Lecture 08. Simplicial/Singular Homology Groups

triangulization

n-Simplex $\Delta^n = [v_0, v_1, \cdots, v_n] = \{(t_0, t_1, \cdots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0,\ \sum_{i=0}^n t_i = 1\}$.
($\sum_{i=0}^n t_i v_i$)

= the smallest convex subset in $\mathbb{R}^{n+1}$ with vertices v_i.

Δ^n is ordered: $v_0, v_1, \cdots, v_n$

$[v_1, v_0, v_2, \cdots, v_n] \neq [v_0, v_1, v_2, \cdots, v_n]$

i-face of Δ^n: $[v_0, v_1, \cdots \hat{v}_i, \cdots, v_n]$ $\quad \hat{v}_i$

$= \{(t_0, t_1, \cdots, t_{i-1}, 0, t_{i+1}, \cdots, t_n) \in \Delta^n\}$

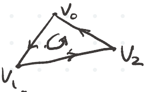
$\partial \Delta^n = \bigcup_{i=0}^n [v_0, v_1, \cdots, \hat{v}_i, \cdots, v_n]$

Example. $n=0$, $\Delta^0 = [v_0]$ $\qquad \partial \Delta^0 = \phi$

$n=1$:

$[v_0, v_1]$ $\qquad \partial[v_0, v_1] = [v_1] - [v_0]$

$n=2$:

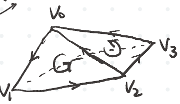


$[v_0, v_1, v_2]$

$\partial[v_0, v_1, v_2] = [v_1, v_2] + [v_0, v_2] + [v_0, v_1]$

$= [v_1, v_2] - [v_2, v_0] + [v_0, v_1]$.

$n=3$



$[v_0, v_1, v_2, v_3]$

$\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$.

In general, $\partial[v_0, v_1, \cdots, v_n] = \sum_{i=0}^n (-1)^i [v_0, v_1, \cdots, \hat{v}_i, \cdots, v_n]$

formal sum

Lemma. $\partial \circ \partial = 0$

proof. $\partial\partial[v_0, v_1, \cdots, v_n] = \partial\left(\sum_{i=0}^n (-1)^i [v_0, v_1, \cdots, \hat{v}_i, \cdots, v_n]\right)$

$= \sum_{i=0}^n (-1)^i \partial[v_0, v_1, \cdots, \hat{v}_i, \cdots, v_n]$

$= \sum_{i=0}^n (-1)^i \left(\sum_{j=0}^{i-1} (-1)^j [v_0, \cdots, \hat{v}_j, \cdots \hat{v}_i, \cdots, v_n] + \sum_{j=i+1}^n (-1)^{j-1} [v_0, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_n] \right)$

$= \sum_{0 \leq j < i} (-1)^{i+j} [v_0, \cdots, \hat{v}_j, \cdots, \hat{v}_i, \cdots, v_n] - \sum_{0 \leq i < j \leq n} (-1)^{i+j} [v_0, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_n]$

$= 0$. $\square$

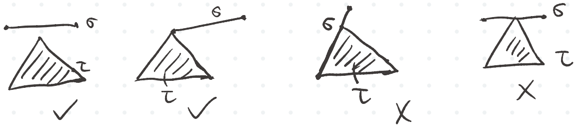
Simplicial Complexes

A simplicial complex K is a finite collection of simplexes such that

(i) If a simplex σ belongs to K (denoted by $\sigma \in K$), then all of its faces belong to K: $\sigma \in K \Rightarrow \sigma_i = [v_0, \cdots, \hat{v}_i, \cdots, v_n] \in K$.

(ii) If two simplexes σ, τ satisfy $\sigma \cap \tau \neq \phi$, then $\sigma \cap \tau$ is a common face of σ and τ.

Examples.



Denote $|K| = \bigcup_{\sigma \in K} \sigma$, viewed as a subspace of $\mathbb{R}^N$, $N \gg 0$.

is called the polyhedron associated to K.

$\dim K = \dim |K| = \max_{\sigma \in K} \{\dim \sigma\}$ is called the dimension of K.

- A space X is called triangulable if there is a homeomorphism $\varphi : |K| \longrightarrow X$.

