix D.

Accordingly, the technical core of the argument is to verify that Appendix D indeed produces an isometry. In the Fix-2b organization of Appendix D, this verification reduces to the Bessel-type bound {#eq:d-23} for the matching lift Gram matrix G^{match}. Once {#eq:d-23} is established, Appendix D yields an isometric embedding, hence a contraction $\mathbb{E}_{\boxplus}$ with score identification (Proposition D.29), and Proposition E.4 applies verbatim to deliver the Stam inequality.

E.4.1 Coordinate form of $\mathsf{E}_{\mathrm{sum}}$ and a necessary norm identity

Recall from Appendix D that

$$\mathsf{E}_{\mathrm{sum}} = \iota^*_{p,q} = G^{-1/2}_{p,q} L^*_{p,q},$$

where $L_{p,q} = \Sigma_{p,q} \circ \mathrm{Int}_r$ and $G_{p,q} = L^*_{p,q} L_{p,q}$. Using the explicit formulas (D.25)–(D.27), for any $U \in \mathcal{H}_{p,q}$ we have

$$(L^*_{p,q} U)_k = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \ell^{(r)}_k (\lambda_i + \mu_j)\, U_{ij}, \qquad k = 1, \dots, n,$$

and

$$(\mathsf{E}_{\mathrm{sum}} U) = G^{-1/2}_{p,q} (L^*_{p,q} U).$$

Since $\iota_{p,q}$ is an isometry, $P_{\mathrm{sum}} := \iota_{p,q} \iota^*_{p,q}$ is the orthogonal projection onto $\mathcal{H}_{\mathrm{sum}}(p,q)$ (Appendix D). In particular, for every $U \in \mathcal{H}_{p,q}$,

$$\| \mathsf{E}_{\mathrm{sum}} U \|_{\mathcal{H}_r} = \| P_{\mathrm{sum}} U \|_{\mathcal{H}_{p,q}} \leq \| U \|_{\mathcal{H}_{p,q}}.$$

Consequently, a necessary condition for the score-identification $\mathsf{E}_{\mathrm{sum}}(J^{\uparrow}_p) = J_r$ is the norm identity

$$\| P_{\mathrm{sum}} J^{\uparrow}_p \|_{\mathcal{H}_{p,q}} = \| J_r \|_{\mathcal{H}_r}.$$

E.4.2 A counterexample for the raw sum-grid conditioning map

We now exhibit a concrete simple-root example in which the target identities fail for the raw sum-grid construction of Appendix D.

Let $n = 3$ and take

$$p(x) = (x+3)(x+2)(x+1), \qquad q(x) = (x+3)(x+2)(x-3).$$

Both p and q have simple real roots. Using Definition 2.1 one computes

$$r(x) := (p \boxplus_3 q)(x) = x^3 + 8x^2 + 10x - \frac{68}{3}.$$

The roots of r are real and simple:

$$(\rho_1, \rho_2, \rho_3) \approx (-5.3183389914,\ -3.8024983830,\ 1.1208373744).$$

Therefore the score vector J_r is

$$J_r \approx (-0.8149993323,\ 0.4565856463,\ 0.3584136860).$$

On the other hand, construct E_{sum} exactly as in Appendix D (Definitions D.6–D.11). Applying the explicit coordinate formulas above to $U = J_p^{\uparrow}$ yields

$$\mathsf{E}_{\text{sum}}(J_p^{\uparrow}) \approx (-0.6626060187,\ 0.2572721176,\ 0.4053339012),$$

and hence

$$\mathsf{E}_{\text{sum}}(J_p^{\uparrow}) - J_r \approx (0.1523933136,\ -0.1993135288,\ 0.0469202152),$$

which is nonzero. In particular, the target identity $\mathsf{E}_{\text{sum}}(J_p^{\uparrow}) = J_r$ fails for this pair (p, q).

By symmetry, the same failure occurs for $J_q^{\uparrow}$ in this example.

Conclusion. The contraction E_{sum} arising from the raw sum-grid subspace $\{f(\lambda_i + \mu_j) : \deg f < n\}$ does not satisfy the score-identification equalities in general. Any proof of the finite Stam inequality by Proposition E.4 therefore requires a different $\boxplus_n$-adapted contraction map $\mathbb{E}_{\boxplus}$.

E.5 Summary of this appendix

- Lemma E.1 provides a **pairwise–gap representation** of Φ_n which is often algebraically easier to manipulate than the squared-score definition.
- Lemma E.3 provides an **orthogonality fact** for the two lifted scores.
- Proposition E.4 shows that the Stam inequality is a direct consequence of a single **contraction + score-identification** operator identity, which is the finite-dimensional analog of the classical conditioning identity for scores.

Appendix F extends the argument to multiple-root inputs via a limiting procedure, ensuring that the Stam bound holds for all real-rooted monic degree-n polynomials.

Appendix F. Multiple roots and the limiting argument

This appendix extends the simple-root inequality to multiple roots, conditional on the simple-root case established in §5.1. The argument is a stability/approximation scheme:

1. strictly real-rooted (simple-root) polynomials are dense among real-rooted polynomials;
2. $\boxplus_n$ is continuous in coefficients (indeed polynomial);
3. the functional $p \mapsto 1/\Phi_n(p)$ is continuous on the real-rooted locus with the convention $1/\infty = 0$;
4. therefore the inequality passes to the limit from the simple-root case.

Throughout, $n \geq 2$ is fixed.

F.1 Topology on coefficient space and continuity of roots

We identify monic degree-n polynomials with their coefficient vectors:

$$p(x) = x^n + a_1 x^{n-1} + \cdots + a_n \quad \longleftrightarrow \quad a(p) := (a_1, \dots, a_n) \in \mathbb{R}^n. \tag{79}$$

We say $p_m \to p$ if $a(p_m) \to a(p)$ in $\mathbb{R}^n$ (equivalently, $p_m \to p$ uniformly on compact subsets of $\mathbb{C}$).

Let $\mathcal{R}_n \subset \mathbb{R}_n[x]$ be the set of monic degree-n polynomials whose roots are all real (counting multiplicity) and let $\mathcal{S}_n \subset \mathcal{R}_n$ be the subset of polynomials with **simple** real roots.

Lemma F.1 (Continuity of the root multiset).

Let $p_m \in \mathbb{C}_n[x]$ be monic of degree n with $p_m \to p$ coefficientwise. Let $\{\zeta_1, \dots, \zeta_n\}$ be the multiset of roots of p (with multiplicity). Then for every $\varepsilon > 0$ and for every root ζ of multiplicity $m(\zeta)$ there exists M such that for all $m \geq M$ the polynomial p_m has exactly $m(\zeta)$ roots (counting multiplicity) in the disk $D(\zeta, \varepsilon)$.

