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DeGroot Opinion Learning Algorithm on Graphs under a General Lp Norm

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Thesis

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Abstract

We study a model of opinion exchange in social networks where a state of the world is realized and every agent receives a zero-mean noisy signal of the realized state.[4] For every step in time, the opinions of the agents update according to a prespecified updating algorithm.

One of the main models in this field is DeGroot dynamics model [10], which has been researched extensively since its introduction. [1, 25, 16, 4] It is known from Golub and Jackson that under DeGroot dynamics agents opinions always converge, and reach a consensus that is close to the state of the world under certain circumstances. [16, p. 124]

In this thesis, we introduce a variant of DeGroot dynamics that we call *p*-DeGroot. We provide some results pertaining to the model on finite graphs, and aim to develop an analogous theory for this opinion exchange dynamics, as known from DeGroot model.

Prominent questions we would address contain the convergence of the model, the influence of stubborn agents, and extensions to various families of graphs regarding the rate and concentration of convergence.

Keywords: DeGroot, graph, social learning

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1. Introduction

1.1. Social Learning

Consider a social network in which people (hereforth referred to as agents) hold a belief or opinion about the state of something in the world, such as the quality of a particular product, the effectiveness of a public policy, or the reliability of a news agency. [1, 25] In all these settings, people learn about the state of the world via observation or communication with others. It is common to assume that more often than not, agents are unaware of the structure of the network, hence they can learn only from their imminent friends. Models of social learning try to formalize these interactions to describe how agents process the information received from their friends in the social network.

In this thesis we introduce a family of social learning models. We answer some of the main questions in the literature about social learning models regarding this family, including whether agents reach a consensus[1], the value to which the network converges in some cases, and how effective stubborn agents can be in belief formation of the entire network.

Following is a short survey of different learning methods.

1.1.1. Bayesian learning

Bayesian learning is a global learning model which assumes that agents update their beliefs using Bayes' rule. Indeed, each agent's belief about different states of the world can be seen as a probability distribution over a set of opinions, and Bayesian updating assumes that this distribution is updated in a statistically optimal manner using Bayes' rule.

More rigorously, let the underlying state be θ. At first, each individual has

[1] We mean consensus in the sense of all agents converge to the same opinion.

a prior probability of θ which is denoted by $P(\theta)$. Then each person updates their belief by receiving some signal s. According to the Bayesian approach, the updating procedure $\theta \to \hat{\theta}$ will follow the rule

$$\hat{\theta} = \arg\max_{\tilde{\theta}} P\left(\tilde{\theta}|s\right) = \arg\max_{\tilde{\theta}} \frac{P\left(s|\tilde{\theta}\right)}{P(s)} \cdot P\left(\tilde{\theta}\right)$$

where the term $P\left(s|\tilde{\theta}\right)$ is the conditional probability over signal space given the true state of the world.

Bayesian learning is often considered the benchmark model for social learning, in which individuals use Bayes' rule to incorporate new pieces of information to their belief. However, it has been shown that such a Bayesian "update" is fairly sophisticated and imposes an unreasonable cognitive load on agents which might not be realistic for human beings. [8] Therefore, scientists have studied simpler non-Bayesian models. One of the earliest models is called Delphi method.

1.1.2. Delphi method

Delphi method [9] is a global learning method that comprises of experts that are asked to give their opinion on the characteristics of possible events. This method involves multiple rounds of anonymous surveys in which experts are asked to provide their opinions and feedback on a particular topic or problem. The responses are then collated and analyzed to identify areas of consensus or disagreement. Through this iterative process, there are theoretical and empirical reasons to believe that a Delphi method learning conducted according to 'ideal' specifications might perform better than the standard laboratory interpretations. [22] A rigorous approach to applying Delphi method was originally published in a research protocol in the BMJ Open in 2015. [21]

This method can be advantageous in several ways, including:

1. Eliminating the influence of dominant or charismatic personalities: The anonymity of Delphi method allows experts to provide their feedback without fear of retribution or social pressure. This helps to eliminate the influence of dominant or charismatic personalities, which can often skew results.
2. Increased objectivity: Since Delphi method involves a group of experts, it provides a more objective view of the problem or topic being addressed. This can be especially beneficial when dealing with complex issues that require multiple perspectives.
3. Flexibility: Delphi method can be adapted to suit different situations, and its iterative nature allows for adjustments to be made as new information becomes available.

Despite its advantages, Delphi method also has some limitations, such as:

1. Time-consuming: The iterative nature of Delphi method can be time-consuming, requiring several rounds of surveys and analysis.
2. Difficulty in selecting experts: Selecting the right experts to participate in Delphi method can be challenging, as it requires finding individuals with the right level of expertise and experience. It requires a rigourous procedure of selecting the experts, and this procedure can be unclear and even controversial in some situations. In mathematical terms, this means that a knowledge of the graph structure is required for this method.
3. Potential for bias: Despite the anonymity of Delphi method, there is still a potential for bias to be introduced, such as if experts are influenced by their own biases or agendas.

It is also worth noting that Delphi method is a global learning model, meaning that it seeks to reach a consensus or at least some sort of agreement among experts. Another method, and perhaps the most notable non-bayesian method, is called DeGroot model, introduced by DeGroot in 1974, which is one of the first models for describing how humans interact with each other in a social network. This is a local learning model that focuses on individual learning and decision-making.

1.1.3. DeGroot model

DeGroot [10] suggested a local learning model which describes how the group learns by pooling their individual opinions. The underlying assumption of the approach is that individuals start from an initial, subjective opinion regarding the state of the world; then, in each time frame, they follow a particular heuristic (most often, a weighted average of the opinions of the neighbors) when revising their opinion, depending on their neighbors and their own opinions. This approach was brought into the economics literature by DeMarzo et al. [11]

Definition 1.1. (DeGroot model) We model the network using a graph $G = (V, E)$. Each vertex v in the graph corresponds to a single agent, and an edge e between vertices means that the two agents can communicate (the graph is undirected). We denote $N(v)$ the set of neighbors of a vertex v, and d_v the degree of agent v. In each time period $t \geq 0$ each agent i obtains a subjective opinion $A_{i,t} \in \mathbb{R}$ regarding the state of the world (i.e., a "noisy" version of the truth value μ). A common assumption is that the initial opinions are distributed according to $A_{i,0} = \mu + \epsilon_i$, where $\{\epsilon_i\}_{i \in V}$ are i.i.d. with $\mathbb{E}(\epsilon_i) = 0$. We also assume that time is discrete, and that the updates occur simultaneously; this is sometimes called the *synchronous model*. As time progresses, the agents simultaneously revise their opinions while relying only on the opinions of their neighbors; for every agent i, each neighbor j is assigned weight $\omega_{i,j}$, where $\sum\limits_{j \in N(i)} \omega_{i,j} = 1$. DeGroot updating rule can be stated as follows:

$$A_{i,t+1} = \sum_{j \in N(i)} \omega_{i,j} A_{j,t}$$

We include a slightly different formulation of DeGroot model.

Definition 1.2. (DeGroot model) The DeGroot updating rule can be stated as follows:

$$A_{i,t+1} = \arg\min \sum_{j \in N(i)} |x - d_i \omega_{i,j} A_{j,t}|^2$$

Note that this is well defined, as the unique solution is $A_{i,t+1} = \sum_{j \in N(i)} \omega_{i,j} A_{j,t}$.

The function that needs to be minimized (i.e. $\sum_{j \in N(i)} |x - d_i \omega_{i,j} A_{j,t}|^2$) is sometimes called the *energy function* of an agent, and the general minimization method is called the *variational method*. This is an important tool, used to show convergence in the model.

From now on, we assume that the weights are equal for each neighbor. This results in a simpler version of the energy function,

$$A_{i,t+1} = \arg\min \sum_{j \in N(i)} |x - A_{j,t}|^2$$

Several classical results about DeGroot model are:

1. Under what conditions does every vertex converge to a limit opinion as $t \to \infty$, which is related to the number of closed communicating, aperiodic classes; [10]
2. Opinion of every vertex as $t \to \infty$, and, in particular, is consensus achieved, which is related to the stationary probability vector of an associated Markov chain; [10]
3. Rate of convergence, which is related to the second largest eigenvalue in magnitude of the adjacency matrix. [17]

2. p-DeGroot model

As written in the introduction, we model the network using a graph $G = (V, E)$. Here, every such graph is undirected, finite and connected. Notations in this chapter are:

- Agents are often i, j, k but may be also x, y, z;
- Opinion of agent i in time t will be $A_{i,t}$ or $B_{i,t}$;
- $M_t = \max_{j \in V} A_{j,t}, \ m_t = \min_{j \in V} A_{j,t}$;
- d_v is the degree of vertex v;
- Time starts from $t = 1$;
- As in Amir et al. [3], we use the energy polynomial of a vertex i in time t, $P_{i,t}(x) = \sum_{j \in N_i} |x - A_{j,t}|^p$, where the value of p is clear from the context.

Definition 2.1. (p-DeGroot model) The p-DeGroot updating rule can be stated as follows, where $1 < p < \infty$:

$$A_{i,t+1} = \arg\min_{x \in \mathbb{R}} \sum_{j \in N(i)} |x - A_{j,t}|^p = \arg\min_{x \in \mathbb{R}} \sum_{j \in N(i)} P_{i,t}(x)$$

This is well defined in the sense that the $\arg\min$ is unique (as p-norm is strictly convex). The minimization process can be interpreted as finding the minimum of the ℓ_p norm of the energy function; notably, when p is equal to 2, the classic DeGroot model is obtained. It is interesting to observe that all p-DeGroot models exhibit similar behavior when the underlying graph forms a cycle, as each vertex in the cycle has only two neighbors.

Remark 2.2. When $p \gg 2$, the above expression can be thought as minimizing the distance from extreme neighbors (since we put more weight to large differences in opinions).

An important difference between the $p = 2$ case and a general $p \neq 2$ is that in the $p = 2$ case, the weighted average $\vec{\pi} \cdot \vec{v}$ (where $\vec{\pi}$ is the stationary vector of the adjacency matrix P) is preserved; this is no longer true for other values of p. This convenient property easily implies that for $p = 2$, if consensus is reached, then its value must be the weighted average of the initial opinions, $\vec{\pi} \cdot \vec{v}$. This value is much less clear for $p \neq 2$.