Examples. ① polyhedra are triangulable.

② All closed/compact surfaces are triangulable.

 $\cong S^2$

Lemma ① $|K| \subseteq \mathbb{R}^N$ is bounded and closed $\Rightarrow$ $|K|$ is compact.

② If $|K|$ is connected, then it is path-connected.

③ $\forall x \in |K|$, $\exists!\ \sigma \in K$, s.t. $x \in \mathring{\sigma} = \mathrm{int}(\sigma)$. (Exercise)

Simplicial homology groups.

Let K be a simplicial complex.

Define $\Delta_n(K) = \mathbb{Z}\langle \text{oriented } n\text{-simplexes of } K \rangle$

$= \text{the free abelian group generated by oriented } n\text{-simplexes}$

$= \{\sum_i k_i \sigma_i \mid \sigma_i \text{ are oriented } n\text{-simplexes}\}$

$\partial_n : \Delta_n(K) \longrightarrow \Delta_{n-1}(K)$ is a homomorphism given by

$$\partial(\underset{\sigma}{\Delta^n} = [v_0, v_1, \cdots, v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, \cdots, \hat{v_i}, \cdots, v_n].$$

$$\partial(\textstyle\sum_i k_i \sigma_i) = \sum_i k_i \partial(\sigma_i)$$

Recall

Lemma: $\partial_n \circ \partial_{n+1} = 0$.

$$0 \to \Delta_m(K) \xrightarrow{\partial_m} \cdots \to \Delta_{n+1}(K) \xrightarrow{\partial_{n+1}} \Delta_n(K) \xrightarrow{\partial_n} \Delta_{n-1}(K) \to \cdots \to \Delta_0(K) \to 0.$$

$m = \dim K$.

$\partial\partial = 0 \Rightarrow \mathrm{Im}\,\partial_{n+1} \subseteq \mathrm{Ker}\,\partial_n$.

The quotient group $H_n(K) := \dfrac{\mathrm{Ker}\,\partial_n}{\mathrm{Im}\,\partial_{n+1}} = \dfrac{Z_n(K)}{B_n(K)}$ is called

the nth simplicial homology group of K. $H_n(K)$ is an abelian group.

$[\alpha] \in H_n(K)$, $[\alpha] = \alpha + \mathrm{Im}\,\partial_{n+1}$ is called a homology class represented by α.

Fact: $H_n(K)$ is a homotopy invariant of simplicial complexes:
if $|K| \simeq |L|$, then $H_n(K) \cong H_n(L)$ for any $n \geq 0$.

$\Delta_0(K) = \mathbb{Z}\langle \text{vertices of } K\rangle$

$\Delta_1(K) = \mathbb{Z}\langle \text{edges of } K\rangle$

$\Delta_2(K) = \mathbb{Z}\langle \text{triangles of } K\rangle$

$$\Delta_2(K) \xrightarrow{\partial_2} \Delta_1(K) \xrightarrow{\partial_1} \Delta_0(K) \xrightarrow{\partial_0} 0$$

$$H_0(K) = \Delta_0(K) \big/ \partial_1(\Delta_1(K))$$

$\forall\, u, v \in \Delta_0(K)$, $[u] = [v] \Leftrightarrow u + \mathrm{Im}\,\partial_1 = v + \mathrm{Im}\,\partial_1$

$\Leftrightarrow u - v = \partial(\sigma)$, $\sigma \in \Delta_1(K)$.

$\Leftrightarrow u - v = \partial([v, v_1] + [v_1, v_2] + \cdots + [v_n, u])$

$\Leftrightarrow$ u and v are two end-vertices of an edge path

$\exists\, \gamma : [0,1]$, $\gamma(0) = u$, $\gamma(1) = v$,

$\Leftrightarrow$ u and v lie in the same (path-)component.

$\Rightarrow$ • if K is (path) connected, then $H_0(K) \cong \mathbb{Z}$.

• if K has m (path) components, then $H_0(K) \cong \mathbb{Z}^m$.

Singular Homology

Let X be a topological space.

