LECTURES ON ATIYAH-SINGER INDEX THEOREM

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0. An overview

Let X be a smooth manifold, E and F be smooth complex vector bundles on X. A *differential operator of order* m is a linear map $P : \Gamma(E) \to \Gamma(F)$, which in local trivializations of E and F can be expressed as a differential operator of order m on Euclidean spaces. One can associate with D a section $\sigma(P) \in (\mathrm{Sym}^m TX) \otimes \mathrm{Hom}(E, F)$, called the *principal symbol* of P. For any $x \in X$ and $\xi \in T_x^* X$, one can evaluate $\sigma(P)$ at ξ to obtain $\sigma_\xi(P) \in \mathrm{Hom}(E_x, F_x)$. We say that P is an *elliptic operator* if $\sigma_\xi(P)$ is invertible for all $x \in X$ and all nonzero ξ.

Theorem 0.1. *Let X be a closed manifold, and $P : \Gamma(E) \to \Gamma(F)$ an elliptic operator. Then P is Fredholm.*

Definition 0.2. We define the *analytic index* of P to be $\mathrm{ind}(P) = \dim \ker(P) - \dim \mathrm{coker}(P)$.

Let still X be a closed manifold, and $P : \Gamma(E) \to \Gamma(F)$ an elliptic operator. Denote by π the projection $TX \to X$. The principal symbol $\sigma(P)$ defines a class

$$[\pi^* E, \pi^* F, \sigma(P)] \in K_{\mathrm{cpt}}(TX),$$

where $K_{\mathrm{cpt}}(TX)$ is the topological K-theory of TX with compact supports. Choose an arbitrary imbedding $f : X \hookrightarrow \mathbb{R}^N$. There are pushforward maps

$$f_! : K_{\mathrm{cpt}}(TX) \to K_{\mathrm{cpt}}(T\mathbb{R}^N), \text{ and } q_! : K_{\mathrm{cpt}}(T\mathbb{R}^N) \to K(\mathrm{pt}) = \mathbb{Z}.$$

Definition 0.3. We define the *topological index* of P to be $\mathrm{ind}_{\mathrm{top}}(P) = q_! f_! [\pi^* E, \pi^* F, \sigma(P)] \in \mathbb{Z}$.

Theorem 0.4 (Atiyah-Singer [AS-I])**.** *Let X be a closed manifold, E and F be complex vector bundles on X. Let D be an elliptic operator from E to F. Then*

$$\mathrm{ind}(D) = \mathrm{ind}_{\mathrm{top}}(D).$$

In this course, we will review basic notions in differential geometry and complex geometry, elliptic operators, and K-theory of complex vector bundles. We will sketch Atiyah-Singer's proof of the index theorem, for which we need to introduce the notion of pseudodifferential operator, a notion more flexible than the differential operators. Two typical applications of the index theorem will be given: the Riemann-Roch-Hirzebruch theorem, and the Weyl character formula.

1. Differentiable manifolds

1.1. **Topological spaces.**

Definition 1.1. A topological space is a pair $(X, \mathcal{U})$, where X is a set, $\mathcal{U} = \{U_i\}_{i \in I}$ is a collection[1] of subsets of X, satisfying the following:

- (i) $\emptyset$ and X belongs to $\mathcal{U}$.
- (ii) if $J \subset I$, then $\bigcup_{j \in J} U_j \in \mathcal{U}$.
- (iii) if $J \subset I$ and J is finite, then $\bigcap_{j \in J} U_j \in \mathcal{U}$.

We also say that $\mathcal{U}$ is a *topology on* X, and an element of $\mathcal{U}$ is called an *open subset* of X.

Remark 1.2. In the categorical viewpoint, the requirement (i) is redundant. In fact, if $J = \emptyset$, then $\bigcup_{j \in J} U_j$ is the *initial object* of the category of subsets of X, thus is the empty subset $\emptyset$ of X, and $\bigcap_{j \in J} U_j$ is the *final object* of the category of subsets of X, thus is X itself.

[1] In set theory, "collection" and "family" both mean "set" (of someting), while "class" (of something) means a collection (of something) in daily English which is not necessarily a set in the mathematical sense.

Definition 1.3. Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be two topological spaces, and $f : X \to Y$ be a map of sets. Then we say that f is *continuous* if for any $V \in \mathcal{V}$, $f^{-1}(V) \in \mathcal{U}$.

Definition 1.4. Let $(X,\mathcal{U})$ be a topological space. A *topological basis* of $(X,\mathcal{U})$ is a subset $\mathcal{V}$ of $\mathcal{U}$ such that any element of $\mathcal{U}$ is of the form

$$U = \bigcup_{j\in J} V_j$$

where $V_j \in \mathcal{V}$.

Definition/Proposition 1.5. *Let X be a set, and $\mathcal{V} = \{V_i\}_{i\in I}$ be a collection of subsets of X. Then there is a* smallest *topology $\mathcal{U}$ containing $\mathcal{V}$, which means that any topology $\mathcal{U}'$ on X containing $\mathcal{V}$ contains $\mathcal{U}$. Moreover, a subset of X belongs to $\mathcal{U}$ if and only if it is a union of subsets of the form*

$$U = \bigcap_{k\in K} V_k$$

where K is a finite subset of I. We say that $\mathcal{U}$ is the topology generated by $\mathcal{V}$*, and $\mathcal{V}$ is a* topological subbasis *of $\mathcal{U}$.*

Here are some typical examples of topologies.

Example 1.6 (Order topology)**.** Let $(X, \preceq)$ be a totally ordered set, i.e. $\preceq$ is a binary relation on X satisfying

(i) for any $x, y \in X$, at least one case of $x \preceq y$ or $y \preceq x$ happens, and they both happen if and only if $x = y$.
(ii) $(x \preceq y$ and $y \preceq z) \iff x \preceq z$.