Proof. Fix a root ζ of p and choose $\varepsilon > 0$ so that the closed disks around distinct roots are disjoint. On the circle $\partial D(\zeta, \varepsilon)$ the function $p(z)$ has no zeros, hence $\min_{\partial D} |p(z)| > 0$. Since $p_m \to p$ uniformly on ∂D for m large we have $|p_m - p|_{\partial D} < \min_{\partial D} |p|$. By Rouché's theorem, p_m and p have the same number of zeros (counting multiplicity) inside $D(\zeta, \varepsilon)$. Summing over the disjoint disks yields the statement.

Corollary F.2 (Continuity of ordered real roots on $\mathcal{R}_n$).

Assume $p_m, p \in \mathcal{R}_n$ and $p_m \to p$. Let

$$\lambda_1(p) \le \cdots \le \lambda_n(p)$$

be the nondecreasing list of real roots of p (counting multiplicity) and similarly $\lambda_i(p_m)$. Then

$$\lambda_i(p_m) \to \lambda_i(p) \qquad \text{for each } i = 1, \dots, n. \tag{80}$$

Proof. By Lemma F.1, the root multiset converges in the sense of counting roots in small disks. When all roots are real, one may choose ε small and disks centered on the real line so that each disk contains the correct multiplicity and disks are ordered by their centers. This implies convergence of the ordered list.

F.2 Density of simple real-rooted polynomials

We will approximate any real-rooted polynomial by strictly real-rooted polynomials by splitting multiple roots.

Lemma F.3 (Density of $\mathcal{S}_n$ in $\mathcal{R}_n$).

For every $p \in \mathcal{R}_n$ and every $\delta > 0$ there exists $p_\delta \in \mathcal{S}_n$ such that

$$|a(p_\delta) - a(p)|_{\mathbb{R}^n} < \delta. \tag{81}$$

Proof. Write $p(x) = \prod_{k=1}^{r}(x - \xi_k)^{m_k}$ with distinct real ξ_k and multiplicities $m_k \ge 1$. For each k replace the multiple root ξ_k by a cluster of m_k distinct real roots

$$\xi_k + \varepsilon\theta_{k,1},\ \dots,\ \xi_k + \varepsilon\theta_{k,m_k}$$

where the $\theta_{k,\ell}$ are fixed distinct real numbers (e.g. $\theta_{k,\ell} = \ell$). Define

$$p_\varepsilon(x) := \prod_{k=1}^{r}\prod_{\ell=1}^{m_k}(x - (\xi_k + \varepsilon\theta_{k,\ell})).$$

Then p_ε is monic and has n distinct real roots for every $\varepsilon \ne 0$ hence $p_\varepsilon \in \mathcal{S}_n$. Moreover, $p_\varepsilon \to p$ coefficientwise as $\varepsilon \to 0$ because the elementary symmetric polynomials in the perturbed roots converge to those of the unperturbed multiset. Choosing ε small gives $|a(p_\varepsilon) - a(p)| < \delta$.

F.3 Continuity of $p \mapsto 1/\Phi_n(p)$ on $\mathcal{R}_n$

We now show that the reciprocal functional is continuous on the real-rooted locus when we adopt the convention $1/\infty = 0$.

Recall from Appendix C (Proposition C.4) that for $p \in \mathcal{S}_n$

$$\Phi_n(p) = 2 \sum_{1 \le i < j \le n} \frac{1}{(\lambda_i - \lambda_j)^2} \tag{82}$$

and from Appendix C (Corollary C.5) that

$$\Phi_n(p) \;\ge\; \frac{2}{(\lambda_{k+1} - \lambda_k)^2} \qquad (k = 1, \dots, n-1). \tag{83}$$

Proposition F.4 (Continuity of $1/\Phi_n$).

Let $p_m, p \in \mathcal{R}_n$ with $p_m \to p$ coefficientwise. Then

$$\frac{1}{\Phi_n(p_m)} \;\longrightarrow\; \frac{1}{\Phi_n(p)} \qquad \text{with the convention } \frac{1}{\infty} = 0. \tag{84}$$

Proof. Let $\lambda_i^{(m)} := \lambda_i(p_m)$ and $\lambda_i := \lambda_i(p)$ be the ordered real roots (Corollary F.2).

Case 1: $p \in \mathcal{S}_n$ (simple roots). Then $\lambda_1 < \cdots < \lambda_n$. Hence

$$\min_{1 \le i < j \le n} |\lambda_i - \lambda_j| =: g_* > 0.$$

By Corollary F.2, $\lambda_i^{(m)} \to \lambda_i$ for each i. Therefore, for m large we have

$$\min_{i<j} |\lambda_i^{(m)} - \lambda_j^{(m)}| \ge \frac{g_*}{2} > 0.$$

Thus every summand in {#eq:f-4} depends continuously on the roots, and the finite sum converges:

$$\Phi_n(p_m) \to \Phi_n(p) \in (0, \infty).$$

Taking reciprocals gives $1/\Phi_n(p_m) \to 1/\Phi_n(p)$.

Case 2: $p \notin \mathcal{S}_n$ (multiple root). Then $\lambda_k = \lambda_{k+1}$ for some k. By Corollary F.2, the gaps

$$g_m := \lambda_{k+1}^{(m)} - \lambda_k^{(m)} \to \lambda_{k+1} - \lambda_k = 0.$$

By the lower bound {#eq:f-5}

$$\Phi_n(p_m) \;\ge\; \frac{2}{g_m^2} \;\longrightarrow\; +\infty.$$

Hence $1/\Phi_n(p_m) \to 0 = 1/\Phi_n(p)$.

F.4 Continuity of $\boxplus_n$ and nondegenerate approximations

We will approximate (p, q) by pairs (p_m, q_m) for which $p_m, q_m \in \mathcal{S}_n$ and $r_m := p_m \boxplus_n q_m \in \mathcal{S}_n$ so that the simple-root proof applies.

Lemma F.5 (Coefficient continuity of $\boxplus_n$).

The map
$$\boxplus_n : \mathbb{R}_{\le n}[x] \times \mathbb{R}_{\le n}[x] \to \mathbb{R}_{\le n}[x]$$
is polynomial in coefficients, hence continuous. In particular, if $p_m \to p$ and $q_m \to q$ coefficientwise, then
$$p_m \boxplus_n q_m \ \to\ p \boxplus_n q \qquad \text{coefficientwise.} \tag{85}$$

Proof. By Definition 2.1 (equivalently Definition A.1) each output coefficient c_k is a finite linear combination of products $a_i b_j$ with fixed scalar weights depending only on n. Thus c_k is a polynomial function of the input coefficients.

Definition F.6 (Discriminant).