A map $\sigma : \Delta^n \to X$ is called a singular n-simplex.

$$S_n(X) := \mathbb{Z}\langle \text{singular } n\text{-simplexes} \rangle = \left\{ \sum_i k_i \sigma_i \;\middle|\; \sigma_i : \Delta^n \to X \right\}.$$

└ the group of singular n-chains.

$$S_{n+1}(X) \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X)$$

$$\tau = [v_0, \cdots, v_n] : \Delta^n \to X$$

$$\partial_n \tau = \sum_{i=0}^n (-1)^i \, \tau|_{[v_0, \cdots, \hat{v}_i, \cdots, v_n]}$$

$$\tau|_{[v_0,\cdots,\hat{v}_i,\cdots,v_n]} : \Delta^{n-1} = \Delta^n_i = [v_0,\cdots,\hat{v}_i,\cdots,v_n] \hookrightarrow \Delta^n \xrightarrow{\tau} X.$$

Lemma: $\partial_n \circ \partial_{n+1} = 0$ (Exercise)

$$H_n(X) := \frac{\operatorname{Ker}\partial_n}{\operatorname{Im}\partial_{n+1}} = \frac{\overset{n\text{-cycle}}{Z_n(X)}}{\underset{n\text{-boundary}}{B_n(X)}}$$ is called the n-th singular homology group of X.

$Z_n(X)$: elements of $\operatorname{Ker}\partial_n$ are called n-cycles

$B_n(X)$: - - - - - $\operatorname{Im}\partial_{n+1}$ - - - - - - n-boundaries.

$[\alpha] = \alpha + \operatorname{Im}\partial_{n+1} \in H_n(X)$ is called a homology class represented by a n-cycle α.

Example. $X = \{x_0\}$

$$S_n(X) = \langle \{ \sigma : \Delta^n \to \{x_0\} \} \rangle = \mathbb{Z} e_n$$

$$S_n(X) \xrightarrow{\partial_n} S_{n-1}(X)$$

$$\partial e_n = \sum_{i=0}^n (-1)^i \underbrace{e_n|_{\Delta^n_i}}_{= e_{n-1}} = \left(\sum_{i=0}^n (-1)^i \right) \cdot e_{n-1} = \begin{cases} 0 & n \text{ is odd} \\ e_{n-1} & n \text{ is even.} \end{cases}$$

$$\begin{array}{ccccccccccc} \cdots \to S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \to \cdots \to & S_2(X) & \xrightarrow{\partial_2} & S_1(X) & \xrightarrow{\partial_1 = 0} & S_0(X) & \to 0 \\ & \wr\| & & \wr\| & & \wr\| & \\ \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \to 0 \end{array}$$

Thus $H_0(X) \cong \mathbb{Z}$, $H_i(X) \cong \mathbb{Z}/\mathbb{Z} = 0$, $\forall i \geq 1$.

Example. If $X = \amalg_\alpha X_\alpha$, then $S_n(X) = S_n(\amalg_\alpha X_\alpha) \cong \oplus_\alpha S_n(X_\alpha)$.

$$\Delta^n \longrightarrow X = \amalg_\alpha X_\alpha$$

$$\begin{array}{ccc} S_n(X) & \xrightarrow{\partial_n} & S_{n-1}(X) \\ \| & & \| \\ \oplus_\alpha S_n(X_\alpha) & \xrightarrow{\oplus_\alpha \partial_{n,\alpha}} & \oplus_\alpha S_{n-1}(X_\alpha) \end{array}, \quad \partial_n = \oplus_\alpha \partial_{n,\alpha}$$

$$\therefore H_n(X) \cong \frac{\operatorname{Ker}\partial_n}{\operatorname{Im}\partial_{n+1}} = \frac{\oplus_\alpha \operatorname{Ker}\partial_{n,\alpha}}{\oplus_\alpha \operatorname{Im}\partial_{n+1,\alpha}} \cong \oplus_\alpha \frac{\operatorname{Ker}\partial_{n,\alpha}}{\operatorname{Im}\partial_{n+1,\alpha}} = \oplus_\alpha H_n(X_\alpha).$$

prop. If X is an nonempty path-connected space, then

$$H_0(X) \cong \mathbb{Z} \Leftrightarrow \tilde{H}_0(X) = 0.$$

If $X = \coprod_{i=1}^m X_i$, X_i are path components, then $H_0(X) \cong \mathbb{Z}^m$

$\tilde{H}_0(X) \cong \mathbb{Z}^{m-1}$.

proof. $\longrightarrow S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$

By definition, $H_0(X) = S_0(X) / \operatorname{Im}\partial_1$

Define $S_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$, $\varepsilon(\sum_i k_i \sigma_i) = \sum_i k_i$. $\varepsilon(\sigma) = 1$ for $\sigma: \Delta^0 \to X$.

