

Lecture 2. Topological properties of spaces.

1. Continuous functions = maps.

(ε, δ) arguments: $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous if $\forall \varepsilon > 0, \exists \delta > 0, \forall x_1, x_2 \in \mathbb{R}, |x_1 - x_2| < \delta, |f(x_1) - f(x_2)| < \varepsilon$.

Defn. A function $f: X \rightarrow Y$ between two topological spaces X, Y is continuous if the preimage of open sets in Y is open in X .

prop. Let $f: X \rightarrow Y$ be a function. The followings are equivalent:

- (i) f is continuous
- (ii) The preimage of closed sets in Y is closed in X .
- (iii) $\forall y \in Y$ and V_y (nbhd of y), $f^{-1}(V_y)$ is a nbhd of x , $f(x)=y$.
- (iv) $f(\overline{A}) \subseteq \overline{f(A)}$ for any set $A \subseteq X$
- (v) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for any $B \subseteq Y$.

proof. (i), (ii), (iii) are equivalent: exercise

Exercise. Let $f: X \rightarrow Y$ be a function and B is a topology basis of Y .

Show that f is continuous iff $\forall B \in \mathcal{B}, f^{-1}(B) \subseteq X$ is open.

Examples. ① $i_x: X \xrightarrow{\text{id}} X, x \mapsto x$, is continuous.

② $e: X \rightarrow \{x\}$, is continuous

③ the diagonal map $\Delta: X \rightarrow X \times X, \Delta(x) = (x, x)$, is continuous.

④ if $A \subseteq X$ is a subspace, then the inclusion map $i: A \rightarrow X, i(a) = a$, is continuous.

⑤ $X \xrightarrow{i_1} X \times Y, Y \xrightarrow{i_2} X \times Y$ are continuous.
 $x \mapsto (x, y_0)$

⑥ $X \times Y \xrightarrow{p_1} X, X \times Y \xrightarrow{p_2} Y$ are continuous.

Prop. Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be continuous functions.

Then the composition $g \circ f: X \rightarrow Z$ is continuous.

Proof. $\forall U \subseteq Z$ be open, $(g \circ f)^{-1}(U) = f^{-1} \circ g^{-1}(U) \subseteq X$ is open. \square

Applications. ① Let $A \hookrightarrow X$ be the inclusion map and $f: X \rightarrow Y$ be a map.

Then $f|_A = f \circ i: A \xrightarrow{i} X \xrightarrow{f} Y$ is continuous.

② $Z \xrightarrow{f} X \times Y \quad | \quad Z \xrightarrow{f \circ f_1, f_2} X \times Y$ is continuous iff f_1 and f_2 are continuous.

$$\begin{array}{l} f_1 = f \circ p_1 \\ f_2 = f \circ p_2 \\ f(Z) = \{ (p_1 f(z), p_2 f(z)) \\ \quad = (f_1(z), f_2(z)) \end{array}$$

\Rightarrow clear

$\Leftarrow Z \xrightarrow{\Delta} Z \times Z \xrightarrow{f_1 \times f_2} X \times Y$ is f .

$$z \mapsto (z, z) \mapsto (f_1(z), f_2(z)) = f(z)$$

$$f = (f_1 \times f_2) \circ \Delta.$$

$$\text{or } f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$$

\square

Exercise. Let $f, g: X \rightarrow \mathbb{R}$ be continuous map. Show that

(1) $f \pm g: X \rightarrow \mathbb{R}$, $(f \pm g)(x) = f(x) \pm g(x)$, is continuous.

(2) $f \cdot g: X \rightarrow \mathbb{R}$, $(f \cdot g)(x) = f(x) \cdot g(x)$, is continuous.

$$(X \xrightarrow{\Delta} X \times X \xrightarrow{+ \cdot g} \mathbb{R} \times \mathbb{R} \xrightarrow{+ \cdot R} \mathbb{R})$$

Theorem (gluing lemma, pasting lemma).

Let $X = \bigcup_{i=1}^n X_i$, and let $f: X \rightarrow Y$ be a function.

If each restriction $f_i := f|_{X_i}: X_i \rightarrow Y$ is continuous,

and each X_i is closed (resp. open), then f is continuous. ($f = \bigcup_{i=1}^n f_i$).

Proof $\forall V \subseteq Y$ is closed, $f^{-1}(V) = X \cap f^{-1}(V) = \left(\bigcup_{i=1}^n X_i \right) \cap f^{-1}(V)$

$$= \bigcup_{i=1}^n X_i \cap f^{-1}(V)$$

$$= \bigcup_{i=1}^n f_i^{-1}(V) \subseteq X \text{ is closed.}$$

Thus f is continuous.