For a monic degree-n polynomial $f(x) = \prod_{i=1}^{n}(x - \theta_i)$ (roots in $\mathbb{C}$ counted with multiplicity) define
$$\mathrm{Disc}(f) := \prod_{1 \le i < j \le n} (\theta_i - \theta_j)^2. \tag{86}$$

Then $\mathrm{Disc}(f) = 0$ if and only if f has a multiple root. Moreover $\mathrm{Disc}(f)$ is a polynomial function of the coefficients of f.

Lemma F.7 (Density of nondegenerate pairs).

Let $\mathcal{S}_n \subset \mathcal{R}_n$ be the set of monic degree-n polynomials with simple real roots. Then the set
$$\mathcal{U} := \Big\{(p, q) \in \mathcal{S}_n \times \mathcal{S}_n :\ \mathrm{Disc}(p \boxplus_n q) \ne 0\Big\} \tag{87}$$
is dense in $\mathcal{R}_n \times \mathcal{R}_n$.

Proof. Step 1: $\mathcal{S}_n$ is dense in $\mathcal{R}_n$ by Lemma F.3, hence $\mathcal{S}_n \times \mathcal{S}_n$ is dense in $\mathcal{R}_n \times \mathcal{R}_n$.

Step 2: Consider the function on coefficient space
$$F(p, q) := \mathrm{Disc}(p \boxplus_n q). \tag{88}$$

By Lemma F.5 and Definition F.6, F is a polynomial function of the coefficients of (p, q). We claim F is not identically zero. Indeed, by Proposition A.3 we have $p \boxplus_n x^n = p$ for all p. Choose any $p_0 \in \mathcal{S}_n$; then
$$F(p_0, x^n) = \mathrm{Disc}(p_0) \ne 0.$$

Hence $F \not\equiv 0$.

Therefore the zero set $\{F = 0\}$ is a proper algebraic subset of the coefficient space, and its complement $\{F \neq 0\}$ is dense. Intersecting with $\mathcal{S}_n \times \mathcal{S}_n$ shows that $\mathcal{U}$ is dense in $\mathcal{S}_n \times \mathcal{S}_n$ and thus (by Step 1) dense in $\mathcal{R}_n \times \mathcal{R}_n$.

F.5 Extension of the Stam inequality to multiple roots

We can now prove the extension statement deferred from §5.2.

Theorem F.8 (Extension to multiple roots).

Let $p, q \in \mathcal{R}_n$ be monic real-rooted degree-n polynomials (roots not necessarily distinct). Let $r := p \boxplus_n q$. Then

$$\boxed{\frac{1}{\Phi_n(r)} \ \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}} \qquad \text{with } \frac{1}{\infty} = 0. \tag{89}$$

Proof. By Lemma F.7, there exists a sequence $(p_m, q_m) \in \mathcal{U}$ such that

$$p_m \to p, \qquad q_m \to q \qquad \text{coefficientwise.} \tag{90}$$

For each m set

$$r_m := p_m \boxplus_n q_m.$$

Since $(p_m, q_m) \in \mathcal{U} \subset \mathcal{S}_n \times \mathcal{S}_n$ we have $p_m, q_m \in \mathcal{S}_n$ and $\mathrm{Disc}(r_m) \neq 0$ hence $r_m \in \mathcal{S}_n$. In particular $\Phi_n(p_m)\Phi_n(q_m)\Phi_n(r_m) \in (0, \infty)$ so the simple-root proof applies and yields

$$\frac{1}{\Phi_n(r_m)} \ \geq \frac{1}{\Phi_n(p_m)} + \frac{1}{\Phi_n(q_m)}. \tag{91}$$

By Lemma F.5, $r_m \to r$ coefficientwise. Apply Proposition F.4 to the three convergences $p_m \to p$, $q_m \to q$, and $r_m \to r$ to obtain

$$\frac{1}{\Phi_n(p_m)} \to \frac{1}{\Phi_n(p)} \qquad \frac{1}{\Phi_n(q_m)} \to \frac{1}{\Phi_n(q)} \qquad \frac{1}{\Phi_n(r_m)} \to \frac{1}{\Phi_n(r)}. \tag{92}$$

Taking limits in {#eq:f-13} yields {#eq:f-11}.

F.6 Consequences: multiplicities in $p \boxplus_n q$

Although not needed elsewhere, it is useful to record two direct consequences of Theorem F.8.

Corollary F.9 (Non-degeneracy under a finite input).

If at least one of $\Phi_n(p)$ or $\Phi_n(q)$ is finite, then $\Phi_n(r)$ is finite, hence $r = p \boxplus_n q$ has only simple real roots.

Proof. If $\Phi_n(p) < \infty$ (or $\Phi_n(q) < \infty$) then $1/\Phi_n(p) > 0$ (or $1/\Phi_n(q) > 0$). Theorem F.8 implies $1/\Phi_n(r) > 0$ hence $\Phi_n(r) < \infty$, which by definition forces r to have distinct roots.

Corollary F.10 (Monotone bound in the infinite-input limit).

If $\Phi_n(p) = +\infty$ and $\Phi_n(q) < \infty$ then

$$\frac{1}{\Phi_n(p \boxplus_n q)} \;\geq\; \frac{1}{\Phi_n(q)} \qquad \text{equivalently} \qquad \Phi_n(p \boxplus_n q) \;\leq\; \Phi_n(q). \tag{93}$$

Proof. This is Theorem F.8 with $1/\Phi_n(p) = 0$.

F.7 Summary

- Lemma F.3 gives density of simple real-rooted polynomials.
- Proposition F.4 shows that $1/\Phi_n$ is continuous on $\mathcal{R}_n$ with $1/\infty = 0$.
- Lemma F.7 ensures we can approximate any $(p, q) \in \mathcal{R}_n \times \mathcal{R}_n$ by pairs for which $p, q, r = p \boxplus_n q$ all have simple real roots.
- Theorem F.8 completes the extension of the conditional Stam inequality to multiple roots, conditional on the simple-root case.

Appendix G. Equality in low degree and a general equality criterion

This appendix provides:

1. a complete proof that **for** $n = 2$ the finite Stam inequality is always an equality (Proposition 6.1 in the main text) and
2. a general Hilbert-space **criterion** that characterizes equality in the simple-root case in terms of the "sum subspace" constructed in Appendix D.

Throughout, p, q are monic real-rooted polynomials of degree n and $r := p \boxplus_n q$.

G.1 The case $n = 2$: the inequality is always an equality

We work with the coefficient convention from Definition 2.1 (and Definition A.1): for $n = 2$

$$p(x) = x^2 + a_1 x + a_2, \qquad q(x) = x^2 + b_1 x + b_2, \tag{94}$$

and $r = p \boxplus_2 q$ is defined by the coefficient rule

$$r(x) = x^2 + c_1 x + c_2, \qquad c_k = \sum_{i+j=k} \frac{(2-i)!\,(2-j)!}{2!\,(2-k)!}\, a_i b_j. \tag{95}$$

Lemma G.1 (Closed form for $p \boxplus_2 q$).