- ε is a group homomorphism.
- ε is surjective. $\forall k \in \mathbb{Z}$, take $\sigma: \Delta^0 \to X$, $\varepsilon(k\sigma) = k$.

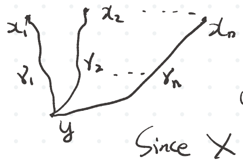
Claim: $\operatorname{Im}\partial_1 = \operatorname{Ker}\varepsilon$ if X is path connected.

$\operatorname{Im}\partial_1 \subseteq \operatorname{Ker}\varepsilon \Leftrightarrow \varepsilon\partial_1 = 0$: $\varepsilon(\partial_1\sigma) = \varepsilon(\sigma|_{[v_1]} - \sigma|_{[v_0]}) = 1 - 1 = 0$.

$\sigma = [v_0, v_1]: \Delta^1 \to X$

$\operatorname{Ker}\varepsilon \subseteq \operatorname{Im}\partial_1$: $\forall$ $\sum_i k_i \sigma_i \in \operatorname{Ker}\varepsilon$, $\sum_i k_i = 0$, $\sigma_i = [v_i]: \Delta^0 \to X$.

$x_i = \sigma_i(v_i)$, $\forall i$.



Take y to be a (base) point that is different from $x_1, x_2, \cdots, x_n$.

Since X is path-connected, there exist path γ_i s.t. $\gamma(0) = y$, $\gamma(1) = x_i$

$\gamma_i: I = \Delta^1 \longrightarrow X$ can be viewed as 1-simplex in X

$\partial\gamma_i = x_i - y$.

Let $\gamma = \sum_i k_i \gamma_i$, then $\partial_1\gamma = \partial(\sum_i k_i \gamma_i) = \sum_i k_i x_i - (\sum_i k_i) y = \sum_i k_i \sigma_i$

Lemma1. (Universal property of quotient groups).

Given a homomorphism $f: G \to H$ and a canonical projection $\pi: G \to G/N$.

If $N \subseteq \operatorname{Ker} f$, then there is a unique homomorphism $\bar{f}: G/N \to H$. s.t. $\bar{f} \circ \pi = f$

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & H \\ G/N \arrow[ur, dashed, "\exists ! \bar{f}"'] & \end{tikzcd}$$

Moreover, ① f is surjective $\Rightarrow$ $\bar{f}$ is surjective

② $\operatorname{Ker} \bar{f} = \operatorname{Ker} f / N$.

proof. Define $\bar{f}([x]) = f(x)$.

$N \subseteq \operatorname{Ker} f \Rightarrow \bar{f}$ is well-defined. □

$$\begin{tikzcd} S_1(X) \arrow[r, "\partial_1"] & S_0(X) \arrow[r, "\varepsilon"] \arrow[d, "\pi"] & \mathbb{Z} \\ & H_0(X) = S_0(X)/\operatorname{Im}\partial_1 \arrow[ur, "\tilde{\varepsilon}"'] & \end{tikzcd}$$

$\operatorname{Im} \partial_1 = \operatorname{Ker} \varepsilon$ if X is path-connected

$\tilde{H}_0(X) = \operatorname{Ker} \tilde{\varepsilon} = \operatorname{Ker} \varepsilon / \operatorname{Im} \partial_1$ $(=0$ if X is path connected$)$

is called the reduced 0-th singular homology group of X.

