1. Pullback bundles

<u>Def</u>. Given a (complex) vector bundle $\pi: E \to B$ and a map $f: X \to B$, there is an induced vector bundle $f^*(E)$ over X, called <u>the pullback of E by f</u>, given by the pullback diagram.

$$\begin{tikzcd} f^*(E) \arrow[r] \arrow[d] \arrow[dr, phantom, "p.b."] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

(Note. $f^*(E)$ is unique up to bundle isomorphism).

Prop (Elementary properties of pullback bundles)

(1) $f^* g^*(E) \cong (gf)^*(E)$.
(2) $1^*(E) \cong E$.

$\Rightarrow$ pullback of bundles is a contravariant functor.

(3) $f^*(E_1 \oplus E_2) \cong f^*(E_1) \oplus f^*(E_2)$
(4) $f^*(E_1 \otimes E_2) \cong f^*(E_1) \otimes f^*(E_2)$

Denote by $\mathrm{Vect}^{\mathbb{C}}(X)$ the set of isomorphism classes of v.bs over X.

Then $\mathrm{Vect}(X)$ is a semi-ring:

$(\oplus, \otimes)$, ① additive identity = zero vector bundle : $E \oplus 0 = E$. $\mathbb{C}^n \oplus \mathbb{C}^0 = \mathbb{C}^n$.
② multiplicative identity = trivial line bundle : $E \otimes \mathbb{C} = E$
$\mathbb{C}^n \otimes \mathbb{C} = \mathbb{C}^n$.

Lem. If B is compact, then for any vector bundle $E \to B$, there exists a bundle $E' \to B$ such that $E \oplus E' \cong \varepsilon^n = B \times \mathbb{C}^n$ for some n.

Theorem. Let $p: E \to B$ be a vector bundle. Suppose that X is compact Hausdorff and that $f_0 \simeq f_1 : X \to B$, then there is an isomorphism

$$f_0^*(E) \cong f_1^*(E).$$

proof. Let $H: X \times I \to B$ be a homotopy between f_0 and f_1; that is $H_0 = f_0$, $H_1 = f_1$.

Then it suffices to show that $H_0^*(E) = H^*(E)|_{X\times\{0\}} \cong H^*(E)|_{X\times\{1\}} = H_1^*(E)$

Since X is compact, then there is a finite cover $\{U_1, \cdots, U_n\}$ of X; moreover, we can asume that $H^*(E)$ is trivial over each $U_i \times I$

By the associated partition of unity, there are functions $f_i : X \to I$ st. $\mathrm{Supp}(f_i) \subseteq U_i$, $\Sigma_{i=1}^n f_i = 1$.

Define $\psi_i = \varphi_1 + \cdots + \varphi_i$, then $\psi_0 = 0$, $\psi_n = 1$. Set $X_i = \{(x, \psi_i(x)) \in X \times I \mid x \in X\}$ $= \mathrm{graph}(\psi_i)$

Then the projection $q_i : X_i \to X_i$, $q_i(x, \psi_i(x)) = (x, \psi_{i-1}(x))$ is a homeomorphism.

$$\begin{tikzcd} E|_{X_i} =: E_i \arrow[r, dashed, "h_i"] \arrow[d, "p_i"] & E_{i-1} \arrow[d, "p_{i-1}"] \\ U_i \times \psi_i(X) \subseteq X_i \arrow[r, "q_i", "\cong"'] & X_{i-1} \end{tikzcd}$$

Since $H^*(E)$ is trivial over $U_i \times I$, we may put $p^{-1}(U_i \times I) = U_i \times I \times \mathbb{C}^n$,

then $h_i(x, \psi_i(x), v) = (x, \psi_{i-1}(x), v)$, $\forall (x, \psi_i(x)) \in p^{-1}(U_i \times I)$

$h_i(x, \underset{0}{\underset{\parallel}{\psi_i(x)}}, v) = (x, 0, v) = \mathrm{id}(x, \psi_i(x), v)$ $\forall (x, \psi_i(x)) \in X_i \setminus U_i \times \psi_i(X)$

