

## Lecture 2. Topological properties of spaces.

### 1. Continuous functions = maps.

( $\epsilon, \delta$ ) arguments:  $f: \mathbb{R} \rightarrow \mathbb{R}$  is continuous if  $\forall \epsilon > 0, \exists \delta > 0, \forall x_1, x_2 \in \mathbb{R}, |x_1 - x_2| < \delta, |f(x_1) - f(x_2)| < \epsilon$ .

Defn. A function  $f: X \rightarrow Y$  between two topological spaces  $X, Y$  is continuous if the preimage of open sets in  $Y$  is open in  $X$ .

prop. Let  $f: X \rightarrow Y$  be a function. The followings are equivalent:

- (i)  $f$  is continuous
- (ii) The preimage of closed sets in  $Y$  is closed in  $X$ .
- (iii)  $\forall y \in Y$  and  $V_y$  (nbhd of  $y$ ),  $f^{-1}(V_y)$  is a nbhd of  $x$ ,  $f(x)=y$ .
- (iv)  $f(\bar{A}) \subseteq \bar{f(A)}$  for any set  $A \subseteq X$
- (v)  $\bar{f^{-1}(B)} \subseteq f^{-1}(\bar{B})$  for any  $B \subseteq Y$ .

proof. (i), (ii), (iii) are equivalent: exercise

Exercise. Let  $f: X \rightarrow Y$  be a function and  $B$  is a topology basis of  $Y$ .

Show that  $f$  is continuous iff  $\forall B \in \mathcal{B}$ ,  $f^{-1}(B) \subseteq X$  is open.

Examples. ①  $i_x: X \xrightarrow{\text{id}} X$ ,  $x \mapsto x$ , is continuous.

②  $e: X \rightarrow \{x\}$ , is continuous

③ the diagonal map  $\Delta: X \rightarrow X \times X$ ,  $\Delta(x) = (x, x)$ , is continuous.

④ if  $A \subseteq X$  is a subspace, then the inclusion map  $i: A \rightarrow X$ ,  $i(a) = a$ , is continuous.

⑤  $X \xrightarrow{i_1} X \times Y$ ,  $Y \xrightarrow{i_2} X \times Y$  are continuous.  
 $x \mapsto (x, y_0)$

⑥  $X \times Y \xrightarrow{p_1} X$ ,  $X \times Y \xrightarrow{p_2} Y$  are continuous.

Prop. Let  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  be continuous functions.

Then the composition  $g \circ f: X \rightarrow Z$  is continuous.

Proof.  $\forall U \subseteq Z$  be open,  $(g \circ f)^{-1}(U) = f^{-1} \circ g^{-1}(U) \subseteq X$  is open.  $\square$

Applications. ① Let  $A \hookrightarrow X$  be the inclusion map and  $f: X \rightarrow Y$  be a map.

Then  $f|_A = f \circ i: A \xrightarrow{i} X \xrightarrow{f} Y$  is continuous.

②  $Z \xrightarrow{f} X \times Y \quad | \quad Z \xrightarrow{f \circ f_1, f_2} X \times Y$  is continuous iff  $f_1$  and  $f_2$  are continuous.

$$\begin{array}{l} f_1 = f \circ p_1 \\ f_2 = f \circ p_2 \\ f(Z) = \{f(p_1(z), f(p_2(z))) \\ = (f_1(z), f_2(z))\} \end{array}$$

$\Rightarrow$  clear

$\Leftarrow Z \xrightarrow{\Delta} Z \times Z \xrightarrow{f_1 \times f_2} X \times Y$  is  $f$ .

$$z \mapsto (z, z) \mapsto (f_1(z), f_2(z)) = f(z)$$

$$f = (f_1 \times f_2) \circ \Delta. \quad \square$$

$$\text{or } f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$$

Exercise. Let  $f, g: X \rightarrow \mathbb{R}$  be continuous map. Show that

(1)  $f \pm g: X \rightarrow \mathbb{R}$ ,  $(f \pm g)(x) = f(x) \pm g(x)$ , is continuous.

(2)  $f \cdot g: X \rightarrow \mathbb{R}$ ,  $(f \cdot g)(x) = f(x) \cdot g(x)$ , is continuous.

$$(X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} \mathbb{R} \times \mathbb{R} \xrightarrow{+} \mathbb{R})$$

Theorem (gluing lemma, pasting lemma).

Let  $X = \bigcup_{i=1}^n X_i$ , and let  $f: X \rightarrow Y$  be a function.

If each restriction  $f_i := f|_{X_i}: X_i \rightarrow Y$  is continuous,

and each  $X_i$  is closed (resp. open), then  $f$  is continuous. ( $f = \bigcup_{i=1}^n f_i$ ).

Proof  $\forall V \subseteq Y$  is closed,  $f^{-1}(V) = X \cap f^{-1}(V) = \left( \bigcup_{i=1}^n X_i \right) \cap f^{-1}(V)$

$$= \bigcup_{i=1}^n X_i \cap f^{-1}(V)$$

$$= \bigcup_{i=1}^n f_i^{-1}(V) \subseteq X \text{ is closed.}$$

Thus  $f$  is continuous.