Then we say $x \prec y$ if $x \preceq y$ and $x \neq y$, and define the *open intervals* in X to be the subsets of the form $(x, y) := \{z \in X : x \prec z \prec y\}$. The *order topology* on X is the topology generated by the open intervals.

Example 1.7 (Norm topology)**.** Let V be a real vector space, not necessarily finite dimensional. A map $\| \cdot \|: V \to \mathbb{R}_{\geq 0}$ is called a *norm on V* if

(i) $\| x \| = 0$ if and only if $x = 0 \in V$;
(ii) $\| x + y \| \leq \| x \| + \| y \|$ for any $x, y \in V$;
(iii) $\| \lambda x \| = |\lambda| \cdot \| x \|$ for any $x \in V$ and $\lambda \in \mathbb{R}$.

Given a norm as such, we define the *open balls* in V to be the subsets of the form

$$B(x, r) := \{y \in V : \| y - x \| < r\}$$

where $x \in V$ and $r \in \mathbb{R}_{>0}$. Then the *norm topology on V* (induced by $\| \cdot \|$) is the topology generated by the open balls.

Exercise 1. The set of real numbers $\mathbb{R}$ is totally ordered by the classical order $\leq$, and is equipped with the norm $| \cdot |$, i.e. the absolute value. Show that the order topology of $(\mathbb{R}, <)$ and the norm topology of $(\mathbb{R}, | \cdot |)$ coincide.

It will turn out that the topologies in this course are induced by the above two types of topologies and the following constructions of *induced topologies*.

Construction 1.8.

(i) Let $(X,\mathcal{U})$ be a topological space, and Y be a subset of X. The induced *subspace topology on Y* is the smallest topology on Y such that the inclusion map $Y \hookrightarrow X$ is continuous. Then $V \subset Y$ is open in the subspace topology if and only if $V = Y \cap U$ for some $U \in \mathcal{U}$.

(ii) Let $(X,\mathcal{U})$ be a topological space, and $\sim$ is an equivalence relation on X, which means that it is a binary relation satisfying

(1) $x \sim x$ for any $x \in X$;
(2) $x \sim y$ if and only if $y \sim x$;
(3) if $x \sim y$ and $y \sim z$ then $x \sim z$.

Then we can make the quotient space $X/\sim$, which means that every element of $X/\sim$ is reperesented by some $x \in X$, denoted by $[x]$, and $[x] = [y]$ if and only if $x \sim y$. There is a projection map

$$\pi : X \to X/\sim$$
$$x \mapsto [x].$$

Then the induced *quotient topology on* $X/\sim$ is the *largest topology* on $X/\sim$ such that π is continuous. Then a subset of $X/\sim$ is open in the quotient topology if and only if $\pi^{-1}(U) \in \mathcal{U}$.

(iii) Let $(X_i, \mathcal{U}_i)$, $i \in I$, be a family of topological spaces. The *product topology* on $\prod_{i \in I} X_i$ is the smallest topology on it such that the projection maps

$$\pi_j : \prod_{i \in I} X_i \to X_j$$
$$(x_i)_{i \in I} \mapsto x_j$$

are continuous, for all $j \in I$. Then if the index set I is finite, we have a topological basis of the product topology, consisting of the subsets of the form

$$\prod_{i \in I} U_i$$

where $U_i \in \mathcal{U}_i$ for $i \in I$. If I is not necessarily finite, we have a topological basis of the product topology, consisting of the subsets of the form

$$\prod_{i \in I} U_i$$

where $U_i \in \mathcal{U}_i$ for $i \in I$, and for all but finitely many $i \in I$, $U_i = X_i$.

(iv) Let $(X_i, \mathcal{U}_i)$, $i \in I$, be a family of topological spaces. The *union topology* on the disjoint union $\coprod_{i \in I} X_i$ is the largest topology on it such that the inclusion maps

$$\iota_j : X_j \hookrightarrow \coprod_{i \in I} X_i$$

are continuous, for all $j \in I$. Then a subset U of $\coprod_{i \in I} X_i$ is open in the union topology if and only if it is of the form

$$U = \coprod_{i \in I} U_i$$

where U_i is open in X_i, for all $i \in I$.

In the following, we shorten the terminology by saying a *topological space* X instead of $(X, \mathcal{U})$.

Construction 1.9.

(v) Let X, Y, Z be topological spaces, and $f : X \to Z$ and $g : Y \to Z$ be continuous maps. The *fiber product* $X \times_Z Y$ is defined, as a set, to be

$$\{(x, y) \in X \times Y | x \in X,\ y \in Y,\ f(x) = g(y)\}.$$

We have the following commutative diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

where p and q are the projection maps, i.e.

$$p(x, y) = x,\ q(x, y) = y.$$

Then the *pullback topology on* $X \times_Z Y$ is the *smallest* topology such that p and q are continuous. It is essentially not a new construction. In fact, we have the inclusion

$$X \times_Z Y \hookrightarrow X \times Y$$
$$(x, y) \mapsto (x, y),$$

and the pullback topology on $X \times_Z Y$ is no other than the subspace topology induced by the product topology on $X \times Y$.

Exercise 2. Let X, Y, Z be topological spaces, and $f : X \to Y$ and $g : X \to Z$ be continuous maps. Recall the notion of *pushout* in a category, and describe what is the pushout of

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Z \\ Y & \end{tikzcd}$$

in the category of topological spaces and continous maps.

Definition 1.10. The *Euclidean topology* on $\mathbb{R}^n$ is the product topology of the order topology on $\mathbb{R}$.