Clearly h_i is a homeomorphism

$\Rightarrow$ $h = h_1 h_2 \cdots h_n : E_n \to E_0$ is then an isomorphism:
$$\begin{tikzcd} H_1^*(E) \arrow[r, "h", "\cong"'] \arrow[d] & H_0^*(E) \arrow[d] \\ X \times \{1\} \arrow[r, "q", "\cong"'] & X \times \{0\} \end{tikzcd}$$

$\square$

Cor. If $f: X \to B$ is a homotopy equivalence between compact spaces,

~~then~~ ① $f^*: \mathrm{Vect}_n(B) \longrightarrow \mathrm{Vect}_n(X)$ is bijection.

In particular, $\mathrm{Vect}_n(B) = \{B \times \mathbb{C}^n\}$ if B is contractible.

② $f^*: \mathrm{Vect}(B) \longrightarrow \mathrm{Vect}(X)$ is an isomorphism of semirings

$\mathrm{HoTop}_{\mathrm{comp.}} \xrightarrow{\mathrm{Vect}^{\mathbb{C}}(-)} \underline{\text{Semi-Rings}}$ is a contravariant functor.

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & \mathrm{Vect}(X) \\
Y \arrow[r, mapsto] & \mathrm{Vect}(Y) \arrow[u, "f^*"']
\end{tikzcd}$$

▲ Theorem. Let X be compact Hausdorff based ~~topological~~ space.

(Fact) ① For any $E \in \mathrm{Vect}(X)$, $\exists!$ $F \in \mathrm{Vect}(X)$ s.t. $E \oplus F = \varepsilon^n \ (= X \times \mathbb{C}^n)$

② $[X, BU(n) = G_n(\mathbb{C}^\infty)] \xrightarrow{1:1} \mathrm{Vect}_n(X)$

$f \longmapsto f^*(EU(n))$

$$\begin{tikzcd}
f^*(EU(n)) \arrow[r] \arrow[d] & EU(n) \arrow[d, "\pi"] \\
X \arrow[r, "f"] & BU(n)
\end{tikzcd}$$

$EU(n) \xrightarrow{\pi} BU(n)$ is the universal classifying fibration.

2. Introduction to complex topological K-theory

- X is ~~any~~ a based topological space.
- $\mathrm{Vect}(X) :=$ the set of isomorphism classes of vector bundles over X

$$\mathrm{Vect}_n(X) := \{\xi \in \mathrm{Vect}(X) \mid \mathrm{rank}_{\mathbb{C}}\xi = n\}$$

$$\mathrm{Vect}_n(X) \subseteq \mathrm{Vect}_{n+1}(X) \rightsquigarrow \mathrm{Vect}(X) = \operatorname{colim}_{n\to\infty} \mathrm{Vect}_n(X).$$
$$\xi \mapsto \xi \oplus \varepsilon.$$

- $(\mathrm{Vect}(X), \oplus)$ is a commutative semigroup $(\oplus)$ / semiring $(\oplus, \otimes)$
- Grothendieck group $K_0(A)$ of a semigroup $(A, \oplus)$
  - $F(A)$: the free abelian group generated by elements of A
  - $E(A)$: the subgroup of $F(A)$ generated by elements of the form: $a + a' - (a \oplus a')$, $\forall a, a' \in A$.

  $\Rightarrow K_0(A) = F(A)/E(A)$, in $K_0(A)$, $a + a' = [a \oplus a']$.

  - Universal property. $\forall\ \alpha : A \to G$ a semigroup homomorphism. $\exists!\ \bar{\alpha} : K_0(A) \to G$ s.t. $\alpha = \bar{\alpha} \circ \iota : A \overset{\iota}{\hookrightarrow} K_0(A) \overset{\bar{\alpha}}{\to} G$.

Examples
- $(A, \oplus) \in$ projective R-mod $\longrightarrow$ algebraic K_0-groups
- $(A, \oplus) \in \mathrm{Vect}(\mathrm{HoTop}_{comp}) \rightsquigarrow$ (complex) topological K-groups.

- Second definition of $K_0(A)$ $(A, \oplus)$ semigroup.