For p, q as in {#eq:g-1} one has

$$\boxed{(p \boxplus_2 q)(x) = x^2 + (a_1 + b_1)x + \Big(a_2 + b_2 + \frac{a_1 b_1}{2}\Big).} \tag{96}$$

Proof. We compute c_0, c_1, c_2 from {#eq:g-2}.

- For $k = 0$: the only term is $i = j = 0$ giving $c_0 = a_0 b_0 = 1$ (monicity).
- For $k = 1$: the pairs are $(i, j) = (0, 1), (1, 0)$. Since

  $$\frac{(2-0)!\,(2-1)!}{2!\,(2-1)!} = 1, \qquad \frac{(2-1)!\,(2-0)!}{2!\,(2-1)!} = 1,$$

  we obtain $c_1 = a_1 + b_1$.
- For $k = 2$: the pairs are $(0, 2), (1, 1), (2, 0)$. Since

  $$\frac{(2-0)!\,(2-2)!}{2!\,(2-2)!} = 1, \qquad \frac{(2-1)!\,(2-1)!}{2!\,(2-2)!} = \frac{1}{2}, \qquad \frac{(2-2)!\,(2-0)!}{2!\,(2-2)!} = 1,$$

  we obtain $c_2 = b_2 + \frac{1}{2} a_1 b_1 + a_2$.

Substituting into $r(x) = x^2 + c_1 x + c_2$ yields {#eq:g-3}.

Lemma G.2 (Φ_2 and the discriminant).

Let $p(x) = x^2 + a_1 x + a_2 = (x - \lambda_1)(x - \lambda_2)$ with $\lambda_1 \neq \lambda_2$. Then

$$\Phi_2(p) = \frac{2}{(\lambda_1 - \lambda_2)^2} = \frac{2}{\Delta(p)} \qquad \frac{1}{\Phi_2(p)} = \frac{\Delta(p)}{2} \tag{97}$$

where the discriminant is

$$\Delta(p) := a_1^2 - 4a_2 = (\lambda_1 - \lambda_2)^2. \tag{98}$$

Proof. By Definition 2.3,

$$\Phi_2(p) = \left(\frac{1}{\lambda_1 - \lambda_2}\right)^2 + \left(\frac{1}{\lambda_2 - \lambda_1}\right)^2 = \frac{2}{(\lambda_1 - \lambda_2)^2}.$$

For a monic quadratic, $(\lambda_1 - \lambda_2)^2$ equals the discriminant $a_1^2 - 4a_2$ proving {#eq:g-4}–{#eq:g-5}.

Theorem G.3 (Equality for $n = 2$).

Let $p, q \in \mathbb{R}_2[x]$ be monic and real-rooted. Set $r := p \boxplus_2 q$. Then

$$\boxed{\frac{1}{\Phi_2(r)} = \frac{1}{\Phi_2(p)} + \frac{1}{\Phi_2(q)}} \qquad \text{with the convention } \frac{1}{\infty} = 0. \tag{99}$$

In particular, if p, q have simple roots, equality holds in Theorem 2.4.

Proof. If p or q has a multiple root, then $\Phi_2(p) = \infty$ or $\Phi_2(q) = \infty$ and {#eq:g-6} reduces to a trivial statement (both sides are ≥ 0).

Assume now p, q have simple real roots, so $\Phi_2(p)\Phi_2(q) < \infty$. Write them as in {#eq:g-1}. By Lemma G.1, $r = x^2 + c_1 x + c_2$ with

$$c_1 = a_1 + b_1, \qquad c_2 = a_2 + b_2 + \frac{a_1 b_1}{2}.$$

Compute the discriminant of r:

$$\Delta(r) = c_1^2 - 4c_2 = (a_1 + b_1)^2 - 4\left(a_2 + b_2 + \frac{a_1 b_1}{2}\right) = (a_1^2 - 4a_2) + (b_1^2 - 4b_2) = \Delta(p) + \Delta(q). \tag{100}$$

Using Lemma G.2 for p, q, r

$$\frac{1}{\Phi_2(r)} = \frac{\Delta(r)}{2} = \frac{\Delta(p) + \Delta(q)}{2} = \frac{\Delta(p)}{2} + \frac{\Delta(q)}{2} = \frac{1}{\Phi_2(p)} + \frac{1}{\Phi_2(q)}.$$

This is exactly {#eq:g-6}.

Corollary G.4 (Proposition 6.1).

Proposition 6.1 in the main text holds: for $n = 2$ equality always holds in Theorem 2.4 whenever $\Phi_2(p)\Phi_2(q) < \infty$.

Proof. This is Theorem G.3.

G.2 A general equality criterion in the simple-root case

We now record a general (abstract) equality criterion that follows directly from the proof in §5.1 of the main text once the projection framework of §4 and Appendix D is in place.

Assume p, q, r have simple real roots and let

$$A := \Phi_n(p) \qquad B := \Phi_n(q) \qquad r := p \boxplus_n q. \tag{101}$$

Recall from §5.1 that the proof uses the estimate (for any $a \in \mathbb{R}$)

$$\Phi_n(r) \le a^2 \Phi_n(p) + (1-a)^2 \Phi_n(q) \tag{102}$$

and then optimizes over a.

Let

$$a_* := \frac{B}{A+B} \qquad v_* := a_* J_p^{\uparrow} + (1-a_*) J_q^{\uparrow} \in \mathcal{H}_{p,q} \tag{103}$$

where the lifts $J_p^{\uparrow}, J_q^{\uparrow}$ are defined in §4.1.

Proposition G.5 (Equality criterion via the $\boxplus_n$-sum subspace).

Assume p, q, r have simple real roots and the $\boxplus_n$ contraction $\mathbb{E}_{\boxplus} = (\iota_{p,q}^{\boxplus})^*$ is defined as in Appendix D. Then the following are equivalent:

1. Equality holds in Theorem 2.4 for (p, q), i.e.

$$\frac{1}{\Phi_n(r)} = \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}. \tag{104}$$

2. Equality holds in the contraction step at the optimizer $a = a_*$ i.e.

$$|J_r|_{\mathcal{H}_r} = |v_*|_{\mathcal{H}_{p,q}}. \tag{105}$$

3. The optimizer vector v_* lies in the $\boxplus_n$-sum subspace:

$$v_* \in \mathcal{H}_{\boxplus}(p,q) = \operatorname{Ran}(\iota_{p,q}^{\boxplus}) \qquad \text{equivalently} \qquad P_{\boxplus} v_* = v_*, \tag{106}$$

   where $P_{\boxplus}$ is the orthogonal projection defined in Appendix D.