$H_0(X) / \tilde{H}_0(X) \cong \mathbb{Z} \iff \boxed{H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}.\quad \tilde{H}_0(X) = 0 \text{ if } X \text{ is path-connected.}}$

Lemma 2. If $G/N \cong \mathbb{Z}$, G and N are abelian groups, then $G \cong N \oplus \mathbb{Z}$

proof. $0 \to N \xrightarrow{i} \underset{g^{t}}{G} \underset{s}{\overset{\pi}{\rightleftarrows}} \mathbb{Z} \to 0$

$\mathbb{Z} = \langle 1 \rangle$, fix $g_0 \in \pi^{-1}(1)$, define $s(1) = g_0$. $\Rightarrow \pi s = \mathrm{id}_{\mathbb{Z}}$.

Define $N \oplus \mathbb{Z} \xrightarrow{(i,s)} G$

$(i,s)(x,y) = i(x) + s(y)$.

(i,s) is injective: If $i(x) + s(y) = 0$, then

$0 = \pi \circ i(x) + \pi \circ s(y) = 0 + y \Rightarrow y = 0$

$\therefore i(x) = 0$. Since i is injective, we have $x = 0$

(i,s) is surjective: $\forall g \in G$, $s \circ \pi(g) \in G$.

$g - s \circ \pi(g) \in \operatorname{Ker}(\pi)$: $\pi(g - s \circ \pi(g)) = 0$

$x = g - s \circ \pi(g) \in N$

$y = \pi(g)$. then $(i,s)(x,y) = g$. □

$$S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial_0 = \varepsilon} \mathbb{Z}$$

reduced homology groups $\tilde{H}_n(X)$ = the homology groups of $\left\{ S_n(X), \partial_n ; \begin{matrix} S_{-1}(X) = \mathbb{Z} \\ \partial_0 = \varepsilon \end{matrix} \right\}$

$$\begin{cases} \tilde{H}_n(X) \cong H_n(X) & \text{for } n > 0. \\ \tilde{H}_0(X) \cong H_0(X)/\mathbb{Z}. \end{cases}$$

· Induced homomorphisms

Let $f: X \to Y$ be map. $\Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y$

Define $f_\#: S_n(X) \to S_n(Y)$, $f(\sigma) = f \circ \sigma$.

Lemma: $\partial f_\# = f_\# \partial$:

$$\begin{tikzcd} S_n(X) \arrow[r, "f_\#"] \arrow[d, "\partial"] & S_n(Y) \arrow[d, "\partial"] \\ S_{n-1}(X) \arrow[r, "f_\#"] & S_{n-1}(Y) \end{tikzcd}$$

proof
$$\begin{aligned} (\partial f_\#)(\sigma) = \partial(f \circ \sigma) &= \textstyle\sum_{i=0}^n (-1)^i (f \circ \sigma)|_{[v_0, \cdots, \hat{v}_i, \cdots, v_n]} \\ &= \textstyle\sum_{i=0}^n (-1)^i f(\sigma|_{[v_0, \cdots, \hat{v}_i, \cdots, v_n]}) \\ &= f \circ \left(\textstyle\sum_{i=0}^n (-1)^i \sigma|_{[v_0, \cdots, \hat{v}_i, \cdots, v_n]}\right) \\ &= f_\#(\partial\sigma) = (f_\# \circ \partial)(\sigma). \end{aligned}$$
$\square$

· $f_\#$: $\operatorname{Ker}\partial_n^X \to \operatorname{Ker}\partial_n^Y$, $\partial\sigma = 0 \Rightarrow \partial(f_\#\sigma) = f_\#(\partial\sigma) = 0$

$\operatorname{Im}\partial_{n+1}^X \to \operatorname{Im}\partial_{n+1}^Y$, $f_\#(\partial\sigma) = \partial(f_\#\sigma)$

Thus $f: X \to Y$ induces a homomorphism $f_*: H_n(X) \to H_n(Y)$, $\forall n \geq 0$.

Moreover, ① $(g \circ f)_\# = g_\# \circ f_\#$: $S_n(X) \to S_n(Y) \to S_n(Z)$

$(id_X)_\# = id_{S_n(X)}$: $S_n(X) \to S_n(X)$

$\Rightarrow (g \circ f)_* = g_* \circ f_*$, $(id_X)_* = id_{H_*(X)}$.

④ If X is contractible, then $\tilde{H}_n(X) = 0$, $\forall n \geq 0$.

