Lecture 2. Topological properties of Spaces.

1. Continuous functions = maps.

(ε, δ) arguments: $f: \mathbb{R} \to \mathbb{R}$ is continuous if $\forall \varepsilon > 0$, $\underset{f(x_1)}{y_1}, \underset{f(x_2)}{y_2} \in \mathbb{R}$

$\exists \delta > 0$, $x_1, x_2 \in \mathbb{R}$, such that

$|x_1 - x_2| < \delta$, $|f(x_1) - f(x_2)| < \varepsilon$.

<u>Defn.</u> A function $f: X \to Y$ between two topological spaces X, Y is <u>continuous</u> if the preimage of open sets in Y is open in X.

prop. Let $f: X \to Y$ be a function. The followings are equivalent:

(i) f is continuous

(ii) The preimage of closed sets in Y is closed in X.

(iii) $\forall y \in Y$ and V_y (nbhd of y), $f^{-1}(V_y)$ is a nbhd of x, $f(x) = y$.

(iv) $f(\overline{A}) \subseteq \overline{f(A)}$ for any set $A \subseteq X$

(v) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for any $B \subseteq Y$.

proof. (i), (ii), (iii) are equivalent: <u>exercise</u>

Exercise. Let $f: X \to Y$ be a function and $\mathcal{B}$ is a topology basis of Y. Show that f is continuous iff $\forall B \in \mathcal{B}$, $f^{-1}(B) \subseteq X$ is open.

Examples. ① $\mathbb{1}_X: X \xrightarrow{1_X / id} X$, $x \mapsto x$, is continuous.

② $e: X \to \{x_0\}$, is continuous

③ the diagonal map $\Delta: X \to X \times X$, $\Delta(x) = (x, x)$, is continuous.

④ if $A \subseteq X$ is a subspace, then the inclusion map $i: A \to X$, $i(a) = a$, is continuous.

⑤ $X \xrightarrow{i_1} X \times Y$, $Y \xrightarrow{i_2} X \times Y$ are continuous.
$x \mapsto (x, y_0)$

⑥ $X \times Y \xrightarrow{P_1} X$, $X \times Y \xrightarrow{P_2} Y$ are continuous.

prop. Let $f: X \to Y$ and $g: Y \to Z$ be continuous functions.

Then the composition $g \circ f: X \to Z$ is continuous.

proof. $\forall\, U \subseteq Z$ be open, $(g \circ f)^{-1}(U) = f^{-1} \circ g^{-1}(U) \subseteq X$ is open. $\square$

Applications. ① Let $A \overset{i}{\hookrightarrow} X$ be the inclusion map. and $f: X \to Y$ be a map.

Then $f|_A = f \circ i: A \xrightarrow{i} X \xrightarrow{f} Y$ is continuous.

② $$\begin{tikzcd} Z \arrow[r, "f"] \arrow[dr, "f_1 = p_1 f"'] & X \times Y \arrow[d, "p_1"] \\ & X \end{tikzcd}$$

$f(z) = (p_1 f(z), p_2 f(z))$
$= (f_1(z), f_2(z))$

$Z \xrightarrow{f = (f_1, f_2)} X \times Y$ is continuous iff f_1 and f_2 are continuous.

$\Rightarrow$ clear

$\Leftarrow$ $Z \xrightarrow{\Delta} Z \times Z \xrightarrow{f_1 \times f_2} X \times Y$ is f.

$z \longmapsto (z, z) \longmapsto (f_1(z), f_2(z)) = f(z)$

$f = (f_1 \times f_2) \circ \Delta$. $\square$

or $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$

Exercise. Let $f, g: X \to \mathbb{R}$ be continuous map. Show that

(1) $f \pm g: X \to \mathbb{R}$, $(f \pm g)(x) = f(x) \pm g(x)$, is continuous.

(2) $f \cdot g: X \to \mathbb{R}$, $(f \cdot g)(x) = f(x) \cdot g(x)$, is continuous.

($X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} \mathbb{R} \times \mathbb{R} \xrightarrow{\pm} \mathbb{R}$)

Theorem (gluing lemma, pasting lemma).

Let $X = \bigcup_{i=1}^n X_i$, and let $f: X \to Y$ be a function.

If each restriction $f_i := f|_{X_i}: X_i \to Y$ is continuous,

and each X_i is closed (resp. open), then f is continuous. $(f = \bigcup_{i=1}^n f_i)$.

proof $\forall\, V \subseteq Y$ is closed, $f^{-1}(V) = X \cap f^{-1}(V) = (\bigcup_{i=1}^n X_i) \cap f^{-1}(V)$

$= \bigcup_{i=1}^n X_i \cap f^{-1}(V)$

$= \bigcup_{i=1}^n f_i^{-1}(V) \subseteq X$ is closed.

Thus f is continuous.

- If each X_i is open, the statement is still true if $n = \infty$.

