Theoretical Ex2 B

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1. Claim For L, a Laplacian matrix, it holds $\forall f$:$f^t \cdot L \cdot f \geq 0$. Meaning L is positive semi-definite.

   Proof Let f$\in 1 \times n$

   $f^t \cdot L \cdot f = f^t \cdot D \cdot f$ - $f^t \cdot W \cdot f = \sum_i d_{ii} f_i^2 - f^t \cdot W \cdot f =$

   $\sum_i d_{ii} f_i^2 - \sum_i \sum_j f_j w_{ij} f_i^2 = \sum_i \sum_j w_{ij} f_i^2 - \sum_i \sum_j f_j w_{ij} f_i =$

   $\frac{1}{2}\left[\sum_i \sum_j w_{ij} f_i^2 - 2\sum_i \sum_j f_j w_{ij} f_i + \sum_i \sum_j w_{ij} f_j^2\right] = \frac{1}{2}\sum_i \sum_j w_{ij}(f_i - f_j)^2$

   Now from definition $0 \leq w_{ij} \leq 1 \rightarrow w_{ij} \geq 0$ and clearly $(f_i - f_j)^2 > 0$

   Therefore $\frac{1}{2}\sum_i \sum_j w_{ij}(f_i - f_j)^2 \geq 0$

   - Now L is a PSD matrix let u be a eigenVector with a corresponding eigenvalue $\lambda \rightarrow u^t \cdot L \cdot u = u^t \cdot \lambda 1_{n \times n} u = \lambda \sum_i u_i^2 \geq 0$

     all it's eigenvalues≥ 0 which clearly means that the minimal eigenvalue we can get is 0.

   - Since the rows of L sum up to zero if we multiply the sorted L by $v \in R^{n \times 1}$, where the first r entries of v are 1, and the next n-r entries are 0, we will get 0. Those we get that for every cluster group a corresponding eigenvector with the eigenvalue $\lambda = 0$.

2. I will show that z_0 =D$^{\frac{1}{2}} \cdot \vec{1}$ is an eigenvector with a eigenvalue of zero(the minimal eigenvector)

   $Ax = \lambda x \Rightarrow D^{\frac{1}{2}} \cdot L \cdot D^{-\frac{1}{2}} \cdot D^{\frac{1}{2}} \overrightarrow{1} = z_0 \cdot D^{\frac{1}{2}} = 0 \cdot D^{\frac{1}{2}}$

   $(D^{-\frac{1}{2}} \cdot D^{\frac{1}{2}})$=1 $\rightarrow D^{\frac{1}{2}} \cdot L \cdot D^{-\frac{1}{2}} \cdot D^{\frac{1}{2}} \overrightarrow{1} = D^{\frac{1}{2}} \cdot L \overrightarrow{1} = 0 \cdot D^{\frac{1}{2}}$ ■

3. Let

   (a) $f_i = \left\{ \begin{array}{ll} \sqrt{\frac{vol(\bar{A})}{vol(A)}} & v_i \in A \\ & \\ \sqrt{\frac{vol(A)}{vol(\bar{A})}} & v_i \in \bar{A} \end{array} \right\}$

(b) $Ncut(A,\bar{A}) = cut(A,\bar{A})\cdot(\frac{1}{vol(A)} + \frac{1}{vol(\bar{A})})$

Where $cut(A,\bar{A}) = \frac{1}{2}\underset{j\in A, i\in\bar{A}}{\Sigma} w_{ij}$

Claim $\quad f^t \cdot L \cdot f = vol(V)\cdot Ncut(A,\bar{A})$

$f^t \cdot L \cdot f = \frac{1}{2}\underset{i}{\Sigma}\underset{j}{\Sigma} w_{ij}(f_i - f_j)^2$ (from the first question)

$\overset{a}{=}\frac{1}{2}\underset{i\in A, j\in\bar{A}}{\Sigma} w_{ij}(\sqrt{\frac{vol(\bar{A})}{vol(A)}} + \sqrt{\frac{vol(A)}{vol(\bar{A})}})^2 + \frac{1}{2}\underset{j\in A, i\in\bar{A}}{\Sigma} w_{ij}(-\sqrt{\frac{vol(\bar{A})}{vol(A)}} - \sqrt{\frac{vol(A)}{vol(\bar{A})}})^2$
$=$

$\frac{1}{2}\underset{i\in A, j\in\bar{A}}{\Sigma} w_{ij}(\frac{\mathbf{vol}(\bar{A})}{vol(A)} + 2\sqrt{\frac{vol(\bar{A})}{vol(A)}}\cdot\sqrt{\frac{vol(A)}{vol(\bar{A})}} + \frac{vol(A)}{vol(\bar{A})}) + \frac{1}{2}\underset{j\in A, i\in\bar{A}}{\Sigma} w_{ij}(\frac{\mathrm{vol}(\bar{A})}{vol(A)} + 2\sqrt{\frac{vol(\bar{A})}{vol(A)}}\cdot\sqrt{\frac{vol(A)}{vol(\bar{A})}} + \frac{vol(A)}{vol(\bar{A})}) =$

$\frac{1}{2}\underset{i\in A, j\in\bar{A}}{\Sigma} w_{ij}(\frac{\mathbf{vol}(\bar{A})}{vol(A)} + 2 + \frac{vol(A)}{vol(\bar{A})}) + \frac{1}{2}\underset{j\in A, i\in\bar{A}}{\Sigma} w_{ij}(\frac{\mathrm{vol}(\bar{A})}{vol(A)} + 2 + \frac{vol(A)}{vol(\bar{A})}) =$

$\underset{i,j}{\Sigma} w_{ij}(\frac{\mathbf{vol}(\bar{A})}{vol(A)} + 2 + \frac{vol(A)}{vol(\bar{A})}) = cut(A,\bar{A})\cdot(\frac{vol(\bar{A})}{vol(A)} + \frac{vol(A)}{vol(A)} + \frac{vol(\bar{A})}{vol(\bar{A})} + \frac{vol(A)}{vol(\bar{A})}) = cut(A,\bar{A})\cdot(\frac{vol(\bar{A})+vol(A)}{vol(A)} + \frac{vol(\bar{A})+vol(A)}{vol(\bar{A})}) =$

$cut(A,\bar{A})\cdot(\frac{vol(V)}{vol(A)} + \frac{vol(V)}{vol(\bar{A})} = vol(V)\cdot Cut(A,\bar{A})\cdot(\frac{1}{vol(A)} + \frac{1}{vol(\bar{A})}) \overset{b}{=} vol(V)\cdot Ncut(A,\bar{A})$ ■