Exercise 3. Show that the Euclidean topology on $\mathbb{R}^n$ coincides with the norm topology on $\mathbb{R}^n$ defined by any of the following norms

(i) $\| x \| = \sqrt{x_1^2 + \cdots + x_n^2}$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, or
(ii) $|x| = |x_1| + \cdots + |x_n|$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Definition 1.11. Let X be a topological space. An open covering of X is a family $\mathcal{V}$ of open subsets of X, such that

$$X = \bigcup_{V \in \mathcal{V}} V.$$

We say that X is *compact*, if any open covering of X has a finite sub-family which is still an open covering of X.

Definition 1.12. A topological space X is *path connected* if for any $x, y \in X$ there exists a continous map $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$, where $[0,1] \subset \mathbb{R}$ is equipped with the subspace topology induced by the Euclidean topology on $\mathbb{R}$.

1.2. **Topological manifolds.**

Definition 1.13. A a topological space X is called a (resp. n-dimensional) *topological manifold* if the following are satisfied:

(1) X is *Hausdorff*, i.e. for $x, y \in X$ and $x \neq y$, there exists open subsets $U_x \ni x$ and $U_y \ni y$ such that $U_x \cap U_y = \emptyset$.
(2) X is *second-countable*, i.e. X bas a countable topological basis, i.e. there exists a countable collection $\mathcal{U} = \{U_\alpha\}$ of open subsets of X, such that any open subset U of X is the union of a sub collection of subsets in $\mathcal{U}$.
(3) X is *locally Euclidean (resp. of dimension n)*, i.e. for any $x \in X$, there exists an open subset $U \ni x$ such that U is homeomorphic to an open subset of $\mathbb{R}^n$ for some n (resp. for the given n).

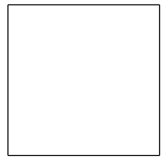
The third condition is the essence of the definition, while the first two are required to avoid certain pathological cases.

Example 1.14. (1) $\mathbb{R}^1$ endowed with the order topology is a 1-dim topological manifold. $\mathbb{R}^n$ equipped with the product topology from the order topology of $\mathbb{R}^1$ is an n-dim topological manifold. We call this topology the *Euclidean topology* on $\mathbb{R}^n$. We tacitly use the Euclidean topology on $\mathbb{R}^n$, and for subspaces $X \subset \mathbb{R}^n$ we tacitly mean that X is equipped with the subspace topology induced from the Euclidean topology of $\mathbb{R}^n$.
(2) For $r > 0$, the ball $\mathbb{B}^n(r) := \{x \in \mathbb{R}^n : |x| < r\} \subset \mathbb{R}^n$ is an n-dim topological manifold.
(3) For $r > 0$, the sphere $\mathbb{S}^n(r) := \{x \in \mathbb{R}^n : |x| = r\} \subset \mathbb{R}^{n+1}$ is an n-dim topological manifold.
(4) The real projective space

$$\mathbb{RP}^n = \frac{\{(x_0, \ldots, x_n) \in (\mathbb{R}^\times)^{n+1}\}}{(x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n) \text{ for } \lambda \in \mathbb{R}^\times}$$

equipped with the quotient topology is a topological manifold.

(5) The "hollow square" in $\mathbb{R}^2$



is a 1-dim topological manifold.

(6)

$$L = \mathbb{R}^1 \underset{\mathbb{R}^1 \setminus \{0\}}{\cup} \mathbb{R}^1$$

(endowed with the pushout topology) is not Hausdorff.

(7) Let ω_1 be the smallest uncountable ordinal. The "long ray" is the union $\omega_1 \times [0,1)$ endowed with the order topology from the lexicographical order. It is Hausdorff, but not second-countable, and is locally Euclidean away from the origin $0 = (\emptyset, 0)$. Then the *long line*

$$\widetilde{L} := \omega_1 \times [0,1) \underset{\{0\}}{\cup} \omega_1 \times [0,1)$$

is locally Euclidean and Hausdorff, but not second-countable.

Remark 1.15. The disjoint union of an m-dim topological manifold and an n-dim topological manifold is a topological manifold, which does not have a global dimension unless $m = n$. A connected topological manifold always has a global dimension. This relies on a non-trivial fact (invariance of topological dimensions): $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$ unless $m = n$.

Exercise 4. $\mathbb{R}^1$ is not homeomorphic to $\mathbb{R}^n$ unless $n = 1$.

Hint. Consider the (path) connectedness of the complements of the points. □

Exercise 5. Show that the union of axes $\{x = 0\} \cup \{y = 0\} \subset \mathbb{R}^2$ is not a topological manifold.

Definition 1.16. A *map* $X \to Y$ between topological manifolds is understood to be just a continuous map between topological spaces $X \to Y$. In other words, the category of topological manifolds is a *full subcategory* of the category of topological spaces.

1.3. Smooth manifolds and smooth maps.

Definition 1.17. Let X be a topological manifold of dimension n.

