Ex3 - Lior Ziv

1. At class we learned that $P_{JC}P(a \underset{t}{\rightarrow} b) = \begin{cases} \frac{1}{4}(1+3e^{-4\alpha t}) & a=b \\ \\ \frac{1}{4}(1-e^{-4\alpha t}) & a\neq \text{b} \end{cases}$,

   we also have $\alpha = \frac{1}{4}$

   we need to perfume some algebraic manipulations.

   $\prod\limits_i \pi_{a_i}[e^{tR}]_{a_i b_i} = \prod\limits_{i\in[a_i \neq b_i]} \prod\limits_i \pi_{a_i}[e^{tR}]_{a_i b_i} \cdot \prod\limits_{i\in[a_i = b_i]} \prod\limits_i \pi_{a_i}[e^{tR}]_{a_i a_i}$

   $\prod\limits_{i\in[a_i \neq b_i]} \frac{1}{4}(1-e^{-4\alpha t}) \prod\limits_{i\in[a_i = b_i]} \frac{1}{4}(1+3e^{-4\alpha t}) =$

   $\frac{1}{4}(1-e^{-4\alpha t})^{\ |a_i\neq b_i|} \frac{1}{4}(1+3e^{-4\alpha t})^{|a_i=b_i|}$

   ** $|a_i \neq b_i|$and $|a_i = b_i|$ represents the number of transitions made from a to a and the number of transition between a to b (when a$\neq b$)

   Now it is easy to see that all we need in order to calculate t is the number of all the occurrences of transitions from $a_i = b_i$ and $a_i \neq b_i$, so those will be our sufficient statistics.

   back to finding the formula to find $\hat{t} = argmax_t \prod\limits_i \pi_{a_i}[e^{tR}]_{a_i b_i} \Rightarrow$ put a log in order to derive more easily log($\frac{1}{4}(1-e^{-4\alpha t})^{\ |a_i\neq b_i|} \frac{1}{4}(1+3e^{-4\alpha t})^{|a_i=b_i|}$) $= |a_i \neq b_i|\log(\frac{1}{4}(1-e^{-t})+|a_i = b_i|\ \log(\frac{1}{4}(1+3e^{-t}))$, (with α =0.25)

   Now we will derive in order to get argmax $\frac{|a_i=b_i|(\frac{1}{4}e^{-t})}{\frac{1}{4}(1-e^{-t})} + \frac{(-\frac{1}{4}\cdot 3e^{-t})|a_i\neq b_i|}{\frac{1}{4}(1+3e^{-t})}$,

   now we can compare to zero and find $\hat{t}$,

   $\frac{|a_i=b_i|(e^{-t})}{1-e^{-t}} + \frac{(3e^{-t})|a_i\neq b_i|}{1+3e^{-t}} = 0$

   $0 = |a_i = b_i|(e^{-t})(1+3e^{-t}) + (-3e^{-t})|a_i \neq b_i|(1-e^{-t})$

   $= |a_i = b_i|(e^{-t}+3e^{-2t}) + |a_i \neq b_i|(-3e^{-t}+3e^{-2t})$

   $= 3e^{-2t}(|a_i = b_i| + |a_i \neq b_i|) + e^{-t}(|a_i = b_i|\text{-}3|a_i \neq b_i|)$

   $\iff 3e^{-2t}(|a_i = b_i| + |a_i \neq b_i|) = e^{-t}(-|a_i = b_i| + 3|a_i \neq b_i|)$

   $\iff e^t \cdot \frac{1}{3} = \frac{(|a_i=b_i|+|a_i\neq b_i|)}{(-|a_i=b_i|+3|a_i\neq b_i|)}$

   $\iff t = \log(\frac{3(|a_i=b_i|-|a_i\neq b_i|)}{(-|a_i=b_i|+3|a_i\neq b_i|)})$

   so we get that the MLE for t is $\log(\frac{3(|a_i=b_i|-|a_i\neq b_i|)}{(-|a_i=b_i|-3|a_i\neq b_i|)})$

2. we can take the definition of $P(a \underset{t}{\rightarrow} b)$= $[e^{tR}]_{ab}$ and get

   P(X$_1$,X_2,X_3, ..X_{2n-1}) $= P(X_{2n-1}) \prod_{(i\rightarrow j)\in T} P(X_i \overset{t_{ij}}{\rightarrow} X_j)$=$P(X_{2n-1})\prod_{(i\rightarrow j)\in T}[e^{t_{ij}R}]_{X_i,X_j} =$

   $\pi(X_{2n-1})\prod_{(i\rightarrow j)\in T}[e^{t_{ij}R}]_{X_i,X_j} \cdot \frac{\pi X_j}{\pi X_j} = \pi(X_{2n-1})\prod_{(i\rightarrow j)\in T} \pi X_j \cdot \frac{[e^{t_{ij}R}]_{X_i,X_j}}{\pi X_j} = (\pi(X_{2n-1})\prod_{(i\rightarrow j)\in T} \pi X_j)$

now lets pay attention that the phrase in brackets, it equals to $\prod_i \pi X_i$, since by definition $\prod_{(i \to j) \in T} \pi X_j$ takes in account all the nodes besides the root.

$\Rightarrow \prod_{(i \to j) \in T} \frac{[e^{t_{ij}R}]_{X_i, X_j}}{\pi X_j} \cdot \prod_i \pi X_i$ and we are done.

3. We can choose a new root r* and set the new tree to be T*, it includes the same nodes and edges besides those how are between to (r*, r) , those nodes goes now to the opposite direction in T*. So we can divide the tree to two separate groups :

- T\T*- the part of the tree that contains different edges.
- T$\cap T^*$−the part of the tree that statyed the same.

  We get P(X$_1$,X_2,X_3, ..X_{2n-1}) $= \prod_i \pi X_i \prod_{(i \to j) \in T \cap T^*} \cdot \frac{[e^{t_{ij}R}]_{X_i, X_j}}{\pi X_j} \prod_{(i \to j) \in T \setminus T^*} \cdot \frac{[e^{t_{ij}R}]_{X_i, X_j}}{\pi X_j}$, Now we can use the detailed balance property($\forall a, b, t \ \pi_a P(a \underset{t}{\to} b) = \pi_b P(b \underset{t}{\to} a) \to \prod_{(i \to j) \in T \cap T^*} \cdot \frac{[e^{t_{ij}R}]_{X_i, X_j}}{\pi X_j} \prod_{(j \to i) \in T* \setminus T} \cdot \frac{[e^{t_{ij}R}]_{X_i, X_j}}{\pi X_j}$,we got all T*$edges \Rightarrow \prod_{(i \to j) \in T^*} \frac{[e^{t_{ij}R}]_{X_i, X_j}}{\pi X_j} \cdot \prod_i \pi X_i$, It is easy to see that the formula we got equals to the one of the original T , with the difference of the annotation of T(T*) only, those we get the changing a trees root doesn't change it's joint distribution.