Proof.

- (1) $\iff$ (2): In §5.1, the only inequality before optimization is the contraction bound

$$|J_r|_{\mathcal{H}_r} = |\mathbb{E}_{\boxplus}(v)|_{\mathcal{H}_r} \le |v|_{\mathcal{H}_{p,q}}$$

  applied to $v = aJ_p^{\uparrow} + (1-a)J_q^{\uparrow}$. The optimization in a is exact, and the minimizing value is a_*. Therefore the overall equality (1) holds if and only if the contraction inequality is an equality at $a = a_*$, which is (2).
- (2) $\iff$ (3): Since $\iota_{p,q}^{\boxplus}$ is an isometry, its adjoint $\mathbb{E}_{\boxplus} = (\iota_{p,q}^{\boxplus})^*$ satisfies

$$|\mathbb{E}_{\boxplus}(v)|_{\mathcal{H}_r} = |(\iota_{p,q}^{\boxplus})^* v|_{\mathcal{H}_r} = |P_{\boxplus} v|_{\mathcal{H}_{p,q}} \le |v|_{\mathcal{H}_{p,q}}$$

  where $P_{\boxplus} = \iota_{p,q}^{\boxplus}(\iota_{p,q}^{\boxplus})^*$ is the orthogonal projection onto $\operatorname{Ran}(\iota_{p,q}^{\boxplus})$ (Appendix D). Equality $|P_{\boxplus} v| = |v|$ holds if and only if $v \in \operatorname{Ran}(\iota_{p,q}^{\boxplus})$, i.e. $v \in \mathcal{H}_{\boxplus}(p,q)$. Applying this to $v = v_*$ gives (2) $\iff$ (3).

This completes the proof.

Remark G.6 (Interpretation). Proposition G.5 shows that equality is controlled entirely by whether the optimally weighted lifted score v_* lies in the $\boxplus_n$-sum subspace $\mathcal{H}_{\boxplus}(p,q)$. In degree $n=2$ this condition is automatically satisfied (Theorem G.3). For higher degrees, a full classification of equality cases is a separate structural problem and is not pursued here.

Appendix H. Worked examples, invariances, and numerical sanity checks

This appendix provides worked examples and computational sanity checks illustrating the finite Stam inequality (Theorem 2.4). These examples help calibrate the functional Φ_n, clarify its scaling behavior, and provide benchmarks against which intermediate identities may be tested during verification of the remaining operator bounds isolated in Appendix D.

Throughout, $n \geq 2$ is fixed, polynomials are monic of degree n and all roots are real (with multiplicities allowed unless stated otherwise).

H.1 Affine invariances and scaling laws

The functional Φ_n depends only on root differences, hence exhibits simple affine behavior. It is often convenient to normalize by translation and dilation before running numerical checks.

Definition H.1 (Affine change of variable preserving monicity).

Let $p \in \mathbb{R}_n[x]$ be monic and let $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}$. Define

$$(\mathcal{A}_{a,b}p)(x) := a^{-n}\, p(ax+b). \tag{107}$$

Then $\mathcal{A}_{a,b}p$ is monic of degree n and its roots are

$$\lambda_i(\mathcal{A}_{a,b}p) = \frac{\lambda_i(p) - b}{a}. \tag{108}$$

Lemma H.2 (Affine scaling of Φ_n).

If $p \in \mathbb{R}_n[x]$ is real-rooted and monic, then

$$\Phi_n(\mathcal{A}_{a,b}p) = a^2\, \Phi_n(p) \qquad \frac{1}{\Phi_n(\mathcal{A}_{a,b}p)} = a^{-2}\, \frac{1}{\Phi_n(p)} \tag{109}$$

with the convention $1/\infty = 0$.

Proof. If λ_i are the roots of p then by {#eq:h-2} the roots of $\mathcal{A}_{a,b}p$ are $\lambda_i' = (\lambda_i - b)/a$. Hence each gap rescales as

$$\lambda_i' - \lambda_j' = \frac{\lambda_i - \lambda_j}{a}$$

so each inverse-squared gap rescales by a^2. Using the pairwise-gap representation (Appendix C, Proposition C.4)

$$\Phi_n(p) = 2\sum_{i<j} \frac{1}{(\lambda_i - \lambda_j)^2}$$

we obtain $\Phi_n(\mathcal{A}_{a,b}p) = a^2\Phi_n(p)$. The reciprocal identity follows.

Lemma H.3 (Affine covariance of $\boxplus_n$).

For any $a \neq 0$, $b \in \mathbb{R}$, and monic $p, q \in \mathbb{R}_n[x]$

$$\mathcal{A}_{a,b}(p \boxplus_n q) = (\mathcal{A}_{a,b}p) \boxplus_n (\mathcal{A}_{a,b}q). \tag{110}$$

Proof. Translation covariance (the parameter b) is Proposition A.8. For dilation (the parameter a) use the differential-operator representation (Theorem A.5): write

$$(p \boxplus_n q)(x) = \frac{1}{n!}\sum_{k=0}^{n} p^{(k)}(x)\, q^{(n-k)}(0).$$

If $\tilde{p} = \mathcal{A}_{a,0}p$ then $\tilde{p}^{(k)}(x) = a^{k-n}p^{(k)}(ax)$. Similarly $\tilde{q}^{(n-k)}(0) = a^{-k}q^{(n-k)}(0)$. Therefore

$$(\tilde{p}\boxplus_n\tilde{q})(x) = \frac{1}{n!}\sum_{k=0}^{n} a^{k-n}p^{(k)}(ax)\cdot a^{-k}q^{(n-k)}(0) = a^{-n}(p\boxplus_n q)(ax) = \mathcal{A}_{a,0}(p\boxplus_n q)(x).$$

Combining with translation covariance yields {#eq:h-4}.

Corollary H.4 (Affine invariance of the Stam inequality).

If Theorem 2.4 holds for (p, q) then it holds for $(\mathcal{A}_{a,b}p, \mathcal{A}_{a,b}q)$ for any $a \neq 0$, $b \in \mathbb{R}$.

Proof. By Lemma H.3, the output polynomial transforms covariantly:

$$(\mathcal{A}_{a,b}p) \boxplus_n (\mathcal{A}_{a,b}q) = \mathcal{A}_{a,b}(p \boxplus_n q).$$

By Lemma H.2, the reciprocal $1/\Phi_n$ scales by a^{-2} for each of p, q, r. Multiplying the inequality by a^2 preserves its direction.

H.2 A closed form for equispaced roots

The simplest nontrivial family is given by equispaced roots.

Definition H.5 (Equispaced-root polynomial).

Let

$$p_n(x) := \prod_{k=0}^{n-1}(x-k). \tag{111}$$

Its roots are $\lambda_k = k$ for $k = 0, 1, \dots, n-1$.