(1) A *chart of* X is a triple (U, V, ϕ) (resp. around $x \in X$) where U is an open subset of X (resp. U is an open neighborhood of x in X), V an open subset of $\mathbb{R}^n$, and $\phi : V \to U$ a homeomorphism.
(2) An *atlas* of X is a collection $\{(U_i, V_i, \phi_i\}_{i \in I}$ of charts such that $\bigcup_{i \in I} U_i = X$.
(3) An atlas $\{(U_i, V_i, \phi_i\}_{i \in I}$ is *smooth* if

$$\phi_j^{-1} \circ \phi_i\big|_{\phi_i^{-1}(U_i \cap U_j)} : \phi_i^{-1}(U_i \cap U_j) \to \phi_j^{-1}(U_i \cap U_j)$$

is a smooth map for any $i, j \in I$. The maps $\phi_j^{-1} \circ \phi_i\big|_{\phi_i^{-1}(U_i \cap U_j)}$ are called *transition functions*.
(4) Two smooth atlas $\{(U_i, V_i, \phi_i\}_{i \in I}$ and $\{(U_i', V_i', \phi_i'\}_{i \in I'}$ of X are *compatible* if their union is still a smooth atlas.
(5) A *smooth manifold (without boundary)* is a pair $(X, \mathcal{A})$ where X is a topological manifold, and $\mathcal{A}$ is an equivalence class of smooth atlas on X; such $\mathcal{A}$ is called a *smooth structure* on X. When $\mathcal{A}$ is tacitly given, we briefly say that X is a smooth manifold, or even a *manifold*.
(6) On a smooth manifold there is a canonical smooth atlas, the *maximal* one, i.e. the union of all the smooth atlas in the given equivalence class. When $\mathcal{A}$ is tacitly given, we briefly say that X is a smooth manifold, or even a *manifold*, equipped with the corresponding maximal atlas.
(7) A *closed manifold* is a compact smooth manifold without boundary.

Definition 1.18. A *smooth map* $f : X \to Y$ between two smooth manifolds $(X, \mathcal{A})$ and $(Y, \mathcal{A}')$ is a continuous map such that for any $(U_i, V_i, \phi_i) \in \mathcal{A}$ and $(U_j', V_j', \phi_j') \in \mathcal{A}'$, the map

$$\phi_j'^{-1} \circ \phi_i : \phi_i^{-1}(U_i \cap f^{-1}U_j') \to \phi_j'^{-1}\big(f(U_i) \cap U_j'\big)$$

is smooth. If a smooth map has a smooth inverse map, then it is called a *diffeomorphism*.

Remark 1.19.
(1) Examples of topological manifolds in Example 1.14 all admits smooth structures.
(2) If a topological manifold X has a smooth structure, it trivially has many non-equivalent smooth structures in the literal sense. However, it happens often that two non-equivalent smooth structures $(X, \mathcal{A})$ and $(X, \mathcal{A}')$ are diffeomorphic. Hence, when we talk about "different smooth structures on X" we usually mean the smooth structures *modulo diffeomorphisms*. For example, Milnor showed that the 7-dimensional sphere admits a smooth structure ("exotic sphere") that is not equivalent to the standard smooth structure in Example 1.14(3).
(3) There do exist topological manifolds which do NOT admit any smooth structures.

Definition 1.20. In Definitions 1.17 and 1.18, replacing $\mathbb{R}$ with $\mathbb{C}$, and smooth maps with holomorphic maps, we obtain the notion of *complex manifolds* and *holomorphic maps* between complex manifolds.

1.4. **Submanifolds.**

Definition 1.21. Let X be a smooth manifold of dimension n. An embedded submanifold (of codimension c) is a subset Y of X such that for any $y \in Y$ there exists a chart of X around y, (U, V, ϕ), where V is an open subset of $\mathbb{R}^n$ with coordinates $x_1, \dots, x_n$, such that

$$\phi^{-1}(Y \cap U) = \{(x_1, \dots, x_n) \in V | x_1 = \cdots = x_c = 0\}.$$

Then Y has an induced smooth structure, hence is a smooth manifold.

Exercise 6.
(1) The ball $\mathbb{B}^n(r)$ and the sphere $\mathbb{S}^{n-1}(r)$ are embedded submanifolds of $\mathbb{R}^n$.
(2) The "hollow square" in Example 1.14(5) has a smooth structure, but is not an embedded submanifold of $\mathbb{R}^2$.

1.5. **Tangent spaces.**

Let X and Y be two (smooth) manifolds, and $x \in X$. Define

$$\mathcal{G}\big((X,x),Y\big) = \frac{(U,f)|\text{open } U \subset X,\ x \in U,\ f : U \to Y \text{ smooth}}{(U \xrightarrow{f} Y) \sim (V \xrightarrow{g} Y) \text{ if } f|_W = g|_W \text{ for some open } \ W \subset U \cap V \text{ s.t. } x \in W}.$$

An element of $\mathcal{G}_p(X, Y)$ is called a *germ of smooth map* from X to Y at x. Similarly we define, for $x \in X$ and $y \in Y$,

$$\mathcal{G}\big((X,x),(Y,y)\big) = \frac{(U,f)|\text{open } U \subset X,\ x \in U,\ f : U \to Y \text{ smooth},\ f(x) = y}{(U \xrightarrow{f} Y) \sim (V \xrightarrow{g} Y) \text{ if } f|_W = g|_W \text{ for some open } \ W \subset U \cap V \text{ s.t. } x \in W}.$$

Now fix a point $x \in X$. Let $W_x = \mathcal{G}\big((\mathbb{R},0),(X,x)\big) = \{\text{"germs of curves" in } X \text{ starting from } x\}$. For $[\gamma_1], [\gamma_2] \in W_x$, we say that $[\gamma_1]$ is *tangent to* $[\gamma_2]$, denoted by $[\gamma_1] \overset{\tan}{\sim} [\gamma_2]$, if the following holds: let γ_i represents $[\gamma_i]$ for $i = 1, 2$, and $\phi : U \to X$ be a chart around x,

$$\begin{array}{ccc} V_1 \xrightarrow{\gamma_1} \phi(U) & \qquad & V_2 \xrightarrow{\gamma_2} \phi(U) \\ \\ 0 \longmapsto x & \qquad & 0 \longmapsto x \end{array}$$

let t be the coordinate on $\mathbb{R}$, we have

$$\frac{\mathrm{d}(\phi^{-1} \circ \gamma_1)}{\mathrm{d}t}\Big|_{t=0} = \frac{\mathrm{d}(\phi^{-1} \circ \gamma_2)}{\mathrm{d}t}\Big|_{t=0}.$$

Lemma 1.22. *The relation* $\overset{\tan}{\sim}$ *is an equivalence relation on* W_x. *Moreover,* $W_x / \overset{\tan}{\sim}$ *has a natural* $\mathbb{R}$*-vector space structure.*

Sketch proof. Since $U \subset \mathbb{R}^n$, where $n = \dim X$, we can identify $T_{\phi^{-1}(x)}U$ to $\mathbb{R}^n$, which has a natural $\mathbb{R}$-vector space structure. By transport the addition operaton through ϕ^{-1}, we define the addition on T_xX. One can show that this is well-defined.