Also, in the $p = 2$ case, we have a very good description of the effect of the opinion of v at time t; this comes from analysing random walks on the graph. There is no clear analog to this in the $p \neq 2$ case. It would be very interesting to develop an analog of the random walk and martinagle methods used to derive important results of DeGroot model [4, 15, 13]; we have not managed to do this in this thesis, yet some main ideas (e.g. harmonic functions) and other methods (e.g. energy function) were used.

In this thesis, we wish to further explore this family of models, and answer a few of the questions written in the introduction:

1. Do the opinions in the graph converge - and if so, whether or not a consensus is reached;
2. How effective can stubborn agents be in the entire network;
3. How fast is the rate of convergence;
4. Further description of the model on several finite graph families, e.g. K_n and $G(n, p)$.

Visualisations and simulations are provided as attachments to this thesis.

2.1. Convergence

2.1.1. $p = 2$

In the context of the DeGroot model, a key interest lies in investigating whether the opinion of each agent converge, and if it does, whether the convergence leads to a uniform value, also known as a consensus. To analyze this, we can represent the model using a normalized form of the adjacency matrix P, where each element $p_{i,j}$ represents the weight that individual i assigns to the opinion of individual j during the revision process. It is important to note that the matrix P is a $k \times k$ stochastic matrix, meaning it can be viewed as the one-step transition probability matrix of a Markov chain with k states. This interpretation allows us to leverage the standard limit theorems from the theory of Markov chains, facilitating further analysis and understanding of the convergence dynamics in the DeGroot model.

Fact 2.3. *In general, any individuals for whom the corresponding states of the Markov chain form a closed communicating, aperiodic class will reach a consensus among themselves. [10]*

A short proof sketch follows. 1 is an eigenvalue of P since

$$P \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Perron-Frobenius theorem gives us that it is its biggest eigenvalue since it is a stochastic matrix. [6, Page 197, Theorem 1.1] It also means that there is a left eigenvector, $\vec{\pi} \cdot P = \vec{\pi}$, and that

$$\lim_{n\to\infty} \frac{P^n}{1^n} = \vec{1}^T \cdot \vec{\pi} = \begin{pmatrix} \pi_1 & \dots & \pi_n \\ \vdots & \vdots & \vdots \\ \pi_1 & \dots & \pi_n \end{pmatrix}$$

so for every $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\lim\limits_{n\to\infty} \frac{P^n}{1^n} \cdot \vec{x} = \vec{1}^T \cdot \vec{\pi} \cdot \vec{x} = \begin{pmatrix} \sum \pi_i x_i \\ \vdots \\ \sum \pi_i x_i \end{pmatrix}$. This eigenvector is of course $\vec{\pi}_i = \frac{d_i}{\sum\limits_{j\in N(i)} d_j}$. Note that for each step, the weighted opinion vector $\vec{\pi}\vec{x}_0$ is conserved, since $\vec{\pi}\vec{x}_{t+1} = \vec{\pi}P\vec{x}_t = \vec{\pi}\vec{x}_t$. Of course $\begin{pmatrix} \sum \pi_i x_i \\ \vdots \\ \sum \pi_i x_i \end{pmatrix}$ can be thought of a harmonic function on the vertices of the graph since it is constant. A similiar proof works for the case with stubborn agents - the matrix P can be thought of as a Markov chain process with the stubborn agents as absorbing states, such that the consensus depends only on the opinions of the stubborn agents. [Theorem 15.3.0.2 26]

If the graph G is bi-partite, a consensus may not be reached. [10] Yet, the limits $\lim\limits_{n\to\infty} P^{2n}$, $\lim\limits_{n\to\infty} P^{2n+1}$ do exist. [4]

2.1.2. $1 < p < \infty$

We present some basic yet important claims that allow us to prove that convergence happens in this model.

Claim 2.4. $\arg\min\limits_{x\in\mathbb{R}} P_{i,t}(x) \in \left[\min\limits_j A_{j,t}, \max\limits_j A_{j,t}\right]$

Proof. Write the energy of the agent i in time t, $P_{i,t}(x) = \sum\limits_{j\in N_i} |x - A_{j,t}|^p$. Then the claim follows from derivative constraints (as both $\frac{\partial P_{i,t}\left(\max\limits_j A_{j,t}\right)}{\partial x} \geq 0$ and

$\frac{\partial P_{i,t}\left(\min\limits_j A_{j,t}\right)}{\partial x} \leq 0)$. □

Corollary 2.5. *The sequence* $\left\{\max\limits_i A_{i,t} \equiv M_t\right\}_{t=1}^{\infty}$ *is monotonically non-increasing and bounded from below, so it converges; we denote* $M_t \searrow M$*. Similarly, the sequence* $\left\{\min\limits_i A_{i,t} \equiv m_t\right\}_{t=1}^{\infty}$ *is monotonically non-decreasing and bounded from above, so it converges; we denote* $m_t \nearrow m$*.*

Lemma 2.6. ***(Monotonicity property of opinions)*** *Fix an agent* i *at time* t *with two sets of opinions of neighbors,* $\eta(i,t) = \{A_{\alpha,t}\}_{\alpha\in N(i)}$*, and* $\zeta(i,t) = \{B_{\alpha,t}\}_{\alpha\in N(i)}$*. Assume that for all* $\alpha \in N(i)$*,* $A_{\alpha,t} \leq B_{\alpha,t}$*. Then,* $A_{i,t+1} \leq B_{i,t+1}$*.*

Remark 2.7. This is also true with strong inequality: if for all $\alpha \in N(i)$, $A_{\alpha,t} < B_{\alpha,t}$ then, $A_{i,t+1} < B_{i,t+1}$.

Proof. It is enough to assume that the two configurations differ only in one coordinate, and then to proceed with changing only one coordinate at a time to get from ζ to η.

Denote $\eta(i,t) = \{y_k\}$, and $\zeta(i,t) = \{x_k\}$. Assume that only $y_1 \leq x_1$, and that otherwise, $y_i = x_i$. Define

$$f(x) = \sum_{i=1}^{d} |x - y_i|^p$$

Hence, $A_{i,t+1} = \arg\min\limits_{x\in\mathbb{R}} f(x)$. We can write

$$f(x) = |x-y_1|^p - |x-x_1|^p + \underbrace{|x-x_1|^p + |x-x_2|^p + \ldots + |x-x_d|^p}_{\equiv g(x)}$$

Since $\ell_p^{(n)}$ is a totally convex space, both $\arg\min f(x)$ and $\arg\min g(x)$ exist and are unique (as both $f(x)$ and $g(x)$ are strictly convex functions). Also, since $p > 1$, $f(x), g(x) \in C^1(\mathbb{R})$. Denote $x_m = B_{i,t+1} = \arg\min(g(x))$, and calculate the sign of $f'(x_m)$. We can differentiate exactly,

$$f'(x_m) = p\cdot\left(|x_m - y_1|^{p-1}\cdot\frac{x_m - y_1}{|x_m - y_1|} - |x_m - x_1|^{p-1}\cdot\frac{x_m - x_1}{|x_m - x_1|} + \underbrace{\cancel{g'(x_m)}}_{x_m \text{ minimizes } g(x)}\right) =$$

$$= p\cdot\Big(|x_m - y_1|^{p-1}\cdot sgn(x_m - y_1) - |x_m - x_1|^{p-1}\cdot sgn(x_m - x_1)\Big)$$

And so there are 3 cases:

1. $y_1 \le x_1 \le x_m$: then $f'(x_m) = p\cdot\Big(|x_m - y_1|^{p-1} - |x_m - x_1|^{p-1}\Big) \ge 0$.
2. $y_1 \le x_m \le x_1$: then $f'(x_m) = p\cdot\Big(|x_m - y_1|^{p-1} + |x_m - x_1|^{p-1}\Big) \ge 0$.
3. $x_m \le y_1 \le x_1$: then $f'(x_m) = p\cdot\Big(-|x_m - y_1|^{p-1} + |x_m - x_1|^{p-1}\Big) \ge 0$.

Finally, $f'(x_m) \ge 0$. Combined with the fact that $f(x) \in C^1(\mathbb{R})$ is a strictly convex function, this means that $A_{i,t+1} = \arg\min_{x\in\mathbb{R}} f(x) \le x_m = B_{i,t+1}$. □

Claim 2.8. Assume that in time t we have an agent i with n neighbors, all of which have opinion A except one neighbor that has opinion B, where $A > B$. Denote $\delta = A - B$. Then, $A_{i,t+1} \le A - \frac{\delta}{1+c}$, where $c = \sqrt[p-1]{d-1}$.

Proof. We write an energy function $f(x) = (n-1)\cdot|x-A|^p + |x-B|^p$. As seen above, $B \le \arg\min f(x) \le A$, so near $\arg\min f(x)$ we can write

$$f(x) = (n-1)\cdot(A-x)^p + (x-B)^p$$

Differentiate and compare to 0,

$$f'(x) = p\left[(n-1)\cdot(A-x)^{p-1}\cdot(-1) + (x-B)^{p-1}\right] = 0$$

$$\Rightarrow A_{i,t+1} = \frac{\sqrt[p-1]{n-1}A + B}{1+\sqrt[p-1]{n-1}} = \frac{\sqrt[p-1]{n-1}A + A + B - A}{1+\sqrt[p-1]{n-1}} =$$

$$= A - \frac{\delta}{1+\sqrt[p-1]{n-1}} \le A - \frac{\delta}{1+\sqrt[p-1]{d-1}} = A - \frac{\delta}{1+c}$$

$\square$

Claim 2.9. If G is not bi-partite, then the graph reaches consensus (i.e. all of the opinions converge to the same value).

Proof. Denote $M - m = \delta$ and let $\epsilon > 0$ be determined going forward (ϵ is a function of δ and G). Since $M_t \searrow M$ (Corollary 2.5), let t be a time from which $\max\limits_{i \in V} A_{i,t} \leq M + \epsilon$. Denote $\arg\min\limits_{i \in V} A_{i,t} = a$ (a is a vertex).