If each X_i is open, the statement is still true if $n = \infty$.

Homeomorphism. $f: X \rightarrow Y$ is a homeomorphism if

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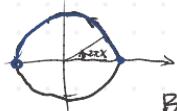
(i) f is continuous

(ii) f is a bijection: the inverse f^{-1} exists.

(iii) The inverse $f^{-1}: Y \rightarrow X$ is continuous.

$$f: X \xrightarrow{\cong} Y$$

Example: ① $f(x) = e^{ixx}: [0, 1] \rightarrow S^1$ is continuous.



$$g: S^1 \rightarrow [0, 1], g(z = e^{i\theta}) = \frac{\theta}{2\pi}.$$

$$fg = 1, gf = 1.$$

But g is not continuous.

$$g([0, \frac{1}{2}]) = \text{a closed curve} \subseteq S^1 \text{ is not open.}$$

$$\textcircled{2} f: (0, 1) \xrightarrow{\cong} \mathbb{R}, f(x) = \frac{x}{1-x}; g: \mathbb{R} \rightarrow (0, 1), g(y) = \frac{y}{1+y}.$$

f is a homeomorphism

$$\textcircled{3} B(0, 1) = \{x \in \mathbb{R}^n \mid |x| < 1\} \xrightarrow{\cong} \mathbb{R}^n$$

$$\textcircled{4} \mathbb{R}^n \setminus \{0\} \xrightarrow{f \cong} \mathbb{R}^n \setminus D^n, D^n = \overline{B(0, 1)}$$



$$f(x) = x + \frac{x}{|x|}$$

$$\textcircled{5} \text{ Stereographic projection: } S^2 \setminus \{N\} \xrightarrow{\Phi \cong} \mathbb{R}^2$$

$$\Phi(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right)$$



A property P is called a topological property of a space if it is preserved by homeomorphisms of the space.

Hausdorff. A space X is Hausdorff if it satisfies the T_2 axiom:
 $\forall x \neq y \in X, \exists$ nbhds U_x and U_y , st. $U_x \cap U_y = \emptyset$.



Example: metric spaces are Hausdorff

prop. Let $f: X \rightarrow Y$ be an injective map. If Y is Hausdorff, then so is X .

proof. $\forall x_1 \neq x_2 \in X, f(x_1) \neq f(x_2)$. Since Y is Hausdorff,
 $\exists V_{f(x_1)} \cap V_{f(x_2)} = \emptyset$.

$$U_{x_1} = f^{-1}(V_{f(x_1)}), U_{x_2} = f^{-1}(V_{f(x_2)})$$

$$\text{Then } U_{x_1} \cap U_{x_2} = \emptyset.$$

□

Cor. Hausdorff property is a topological property.

prop. The following hold:

(1) points of a Hausdorff space are closed

(2) Subspaces of a Hausdorff space are Hausdorff

(3) product spaces of two Hausdorff spaces are Hausdorff.

proof. Exercise.

prop. Let $f: X \rightarrow Y$ be a map with Y Hausdorff. Then

(1) The graph $G_f = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$ is closed.

(2) Let $g: X \rightarrow Y$ be another map. Then $eq(f, g) = \{x \in X \mid f(x) = g(x)\}$
is a closed subset of X . (Exercise)

proof. (1) $Z = X \times Y - G_f$. $\forall (x, y) \in Z, y \neq f(x)$.

Since Y is Hausdorff, there exist V_y and $V_{f(x)}$, st. $V_y \cap V_{f(x)} = \emptyset$.

$U_x = f^{-1}(V_{f(x)})$ is a nbhd of x . Then $U_x \times V_y \subseteq Z = X \times Y - G_f$.
 $\therefore Z$ is open. □

Connectedness.

A topological space X is connected if one of the following equivalent conditions holds:

(1) X cannot be expressed as a union of two disjoint non-empty open sets.
 (2) X - - - - - - - - - - - - - closed sets

(3) The subsets of X that are both open and closed are \emptyset, X .

Exercise. Show the above three statements are equivalent.

Prop. Let $f: X \rightarrow Y$ be a surjective map with X connected.

then $Y=f(X)$ is connected. (the image of connected space is connected)
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Proof. Let V be an open and closed subset of Y .

$f^{-1}(V)$ is both open and closed as a subset of X .

Since X is connected, $f^{-1}(v) = \emptyset$, or $f^{-1}(v) = X$.

Thus $V = \emptyset$ or $V = f(X) = Y$.

That is, γ is connected.

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Cor. Connectedness is a topological property.