The scalar multiplication of $r \in \mathbb{R}$ on $[\gamma]$ is given by $[\gamma \circ (r\cdot)]$, where $r\cdot$ is the multiplication on $\mathbb{R}$, i.e. the domain of a curve germ representing $[\gamma]$. $\square$

Definition 1.23. We define the *tangent space* of X at x, $T_xX := W_x/\overset{\tan}{\sim}$.

Let $f : X \to Y$ be a smooth map, and $x \in X$. Then there is a linear map,

$$T_xf : T_xX \to T_{f(x)}Y, \ [\gamma] \mapsto [f \circ \gamma],$$

called the *tangent map* of f at x.

1.6. **Regular values.** With the notations as above, we say $y \in Y$ is a *regular value* of f if for any $x \in f^{-1}(x)$, the tangent map ${}_xf : T_xX \to T_yY$ is surjective. In particular if $y \notin f(X)$ then y is a regular value. Since having a maximal rank is an open condition for a continuous family of linear maps, the set of regular values is an open subset of Y if f is a proper map (i.e. $f^{-1}(V)$ is compact for any compact subset $V \subset Y$). Moreover:

Proposition 1.24. *If y is a regular value of f, then $f^{-1}(y)$ is a submanifold of X of dimension* $\dim X - \dim Y$*.*

1.7. **Lie groups and group actions.**

Definition 1.25. A *Lie group* is a smooth manifold G with a distinguished point e with inclusion $\iota : \{e\} \hookrightarrow G$, and smooth maps $\mu : G \times G \to G$ and $\mathrm{inv} : G \to G$ satisfying the axioms for an abstract group, i.e. the following commutative diagrams:

$$\begin{tikzcd}
G \times G \arrow[r, "\mu"] & G \\
\{e\} \times G \arrow[u, hook, "\iota\times\mathrm{id}"] \arrow[ur, equal] \arrow[ur, phantom, shift left=4, "\circlearrowleft"]
\end{tikzcd}
,\qquad
\begin{tikzcd}
G \times G \times G \arrow[r, "\mu\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\mu"'] \arrow[dr, phantom, "\circlearrowleft"] & G \times G \arrow[d, "\mu"] \\
G \times G \arrow[r, "\mu"] & G,
\end{tikzcd}$$

$$\begin{tikzcd}
G \times G \arrow[r, "\mathrm{id}\times\mathrm{inv}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & G \times G \arrow[d, "\mu"] \\
\{e\} \arrow[r, hook, "\iota"] & G.
\end{tikzcd}$$

Let $M_n(\mathbb{R})$ be the set of $n \times n$ matrices with entries in $\mathbb{R}$. Via an identification $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, it acquires a smooth manifold structure. Let $\mathrm{GL}_n(\mathbb{R})$ be the subset of $M_n(\mathbb{R})$ consisting of invertible matrices. Since $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is continuous, $\mathrm{GL}_n(\mathbb{R})$ is an open submanifold of $M_n(\mathbb{R})$. We can show that the classical groups are Lie groups using Proposition 1.24.

Proposition 1.26.

(1) $\mathrm{SL}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) | \det(A) = 1\}$ *is a closed submanifold of* $\mathrm{GL}_n(\mathbb{R})$ *of dimension* $n^2 - 1$*.*
(2) $\mathrm{O}(n) = \{A \in \mathrm{GL}_n(\mathbb{R}) | A^\top A = I\}$ *is a closed submanifold of* $\mathrm{GL}_n(\mathbb{R})$ *of dimension* $\frac{n^2-n}{2}$*.*

This follows from the following two lemmas.

Lemma 1.27. *1 is a regular value of* $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}$*.*

Proof. Let $A \in \mathrm{GL}_n(\mathbb{R})$. Consider the curve $t \mapsto A + tB$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\det(A + tB) = \det(A)\frac{\mathrm{d}}{\mathrm{d}t}\det(1 + tA^{-1}B).$$

Since $\det(A) \neq 0$, it suffices to consider the case $A = I$. Then since $\frac{\mathrm{d}}{\mathrm{d}t}\det(1 + tB) = \frac{\mathrm{d}}{\mathrm{d}t}\det(1 + tB')$ if B is similar to B', we can assume that B is upper triangular. Suppose the eigenvalues of B are $\lambda_1, \dots, \lambda_n$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\det(1 + tB)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\prod_{i=1}^{n}(1 + t\lambda_i)\Big|_{t=0} = \sum_{i=1}^{n}\lambda_i.$$

For any $\lambda \in \mathbb{R}$ we can take B with $\sum_{i=1}^n \lambda_i = \lambda$, hence $\det(A)$ is a regular value of det. □

Lemma 1.28. *Let* $\mathrm{Sym}_n(\mathbb{R})$ *be the set of symmetric matrices of size* $n \times n$*, which is a linear submanifold of* $M_n(\mathbb{R})$ *of dimension* $\frac{n^2+n}{2}$*. Let* $F : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ *be the map* $A \mapsto A^\top A$*. Then* I *is a regular value of* F*.*