Let $b \in N(a)$. Then b has a neighbor a where $A_{a,t} \leq m$, and some other neighbors with $A_{j,t} \leq M + \epsilon$. Claim 2.8 and monotonicity property of opinions gives us that $A_{b,t+1} \leq M + \epsilon - \frac{M+\epsilon-m}{1+c} \leq M + \epsilon - \frac{\delta}{1+c}$.

Now, at time $t+1$ the neighbors of b will have a neighbor with $A_{b,t+1} \leq M + \epsilon - \underbrace{\frac{\delta}{1+c}}_{\delta'}$, and some other neighbors with $A_{j,t+1} \leq M + \epsilon$. Using Claim 2.8 and monotonic property of opinions again, this gives us that for every $c \in N(b)$, $A_{c,t+2} \leq M + \epsilon - \frac{\delta'}{1+c} = M + \epsilon - \frac{\delta}{(1+c)^2}$.

Since the graph is not bi-partite (and, as a remainder, is connected and undirected) then there is an N such that there is a path of length N from every agent to every agent. [2, Proposition 1.7] We take that N for our graph, and (by induction) $\forall i \in V : A_{i,N} \leq M + \epsilon - \frac{\delta}{(1+c)^N}$. Choose $\epsilon = \frac{\delta}{2(1+c)^N}$, then the maximum in the graph is $M - \frac{\delta}{2(1+c)^N}$, so it must be that $\delta = 0$. It follows that $m = M$ meaning opinions converge to a consensus. $\square$

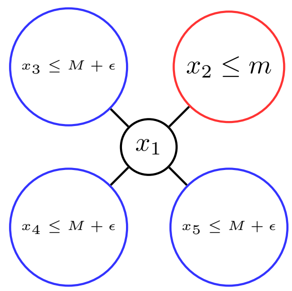

Figure 2.1: Illustration of an agent with 4 neighbors, 3 of which are $\leq M + \epsilon$, and one of which is $\leq m$

2.1.3. $p = \infty$

When the value of p is set to infinity ($p = \infty$), the definition of the energy function changes from the previous formulation. In this case, we define the energy function in a manner similar to the ∞-norm of the gradient, denoted as $\|x\|_{\ell_\infty} = \max |x_i|$. It is worth noting that this formulation can be viewed as the limit of the p-DeGroot updating rule as p approaches infinity ($p \to \infty$).

Definition 2.10. (∞-DeGroot Model) The ∞-DeGroot updating rule can be stated as follows:

$$A_{i,t+1} = \arg\min_{x \in \mathbb{R}} \max_{j \in N(i)} |x - A_{j,t}|$$

Remark 2.11. Assume we have an agent i with neighbors $\eta(i,t) = \{A_{\alpha,t}\}_{\alpha \in N(i)}$. Then $A_{i,t+1} = \frac{\max \eta(i,t) + \min \eta(i,t)}{2}$.

The monotonicity property of opinions easily carries on to this model; Claim 2.8 also applies, with a significantly better bound of $A_{i,t+1} \leq A - \frac{\delta}{2}$. Hence, if G is not bi-partite, then the graph reaches consensus.

2.1.4. $p = 1$

When the value of p is set to 1, the energy function exhibits different properties compared to the previous cases. This is because the $\ell_1^{(n)}$ space is not

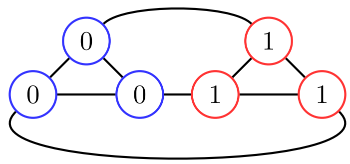

Figure 2.2: Illustration of a graph that converges under 1-DeGroot dynamics, but does not reach consensus

strictly convex. In fact, the energy function is not strictly convex either. A notable consequence of this is that when an agent has an even number of neighbors, represented by $x_1, \ldots, x_{2n}$, the minimum of the energy function can be achieved at any point between the median values $[x_n, x_{n+1}]$. In situations where the degree is even, to ensure a unique definition of the median, the agent's own opinion is considered as part of the pool; an alternative approach to handle the even-degree case is to use coin flips to randomly select between the two median values, effectively creating a median-based version of the zero-temperature Glauber dynamics. [3]

Remark 2.12. Even if G is not bi-partite and the degree of each vertex is odd, then consensus is not always reached.

Example 2.13. Take $G = K_4$, with initial set of opinions $1, 0, 1, 0$. Then the opinions switch from 0 to 1 and vice versa, ad infinitum.

Example 2.14. Take G to be two triangles connected vertex to vertex, as shown in Figure 2.2. Convergence is achieved, but not to a consensus.

Remark 2.15. When the initial opinions are taken from the set $\{0, 1\}$, this process becomes majority dynamics, and the method ties are settled in the 1-DeGroot model is isomorphic to the method ties are settled in majority dynamics.

2.2. Convergence in the presence of stubborn agents

As seen in 2.1.4, the case $p = 1$ arises some deep troubles. Therefore, we venture ahead with no fear, ready to conquer our problem in the realm of the case $1 < p \leq \infty$.

2.2.1. Single stubborn agent

Say that one of the agents is 'stubborn'; it starts with a certain opinion and never changes it. Running DeGroot dynamics with such a stubborn agent results in everyone adopting the opinion of the stubborn agent in any finite connected graph. [26]

Claim 2.16. Monotonicity of $\left\{\max_i A_{i,t}\right\}_{t=1}^{\infty}$ and $\left\{\min_i A_{i,t}\right\}_{t=1}^{\infty}$ occurs with a stubborn agent, since $\forall i \forall t : A_{i,t+1} \in \left[\min_{j\in N(i)\bigcup\{i\}} A_{j,t}, \max_{j\in N(i)\bigcup\{i\}} A_{j,t}\right] \subseteq [m_t, M_t]$.

Claim 2.17. Consider a connected finite graph $G = (V, E)$ and a unique stubborn agent $i_0 \in V$ that sets its opinion to be constant c. Then for every agent $i \in V$ it holds that $\lim_{t\to\infty} A_{i,t} = c$, for every p-DeGroot model

Proof. Both $\max_i A_{i,t}$ and $\min_i A_{i,t}$ converge, say to M and to m, respectively. Note that $\forall t : c \leq M_t$ so $c \leq M$. Denote $\delta = M - c \neq 0$, and as in the proof of convergence without a stubborn agent, follow the stubborn agent i_0 instead of $\arg\min_{i\in V} A_{i,t}$. Then, for a suitable choice of ϵ and t, we get that after a finite amount of time every agent has an opinion $A_{j,t} < M$. Therefore, we must have that $c = M$. By considering the same graph G with the same initial opinions but with opposite signs, we find the same dynamics but in opposite signs, i.e. m and M swap their roles, so we find that $-c = -m$. Finally, $M = m = c$, so for every agent i, $\lim_{t\to\infty} A_{i,t} = c$. $\square$

2.2.2. Multiple stubborn agents

This problem shares similarities with previous works such as [23, 7, 12, 14]. When $p = 2$, the opinions converge to a harmonic function with boundary conditions (which are stubborn vertices), resulting in a unique convergence value. This convergence can be interpreted as an expectation of the aforementioned function's value at the location where a random walk starting from vertex v reaches the boundary.

In the context of semi-supervised machine learning, the problem involves a set of vertices with an a-priori known label; we wish to learn the label of the other unlabeled vertices, with the labeling having the lowest possible energy (i.e. "low amount of differences"). The astute reader would have probably guessed by now that in this case a consensus can not be reached; therefore, for the case with multiple stubborn agents, we are bound to use a different technique, as shown in [3]. Here, the energy polynomial of a vertex i in time t is defined as $P_{i,t}(x) = \sum\limits_{j \in N_i} |x - A_{j,t}|^p$, allowing us to define the total energy of the graph as

$$E(\vec{x}, t) : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}, E(\vec{x}, t) = \sum_{i \in V} P_{i,t}(\vec{x}_i)$$

where $\vec{x}$ represents a vector comprised of the vertices' opinions, and $\vec{x}_i$ denotes the i-th coordinate of $\vec{x}$.

We now establish a few important claims about the energy.

Fact 2.18. *The energy is convex in $\vec{x}$ – this is easily derived from the convexity of $f(x) = |x|^p$ (norm of affine function is convex).*

Consequently, any point where the divergence of $E(\vec{x})$ is zero (denoted as$\vec{\nabla} \cdot E(\vec{x}) = \vec{0}$) is a global minima of the energy.[2] Additionally, at these minimum

[2]It is important to note that the gradient is taken as gradient with respect to $\vec{x}$, and not with respect to the graph structure.

points, it holds true that $\frac{\partial P_{i,t}(x_i)}{\partial x_i} = 0$, indicating that each vertex achieves a minimum energy state. In the absence of stubborn agents, the minima of the energy is 0, which can be attained by any vector of the form $\vec{x} = c \cdot \begin{pmatrix} 1 & \dots & 1 \end{pmatrix}$. However, there are no additional minimum points, since any other vector would yield a positive energy value. Conversely, when stubborn agents are present, the energy function becomes strictly convex, as described in the following claim.

Claim 2.19. $\arg\min E(\vec{x})$ is unique.

Proof. This stems from strict convexity of the energy funciton in the presence of multiple stubborn agents:

$$E(t\vec{x} + (1-t)\vec{y}) = \sum_{i\in V} \sum_{j\in N_i} |t(x(i) - x(j)) + (1-t)(y(i) - y(j))|^p \leq$$

$$\leq \sum_{i\in V} \sum_{j\in N_i} t \cdot |(x(i) - x(j))|^p + (1-t) \cdot |y(i) - y(j)|^p = t \cdot E(\vec{x}) + (1-t) \cdot E(\vec{y})$$

This inequality is an equality if and only if for every $i \sim j$, $x(i) - x(j) = y(i) - y(j) \iff x(i) - y(i) = x(j) - y(j)$; pick i to be any stubborn agent, and then observe that for every neighbor $j \sim i$, $x(j) = y(j)$, and by connectivity $\vec{x} = \vec{y}$. Hence, $E(\vec{x})$ is strictly convex and $\arg\min E(\vec{x})$ is unique. $\square$

Definition 2.20. We define a stationary vector (of opinions) to be a set of opinions $\vec{s}$ which satifies for every non-stubborn vertex i

$$\vec{s}(i) = \arg\min_{x\in\mathbb{R}} \sum_{j\in N(i)} |x - \vec{s}(j)|^p$$

Claim 2.21. The stationary vector is unique.