Proposition H.6 (Closed form for $\Phi_n(p_n)$).

For p_n defined by {#eq:h-5}

$$\boxed{\Phi_n(p_n) = 2\sum_{d=1}^{n-1}\frac{n-d}{d^2} = 2\Big(n, H_{n-1}^{(2)} - H_{n-1}\Big)} \tag{112}$$

where $H_m := \sum_{d=1}^{m}\frac{1}{d}$ is the harmonic number and $H_m^{(2)} := \sum_{d=1}^{m}\frac{1}{d^2}$.

Proof. By the pairwise-gap formula (Appendix C, Proposition C.4)

$$\Phi_n(p_n) = 2\sum_{0\le i<j\le n-1}\frac{1}{(j-i)^2}.$$

For a fixed gap $d = j-i \in \{1, \dots, n-1\}$ there are exactly $n-d$ pairs (i, j) with $j - i = d$. Therefore

$$\Phi_n(p_n) = 2\sum_{d=1}^{n-1}(n-d)\frac{1}{d^2} = 2\left(n\sum_{d=1}^{n-1}\frac{1}{d^2} - \sum_{d=1}^{n-1}\frac{1}{d}\right) = 2(nH_{n-1}^{(2)} - H_{n-1}).$$

H.3 Worked examples in degree $n = 3$

For $n = 3$ the functional has the particularly simple form

$$\Phi_3(p) = 2\left(\frac{1}{(\lambda_2-\lambda_1)^2} + \frac{1}{(\lambda_3-\lambda_2)^2} + \frac{1}{(\lambda_3-\lambda_1)^2}\right) \tag{113}$$

and the convolution $p \boxplus_3 q$ is explicitly computable from coefficients.

Example H.7 (A symmetric cubic yielding equality).

Let

$$p(x) = x^3 - x = (x+1)x(x-1) \qquad q(x) = p(x). \tag{114}$$

Then $p \boxplus_3 q$ is computed from Definition 2.1: writing $p(x) = x^3 + a_1 x^2 + a_2 x + a_3$ gives $a_1 = 0, a_2 = -1, a_3 = 0$ and similarly $b_1 = 0, b_2 = -1, b_3 = 0$. For $n = 3$

$$c_1 = a_1 + b_1 = 0, \quad c_2 = a_2 + b_2 + \frac{2}{3}a_1 b_1 = -2, \quad c_3 = a_3 + b_3 + \frac{1}{3}(a_1 b_2 + a_2 b_1) = 0,$$

hence

$$r(x) := (p \boxplus_3 q)(x) = x^3 - 2x = x(x - \sqrt{2})(x + \sqrt{2}). \tag{115}$$

Compute the Φ_3 values using {#eq:h-7}:

- p has roots $-1, 0, 1$ gaps (1,1,2) hence

$$\Phi_3(p) = 2\left(1 + \frac{1}{4} + 1\right) = \frac{9}{2}. \tag{116}$$

- r has roots $-\sqrt{2}, 0, \sqrt{2}$ gaps $\sqrt{2}, \sqrt{2}, 2\sqrt{2}$ hence

$$\Phi_3(r) = 2\left(\frac{1}{2} + \frac{1}{8} + \frac{1}{2}\right) = \frac{9}{4}. \tag{117}$$

Therefore

$$\frac{1}{\Phi_3(r)} = \frac{4}{9} = \frac{2}{9} + \frac{2}{9} = \frac{1}{\Phi_3(p)} + \frac{1}{\Phi_3(q)}$$

so the Stam inequality holds with equality for this example.

Example H.8 (A generic cubic with strict inequality).

Let

$$p(x) = (x)(x-1)(x-3) = x^3 - 4x^2 + 3x, \qquad q(x) = (x)(x-2)(x-5) = x^3 - 7x^2 + 10x. \tag{118}$$

From Definition 2.1 with $n = 3$ we obtain

$$r(x) = (p \boxplus_3 q)(x) = x^3 + (a_1 + b_1)x^2 + \left(a_2 + b_2 + \frac{2}{3}a_1 b_1\right)x + \left(a_3 + b_3 + \frac{1}{3}(a_1 b_2 + a_2 b_1)\right) \tag{119}$$

and substituting $a_1 = -4, a_2 = 3, a_3 = 0$ and $b_1 = -7, b_2 = 10, b_3 = 0$ yields

$$r(x) = x^3 - 11x^2 + \frac{95}{3}x - \frac{61}{3}. \tag{120}$$

The roots of r are real and distinct; numerically,

$$\rho(r) \approx (0.9010031488,\ 3.3377754204,\ 6.7612214307). \tag{121}$$

Now compute Φ_3 from the roots:

- p has roots $(0, 1, 3)$ so

$$\Phi_3(p) = 2\left(1 + \frac{1}{4} + \frac{1}{9}\right) = \frac{49}{18} \approx 2.72222222. \tag{122}$$

- q has roots $(0, 2, 5)$ so

$$\Phi_3(q) = 2\left(\frac{1}{4} + \frac{1}{9} + \frac{1}{25}\right) = \frac{361}{450} \approx 0.80222222. \tag{123}$$

- r has approximate roots {#eq:h-15} hence

$$\Phi_3(r) \approx 0.5657079097, \qquad \frac{1}{\Phi_3(r)} \approx 1.7676966908. \tag{124}$$

On the other hand,

$$\frac{1}{\Phi_3(p)} + \frac{1}{\Phi_3(q)} \approx \frac{1}{2.72222222} + \frac{1}{0.80222222} \approx 1.6138843349. \tag{125}$$

Thus, in this example,

$$\frac{1}{\Phi_3(r)} \approx 1.7677 \; > \; 1.6139 \approx \frac{1}{\Phi_3(p)} + \frac{1}{\Phi_3(q)}$$

so the inequality holds strictly.

H.4 A reproducible numerical check protocol

For computational verification on random instances, it is convenient to use the pairwise-gap formula (Appendix C, Proposition C.4) since it avoids subtraction-cancellation in the original score definition.

Definition H.9 (Computing Φ_n from roots).

Given distinct real roots $\lambda_1, \dots, \lambda_n$ define

$$\Phi_n(\lambda_1, \dots, \lambda_n) := 2 \sum_{1 \le i < j \le n} \frac{1}{(\lambda_i - \lambda_j)^2}. \tag{126}$$

This equals $\Phi_n(p)$ when $p(x) = \prod_{i=1}^{n}(x - \lambda_i)$.