Proof. Let $A \in M_n(\mathbb{R})$ such that $A^\top A = I$. We compute

$$\frac{\mathrm{d}}{\mathrm{d}t}(A+tB)^\top(A+tB)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(A^\top A + tA^\top B + tB^\top A + t^2 B^\top B)\Big|_{t=0} = A^\top B + B^\top A.$$

For any $C \in \mathrm{Sym}_n(\mathbb{R})$, we take $B = \frac{1}{2}AC$, so that $\frac{\mathrm{d}}{\mathrm{d}t}(A+tB)^\top(A+tB) = C$, which shows that C is in the image of the tangent map $T_A F$. So I is a regular value of F. □

Exercise 7. Let U_n be the unitary group of $n \times n$ unitary matrices. Show that it is a Lie group and compute its dimension. Also, do the same for the symplectic groups Sp_n. (Don't use Cartan's theorem in the following, unless you can present a proof of it.)

Definition 1.29. Let G be a Lie group and X be a smooth manifold. An (smooth) *group action* of G on X is a smooth map $\alpha : G \times X \to X$ which satisfies the usual requirement of abstract group actions. For simplicity we often write $g.x = \alpha(g, x)$ when α is understood in the context. We have the following notions:

(1) α is *transitive* if for any $x, y \in X$ there exists $g \in G$ such that $g.x = y$.
(2) For $x \in X$, the *stabilizer* of x is the subgroup $G_x = \{g \in G | g.x = x\}$.
(3) α is *effective* if the induced map $G \to \mathrm{Aut}(X)$ as a map of abstract groups is an injective map. This is equivalent to $\bigcap_{x\in X} G_x = \{e\}$.

Example 1.30. $O(n+1)$ acts on $\mathbb{R}^{n+1}$ by the multiplication on vectors, and acts on the unit sphere $S^n \in \mathbb{R}^{n+1}$ transitively. Hence to show that S^n is a smooth manifold, we can show it around the special point $(1, 0, \dots, 0)$.

Theorem 1.31 (Élie Cartan)**.** *A closed subgroup of a Lie group is an embedded submanifold.*

Corollary 1.32. *Let* $\mu : G \times X \to X$ *be a group action. For any* $x \in X$*, the stabilizer* G_x *is a Lie group.*

Proof. By definition we have $G_x = \alpha^{-1}(x) \cap (G \times \{x\})$, which is a closed subset. So by Cartan's theorem, G_x is a Lie group. □

Proposition 1.33. *Let* G *be a Lie group, and* H *be a closed subgroup of* G*, then the quotient space* G/H *has a smooth manifold structure such that the natural action of* G *on* G/H *is a smooth group action. Conversely, any transitive group action is of this form.*

Proof. The second statement follows from the previous corollary. For the first statement see [Lee13, Theorem 21.17]. □

2. Vector bundles

Definition 2.1. Let X be a topological space. A *(real) vector bundle of rank* n over X is a topological space E together with a continuous map $\pi : E \to X$ satisfying the following:

(i) there exists an open coving $\{U_i\}_{i\in I}$ and homeomorphisms (called trivializations) $\phi_i : \pi^{-1}U_i \xrightarrow{\sim} U_i \times \mathbb{R}^n$;
(ii) for $i, j \in I$, the transition map $\phi_j|_{U_i\cap U_j} \circ \phi_i^{-1}|_{U_i\cap U_j} : (U_i \cap U_j) \times \mathbb{R}^n \to (U_i \cap U_j) \times \mathbb{R}^n$ is given by a continous family of linear isomorphisms of $\mathbb{R}^n$, i.e. there exists a continous map $\phi_{i,j} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{R})$ such that $\phi_j|_{U_i\cap U_j} \circ \phi_i^{-1}|_{U_i\cap U_j}(x) = \phi_{i,j}$.

Given a vector bundle as such, for $x \in X$, $E_x := \pi^{-1}(x)$ is called the fiber of E over x, and n is called the *rank* of E. For any subset Y of X, a continuous map $s : Y \to \pi^{-1}(Y)$ such that $\pi \circ s = \mathrm{id}_Y$ is called a *section of* E *over* Y.

We have the following variants:

(1) if $\mathbb{R}$ is replaced by $\mathbb{C}$, E is a *complex vector bundle*;

(2) if X is a C^∞-manifold and the transition maps $\phi_{i,j}$ are C^∞-maps, E is a C^∞ *vector bundle*;
(3) if X is a complex manifold, $\mathbb{R}$ is replaced by $\mathbb{C}$, and and the transition maps $\phi_{i,j}$ are holomorphic maps, then E is a *holomorphic vector bundle*.

Definition 2.2. Let X_1 and X_2 be a smooth manifold, and $\pi_1 : E_1 \to X_1$ and $\pi_2 : E_2 \to X_2$ be smooth vector bundles over X and Y respectively. A *bundle homomorphism* $F : E_1 \to E_2$ is a smooth map $E_1 \to E_2$ satisfying the following:

(1) there exists a smooth map $f : X_1 \to X_2$ such that the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ X_1 \arrow[r, "f"] & X_2 \end{tikzcd}$$

Note that such f is unique if it exists.
(2) for every $x \in X_1$, the restriction $F_x : E_{1,x} \to E_{2,f(x)}$ is a linear map.

A bundle homomorphism is an *isomorphism* if it has an inverse. A vector bundle over X isomorphic to $X \times \mathbb{R}^n$ for some n is called a *trivial vector bundle*.

Exercise 8. A vector bundle E of rank n over X is trivial iff there are sections $s_1, \ldots, s_n$ over X which are linearly independent at every point x of X.