Proof. The stationary vector satisfies $\frac{\partial P_{i,t}(x_i)}{\partial x_i} = 0$, and by the strict convexity of the energy we deduce that it is a minima of the energy, which is then unique by Claim 2.19. $\square$

Claim 2.22. The graph converges to the stationary vector.

Proof. Let $\vec{x}\,(t)$ be a p-DeGroot process. We prove that every partial limit of $\vec{x}\,(t)$ is $\vec{s}$. Say that we have a partial limit of $\vec{x}\,(t)$, characterised by its time sequence $\{t_k\}_{k=1}^{\infty}$, such that $\lim\limits_{k\to\infty} \vec{x}\,(t_k) = \vec{v} \neq \vec{s}$. In particular, since the stationary vector is unique, let $i \in V$ such that

$$\left|\arg\min_{x\in\mathbb{R}} \sum_{j\in N(i)} |x-\vec{v}\,(j)|^p - \vec{v}\,(i)\right| = \delta \neq 0 \tag{2.1}$$

By the continuity of $f\,(\vec{x}) = \arg\min\limits_{x\in\mathbb{R}} \sum\limits_{j\in N(i)} |x-\vec{x}\,(j,t)|^p$, there is δ' such that if for every $l \in V$ it is true that $|\vec{x}\,(l,t)-\vec{v}\,(l)| \leq \delta'$, then[3]

$$\left|\arg\min_{x\in\mathbb{R}} \sum_{j\in N(i)} |x-\vec{v}\,(j)|^p - \vec{x}\,(i,t+1)\right| \leq \frac{\delta}{4}$$

Let $N \in \mathbb{N}$ such that for every $N \leq k$ and for every $l \in V$, $|\vec{x}\,(l,t_k)-\vec{v}\,(l)| \leq \min\left\{\frac{\delta}{4},\delta'\right\}$.We get

$$\left|\arg\min_{x\in\mathbb{R}} \sum_{j\in N(i)} |x-\vec{v}\,(j)|^p - \vec{v}\,(i)\right| \leq$$

$$\leq \left|\arg\min_{x\in\mathbb{R}} \sum_{j\in N(i)} |x-\vec{v}\,(j)|^p - \vec{x}\,(i,t_{N+1})\right| + |\vec{x}\,(i,t_{N+1})-\vec{v}\,(i)| \leq \frac{\delta}{4}+\frac{\delta}{4} < \delta$$

This contradicts eq. 2.1, hence $\vec{v}=\vec{s}$. Since $\vec{x}\,(t)$ is a bounded sequence in a compact set, and has only one partial limit $\vec{s}$, it converges to that limit. □

[3]Since the exact solution is $\arg\min\limits_{x\in\mathbb{R}} \sum\limits_{j\in N(i)} |x-\vec{x}\,(j,t)|^p = \vec{x}\,(i,t+1)$.

Let us now consider the case when p is set to infinity ($p = \infty$). In this case, we define the energy of the graph

$$E\left(\vec{x}, t\right) = \max_{i \in V} P_{i,t}\left(A_{i,t}\right) = \max_{(i,j) \in E} \left|A_{i,t} - A_{j,t}\right|$$

Note that there can be infinite minimizers to the energy, as it is based solely on the extermal values of the energy of the vertices (e.g., you can "nudge" a vertex that does not belong to $\arg \max_{(i,j) \in E} \left|A_{i,t} - A_{j,t}\right|$, and the energy of the graph would remain the unchanged). Hence, in this case, we require a method that does not rely on convexity to establish the uniqueness of the stationary distribution. Still, we can directly demonstrate that there exists a unique stationary distribution to which the graph converges. Recall that we define a stationary distribution (of opinions) to be a set of opinions $\vec{s}$ which satifies for every non-stubborn vertex i

$$2\vec{s}\left(i\right) = \vec{s}\left(\text{max neighbour of i in u}\right) + \vec{s}\left(\text{min neighbour of i in u}\right)$$

Claim 2.23. The minimum and the maximum opinion in a stationary distribution occur in a stubborn agent.

Proof. We prove this for the maximum, as the proof for the minimum is analogous. Denote by i the agent that has the maximal opinion M. If she is not stubborn, then each of her neighbors must have opinion M. We can continue this chain of neighbors with opinion M until we arrive at a stubborn agent. □

Claim 2.24. When $p = \infty$, the stationary distribution is unique.

Proof. Let $\vec{u}, \vec{v}$ be stationary distributions, $m \in \arg \max_{i \in V} \left(\vec{u}\left(i\right) - \vec{v}\left(i\right)\right)$, and $a =$

$\max_{i \in V} (\vec{u}(i) - \vec{v}(i))$. Thus,

$$2\vec{u}(m) = \vec{u}(\text{max neighbour of m in u}) + \vec{u}(\text{min neighbour of m in u})$$

$$2\vec{v}(m) = \vec{v}(\text{max neighbour of m in v}) + \vec{v}(\text{min neighbour of m in v})$$

And,

$$2a = 2\vec{u}(m) - 2\vec{v}(m) = \vec{u}(\text{max neighbour of m in u}) - \vec{v}(\text{max neighbour of m in v}) +$$

$$+\vec{u}(\text{min neighbour of m in u}) - \vec{v}(\text{min neighbour of m in v})$$

Note that,

$$\vec{u}(\text{max neighbour of m in u}) - \vec{v}(\text{max neighbour of m in v}) \leq$$

$$\leq \vec{u}(\text{max neighbour of m in u}) - \vec{v}(\text{max neighbour of m in u}) \leq a$$

And,

$$\vec{u}(\text{min neighbour of m in u}) - \vec{v}(\text{min neighbour of m in v}) \leq$$

$$\leq \vec{u}(\text{min neighbour of m in v}) - \vec{v}(\text{min neighbour of m in v}) \leq a$$

So it must be that,

$$\vec{u}(\text{max neighbour of m in u}) - \vec{v}(\text{max neighbour of m in v}) = a =$$

$$= a = \vec{u}(\text{min neighbour of m in u}) - \vec{v}(\text{min neighbour of m in v})$$

The inequalities above also amount to,

$$\vec{u}\,(\text{max neighbour of m in u}) - \vec{v}\,(\text{max neighbour of m in u}) = a =$$

$$= a = \vec{u}\,(\text{min neighbour of m in v}) - \vec{v}\,(\text{min neighbour of m in v})$$

We see that (max neighbour of m in u) $\in \arg\max_{i \in V} (\vec{u}\,(i) - \vec{v}\,(i))$, and so we may continue by induction (since the graph is connected) until we reach the agent with the maximal opinion in $\vec{u}$, which is a stubborn agent (by Claim 2.23), and this means $a = 0$. □

Henceforth we shall address the stationary distribution, marked by $\vec{s}$.

Claim 2.25. The graph converges to the stationary distribution.

Proof. Let $\vec{x}\,(t)$ be a ∞-DeGroot process. We prove that every partial limit of $\vec{x}\,(t)$ is $\vec{s}$. Say that we have a partial limit of $\vec{x}\,(t)$, characterised by its time sequence $\{t_k\}_{k=1}^{\infty}$, such that $\lim_{i \to \infty} \vec{x}\,(t_k) = \vec{v} \neq \vec{s}$. We write $m\,(\vec{v}, i)$ for the value of i's minimal neighbor in $\vec{v}$, and $M\,(\vec{v}, i)$ for the maximal. In particular, since the stationary vector is unique, let $i \in V$ such that

$$\left| \frac{M\,(\vec{v}, i) + m\,(\vec{v}, i)}{2} - \vec{v}\,(i) \right| = \delta \neq 0 \tag{2.2}$$

Let $N \in \mathbb{N}$ such that for every $N \leq k$ and for every $j \in V$, $|\vec{x}\,(j, t_k) - \vec{v}\,(j)| \leq \frac{\delta}{4}$. Hence, every neighbor j of i is located at most $\frac{\delta}{4}$ from $\vec{v}\,(j)$. We get that at $t = N + 1$ it is true that $\left| \frac{M(\vec{v},i)+m(\vec{v},i)}{2} - \vec{x}\,(i, t_{N+1}) \right| \leq \frac{\delta}{4}$ by the updating rule, and so

$$\left| \frac{M\,(\vec{v}, i) + m\,(\vec{v}, i)}{2} - \vec{v}\,(i) \right| \leq$$

$$\leq \left| \frac{M\,(\vec{v}, i) + m\,(\vec{v}, i)}{2} - \vec{x}\,(i, t_{N+1}) \right| + |\vec{x}\,(i, t_{N+1}) - \vec{v}\,(i)| \leq \frac{\delta}{4} + \frac{\delta}{4} < \delta$$

This contradicts eq. 2.2, hence $\vec{v} = \vec{s}$. Since $\vec{x}\,(t)$ is a bounded sequence in a compact set, and has only one partial limit $\vec{s}$, it converges to that limit $\square$

Remark 2.26. One may question the exact nature of the stationary distribution in the case of multiple stubborn agents. In the case $p = \infty$, following [19, p. 26-29], we can adapt a polynomial time algorithm to determine the unique stationary distribution. To begin, we define the slope of a path $P = \{v_0, v_1, \ldots, v_k\}$ to be $s_p = \frac{A_{v_k} - A_{v_0}}{k}$, where A_i is the opinion of agent i.

Data: The graph G
Result: The unique stationary distribution for G
$V' := \{\text{stubborn agents}\}$;
while $V' \neq V$ **do**
    Find the path P with the biggest slope s such that it begins and ends in V' and all of its other vertices are in $V \backslash V'$, and enumerate $P = \{v_1, \ldots, v_k\}$ where $A_{v_1} \leq A_{v_k}$;
    for i *in* P **do**
        $A_i = A_1 + s \cdot (i-1)$
    end
    $V' := V' \bigcup P$
end

Algorithm 1: Finding the unique stationary distribution in $p = \infty$

We can see that in each step, a new set of vertices joins V' via linear interpolation, which represents the optimal approach in the context of $p = \infty$. To show correctness, note that the sequence of slopes of connecting paths occurring in the algorithm is non-increasing. [19, p. 28]

We now prove by induction that the maximal and the minimal neighbors of a vertex lie upon the path P that added the vertex to V'. Let's assume that this is not the case for the maximal neighbor of the vertex (the minimal neighbor follows a similar argument).