Proposition H.10 $Sanity-checkworkflow$**.**

To numerically test the Stam inequality for given $p, q \in \mathbb{R}_n[x]$:

1. Compute the coefficients of $r := p \boxplus_n q$ via Definition 2.1.
2. Compute the real roots of p, q, r (sorted).

3. Compute (_n(p) _n(q) _n(r)) using {#eq:h-20}.
4. Check whether
$$\frac{1}{\Phi_n(r)} \;\geq\; \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}. \tag{127}$$

In near-degenerate regimes (very small root gaps) step 3 should be performed with extended precision, and one may additionally monitor the lower bound

$$\Phi_n(p) \;\geq\; \frac{2}{\min_i(\lambda_{i+1} - \lambda_i)^2}$$

(Corollary C.5) as a conditioning indicator.

Proof. This is a procedural statement: steps (1)–(3) implement the definitions, and {#eq:h-21} is the target inequality.

H.5 Summary

- Φ_n scales quadratically under dilation and is invariant under translation (Lemma H.2).
- The convolution $\boxplus_n$ is affine-covariant (Lemma H.3) hence the inequality is affine-invariant (Corollary H.4).
- Explicit worked cubic examples show both equality and strict inequality can occur (Examples H.7–H.8).
- A simple and robust numerical workflow is given in Proposition H.10.

Appendix I. The U-transform viewpoint: a "Fourier–Laplace" linearization of $\boxplus_n$ and its conceptual link to Stam's argument

This appendix is **optional** and is **not used** in the proofs of Theorem 2.4. Its purpose is conceptual: it records an explicit linear transform on coefficients that turns the symmetric additive finite free convolution $\boxplus_n$ into a truncated product, analogous to how a Fourier/Laplace transform turns classical additive convolution into multiplication. This viewpoint clarifies why a projection-and-optimization proof of a Stam-type inequality is natural in the finite free setting.

I.1 A coefficient transform that linearizes $\boxplus_n$

We adopt the coefficient convention from Definition 2.1: for $p \in \mathbb{R}_{\leq n}[x]$ write its degree-n padded form as

$$p(x) = \sum_{k=0}^{n} a_k x^{n-k}. \tag{128}$$

We will map such polynomials to **degree-**$\le n$ polynomials in a new indeterminate t.

Definition I.1 (The normalized coefficient transform $\mathcal{U}_n$).

Define the linear map
$$\mathcal{U}_n : \mathbb{R}_{\le n}[x] \;\longrightarrow\; \mathbb{R}_{\le n}[t]$$
by
$$\boxed{(\mathcal{U}_n p)(t) \;:=\; \sum_{k=0}^{n} \frac{(n-k)!}{n!}\, a_k\, t^k, \qquad \text{for } p(x) = \sum_{k=0}^{n} a_k x^{n-k}.} \tag{129}$$

Define also the truncation operator
$$\Pi_{\le n} : \mathbb{R}[t] \to \mathbb{R}_{\le n}[t], \qquad \Pi_{\le n}\Big(\sum_{k\ge 0} \alpha_k t^k\Big) := \sum_{k=0}^{n} \alpha_k t^k. \tag{130}$$

Proposition I.2 (Invertibility of $\mathcal{U}_n$).

The map $\mathcal{U}_n$ is a linear isomorphism. Its inverse is given by: if
$$(\mathcal{U}_n p)(t) = \sum_{k=0}^{n} A_k t^k, \tag{131}$$
then
$$p(x) = \sum_{k=0}^{n} \frac{n!}{(n-k)!}\, A_k\, x^{n-k}. \tag{132}$$

Proof. The coefficient relation in {#eq:i-2} is $A_k = \frac{(n-k)!}{n!} a_k$ hence $a_k = \frac{n!}{(n-k)!} A_k$, which is exactly {#eq:i-5}.

Theorem I.3 (Truncated multiplicative linearization of $\boxplus_n$).

For all $p, q \in \mathbb{R}_{\le n}[x]$
$$\boxed{\mathcal{U}_n(p \boxplus_n q) \;=\; \Pi_{\le n}\big((\mathcal{U}_n p)\cdot(\mathcal{U}_n q)\big).} \tag{133}$$

Proof. Write
$$p(x) = \sum_{i=0}^{n} a_i x^{n-i} \qquad q(x) = \sum_{j=0}^{n} b_j x^{n-j}.$$

Let $r = p \boxplus_n q$ and write $r(x) = \sum_{k=0}^{n} c_k x^{n-k}$. By Definition A.1 (equivalently Definition 2.1)
$$c_k = \sum_{i+j=k} \frac{(n-i)!\,(n-j)!}{n!\,(n-k)!}\, a_i b_j, \qquad k = 0, 1, \dots, n. \tag{134}$$

Define $A_i := \frac{(n-i)!}{n!} a_i$ and $B_j := \frac{(n-j)!}{n!} b_j$. Then the coefficient identity {#eq:i-7} becomes

$$\frac{(n-k)!}{n!} c_k = \sum_{i+j=k} \left(\frac{(n-i)!}{n!} a_i \right) \left(\frac{(n-j)!}{n!} b_j \right) = \sum_{i+j=k} A_i B_j. \tag{135}$$

But the right-hand side of {#eq:i-8} is exactly the coefficient of t^k in the product

$$\left(\sum_{i=0}^{n} A_i t^i \right) \left(\sum_{j=0}^{n} B_j t^j \right)$$

and the left-hand side is the coefficient of t^k in $\mathcal{U}_n r$ by Definition I.1. Since the identity holds for each $k \le n$ we obtain

$$\mathcal{U}_n r = \Pi_{\le n}((\mathcal{U}_n p)(\mathcal{U}_n q))$$

which is {#eq:i-6}.

I.2 A derivative-at-zero representation

The transform $\mathcal{U}_n$ can be expressed in terms of derivatives at the origin. This matches the differential-operator calculus of Appendix A.

Lemma I.4 (Derivative representation).

Let $p \in \mathbb{R}_{\le n}[x]$ be written as in {#eq:i-1}. Then for $k = 0, 1, \dots, n$

$$p^{(n-k)}(0) = (n-k)!\, a_k, \tag{136}$$

and consequently

$$\boxed{(\mathcal{U}_n p)(t) = \frac{1}{n!} \sum_{k=0}^{n} p^{(n-k)}(0)\, t^k.} \tag{137}$$

Proof. Since $\frac{d^m}{dx^m} x^{n-k} \big|_{x=0} = 0$ unless $m = n-k$ and equals $(n-k)!$ when $m = n-k$ we obtain {#eq:i-9} immediately from {#eq:i-1}. Substituting into {#eq:i-2} yields {#eq:i-10}.

I.3 "Truncated Laplace transform" interpretation under a moment model

Theorem I.3 formally mirrors the multiplicativity of characteristic functions (or Laplace transforms) under sums of independent random variables. This becomes literal if one represents polynomials as "n-th moment polynomials" of a random variable.

Definition I.5 (Moment-polynomial representation).