② $f \simeq g: X \to Y$. $f \circ \sigma \simeq g \circ \sigma \Rightarrow f_\# = g_\# \Rightarrow f_* = g_*: H_n(X) \to H_n(Y)$, $\forall n$

③ If $X \simeq Y$, then $H_n(X) \cong H_n(Y)$. ($H_*(X)$ is a htp invariant.)

$\square$

Lecture 09. Singular Homology Groups II.

Review of Singular homology groups.

- Singular n-Simplex $\sigma: \Delta^n = [v_0, \cdots, v_n] \longrightarrow X$

  $S_n(X) = \mathbb{Z}\langle \sigma : \Delta^n \to X \rangle$, the free abelian group generated by singular n-simplexes.
- $\partial\partial = 0$. $\cdots \to S_{n+1}(X) \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \to \cdots \xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial_0} 0$

  $H_n(X) = \ker\partial_n / \mathrm{Im}\,\partial_{n+1}$. quotient group / module over $\mathbb{Z}$.

History: Poincaré $\longrightarrow$ Nöether (诺特)

- Induced homomorphism: $f: X \to Y$ induces $S_n(X) \xrightarrow{f_\#} S_n(Y)$ s.t. $\partial f_\# = f_\# \partial$.

  and hence induces $f_*: H_n(X) \longrightarrow H_n(Y)$. (recall the universal property of quotient group)

  $(gf)_* = g_* \cdot f_*$, $(id_X)_* = id_{H_n(X)}$, $\forall n$.

- Homotopy invariance: $f \overset{F}{\simeq} g: X \to Y$, then $f_* = g_*: H_n(X) \to H_n(Y)$, $\forall n \geq 0$.

Correction. The fact ~~that~~ "$f \simeq g: X \to Y$ implies that $f_* = g_*: H_i(X) \to H_i(Y), \forall i \geq 0$"

$\Rightarrow f_\# - g_\# = \partial P + P\partial$, $P: C_i(X) \to C_{i+1}(Y)$

is less trivial. For the detailed proof, see Hatcher, AT, page 112.

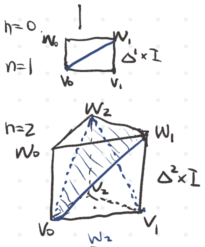
Sketch proof of Homotopy invariance.

$f \xrightarrow[\simeq]{F} g: X \to Y$. $g_\#, f_\#: S_n(X) \longrightarrow S_n(Y)$

$F: X \times I \longrightarrow Y$ $\qquad \underset{\dim = n+1}{\Delta^n \times I} \xrightarrow{\sigma \times id} X \times I \xrightarrow{F} Y$

$\sigma: \Delta^n \longrightarrow X$

Δ^n



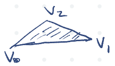
For $\Delta^n = [v_0, \cdots, v_n]$,

Set Δ^n_- as the bottom of $\Delta^n \times I$.

define $\Delta^n_+ = [w_0, \cdots, w_n]$ to be the top n-simplex of Δ^n s.t. the projection $\Delta^n_+ \longrightarrow \Delta^n_-$ satisfying

$w_i \longmapsto v_i$, $i = 0, \cdots, n$.

φ_2:



φ_{-1}:

For each i, define $\varphi_i: \Delta^n = [v_0, \cdots, v_n] \to I$

$\Delta^n = \{(t_0, t_1, \cdots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_i t_i = 1\}$

$\varphi_i(t_0, t_1, \cdots, t_n) = t_{i+1} + \cdots + t_n$.

$n=2$, $\Delta^n_- = [v_0, v_1, v_2]$

$\varphi_{-1}: \Delta^n_- \to 1$

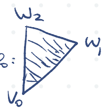
$\varphi_0(t_0, t_1, t_2) = t_1 + t_2$

$(0,1,0) \longmapsto 1$

$(1,0,0) \longmapsto 0$

$(0,0,1) \longmapsto 1$

φ_0:



$\varphi_{-1}(t_0, t_1, \cdots, t_n) = t_0 + t_1 + \cdots + t_n = 1$.