In the first step of the algorithm, there is nothing to prove. Assuming the claim holds true until step n, we proceed to verify it for step $n+1$. Suppose that the addition of the path in step $n+1$ altered the maximum neighbor of some vertex x in the graph. Denote $P = \{v_1, \ldots, x, \ldots, v_k\}$ the path that added x to V' with s_P its slope, and $P' = \{w_1, \ldots, w_m\}$ the vertices added in step $n+1$

with $s_{P'}$ its slope, and note that $s_P > s_{P'}$.

Assume that the maximal neighbor M_x of x has changed - then it must be that $M_x = w_j \in P'$, and also $A_{M_x} - A_x > s_P$. Now, consider the path $\hat{P} = \{x, M_x = w_j, w_{j+1}, \ldots, w_m\}$. which gives

$$s_{\hat{P}} = \frac{A_{w_m} - A_x}{m-j} = \frac{A_{w_m} - A_{M_x}}{m-j} + \frac{A_{M_x} - A_x}{m-j} =$$

$$= \underbrace{\frac{m-j-1}{m-j}}_{\alpha} \cdot \underbrace{\frac{A_{w_m} - A_{M_x}}{m-j-1}}_{=s_{P'}} + \underbrace{\frac{1}{m-j}}_{1-\alpha} \cdot \underbrace{\frac{A_{M_x} - A_x}{1}}_{>s_P}$$

Therefore,

$$s_{\hat{P}} > \alpha \cdot s_{P'} + (1-\alpha) \cdot s_P \geq s_{P'}$$

Hence, we should have chosen $\hat{P}$ in step $n+1$, leading to a contradiction.

2.3. Convergence rate and concentration

In this section, we will explore a modest yet general bound for the convergence time in various graph families. Subsequently, we will present more precise bounds for specific graph families in the subsequent sections.

Let M_t and m_t represent the maximum and minimum opinion values at time t, respectively. We define a stopping time (mixing time) to be

$$\tau_\epsilon = \min_{t\in\mathbb{N}} \{|M_t - m_t| \leq \epsilon\}$$

Recall that the proof for convergence relied on a path of length N from every agent to every agent. In general, an upper bound is known for N, $N \leq n^2 - 2n + 2$, where n is the number of vertices in the graph. This bound is known to be sharp. [20, Claims 8.2.11-15, Exercises 8.2.5,7,9] For the case when $p < \infty$, by an argument similiar to Claim 2.8, we can derive the following result.

Claim 2.27. Assume that in time t we have an agent i with n neighbors, which all have opinion B except one neighbor that has opinion A, where $A > B$. Denote $\delta = A - B$. Then, $B + \frac{\delta}{1+c} \leq A_{i,t+1}$, where $c = \sqrt[p-1]{d-1}$.

The proof is similiar to the proof of Claim 2.8.

Now we can describe more carefully the rate of convergence for a general graph when $1 < p < \infty$. Remember that since the graph is not bi-partite then there is an N such that there is a path of length N from every agent to every agent. [2, Proposition 1.7] Let M_t and m_t be the maximal and minimal opinions in time t, [4] and let $\delta_t = M_t - m_t$. At time $t + N$, we get that for all $i \in V$,

[4]Following the proof of Claim 2.9 when taking a large enough time t for $\epsilon = \frac{\delta_t}{2(1+c)^N}$

$m_t + \frac{\delta_t}{2(1+c)^N} \leq A_{i,t+N} \leq M_t - \frac{\delta_t}{2(1+c)^N}$, hence

$$\delta_{t+N} \leq M_t - \frac{\delta_t}{2\left(1+c\right)^N} - \left(m_t + \frac{\delta_t}{2\left(1+c\right)^N}\right) = \delta_t \cdot \left(1 - \frac{1}{\left(1+c\right)^N}\right)$$

Which means that

$$\frac{\delta_{t+N}}{\delta_t} \leq \frac{\left(1+c\right)^N - 1}{\left(1+c\right)^N}$$

Hence, in order to receive $\delta_t \leq \epsilon$, it is enough to perform kN steps of learning, where

$$\left(\frac{\left(1+c\right)^N - 1}{\left(1+c\right)^N}\right)^k = \epsilon \iff k = \frac{\log \frac{1}{\epsilon}}{\log\left(\frac{(1+c)^N}{(1+c)^N - 1}\right)} \tag{2.3}$$

We now face to deal with the case in which the initial opinions are taken from a uniform distribution $U\left([0,1]\right)$ and are i.i.d, in the case $p = \infty$. A simple calculation demonstrates that, in this case,

$$P\left(\left|A_{i,2} - \frac{1}{2}\right| < \epsilon\right) \geq P\left(|M_i - 1| < \epsilon \wedge |m_i - 0| < \epsilon\right) \geq$$

$$\geq 1 - \left(P\left(|M_i - 1| \geq \epsilon\right) + P\left(|m_i - 0| \geq \epsilon\right)\right) \geq 1 - 2 \cdot \left(1 - \epsilon\right)^d$$

A perhaps more sophisticated lower bound can be found by directly computing probabilities. Denote $X_i = \max_{j \sim i} \{U_j\}$ and $Y_i = \min_{j \sim i} \{U_j\}$, and $Z = X + Y$. Then the joint probability density function of x, y is

$$f_{x,y}\left(x, y\right) = n \cdot \left(n-1\right) \cdot \left(x - y\right)^{n-2}$$

And the cumulative distribution function of Z is

$$P\left(Z \le z\right) = \begin{cases} 0 & z < 0 \\ \frac{z^n}{2} & z \in [0,1) \\ 1 - \frac{(2-z)^n}{2} & z \in [1,2) \\ 1 & z \ge 2 \end{cases}$$

Which gives

$$P\left(\left|A_{i,2} - \frac{1}{2}\right| \le z\right) = P\left(\left|\frac{Z-1}{2}\right| \le z\right) = P\left(1 - 2z \le Z \le 2z + 1\right) =$$

$$= P\left(Z \le 2z + 1\right) - P\left(Z \le 1 - 2z\right) =$$

$$= \begin{cases} 0 & 2z+1 < 0 \\ \frac{(2z+1)^n}{2} & 2z+1 \in [0,1) \\ 1 - \frac{(2-(2z+1))^n}{2} & 2z+1 \in [1,2) \\ 1 & 2z+1 \ge 2 \end{cases} - \begin{cases} 0 & 1-2z < 0 \\ \frac{(1-2z)^n}{2} & 1-2z \in [0,1) \\ 1 - \frac{(2-(1-2z))^n}{2} & 1-2z \in [1,2) \\ 1 & 1-2z \ge 2 \end{cases}$$

$$= \begin{cases} 0 & z < -\frac{1}{2} \\ \frac{(2z+1)^n}{2} & z \in \left[-\frac{1}{2}, 0\right) \\ 1 - \frac{(1-2z)^n}{2} & z \in \left[0, \frac{1}{2}\right) \\ 1 & z \ge \frac{1}{2} \end{cases} - \begin{cases} 0 & \frac{1}{2} < z \\ \frac{(1-2z)^n}{2} & z \in \left(0, \frac{1}{2}\right] \\ 1 - \frac{(1+2z)^n}{2} & z \in \left(-\frac{1}{2}, 0\right] \\ 1 & z \le -\frac{1}{2} \end{cases} \underbrace{=}_{z>0} \begin{cases} 1 - (1-2z)^n & z \in \left[0, \frac{1}{2}\right) \\ 1 & z > \frac{1}{2} \end{cases}$$

Then, if $0 \le \epsilon \le \frac{1}{2}$, the probability that each vertex will be at most ϵ-away from $\frac{1}{2}$ is bounded below by

$$P\left(\bigcap_{i \in V} \left|A_{i,2} - \frac{1}{2}\right| \le \epsilon\right) \ge 1 - \sum_{i \in V} P\left(\left|A_{i,2} - \frac{1}{2}\right| > \epsilon\right) = 1 - \sum_{i \in V} (1 - 2\epsilon)^{d_i} \quad (2.4)$$

2.3.1. Erdős–Rényi Graph $G(n, p)$

In this section, we refer to the model as q-DeGroot model, in order to avoid confusion with the probability p of the graph.

When working with $G(n, p)$, a pre-condition for analysing p-DeGroot models is to peruse the regime where $G(n, p)$ is connected w.h.p., i.e. $p > \frac{\ln n}{n}$; in this regime, one can obtain good bounds for the diameter and for N.

Based on [5], we can give bounds on the convergence rate by estimating the order of magnitude for N; $N = 2$ w.h.p. if $p > \sqrt{2}\frac{\sqrt{\ln n}}{\sqrt{n}}$, since that diameter 2 means $N = 2$. This allows us to utilize eq. 2.3 from the previous section. Alternatively, we can approach the convergence rate and concentration of opinions after a single step in a different (and potentially better) way. By leveraging the fact that w.h.p. all vertices have a large degree, the opinions are already concentrated around $\frac{1}{2}$ after a single step. The following proposition offers a qualitative bound using this approach for the $q = \infty$ case.

Proposition 2.28. *In ∞-DeGroot model, for an Erdős–Rényi Graph $G(n, p)$, if $\forall i, A_{i,1} \sim U([0,1])$ i.i.d., then after one step*

$$\mathbb{P}\left(\bigcap_{i\in V}\left|A_{i,2} - \frac{1}{2}\right| \leq \epsilon\right) \geq 1 - n \cdot \left(e^{-\frac{np}{8}} + e^{-\epsilon np}\right)$$

Taking $p > \frac{\ln n}{n} \cdot \frac{8}{\epsilon}$ will give $\mathbb{P}\left(\bigcap_{i\in V}\left|A_{i,2} - \frac{1}{2}\right| \leq \epsilon\right) = 1$ w.h.p.

