A Tale of Two Cities

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Abstract

We consider the following problem to achieve a basic understanding of taxi driver on-line decisions: We have two cities and jobs arrive one after another and the taxi driver can decide for each job if he wants to pick up the passenger or forfeit the job. Accepting a job yields the driver a reward of 1. However, accepting a job in a different city means that the driver will have to drive to the other city and pay 1 for the relocation - effectively earning nothing for the fare.

1 Notation

We will denote the following:

1. T - The number of total jobs received.
2. L, R - The two cities.
3. $p_{t,L}$ - The probability of the taxi being at city L at time t of our algorithm.
4. $p_{t,R}$ - The probability of the taxi being at city R at time t of our algorithm.

We will notice that always $p_{t,L} + p_{t,R} = 1$.

5. A turn is defined by first receiving a job request and then the decision as to how to act upon that request. We will denote the city the driver decided to go to in time t as d_t.
6. We will denote the job received at time t as $J_t \in \{L, R\}$

2 Deterministic Algorithm

The simple problem of a deterministic algorithm is largely uninteresting in this case since we can at each point in the algorithm a job can hail from the other city meaning that the ALG will receive 0 profit. OPT however will get at the very least a profit of $\frac{T}{2}$ at least by simply choosing the city that has more jobs in it and staying there. This means that any deterministic algorithm can't achieve any finite competitive ratio.

3 Online simple strategies

We will discuss the following two simple strategies:

1. $STAY$ - At the beginning we will use a coin to decide on a city and simply stay there for the rest of the algorithm. It is easy to see that this algorithm is expected to earn $\frac{T}{2}$.

2. $RAND\frac{1}{2}$ - Every turn of the algorithm if the job is in a different city we will flip a coin and move to that city with probability $\frac{1}{2}$. If the job is in the city we are already in we will of course accept it along with our reward.

4 Rand Analysis

Let's look at a possible way of analyzing the profit of the $RAND\frac{1}{2}$ algorithm. At time t we have a probability of p of being in R at the beginning of the turn and probability of $1-p$ of being in L at the beginning of the turn. This means that we can expect to earn p and update the probability of being at R at time $t+1$ to $p+\frac{1}{2}(1-p)=\frac{1}{2}+\frac{1}{2}p$ and the probability of being at L to $\frac{1}{2}(1-p)$. In the case of consistently alternating jobs (the job sequence consisting of $L, R, L, R, L, \ldots$) we can see that a balance is achieved when $p=\frac{1}{3}$. This can be easily demonstrated by the fact that through simple calculation we can see that the probabilities are swapped after the job arrives at R and thus they can continuously swap and remain balanced that way. This means that in every step we will earn exactly $\frac{1}{3}$ achieving a total expected revenue of $\frac{T}{3}$ compared to the $\frac{T}{2}$ of OPT.

5 Rand Worse Case

The following sequence can get a worse competitive ratio for the $RAND\frac{1}{2}$ algorithm: $J_0=0, J_1=0, J_2=1$. This sequence is repeated many time in order to create the real sequence. It is easy to see that OPT can get $\frac{2}{3}T$. However, in an analysis similar to the alternating case we can see that $p_t, 0=\frac{3}{7}$ at the beginning of the sequence. Then $p_(t+1).0=\frac{5}{7}$ and then $p_(t+2), 1=\frac{1}{7}$ yielding a total of $\frac{9}{7}$ per 3 turns giving us a competitive ratio of $\frac{14}{9}$.

6 Random Walk Analysis

It is easy to see that if we generate jobs with equal probability for each city any ON will get exactly $\frac{T}{2}$ in expectancy. If we wish to analyze OPT we have to assume that with probability $\frac{1}{2}$ the next job will be at the city we're already at and with probability $\frac{1}{2}$ the job will be at the other city. If the job is in the other city it means that we will not get any reward from it however we will get the reward for the job after that in OPT since if the job after is in the other city we will move there and get it (TO DO: prove that there is an OPT that does this and is equal to any other OPT) and if the job is in the city we're at we can just stay there and pass on the job. This means that with probability $\frac{1}{2}$ we will gain 1 in the next 2 turns. Now, to calculate the average we realize that with probability $\frac{2}{3}$ we are in one of the 2-step cycles (since they last for 2 steps instead of 1) and with probability $\frac{1}{3}$ we are in the 1-step cycle. This nets us a total of $\frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 1 = \frac{2}{3}$. A total of $\frac{2}{3} \cdot T$.