$\varphi_0(t_0, t_1, \cdots, t_n) = t_1 + \cdots + t_n$

$\varphi_1(t_0, t_1, \cdots, t_n) = t_2 + \cdots + t_n$

$\vdots$

$\varphi_n(t_0, t_1, \cdots, t_n) = 0$

$\varphi_1(v_0 = (1,0,0)) = 0$

$\varphi_1(v_1 = (0,1,0)) = 0$, $\varphi_2(v_2) = 1$

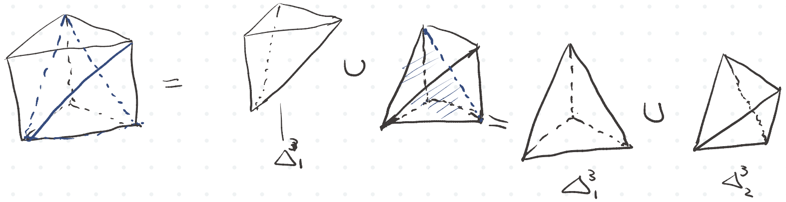
φ_1:

In general, the followings hold.

(i) $1 = \varphi_{-1} \geq \varphi_0 \geq \cdots \geq \varphi_n = 0$.

(ii) the image of φ_i is the n-simplex $[v_0, \cdots, v_i, w_{i+1}, \cdots, w_n]$ and $[v_0, \cdots, v_i, w_{i+1}, \cdots, w_n]$ lies below the n-simplex $[v_0, \cdots, v_{i-1}, w_i, \cdots, w_n]$

(iii) the region between $[v_0, \cdots, v_i, w_{i+1}, \cdots, w_n]$ and $[v_0, \cdots, v_{i-1}, w_i, \cdots, w_n]$ forms an $(n+1)$-simplex $[v_0, \cdots, v_{i-1}, w_{i-1}, w_i, \cdots, w_n]$



In general, $\Delta^n \times I = \bigcup_{i=0}^{n} [v_0, \cdots, v_i, w_i, w_{i+1}, \cdots, w_n]$

any two $(n+1)$-simplexes intersect on one n-simplex

Define $P : S_n(X) \longrightarrow S_{n+1}(Y)$

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i \, F \circ (\sigma \times id)\Big|_{[v_0, \cdots, v_i, w_i, \cdots, w_n]} \in S_{n+1}(Y).$$

$$\Delta^{n+1} \xrightarrow{\ \ } \Delta^n \times I \xrightarrow{\sigma \times id} X \times I \xrightarrow{F} Y.$$

Check that $g_\# - f_\# = \partial P + P\partial$. (Exercise)

Thus $g_* - f_* = 0$, i.e. $g_* = f_* : H_n(X) \longrightarrow H_n(Y)$, $\forall n \geq 0$. $\square$

Cor. If $X \simeq Y$, then $H_n(X) \cong H_n(Y)$, $\forall n$.

eg. $X \simeq \{*\}$ then $H_n(X) \cong \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}$

$D^n, \mathbb{R}^m, S^\infty$

Exact Sequences. 正合序列

A chain of (abelian) groups and homomorphism

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\phi_{n+1}} A_n \xrightarrow{\phi_n} A_{n-1} \xrightarrow{\phi_{n-1}} \cdots$$

is exact if $\operatorname{Im}\phi_{n+1} = \operatorname{Ker}\phi_n$, $\forall n$. ; or equivalent it is exact at all A_n.

exact at A_n if $\operatorname{Im}\phi_{n+1} = \operatorname{Ker}\phi_n$.

Examples. ① $A \xrightarrow{0} B \xrightarrow{b} C$ is exact $\Leftrightarrow$ b is mono.

② $B \xrightarrow{b} C \xrightarrow{0} D$ is exact $\Leftrightarrow$ b is epi.