Let X be a real-valued random variable with finite moments up to order n. Define the monic degree-n polynomial

$$p_X(x) := \mathbb{E}\big[(x-X)^n\big]. \tag{138}$$

Proposition I.6 (Moment generating polynomial equals $\mathcal{U}_n$).

Let X be as in Definition I.5 and let p_X be defined by {#eq:i-11}. Then

$$\boxed{(\mathcal{U}_n p_X)(t) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}\,\mathbb{E}[X^k]\,t^k.} \tag{139}$$

Equivalently, $(\mathcal{U}_n p_X)(t)$ is the degree-$\le n$ Taylor polynomial at $t=0$ of the Laplace transform $t \mapsto \mathbb{E}[e^{-tX}]$.

Proof. Expand $(x-X)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}(-X)^k$ and take expectations:

$$p_X(x) = \sum_{k=0}^{n}\binom{n}{k}(-1)^k\mathbb{E}[X^k]\,x^{n-k}. \tag{140}$$

Thus in the padded coefficient notation {#eq:i-1} we have

$$a_k = \binom{n}{k}(-1)^k\mathbb{E}[X^k]. \tag{141}$$

Apply Definition I.1:

$$(\mathcal{U}_n p_X)(t) = \sum_{k=0}^{n}\frac{(n-k)!}{n!}\,a_k\,t^k = \sum_{k=0}^{n}\frac{(n-k)!}{n!}\binom{n}{k}(-1)^k\mathbb{E}[X^k]\,t^k = \sum_{k=0}^{n}\frac{(-1)^k}{k!}\,\mathbb{E}[X^k]\,t^k,$$

since $\binom{n}{k}(n-k)!/n! = 1/k!$. This proves {#eq:i-12}.

Corollary I.7 (Independence gives multiplicativity).

Let X, Y be independent real-valued random variables with finite moments up to order n. Then

$$\boxed{\mathcal{U}_n(p_{X+Y}) = \Pi_{\le n}\big((\mathcal{U}_n p_X)\cdot(\mathcal{U}_n p_Y)\big).} \tag{142}$$

Proof. By Proposition I.6, $\mathcal{U}_n p_X$ and $\mathcal{U}_n p_Y$ are the degree-$\le n$ Taylor polynomials of $\mathbb{E}[e^{-tX}]$ and $\mathbb{E}[e^{-tY}]$. Independence gives $\mathbb{E}[e^{-t(X+Y)}] = \mathbb{E}[e^{-tX}]\mathbb{E}[e^{-tY}]$. Truncating both sides at degree $\le n$ yields {#eq:i-15}.

I.4 Relation to the finite free convolution and conceptual link to the projection proof

Theorem I.3 shows that $\boxplus_n$ is a "convolution" whose transform $\mathcal{U}_n$ is multiplicative (up to truncation). This parallels the role of characteristic functions in classical probability.

A stronger statement—existence of a concrete probability model such that $p \boxplus_n q$ is realized as a genuine sum $X+Y$ at the level of moment polynomials—has been developed in the finite free probability literature under the name "U-transform". We record the following as **background** (not used elsewhere).

Theorem I.8 (Background: U-transform model, informal formulation).

There exists a transform (commonly called the U-transform) that associates to each monic real-rooted $p \in \mathbb{R}_n[x]$ a finite-support real random variable X_p (or an equivalent finite-support measure) such that:

1. $p(x) = \mathbb{E}[(x - X_p)^n]$ and
2. for independent X_p, X_q one has

$$(p \boxplus_n q)(x) = \mathbb{E}\big[(x - (X_p + X_q))^n\big]. \tag{143}$$

A detailed construction and proof are given in the references listed in §I.6.

Remark I.9 (Conceptual bridge). Theorem I.8 should be read as a conceptual bridge: it explains why the finite free convolution admits a "sum-of-independent-variables" interpretation at a fixed truncation order n.

The projection proof of Theorem 2.4 in the main text is formally identical to the classical proof of Stam's inequality once one recognizes the following Hilbert-space fact.

Lemma I.10 (Conditional expectation is orthogonal projection).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. Then the conditional expectation operator

$$\mathbb{E}[\cdot \mid \mathcal{G}] : L^2(\Omega) \to L^2(\Omega)$$

is the orthogonal projection onto the closed subspace $L^2(\mathcal{G}) \subset L^2(\Omega)$.

Proof. Standard: $L^2(\mathcal{G})$ is closed, $\mathbb{E}[\cdot \mid \mathcal{G}]$ is L^2-contractive, $\mathbb{E}[U \mid \mathcal{G}] = U$ for $U \in L^2(\mathcal{G})$, and $\langle V - \mathbb{E}[V \mid \mathcal{G}], U \rangle = 0$ for all $U \in L^2(\mathcal{G})$ by the defining property of conditional expectation.

Remark I.11 (How this mirrors Appendix D). In the classical Stam argument, one works in $L^2(X, Y)$, identifies the "sum subspace" $L^2(X + Y)$, and uses the orthogonal projection $\mathbb{E}[\cdot \mid X+Y]$. Appendix D constructs the finite-dimensional analog:

- the product space $\mathcal{H}_{p,q}$ corresponds to $L^2(X,Y)$
- the sum subspace $\mathcal{H}_{\boxplus}(p,q)$ corresponds to $L^2(X+Y)$
- the contraction $\mathbb{E}_{\boxplus} = (\iota_{p,q}^{\boxplus})^*$ corresponds to conditional expectation.

Under a concrete U-transform probabilistic model (Theorem I.8) one may view Appendix D's construction as an explicit Gram–Schmidt/polar-decomposition implementation of the orthogonal projection onto functions of the $\boxplus_n$-sum variable.

I.5 Summary

- The transform $\mathcal{U}_n$ defined in {#eq:i-2} is an explicit coefficient-level "Fourier/Laplace" transform for $\boxplus_n$: it converts $\boxplus_n$ into truncated multiplication (Theorem I.3).
- If a polynomial arises as a moment polynomial $p_X(x) = \mathbb{E}[(x-X)^n]$ then $\mathcal{U}_n p_X$ is the truncated Laplace transform $\Pi_{\le n}\mathbb{E}[e^{-tX}]$ (Proposition I.6) and independence yields multiplicativity (Corollary I.7).
- Existing U-transform theory provides a conceptual model that realizes $p \boxplus_n q$ as the moment polynomial of a sum $X_p + X_q$ (Theorem I.8, background). In such a model, the projection step in the main text is the finite-dimensional shadow of conditional expectation as orthogonal projection (Lemma I.10).

I.6 References (Appendix I)

1. A. W. Marcus, *Finite free convolutions and the U-transform* (see author's manuscript / preprint).
2. A. W. Marcus, D. A. Spielman, N. Srivastava, *Finite free convolutions of polynomials*, arXiv:1504.00350.