Proof. Using eq. 2.4, we get

$$\mathbb{P}\left(\bigcap_{i\in V}\left|A_{i,2} - \frac{1}{2}\right| \leq \epsilon\right) \geq 1 - \sum_{i\in V}\mathbb{P}\left(\left|A_{i,2} - \frac{1}{2}\right| > \epsilon\right)$$

Observe

$$\mathbb{P}\left(\left|A_{i,2}-\frac{1}{2}\right|>\epsilon\right)\underbrace{\leq}_{\text{Law of Total Probability}}\mathbb{P}\left(d_i<\frac{np}{2}\right)+\mathbb{P}\left(\left|A_{i,2}-\frac{1}{2}\right|>\epsilon|d_i\geq\frac{np}{2}\right)$$

$$\underbrace{\leq}_{\text{Chernoff Bound, Conditional Probability}} e^{-\frac{np}{8}}+\sum_{k=\frac{np}{2}}^{\infty}P\left(d_i=k|d_i\geq\frac{np}{2}\right)\cdot\mathbb{P}\left(\left|A_{i,2}-\frac{1}{2}\right|>\epsilon|d_i=k\right)$$

Where the Chernoff Bound is based on [24]. Note that by eq. 2.4 we obtain $P\left(\left|A_{i,2}-\frac{1}{2}\right|>\epsilon|d_i=k\right)=(1-2\epsilon)^k\leq e^{-2\epsilon\cdot k}\leq e^{-\epsilon\cdot np}$, so

$$\mathbb{P}\left(\left|A_{i,2}-\frac{1}{2}\right|>\epsilon\right)\leq e^{-\frac{np}{8}}+e^{-\epsilon np}$$

And then

$$\mathbb{P}\left(\bigcap_{i\in V}\left|A_{i,2}-\frac{1}{2}\right|\leq\epsilon\right)\geq 1-n\cdot\left(e^{-\frac{np}{8}}+e^{-\epsilon np}\right)$$

□

In order to generalize prop. 2.28 given just above, to the q-DeGroot model for $1<q<\infty$, we need only to find an upper bound to $\mathbb{P}\left(\left|A_{i,2}-\frac{1}{2}\right|>\epsilon|d_i=k\right)$. The next claim is an interim step in finding such an upper bound.

Claim 2.29. Denote $f_{k,j}(a)=\frac{1}{k}\left|a-x_j\right|^q, f_k(a)=\sum_{j=1}^{k}f_{k,j}(a)=\frac{1}{k}\sum_{j=1}^{k}\left|a-x_j\right|^q$. Then $\mathbb{P}\left(\left|f_k\left(\frac{1}{2}+\epsilon\right)-\mu_{\frac{1}{2}+\epsilon}\right|<\delta\bigcap\left|f_k\left(\frac{1}{2}-\epsilon\right)-\mu_{\frac{1}{2}-\epsilon}\right|<\delta\right)\geq 1-4e^{-2\delta^2k}$

Proof. A simple calculation gives that $\mu_a\equiv\mathbb{E}_{x_1,\ldots,x_k}\left(f_k(a)\right)=\frac{a^{q+1}+(1-a)^{q+1}}{q+1}$. Observe that due to convexity constraints, $f_k\left(\frac{1}{2}\right)<f_k\left(\frac{1}{2}\pm\epsilon\right)\Rightarrow\left|A_{i,1}-\frac{1}{2}\right|<\epsilon$, note that $\mu_{\frac{1}{2}+\epsilon}=\mu_{\frac{1}{2}-\epsilon}$ and $f_k\left(\frac{1}{2}\pm\epsilon\right)-f_k\left(\frac{1}{2}\right)=\left(f_k\left(\frac{1}{2}\pm\epsilon\right)-\mu_{\frac{1}{2}\pm\epsilon}\right)+\left(\mu_{\frac{1}{2}\pm\epsilon}-\mu_{\frac{1}{2}}\right)+\left(\mu_{\frac{1}{2}}-f_k\left(\frac{1}{2}\right)\right)$, hence if we choose $\delta=\frac{\mu_{\frac{1}{2}+\epsilon}-\mu_{\frac{1}{2}}}{2}$ we need to find a lower bound to $\mathbb{P}\left(\left\{\left|f_k\left(\frac{1}{2}+\epsilon\right)-\mu_{\frac{1}{2}+\epsilon}\right|<\delta\right\}\bigcap\left\{\left|f_k\left(\frac{1}{2}-\epsilon\right)-\mu_{\frac{1}{2}-\epsilon}\right|<\delta\right\}\right)$.

This is accomplished by using Hoeffding's inequality [18][5], as $0 \leq f_{k_j}(a) \leq \frac{1}{k}$

$$\mathbb{P}\left(\left|f_k\left(\frac{1}{2}+\epsilon\right)-\mu_{\frac{1}{2}\pm\epsilon}\right|<\delta\right)\geq 1-2e^{-2\delta^2 k}$$

Hence,

$$\mathbb{P}\left(\left|f_k\left(\frac{1}{2}+\epsilon\right)-\mu_{\frac{1}{2}+\epsilon}\right|<\delta\bigcap\left|f_k\left(\frac{1}{2}-\epsilon\right)-\mu_{\frac{1}{2}-\epsilon}\right|<\delta\right)\geq 1-4e^{-2\delta^2 k}$$

And since

$$\mathbb{P}\left(\left|A_{i,2}-\frac{1}{2}\right|<\epsilon|d_i=k\right)\geq\mathbb{P}\left(\left|f_k\left(\frac{1}{2}+\epsilon\right)-\mu_{\frac{1}{2}+\epsilon}\right|<\delta\bigcap\left|f_k\left(\frac{1}{2}-\epsilon\right)-\mu_{\frac{1}{2}-\epsilon}\right|<\delta\right)$$

We arrive at

$$\mathbb{P}\left(\left|A_{i,2}-\frac{1}{2}\right|<\epsilon|d_i=k\right)\geq 1-4e^{-2\delta^2 k}$$

□

Therefore, an upper bound to $\mathbb{P}\left(\left|A_{i,2}-\frac{1}{2}\right|>\epsilon|d_i=k\right)$ is $4e^{-2\delta^2 k}$. For brevity, we denote $g(\epsilon)=2\delta^2$. When plugging it in prop. 2.28 we get the following.

Proposition 2.30. *In q-DeGroot model for $1<q<\infty$, for an Erdős–Rényi Graph $G(n,p)$, if $\forall i, A_{i,0}\sim U([0,1])$ i.i.d., then after one step*

$$\mathbb{P}\left(\bigcap_{i\in V}\left|A_{i,2}-\frac{1}{2}\right|\leq\epsilon\right)\geq 1-n\cdot\left(e^{-\frac{np}{8}}+4e^{-g(\epsilon)np}\right)$$

Where $g(\epsilon)=2\delta^2$, $\delta=\frac{\mu_{\frac{1}{2}\pm\epsilon}-\mu_{\frac{1}{2}}}{2}$, $\mu_a=\frac{a^{q+1}+(1-a)^{q+1}}{q+1}$. Taking $p>\frac{\ln n}{n}\cdot\frac{8}{g(\epsilon)}$ will give $\mathbb{P}\left(\bigcap_{i\in V}\left|A_{i,2}-\frac{1}{2}\right|\leq\epsilon\right)=1$ w.h.p.

[5] Also by $\mathbb{P}(A\bigcap B)\geq 2\min\{\mathbb{P}(A),\mathbb{P}(B)\}-1$

Remark 2.31. After a single step, we can no longer apply the above propositions, since the opinions are not i.i.d. anymore. However, we can still analyze the convergence rate by utilizing convergence rate bounds that are based on the value of N (as discussed in the previous section). These bounds are not dependent on the specific distribution of the opinions, and provide a way to estimate the convergence rate in subsequent steps.

2.3.2. Complete graph K_n

We continue by analysing the convergence rate of quite a convenient family of graphs, K_n, in the case $p = \infty$.[6] In order to give a more accurate estimation to convergence in this family, we also denote the one below maximum and one above minimum opinion values at time t as $\tilde{M}_t$ and $\tilde{m}_t$, accordingly.

In $t = 2$, almost every vertex in the graph will have an opinion $\frac{M_1+m_1}{2}$, but the maximal vertex from time $t = 1$ will have opinion $A = \frac{\tilde{M}_1+m_1}{2} \leq \frac{M_1+m_1}{2}$, and the minimal vertex from time $t = 1$ will have opinion $B = \frac{M_1+\tilde{m}_1}{2} \geq \frac{M_1+m_1}{2}$. Notice that

$$B - A = \frac{M_1 + \tilde{m}_1}{2} - \frac{\tilde{M}_1 + m_1}{2} = \frac{M_1 - m_1 - \left(\tilde{M}_1 - \tilde{m}_1\right)}{2} \leq \frac{M_1 - m_1}{2}$$

The key insight in this case would be to notice the fact that in each time t we have three opinions: m_t, M_t and some other (quite common) $C_t, m_t \leq C_t \leq M_t$. We can write then, for each $t > 2$, $M_t = \frac{M_{t-1}+C_{t-1}}{2}$, $m_t = \frac{m_{t-1}+C_{t-1}}{2}$, and so $M_t - m_t = \frac{M_{t-1}-m_{t-1}}{2}$. To account for the case of $t = 2$, we can simply replace the equality by an inequality:

$$M_t - m_t \leq \frac{M_{t-1} - m_{t-1}}{2}$$

Seeing as $\max\limits_{t} |M_t - m_t| = 1$, we can further subtitute $M_t - m_t \leq \frac{1}{2^t}$, and so $\tau_\epsilon \leq \frac{\log\left(\frac{1}{\epsilon}\right)}{\log(2)}$.

The concentration of the convergence value can be calculated by Markov chain methods - one can easily see that if $\vec{A}_n$ is a vector comprised of the three

[6]Of course, when we consider K_n with loops, we get convergence after a single step. If $p = \infty$, we can even deduce the precise value of the convergence, $\frac{\max\limits_{i \in V} A_{i,1} + \min\limits_{i \in V} A_{i,1}}{2}$

different opinions in time t (for $t \geq 2$), then

$$T = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \vec{A}_t \cdot T = \vec{A}_{t+1}$$

So standard Markov chain methods can be used in order to find the concentration.

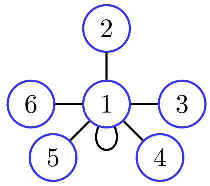

Figure 2.3: The star graph S_6. Note the loop of edge 1.