If each  $X_i$  is open, the statement is still true if  $n = \infty$ .

Homeomorphism.  $f: X \rightarrow Y$  is a homeomorphism if

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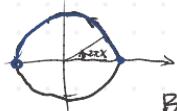
(i)  $f$  is continuous

(ii)  $f$  is a bijection: the inverse  $f^{-1}$  exists.

(iii) The inverse  $f^{-1}: Y \rightarrow X$  is continuous.

$$f: X \xrightarrow{\cong} Y$$

Example: ①  $f(x) = e^{ixx}: [0, 1] \rightarrow S^1$  is continuous.



$$g: S^1 \rightarrow [0, 1], g(z = e^{i\theta}) = \frac{\theta}{2\pi}.$$

$$fg = 1, gf = 1.$$

But  $g$  is not continuous.

$$g([0, \frac{1}{2}]) = \text{a closed curve} \subseteq S^1 \text{ is not open.}$$

$$\textcircled{2} f: (0, 1) \xrightarrow{\cong} \mathbb{R}, f(x) = \frac{x}{1-x}; g: \mathbb{R} \rightarrow (0, 1), g(y) = \frac{y}{1+y}.$$

$f$  is a homeomorphism

$$\textcircled{3} B(0, 1) = \{x \in \mathbb{R}^n \mid |x| < 1\} \xrightarrow{\cong} \mathbb{R}^n$$

$$\textcircled{4} \mathbb{R}^n \setminus \{0\} \xrightarrow{f \cong} \mathbb{R}^n \setminus D^n, D^n = \overline{B(0, 1)}$$



$$f(x) = x + \frac{x}{|x|}$$

$$\textcircled{5} \text{ Stereographic projection: } S^2 \setminus \{N\} \xrightarrow{\Phi \cong} \mathbb{R}^2$$

$$\Phi(x, y, z) = \left( \frac{x}{1-z}, \frac{y}{1-z} \right)$$



A property  $P$  is called a topological property of a space if it is preserved by homeomorphisms of the space.

Hausdorff. A space  $X$  is Hausdorff if it satisfies the  $T_2$  axiom:  
 $\forall x \neq y \in X, \exists$  nbhds  $U_x$  and  $U_y$ , st.  $U_x \cap U_y = \emptyset$ .



Example: metric spaces are Hausdorff

prop. Let  $f: X \rightarrow Y$  be an injective map. If  $Y$  is Hausdorff, then so is  $X$ .

proof.  $\forall x_1 \neq x_2 \in X, f(x_1) \neq f(x_2)$ . Since  $Y$  is Hausdorff,  
 $\exists V_{f(x_1)} \cap V_{f(x_2)} = \emptyset$ .

$$U_{x_1} = f^{-1}(V_{f(x_1)}), U_{x_2} = f^{-1}(V_{f(x_2)})$$

$$\text{Then } U_{x_1} \cap U_{x_2} = \emptyset.$$

□

Cor. Hausdorff property is a topological property.

prop. The following hold:

(1) points of a Hausdorff space are closed

(2) Subspaces of a Hausdorff space are Hausdorff

(3) product spaces of two Hausdorff spaces are Hausdorff.

proof. Exercise.

prop. Let  $f: X \rightarrow Y$  be a map with  $Y$  Hausdorff. Then

(1) The graph  $G_f = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$  is closed.

(2) Let  $g: X \rightarrow Y$  be another map. Then  $eq(f, g) = \{x \in X \mid f(x) = g(x)\}$   
is a closed subset of  $X$ . (Exercise)

proof. (1)  $Z = X \times Y - G_f$ .  $\forall (x, y) \in Z, y \neq f(x)$ .

Since  $Y$  is Hausdorff, there exist  $V_y$  and  $V_{f(x)}$ , st.  $V_y \cap V_{f(x)} = \emptyset$ .

$U_x = f^{-1}(V_{f(x)})$  is a nbhd of  $x$ . Then  $U_x \times V_y \subseteq Z = X \times Y - G_f$ .  
 $\therefore Z$  is open. □

### Connectedness.

A topological space  $X$  is connected if one of the following equivalent conditions holds:

- (1)  $X$  cannot be expressed as a union of two disjoint non-empty open sets.
- (2)  $X$  - - - - - closed sets
- (3) The subsets of  $X$  that are both open and closed are  $\emptyset, X$ .

Exercise. Show the above three statements are equivalent.

Prop. Let  $f: X \rightarrow Y$  be a surjective map with  $X$  connected.

Then  $Y = f(X)$  is connected.

Proof. Let  $V$  be an open and closed subset of  $Y$ .

$f^{-1}(V)$  is both open and closed as a subset of  $X$ .

Since  $X$  is connected,  $f^{-1}(V) = \emptyset$ , or  $f^{-1}(V) = X$ .

Thus  $V = \emptyset$  or  $V = f(X) = Y$ .

That is,  $Y$  is connected. □

Cor. Connectedness is a topological property.