$\Delta: A \to A\times A$ the diagonal map, homomorphism, $a \mapsto (a,a)$

$$K_0(A) := A\times A/\Delta(A), \quad id = \Delta(A)$$

$$\overline{(a,b)}^{-1} = \overline{(b,a)} \Rightarrow K_0(A) \text{ is a group (abelian)}$$

We have: $\iota_A: A \xrightarrow{i_1} A\times A \twoheadrightarrow K_0(A)$. By the universal property of quotient groups,
$a \mapsto (a,0)$

Lemma: For any semihomomorphism $\alpha: A \to B$, there is a comm. diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\iota_A"] & B \arrow[d, "\iota_B"] \\ K_0(A) \arrow[r, "K_0(\alpha)"] & K_0(B) \end{tikzcd}$$

$(K_0(A), \iota_A)$ is a functor
SemiGrps $\to$ Ab.

In particular, if A is a group, then ι_A is an isomorphism.

Thus $K_0(A) = A\times A/\Delta(A)$ satisfies the required universal property.

For any based topological space, $K(X) = K_0(\mathrm{Vect}(X))$, ring.

notation: For any $E \in \mathrm{Vect}(X)$, write $[E] = \iota_{\mathrm{Vect}(X)}(E) \in K(X)$.

- By definition, every element of $K(X)$ is of the form $[E]-[F]$.
- Let $G \in \mathrm{Vect}(X)$ s.t. $F\oplus G = \varepsilon^n$, then

$$[E]-[F] = [E]+[G]-([F]+[G]) = [E\oplus G]-[\varepsilon^n] = [H]-[\varepsilon^n]$$

That is, every element of $K(X)$ is of the form $[H]-[\varepsilon^n]$
$\Downarrow$
$K(X) \cong \mathrm{Vect}(X)\oplus \mathbb{Z}$

eg. $K(pt) \cong \mathbb{Z}$.

- $[E]=[F] \Leftrightarrow E\oplus G = F\oplus G$ for some $G\in \mathrm{Vect}(X)$
$\Leftrightarrow E\oplus \varepsilon^n = F \oplus \varepsilon^n$ for some n.
(i.e. E and F are <u>stably equivalent</u>).

- Any map $f: X \to Y \overset{\text{pullback}}{\rightsquigarrow} f^*: \mathrm{Vect}_n(Y) \to \mathrm{Vect}_n(X)$, $\forall\, n \geq 1$.

$\overset{\text{ring homo}}{\rightsquigarrow} f^*: K(Y) \to K(X)$.

moreover, $f \simeq g \Rightarrow f^* = g^*$.

- Homotopical definition of $K(X)$.

$U = \operatorname{colim}_n U(n)$
$BU(n) = Gr_n(\mathbb{C}^\infty)$
$BU = \operatorname{colim}_n BU(n)$.

<u>X compact</u>: $\tilde{K}(X) = [X, BU]$

$= [X, \operatorname{colim}_n BU(n)]$

$= \operatorname{colim}_n [X, BU(n)]$

$= \operatorname{colim}_n \mathrm{Vect}_n(X) = \mathrm{Vect}(X)$.

$\Rightarrow$ Lem. $K(X) \cong [X, \mathbb{Z} \times BU] \cong \tilde{K}(X) \oplus \mathbb{Z} \cong \tilde{K}(X) \oplus K(pt) \cong \tilde{K}(X_+)$.

$0 \to \tilde{K}(X) \to K(X) \xrightarrow{\epsilon = i^*} \underset{K(pt)}{\overset{\parallel}{\mathbb{Z}}} \to 0$ $\quad i: pt \hookrightarrow X$. splitting exact seq.

Lem. $\tilde{K}(\Sigma X) = [\Sigma X, BU] = [X, \Omega BU] = [X, U]$.

$\tilde{K}^{-n}(X) := \tilde{K}(\Sigma^n X) = [\Sigma^n X, BU] = [X, \Omega^n BU] = [X, \Omega^{n-1} U]$, $\forall n \geq 1$.

- Relative complex K-group. $K(X, Y) \cong \tilde{K}(X/Y)$. $K(X, \phi) = K(X)$.

compact pairs.