Homeomorphism. 同胚.

$f: X \to Y$ is a homeomorphism if

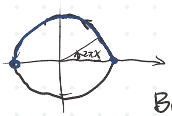
(i) f is continuous

(ii) f is a bijection: the inverse f^{-1} exists.

(iii) The inverse $f^{-1}: Y \to X$ is continuous.

$f: X \xrightarrow{\cong} Y$

Example: ① $f(x) = e^{i2\pi x}: [0,1) \longrightarrow S^1$ is continuous.



$g: S^1 \longrightarrow [0,1)$, $g(z = e^{i\theta}) = \frac{\theta}{2\pi}$. $0 \le \theta < 2\pi$

$fg = \mathbb{1}$, $gf = \mathbb{1}$.

But g is not continuous.

$g^{-1}([0, \frac{1}{2})) =$  $\subseteq S^1$ is not open.

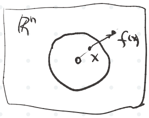
② $f: (0,1) \xrightarrow{\cong} \mathbb{R}$, $f(x) = \frac{x}{1-x}$; $g: \mathbb{R} \longrightarrow (0,1)$, $g(y) = \frac{y}{1+y}$.

f is a homeomorphism

③ $B(0,1) = \{x \in \mathbb{R}^n \mid |x| < 1\} \xrightarrow{\cong} \mathbb{R}^n$

④ $\mathbb{R}^n \setminus \{0\} \xrightarrow[\cong]{f} \mathbb{R}^n \setminus D^n$, $D^n = \overline{B(0,1)}$

$f(x) = x + \frac{x}{\|x\|}$



⑤ Stereographic projection: $S^2 \setminus \{N\} \xrightarrow[\cong]{\phi} \mathbb{R}^2$

$\phi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$

A property P is called a topological property of a space if it is preserved by homeomorphisms of the space.

Hausdorff. A space X is Hausdorff if it satisfies the T_2 axiom:

$\forall\, x \neq y \in X$, $\exists$ nbhds U_x and U_y s.t. $U_x \cap U_y = \phi$.

Example: metric spaces are Hausdorff

prop. Let $f: X \to Y$ be an injective map. If Y is Hausdorff, then so is X.

proof. $\forall\, x_1 \neq x_2 \in X$. $f(x_1) \neq f(x_2)$. Since Y is Hausdorff,

$\exists\, V_{f(x_1)} \cap V_{f(x_2)} = \phi$.

$U_{x_1} = f^{-1}(V_{f(x_1)})$, $U_{x_2} = f^{-1}(V_{f(x_2)})$.

Then $U_{x_1} \cap U_{x_2} = \phi$. $\square$

Cor. Hausdorff property is a topological property.

prop. The followings hold:

(1) points of a Hausdorff space are closed

(2) Subspaces of a Hausdorff space are Hausdorff.

(3) product spaces of two Hausdorff spaces are Hausdorff.

proof. Exercise.

prop. Let $f: X \to Y$ be a map with Y Hausdorff. Then

(1) The graph $G_f = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$ is closed.

(2) Let $g: X \to Y$ be another map. Then $eq(f,g) = \{x \in X \mid f(x) = g(x)\}$ is a closed subset of X. (exercise)

proof. (1) $Z = X \times Y - G_f$. $\forall\, (x, y) \in Z$. $y \neq f(x)$.

Since Y is Hausdorff, there exist V_y and $V_{f(x)}$, s.t. $V_y \cap V_{f(x)} = \phi$.

$U_x = f^{-1}(V_{f(x)})$ is a nbhd of x. Then $U_x \times V_y \subseteq Z = X \times Y - G_f$.

$\therefore Z$ is open $\square$

Connectedness.

A topological space X is <u>connected</u> if one of the following equivalent conditions holds:

(1) X cannot be expressed as a union of two disjoint non-empty open sets.

(2) X - - - - - - - - - - - - - - - - - - - closed sets.

$\boxed{(3)}$ The subsets of X that are both open and closed are ϕ, X.

Exercise. Show the above three statements are equivalent.

prop. Let $f: X \to Y$ be a surjective map with X connected.

Then $Y = f(X)$ is connected. (the image of connected space is connected)

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proof. Let V be an open and closed subset of Y.

$f^{-1}(V)$ is both open and closed as a subset of X.

Since X is connected, $f^{-1}(V) = \phi$, or $f^{-1}(V) = X$.

Thus $V = \phi$ or $V = f(X) = Y$.

That is, Y is connected. $\square$

Cor. Connectedness is a topological property.

- Want: If X and Y are connected, then $X \times Y$ is connected.

$(x, y) \in (X \times \{y\}) \cup (\{x\} \times Y)$

$$X \times Y = \bigcup_{x \in X, y \in Y} (X \times \{y\} \cup \{x\} \times Y).$$

<u>Lemma.</u> If $A \subseteq X$ is both open and closed, and $B \subseteq X$ is connected, then either $A \cap B = \phi$, or $B \subseteq A$.

proof. Assume that $A \cap B \neq \phi$.