2.3.3. Star graph $\tilde{S}_n$

The star graph S_n itself is a bipartite graph (take the center to be one side, and the outer vertices to be the other side). Adding a single loop in the center makes the graph non-bipartite, and we can almost instantly apply Claim 2.8. We denote the graph with the loop as $\tilde{S}_n$.

Without further ado, let us denote the graph vertices from 1 to n, where 1 is the center and the rest are the outer vertices. Write the vector comprised of the different opinions in initial time,

$$\begin{pmatrix} A_{1,1} \\ A_{2,1} \\ \vdots \\ A_{n,1} \end{pmatrix}$$

In time $t = 2$, the vector will be

$$\begin{pmatrix} A_{1,2} \\ A_{2,2} = A_{1,1} \\ \vdots \\ A_{n,2} = A_{1,1} \end{pmatrix}$$

Let $c = \sqrt[p-1]{n-1}$. Now, directly applying the procedure given in Claim 2.8, we obtain that, for $t \geq 1$,

$$\delta_{t+1} = \delta_t - \frac{\delta_t}{1+c}$$

And so,

$$\frac{\delta_{t+1}}{\delta_t} = \frac{c}{1+c} \iff \delta_{t+1} = \left(\frac{c}{1+c}\right)^t \delta_1$$

Seeing as $\delta_1 \leq 1$, we arrive at $\tau_\epsilon \leq \frac{\log\left(\frac{1}{\epsilon}\right)}{\log\left(\frac{1+c}{c}\right)}$.

3. Simulations

Simulations were conducted for demonstrating the convergence of p-DeGroot model in the torus lattice graph $T_{n,m}$. This model pertains to an $m \times n$ grid of the two-dimensionl space $\mathbb{Z}^2$, $T_{n,m}$, where each vertex is connected to all of the eight neighbors in its immediate vicinity, while the boundary is periodic, forming a toroidal shape. The parameters chosen for the simulations were $n = m = 30$, $p = 1.1, 1.5, 2, 3, 5, \infty$. The time steps considered were $t = 1, 5, 10, 50, 100$.

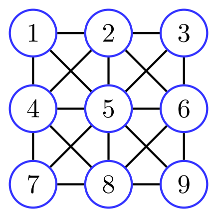

Figure 3.1: The lattice graph of size 3×3. Note that the periodic boundary conditions are not shown.

The simulations revealed that as the value of p increases, the graph tends to become smoother as t approaches infinity. This implies that the transitions between adjacent vertices' opinions are minimized, consistent with the penalty imposed by the energy function. This effect is especially pronounced when examining the simulations involving stubborn agents. These observations align with the results obtained in algorithm 1, which indicate that as p increases, the model progressively resembles $p = \infty$, where equilibrium is achieved through linear interpolation.

In the absence of stubborn agents, running the model leads to opinion convergence and a consensus among the individuals (claim 2.9), resulting in the energy approaching zero. Exploring the rate of this convergence becomes intriguing. We conjectured that the decay follows the form $E(t) = \beta t^{-\alpha}$. Additionally, we performed an experimental fit to estimate the values of α and β.

On the other hand, when stubborn agents are introduced, the opinions converge towards a minimizer of the energy (claims 2.22, 2.25), causing the energy to converge to a constant value. This constant value corresponds to the energy of the unique minimizer, denoted as E_∞. It is therefore of interest to investigate the behavior of $E(t) - E_\infty$. In this case, we hypothesize that the energy function can be represented as $E(t) = E_\infty + \beta t^{-\alpha}$. We conducted an experimental fit to estimate the values of α and β, using the last observed energy as an approximation of E_∞. An additional graph of p vs a was provided (where $a \equiv -\alpha$).

The illustrations of the simulations can be found in Appendix A. The code used for model simulation is available at the following GitHub repository:
https://github.com/liorp/degroot_model

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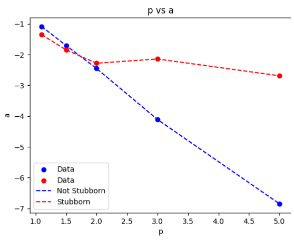
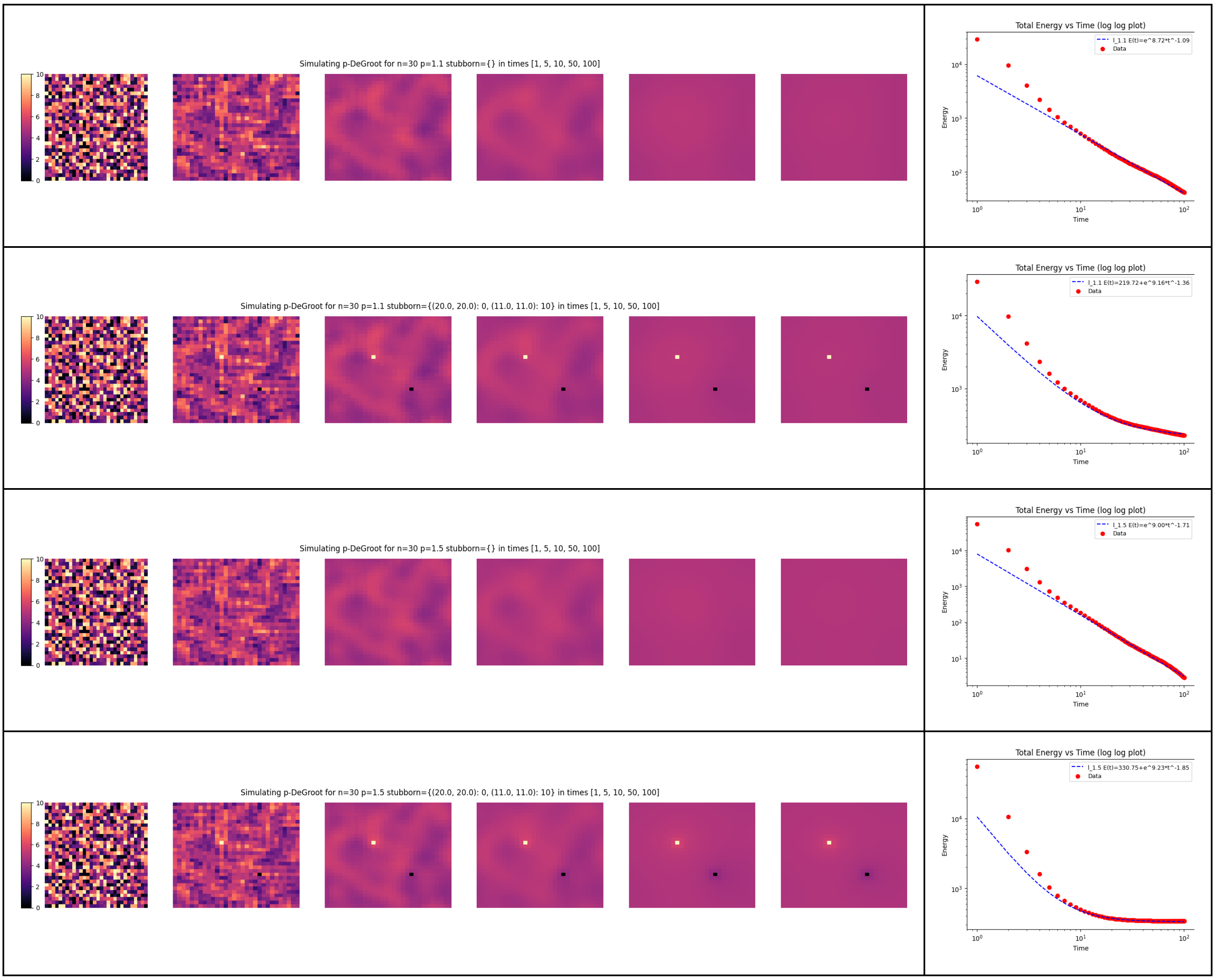
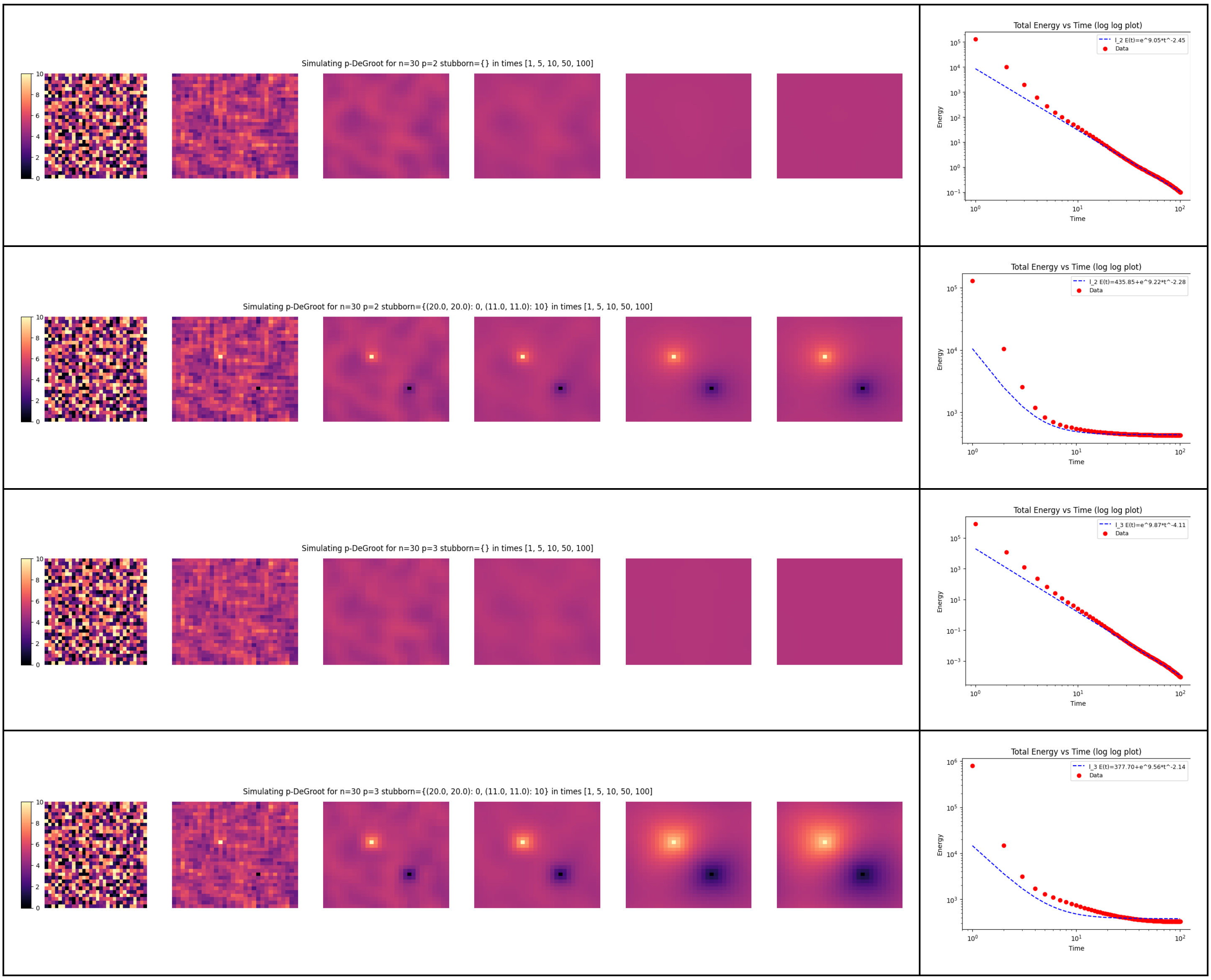
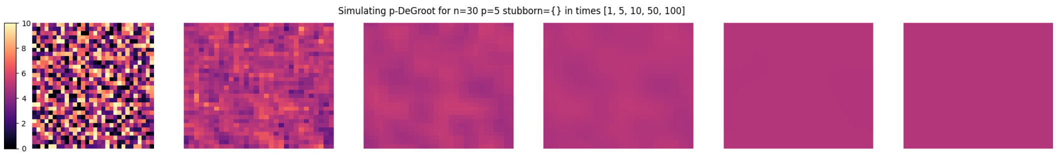
Appendix A. Simulations