<u>Def</u>. $\tilde{K}^{-n}(X) := \tilde{K}(\Sigma^n X)$

$K^{-n}(X, Y) := \tilde{K}^{-n}(X/Y) = \tilde{K}(\Sigma^n(X/Y))$.

$K^{-n}(X) := K^{-n}(X, \phi) = \tilde{K}(\Sigma^n(X_+))$.

- $K^{\bullet}$ is a contravariant functor.
- Fact (Bott Periodicity theorem): $U \simeq \Omega^2 U \Rightarrow \tilde{K}^{-2}(X) \cong K(X)$.

Lem. For any compact pair (X,Y), there is an exact seq.

$$K(X,Y) \xrightarrow{j^*} K(X) \xrightarrow{i^*} K(Y),$$

where $i: Y \hookrightarrow X$ and $j: (X,\cancel{\phi}) \to (X,Y)$ are the inclusions.

proof. $j \circ i: Y \hookrightarrow X \hookrightarrow (X,Y) \rightsquigarrow i^* \circ j^* = (j \circ i)^* = 0$, that is, $\mathrm{Im} j^* \subset \ker i^*$.

$\ker i^* \subseteq \mathrm{Im} j^*$: Let $\xi \in \ker i^*$, $\xi = [E] - [\varepsilon^n]$ for some n.

$$0 = i^* \xi \Rightarrow [E|_Y] = [\varepsilon^n_{|Y}] \in K(Y).$$

$$\Leftrightarrow (E \oplus \varepsilon^m)|_Y = \varepsilon^n \oplus \varepsilon^m \quad \text{for some } n$$

$\Rightarrow E \oplus \varepsilon^m$ is a vector bundle over X/Y.

$$\rightsquigarrow \eta = [E \oplus \varepsilon^m] - [\varepsilon^n \oplus \varepsilon^m] \in \tilde{K}(X/Y) = K(X,Y)$$

$$j^*(\eta) = [E \oplus \varepsilon^m] - [\varepsilon^n \oplus \varepsilon^m] = [E] - [\varepsilon^n] = \xi.$$

$\square$

Cor. $K(X,Y) \xrightarrow{j^*} \tilde{K}(X) \xrightarrow{i^*} \tilde{K}(Y)$ is exact, since

$$K(X) = \tilde{K}(X) \oplus K(x_0).$$

prop. There is a natural exact seq:

$$\cdots \longrightarrow K^{-2}(Y) \xrightarrow{\delta} K^{-1}(X,Y) \xrightarrow{j^*} K^{-1}(X) \xrightarrow{i^*} K^{-1}(Y) \xrightarrow{\delta} K^0(X,Y)$$
$$\xrightarrow{j^*} K^0(X) \xrightarrow{i^*} K^0(Y).$$

See Prop. 2.2.5 of Atiyah's book.

Cor. if Y is a retract of X, then $K^{-n}(X) \cong K^{-n}(Y) \oplus K^{-n}(X,Y)$, $\forall n \geq 0$.

Theorem: For any X and any $n \leq 0$, there is an isomorphism

$$K^{-n}(X) \cong K^{-2}(pt) \otimes K^{-n}(X) \xrightarrow{\cong} K^{-n-2}(X).$$

Exercise 1. Show ~~that~~

① $\tilde{K}^{-n}(X \times Y) \cong \tilde{K}^{-n}(X \wedge Y) \oplus \tilde{K}^{-n}(X) \oplus \tilde{K}^{-n}(Y)$, $\forall n \geq 0$

② $\tilde{K}(S^2) \cong \mathbb{Z}$ $(\cong K^{-2}(pt))$, $\tilde{K}(S^1 = \phi) = 0$.

$$\Rightarrow \tilde{K}(S^m) = \begin{cases} \mathbb{Z} & m \text{ even} \\ 0 & m \text{ odd.} \end{cases}$$

2. Compute $\tilde{K}(\mathbb{C}P^2)$.

3. Let $P^{n+1}(k) = S^n \cup_k e^{n+1}$ be the mod k Moore space. $k \geq 2$, compute $\tilde{K}(P^{n+1}(k))$.