Construction 2.3. Let V be an open subset of $\mathbb{R}^n$. We define the *tangent bundle* TV of V to be the trivial vector bundle $V \times \mathbb{R}^n$, and the latter factor $\mathbb{R}^n$ is endowed with a basis of formal symbols $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ (frames). If V_1 is an open subset of $\mathbb{R}^n$ (with coordinates $x_1, \ldots, x_n$) and V_2 is an open subset of $\mathbb{R}^n$ (with coordinates $y_1, \ldots, y_n$), and $f : V_1 \to V_2$ is a diffeomorphism, there is an induced bundle isomorphism $Tf : TV_1 \to TV_2$, such that, if $\phi = (\phi_1, \ldots, \phi_n)$, then for $i = 1, \ldots, n$,

$$T\phi(\frac{\partial}{\partial x_i}) = \sum_{j=1}^{n} \frac{\partial \phi_j}{\partial x_i} \frac{\partial}{\partial y_j}.$$

Now let X be a smooth manifold. We glue the tangent bundle TV_i for V_i running over the charts in the atlas of X, and the resulting vector bundle is the *tangent bundle of* X, denoted by TX. A section of the tangent bundle TX is called a *vector field* on X.

Construction 2.4. Standard linear algebra constructions, such as duals, direct sums, tensor products, symmetric products, generalize to vector bundles. For example, the dual of TX is the *cotangent bundle* T^*X.

Example 2.5. For brevity, let us keep the notations as Definition 2.1.

(1) Let E be a vector bundle on X with transition functions $\phi_{i,j} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{R})$. Then the *dual vector bundle* $E^\vee$ is the vector bundle on X with transition functions $((\phi_{i,j})^{-1})^\top$.
(2) Let E be a vector bundle on X with transition functions $\phi_{i,j} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{R})$, and F be a vector bundle on X with transition functions (with the same atlas) $\psi_{i,j} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{R})$. Then
   (i) the *tensor product bundle* $E \otimes F$ is the vector bundle on X with transition functions $\phi_{i,j} \otimes \psi_{i,j}$,
   (ii) The *hom bundle* $\mathrm{Hom}(E, F)$ is the he vector bundle on X with transition functions $((\phi_{i,j})^{-1})^\top \otimes \psi_{i,j}$.

Exercise 9. For a vector bundle E over a smooth manifold X, give an explicit description of the symmetric product $\mathrm{Sym}^2(E)$ in terms of the transition functions.

Theorem 2.6. *Let E_1 and E_2 be two vector bundles on a smooth manifold X, and $F : E_1 \to E_2$ be a bundle map. Assume that there exists $r \in \mathbb{Z}_{\geq 0}$ such that* $\operatorname{rank} F_x = r$ *at every point $x \in X$. Then F has image, kernel, and cokernel in the additive category of vector bundles.*

2.1. **Partition of unity.**

Definition 2.7. Let X be a topological space. A family of subsets $\mathcal{U}$ of X is *locally finite* if for every $x \in X$, there are only finitely many elements $U \in \mathcal{U}$ such that $x \in U$.

Definition 2.8. Let X be a topological space. Let $\mathcal{U}$ and $\mathcal{V}$ be two open covering of X. We say that $\mathcal{V}$ *refines* $\mathcal{U}$, or $\mathcal{V}$ *is a refinement of* $\mathcal{U}$, if for every $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $V \subset V$.

Proposition 2.9. *Let X be a smooth manifold, and $\mathcal{U}$ be an open covering of X. Then there exists an open covering V which refines $\mathcal{U}$.*

Theorem 2.10 (Partition of unity)**.** *Let X be a smooth manifold, and $\mathcal{U}$ be an open covering of X. Then there exists a family of smooth functions $\{f_i\}_{i\in I}$ satisfying the following:*

(i) *for every $i \in I$, $\mathrm{Supp}(f_i)$ is contained in some $U \in \mathcal{U}$.*
(ii) *$\{\mathrm{Supp}(f_i)\}_{i\in I}$ is locally finite.*
(iii) *$\sum_{i\in I} f_i = 1$ (the left-hand side makes sense because of (ii)).*

2.2. **Metrics.**

Definition 2.11. Let X be a smooth manifold, and E a vector bundle on X. A *bundle metric* on E is an assignment of positively definite symmetric bilinear form on every E_x, such that if U is any open subset of X and e_1, e_2 are smooth sections of E on U, then $g(e_1, e_2)$ is a smooth function on U.

Definition 2.12. Let X be a smooth manifold. A bundle metric on TX is called a *riemannian metric* on X.

Proposition 2.13. *Let X be a smooth manifold, and E a vector bundle on X. Then there exists a bundle metric on E.*

Proof. Let $r = \mathrm{rank}\, E$. Let $\{U_i\}_{i\in I}$ be an open covering and on every U_i, there is given a trivialization $E|_{U_i} \cong U_i \times \mathbb{R}^r$. Then on U_i we take a bundle metric g_i on $E|_{U_i}$ that is induced by the metric on $\mathbb{R}^r$. Then we take a partition of unity $\{f_j\}_{j\in J}$ and WLOG we assume $I = J$ and $\mathrm{Supp}(f_i) \subset U_i$. Then

$$g = \sum_{i\in I} f_i g_i$$

is a bundle metric on E. □

Corollary 2.14. *Any short exact sequence of smooth vector bundles*

$$0 \to E_1 \to E_2 \to E_3 \to 0$$

is splittable.