Want: If X and Y are connected, then $X \times Y$ is connected.

$$(x,y) \in (X \times \{y\}) \cup (\{x\} \times Y)$$

$$X \times Y = \bigcup_{x \in X, y \in Y} (X \times \{y\} \cup \{x\} \times Y).$$

Lemma. If $A \subseteq X$ is both open and closed, and $B \subseteq X$ is connected, then either $A \cap B = \emptyset$, or $B \subseteq A$.

proof. Assume that $A \cap B \neq \emptyset$.

$\emptyset \neq A \cap B \subseteq B$ is both open and closed.

Since B is connected, $A \cap B = B$, $B \subseteq A$. \square

Theorem. Let $\{X_\alpha\}$ be a collection of connected subspaces of X .

If $X_\alpha \cap X_\beta \neq \emptyset, \forall \alpha, \beta$, then $Y = \bigcup_{\alpha, \beta} X_\alpha \subseteq X$ is connected.

Proof. Let $V \subseteq Y$ is both open and closed. Then $V \cap X_\alpha \subseteq X_\alpha$ is both open and closed. Since X_α is connected, either $V \cap X_\alpha = \emptyset$ or $V \cap X_\alpha = X_\alpha$.

Suppose that $V \neq \emptyset$, and $x \in V$. $\exists \beta$ s.t. $x \in V \cap X_\beta$, $X_\beta \subseteq V$.

Since $X_\alpha \cap X_\beta \neq \emptyset$, we have $V \cap X_\alpha \neq \emptyset$, $X_\alpha \subseteq V$ for $\forall \alpha$.

Thus $Y = \bigcup_\alpha X_\alpha \subseteq V$, and therefore $Y = V$, is connected. \square

Cor. If X and Y are connected spaces, then $X \times Y$ is connected. \square

Cor. $\forall x \in X$, $C(x) = \bigcup_\alpha X_\alpha$, X_α are connected spaces that contain x .

Then $C(x)$ is connected, and $\forall y \in C(x)$, $V_y \subseteq C(y)$.

$C(x)$ is the largest connected subset that contains points of $C(x)$,

We call $C(x)$ the connected component of X containing x .

Exercise. Connected components of X are closed subsets. (\bar{C} is connected \Leftrightarrow C is connected).

Fact. $A \subseteq \mathbb{R}$ is connected iff A is an interval.

Path-connectedness.

A space X is path connected if every two points of X can be joined by a path. $\exists \gamma: [0,1] \rightarrow X$, $\gamma(0) = x$, $\gamma(1) = y$.



$$[0,1] \xrightarrow{\cong} [a,b] \xrightarrow{\gamma} X$$

$a < b$

Prop. Let $f: X \rightarrow Y$ be a surjective map with X path-connected, then

Y is path-connected.

proof. $\forall y_1 \neq y_2 \in Y$, let $y_1 = f(x_1)$, $y_2 = f(x_2)$, then $x_1 \neq x_2$.

Since X is path-connected, $\exists \gamma: [0, 1] \rightarrow X$, s.t. $\gamma(0) = x_1$, $\gamma(1) = x_2$.

Then $f \circ \gamma: [0, 1] \rightarrow Y \rightarrow Y$ is a path connecting y_1 and y_2 .

Thus Y is path-connected. \square

Cor. Path-connectedness is a topological property.

Prop 1. Path-connected spaces are connected.

proof. Let X be path-connected and let $A \subseteq X$ is both open and closed.

Assume $A \neq \emptyset$, and $A \neq X$, then $\exists a \in A$ and $x \in X - A$.

Since X is path-connected, $\exists \gamma: [0, 1] \rightarrow X$, s.t. $\gamma(0) = a$, $\gamma(1) = x$.

$\phi \neq \gamma^{-1}(A) \subseteq [0, 1]$ is both open and closed, $\gamma^{-1}(A) = [0, 1]$, contradiction.

Thus $A = X$, X is connected. \square

Prop 2. If $X \subseteq \mathbb{R}^n$, then X is connected iff X is path-connected.

(Proof is omitted here.)

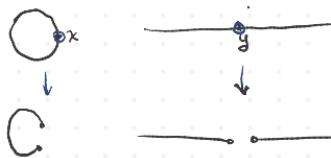
Def. $\forall x \in X$, $P(x) := \{y \in X \mid \exists \gamma: [0, 1] \rightarrow X, \text{ s.t. } \gamma(0) = x, \gamma(1) = y\} \subseteq X$.

Then $P(x)$ is path-connected, and maximal: If $x \in A \subseteq X$ is path-connected,

then $A \subseteq P(x)$.

We call $P(x)$ the path-connected component of X containing x .

Applications. ① $S^1 \not\cong \mathbb{R}^1$:



② S^1 and S^n ($n \geq 2$) are not homeo.