$\phi \neq A \cap B \subseteq B$ is both open and closed.

Since B is connected, $A \cap B = B$, $B \subseteq A$. $\square$

Theorem. Let $\{X_\alpha\}$ be a collection of connected subspaces of X.
If $X_\alpha \cap X_\beta \neq \phi$, $\forall \alpha, \beta$, then $Y = \bigcup_{\alpha,\beta} X_\alpha \subseteq X$ is connected.

proof. Let $V \subseteq Y$ is both open and closed. Then $V \cap X_\alpha \subseteq X_\alpha$ is both open and closed. Since X_α is connected, either $V \cap X_\alpha = \phi$ or $V \cap X_\alpha = X_\alpha$.
Suppose that $V \neq \phi$, and $x \in V$. $\exists\, \beta_0$ s.t. $x \in V \cap X_{\beta_0}$, $X_{\beta_0} \subseteq V$.
Since $X_\alpha \cap X_{\beta_0} \neq \phi$, we have $V \cap X_\alpha \neq \phi$, $X_\alpha \subseteq V$ for $\forall \alpha$.
Thus $Y = \bigcup_\alpha X_\alpha \subseteq V$, and therefore $Y = V$, is connected. $\square$

Cor. If X and Y are connected spaces, then $X \times Y$ is connected. $\square$

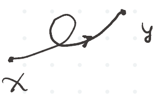
Cor. $\forall x \in X$, $C(x) = \bigcup_\alpha X_\alpha$, X_α are connected spaces that contain x.
Then $C(x)$ is connected, and $\forall y \in C(x)$, $U_y \subseteq C(x)$.
$C(x)$ is the largest connected subset that contains points of $C(x)$,
We call $C(x)$ the <u>connected component</u> of X containg x.

Exercise. Connected components of X are closed subsets. ($\bar{C}$ is connected if C is connected).

Fact. $A \subseteq \mathbb{R}$ is connected iff A is an interval.

Path-connectedness.

A space X is <u>path connected</u> if every two points x, y of X can be joined by a <u>path</u>: $\exists\ \gamma : [0,1] \longrightarrow X$, $\gamma(0) = x$, $\gamma(1) = y$.



$[0,1] \cong [a,b] \xrightarrow{\gamma} X$
$a<b$

Prop. Let $f: X \to Y$ be a surjective map with X path-connected, then Y is path-connected.

proof. $\forall\, y_1 \neq y_2 \in Y$, let $y_1 = f(x_1)$, $y_2 = f(x_2)$, then $x_1 \neq x_2$.

Since X is path-connected, $\exists\, \gamma: [0,1] \to X$, s.t. $\gamma(0) = x_1$, $\gamma(1) = x_2$.

Then $f \circ \gamma: [0,1] \to X \to Y$ is a path connecting y_1 and y_2.

Thus Y is path-connected. $\square$

Cor. Path-connectedness is a topological property.

prop 1. Path-connected spaces are connected.

proof. Let X ~~be~~ path-connected and let $A \subseteq X$ is both open and closed.

Assume $A \neq \emptyset$, and $A \neq X$, then $\exists\, a \in A$ and $x \in X - A$.

Since X is path-connected, $\exists\, \gamma: [0,1] \to X$, s.t. $\gamma(0) = a$, $\gamma(1) = x$.

$\emptyset \neq \gamma^{-1}(A) \subsetneq [0,1]$ is both open and closed, $\gamma^{-1}(A) = [0,1]$, contradiction.

Thus $A = X$, X is connected. $\square$

prop 2. If $X \subseteq \mathbb{R}^n$, then X is connected iff X is path-connected.

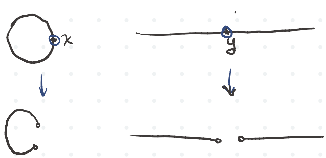
(proof is omitted here.)

Def. $\forall\, x \in X$, $P(x) := \{ y \in X \mid \exists\, \gamma: [0,1] \to X, \text{ s.t. } \gamma(0) = x, \gamma(1) = y \} \subseteq X$.

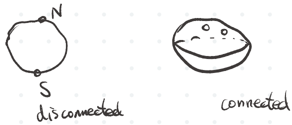
Then $P(x)$ is path-connected, and maximal: If $x \in A \subseteq X$ is path-connected, then $A \subseteq P(x)$.

We call $P(x)$ the path-connected component of X containing x.

Applications. ① $S^1 \not\cong \mathbb{R}^1$:



② S^1 and S^n ($n \geq 2$) are not homeo.



$\mathbb{R}^1 \not\cong \mathbb{R}^n$ ($n \geq 2$)

$S^1 \not\cong 8$.