Proof. Take a bundle metric on E_2. Then there is an induced decomposition $E_2 \cong E_1 \oplus E_1^{\perp}$, where $E_1^{\perp}$ is the *orthogonal complement* of E_1 in E_2. Thus the composition $E_1^{\perp} \hookrightarrow E_2 \to E_3$ is an isomorphism, hence the splitting $E_3 \to E_2$ exists. □

2.3. **Holomorphic vector bundles.** Let $\mathbb{P}^1 = (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*$ be the complex projective line. We think of $\mathbb{P}^1$ as the space of 1-dimensional complex subspaces of $\mathbb{C}^2$. There is a *tautological line bundle* $\mathcal{O}_{\mathbb{P}^1}(-1)$ on $\mathbb{P}^1$, such that for every $x \in \mathbb{P}^1$, the fiber over x is naturally identified with the 1-dimensional complex subspace represented by x.

Remark 2.15. The bundle space of $\mathcal{O}_{\mathbb{P}^1}(-1)$ is close to but not equal to the underlying $\mathbb{C}^2$: they differ only at the origin 0, in that every 1-dimensional complex subspace of $\mathbb{C}^2$, when regarded as a fiber in $\mathcal{O}_{\mathbb{P}^1}(-1)$, has *its own origin* which is identified with the base point of this fiber. The total space of $\mathcal{O}_{\mathbb{P}^1}(-1)$ has another identification, i.e $\mathrm{Bl}_0\mathbb{C}^2$, the blowing up of $\mathbb{C}^2$ at 0.

Let z_0, z_1 be linear coordinats of $\mathbb{C}^2$. A 1-dimensional complex subspace of $\mathbb{C}^2$ is defined by a linear equation $\lambda z_0 = \mu z_1$. We take the proportion $[\lambda : \mu]$ as the homogeneous coordinate on $\mathbb{P}^1$. On the chart $U_0 := \{\lambda \neq 0\}$, we take the non-vanishing section of $\mathcal{O}(-1)$

$$e_0 = (z_0, z_1) = (\frac{\mu}{\lambda}, 1) \text{ over } [\lambda : \mu].$$

On the chart $U_1 := \{\lambda \neq 0\}$, we take the non-vanishing section

$$e_1 = (z_0, z_1) = (1, \frac{\lambda}{\mu}) \text{ over } [\lambda : \mu].$$

Then the transition function is

$$\varphi_{0,1} : U_0 \cap U_1 \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*, \ [\lambda : \mu] \mapsto \frac{\lambda}{\mu},$$

i.e.

$$e_1 = \frac{\lambda}{\mu} e_0 \text{ on } U_0 \cap U_1.$$

We compute the space of global *holomorphic* sections $\Gamma_{\mathrm{hol}}(\mathbb{P}^1, \mathcal{O}(-1))$. Let $a \in \Gamma(\mathbb{P}^1, \mathcal{O}(-1))$. Then on U_0, $a = f(\frac{\mu}{\lambda})e_0$, where $f(z)$ is an *entire function* of z. On U_1, the definition of holomorphic sections forces $f(\frac{\mu}{\lambda})\frac{\mu}{\lambda}$ to be an entire function of $\frac{\lambda}{\mu}$. So the problem amounts to the following: find all entire functions $f(z)$ such that $zf(z)$ is an entire function of $\frac{1}{z}$. But note that the latter requirement implies that $f(z)$ is bounded, hence a constant. The only constant function satisfies the latter requirement is 0. Hence

$$\Gamma_{\mathrm{hol}}(\mathbb{P}^1, \mathcal{O}(-1)) = 0.$$

Denote

$$\mathcal{O}_{\mathbb{P}^1}(k) = \begin{cases} \mathcal{O}_{\mathbb{P}^1}(-1)^{\otimes(-k)}, & \text{if } k < 0 \\ (\mathcal{O}_{\mathbb{P}^1}(-1)^\vee)^{\otimes k}, & \text{if } k > 0 \\ \text{the trivial holomorphic line bundle}, & \text{if } k = 0 \end{cases}.$$

Proposition 2.16. $\Gamma_{\mathrm{hol}}(\mathbb{P}^1, \mathcal{O}(1))$ *is 2-dimensional, and is spanned by* e_0 *and* $\frac{\mu}{\lambda}e_0$.

Exercise 10. Compute the space $\Gamma_{\mathrm{hol}}(\mathbb{P}^1, \mathcal{O}(k))$ (not only its dimension).

2.4. **Elliptic operators.** Let X be a complex manifold, and E be a holomorphic vector bundles. We define the *Cauchy-Riemann* operator

$$\overline{\partial} : \Gamma(X, E) \to \Gamma(X, E),$$

such that if $e_1, \dots, e_r$ is a holomorphic basis of E on a holomorphic chart U and $f_1, \dots, f_r$ are smooth functions on U, then

$$\overline{\partial}(\sum_{i=1}^r f_i e_i) = \sum_{i=1}^r \overline{\partial}(f_i) e_i.$$

This is well defined because the transition maps of E are holomorphic functions. We have

$$\Gamma_{\mathrm{hol}}(X, E) = \mathrm{Ker}(\overline{\partial}).$$

Definition 2.17. Let X be a smooth manifold, E and F be smooth complex vector bundles on X. A *differential operator of order* m is a linear map $P : \Gamma(E) \to \Gamma(F)$, which in local trivializations of E and F can be expressed as a differential operator of order m on Euclidean spaces. One can associate with D a section $\sigma(P) \in (\mathrm{Sym}^m TX) \otimes \mathrm{Hom}(E, F)$, called the *principal symbol* of P. For any $x \in X$ and $\xi \in T_x^*X$, one can evaluate $\sigma(P)$ at ξ to obtain $\sigma_\xi(P) \in \mathrm{Hom}(E_x, F_x)$. We say that P is an *elliptic operator* if $\sigma_\xi(P)$ is invertible for all $x \in X$ and all nonzero ξ.

Then $\overline{\partial}$ is a differential operator, and we check that it is an elliptic operator as follows.

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