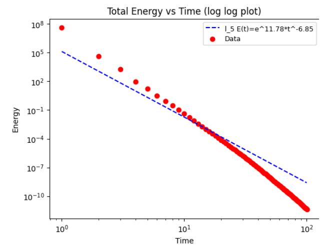

p vs a
a
p
Data
Data
Not Stubborn
Stubborn

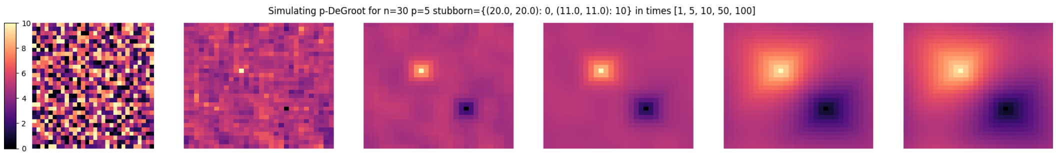

Simulating p-DeGroot for n=30 p=1.1 stubborn={} in times [1, 5, 10, 50, 100]
Total Energy vs Time (log log plot)
l_1.1 E(t)=e^8.72*t^-1.09
Data
Energy
Time
Simulating p-DeGroot for n=30 p=1.1 stubborn={(20.0, 20.0): 0, (11.0, 11.0): 10} in times [1, 5, 10, 50, 100]
Total Energy vs Time (log log plot)
l_1.1 E(t)=219.72+e^9.16*t^-1.36
Data
Energy
Time
Simulating p-DeGroot for n=30 p=1.5 stubborn={} in times [1, 5, 10, 50, 100]
Total Energy vs Time (log log plot)
l_1.5 E(t)=e^9.00*t^-1.71
Data
Energy
Time
Simulating p-DeGroot for n=30 p=1.5 stubborn={(20.0, 20.0): 0, (11.0, 11.0): 10} in times [1, 5, 10, 50, 100]
Total Energy vs Time (log log plot)
l_1.5 E(t)=330.75+e^9.23*t^-1.85
Data
Energy
Time

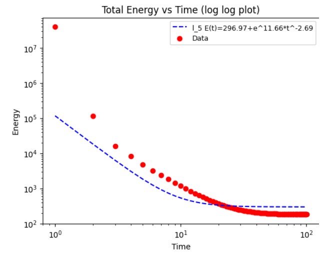


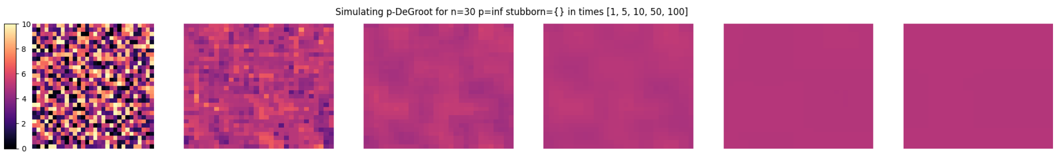
Simulating p-DeGroot for n=30 p=2 stubborn={} in times [1, 5, 10, 50, 100]
Total Energy vs Time (log log plot)
l_2 E(t)=e^9.05*t^-2.45
Data
Energy
Time
Simulating p-DeGroot for n=30 p=2 stubborn={(20.0, 20.0): 0, (11.0, 11.0): 10} in times [1, 5, 10, 50, 100]
Total Energy vs Time (log log plot)
l_2 E(t)=435.85+e^9.22*t^-2.28
Data
Energy
Time
Simulating p-DeGroot for n=30 p=3 stubborn={} in times [1, 5, 10, 50, 100]
Total Energy vs Time (log log plot)
l_3 E(t)=e^9.87*t^-4.11
Data
Energy
Time
Simulating p-DeGroot for n=30 p=3 stubborn={(20.0, 20.0): 0, (11.0, 11.0): 10} in times [1, 5, 10, 50, 100]
Total Energy vs Time (log log plot)
l_3 E(t)=377.70+e^9.56*t^-2.14
Data
Energy
Time

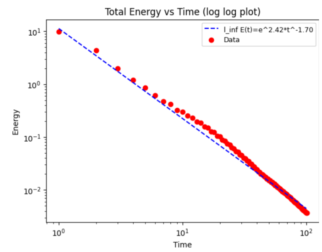

Simulating p-DeGroot for n=30 p=5 stubborn={} in times [1, 5, 10, 50, 100]
10
8
6
4
2
0

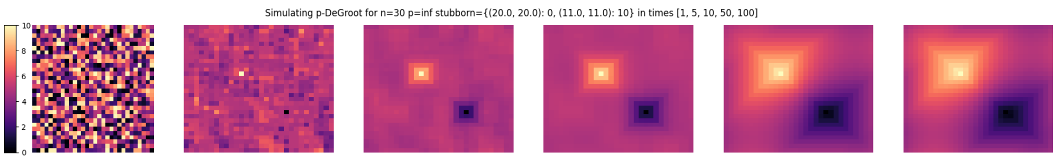

Total Energy vs Time (log log plot)
l_5 E(t)=e^11.78*t^-6.85
Data
Energy
Time

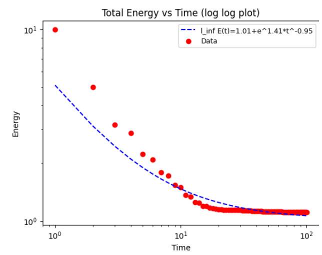

Simulating p-DeGroot for n=30 p=5 stubborn={(20.0, 20.0): 0, (11.0, 11.0): 10} in times [1, 5, 10, 50, 100]
10
8
6
4
2
0


Total Energy vs Time (log log plot)
l_5 E(t)=296.97+e^11.66*t^-2.69
Data
Energy
Time


Simulating p-DeGroot for n=30 p=inf stubborn={} in times [1, 5, 10, 50, 100]
10
8
6
4
2
0


Total Energy vs Time (log log plot)
l_inf E(t)=e^2.42*t^-1.70
Data
Energy
Time


Simulating p-DeGroot for n=30 p=inf stubborn={(20.0, 20.0): 0, (11.0, 11.0): 10} in times [1, 5, 10, 50, 100]
10
8
6
4
2
0


Total Energy vs Time (log log plot)
l_inf E(t)=1.01+e^1.41*t^-0.95
Data
Energy
Time

תקציר

אנו חוקרים מודל של חילופי דעות ברשתות חברתיות, שבו ישנו מצב נתון של העולם, וכל סוכן בעולם מקבל אות רועש של המצב הנתון (אמיר ,2021) בתור דעה התחלתית. בכל צעד בזמן, הדעות של הסוכנים מתעדכנים לפי אלגוריתם שנקבע מראש.

אחד המודלים המרכזיים בתחום זה הוא מודל הדינמיקה של DeGroot (Degroot, 1974), אשר נחקר רבות מאז הצגתו
(Acemoglu and Ozdaglar, 2010; Wikipedia, 2021; Golub and Jackson, 2010; Amir et al., 2021).
מעבודתם של Golub and Jackson, ידוע שתחת מודל הדינמיקה של DeGroot הדעות תמיד מתכנסות ומגיעות לקונצנזוס שקרוב למצב של העולם בנסיבות מסוימות (Golub and Jackson, 2010).

בתזה זו אנו מציגים וריאנט של דינמיקת DeGroot שאנו מכנים p-DeGroot. אנו מספקים כמה תוצאות הנוגעות למודל על גרפים סופיים, ושואפים לפתח תיאוריה מקבילה לדינמיקה זו של חילופי דעות, כפי שידוע ממודל DeGroot.

שאלות בולטות להן נתייחס הן, בין השאר, קצב ההתכנסות של המודל, השפעתם של סוכנים עקשניים וניתוח כמותני למשפחות שונות של גרפים לגבי קצב וריכוז ההתכנסות.

עבודה זו נעשתה תחת הדרכתו של

פרופ' גדעון עמיר

המחלקה למתמטיקה של אוניברסיטת בר-אילן

אוניברסיטת בר אילן

מודל דה-גרוט ללמידה על גרפים תחת נורמת lp כללית

ליאור פולק

עבודה זו מוגשת כחלק מהדרישות לשם קבלת תואר מוסמך
במחלקה למתמטיקה
של אוניברסיטת בר-אילן

רמת גן תשפ"ג