③ $A \xrightarrow{0} B \xrightarrow{b} C \xrightarrow{c} D \xrightarrow{0} E$ is exact $\Leftrightarrow$ $\begin{cases} b \text{ is mono, } c \text{ is epi.} \\ C/\operatorname{Im} b \cong D. \end{cases}$

$\Updownarrow$

$0 \longrightarrow B \xrightarrow{b} C \xrightarrow{c} D \longrightarrow 0$ is exact.
(Short exact)

• note: Long exact sequenses induces short exact Seq:

$$\begin{tikzcd}[column sep=small]
 & & & \operatorname{Im}\alpha_2 = \ker\alpha_3 \arrow[dr] & & & \\
\cdots \arrow[r] & A_0 \arrow[r] & A_1 \arrow[r, "\alpha_1"] \arrow[dr] & A_2 \arrow[r, "\alpha_2"] \arrow[u] & A_3 \arrow[r, "\alpha_3"] \arrow[dr] & A_4 \arrow[r] & \cdots \\
 & & & \operatorname{Im}\alpha_1 = \operatorname{Ker}\alpha_2 \arrow[u] & & \operatorname{Im}\alpha_3 \arrow[u] &
\end{tikzcd}$$

exact seq.

$$0 \to \operatorname{Ker}\alpha_2 = \operatorname{Im}\alpha_1 \xrightarrow{\subseteq} A_2 \xrightarrow{\alpha_2} \operatorname{Im}\alpha_2 \longrightarrow 0$$

Axioms of (oridinary) homology theory / Eilenberg–Steenrod Axioms. 1945.

" Axiomatic approach to homology theory ".

A_1: induced homomorphism, $f: X \to Y$ induces $f_*: H_n(X) \to H_n(Y)$.

A_2: $g_* f_* = (g \circ f)_*: H_n(X) \longrightarrow H_n(Y) \longrightarrow H_n(Z)$

A_3: $(id_X)_* = id: H_n(X) \longrightarrow H_n(X)$.

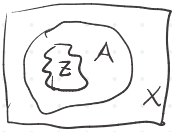
A_4: If $f \simeq g: X \to Y$, then $f_* = g_*: H_n(X) \to H_n(Y)$.

A_5. Long exact sequences. Let ($A \subseteq X$ be a closed subset which is a deformation retraction of nbhd $V \subseteq X$: $V \underset{i}{\overset{r}{\rightleftarrows}} A$) — (X,A) — good pair. Then there is an exact seq:

reduced version $\longrightarrow \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \cdots \to \tilde{H}_0(X/A) \to 0$

unreduced version $\to H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \longrightarrow H_{n-1}(A) \to \cdots \to H_0(X,A) \to 0$

A_6: Excision: Given subspaces $Z \subseteq A \subseteq X$, s.t. $\overline{Z} \subseteq \mathrm{int}(A)$, then the inclusion $(X-Z, A-Z) \hookrightarrow (X, A)$ induces an isomorphism $H_n(X-Z, A-Z) \xrightarrow{\cong} H_n(X, A)$, $\forall n \geq 0$.



another version: Set $B = X-Z$, $Z = X-B$, then $A - Z = A \cap B$
and $\overline{Z} \subseteq \mathrm{int}(A) \Leftrightarrow X = \mathrm{int}(A) \cup \mathrm{int}(B)$ (check).

for subspace $A, B \subseteq X$ st. $X = \mathrm{int}(A) \cup \mathrm{int}(B)$, then the inclusion $(B, A\cap B) \longrightarrow (X, A)$ induces an isomorphism $H_n(B, A\cap B) \xrightarrow{\cong} H_n(X, A)$, $\forall n \geq 0$.

A_7: dimension axiom $H_n(\{p\}) = 0$ for $n > 0$.

- Moreover, any two homology theory h, H satisfy the above 7 axioms,
  Uniqueness then $h = H$: $h_n(X) \cong H_n(X)$, $\forall X \in \mathrm{Top}_*$
- If A_7 axiom fails, then in general, $h_n(X) \ncong H_n(X)$.

Theorem: $\longrightarrow \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{j_*} \widetilde{H}_n(X/A) \xrightarrow{\partial} \widetilde{H}_{n-1}(A) \longrightarrow \cdots \longrightarrow \widetilde{H}_0(X/A) \longrightarrow 0$

Recall $\widetilde{H}_k(D^n) = 0$, $\forall k \geq 0$.

Example: $S^n \cong D^n/\partial D^n = S^{n-1}$. $n=2$

$(X, A) = (D^n, S^{n-1})$:

$$\widetilde{H}_i(S^{n-1}) \longrightarrow \underset{\overset{\parallel}{0}}{\widetilde{H}_i(D^n)} \xrightarrow{0} \underset{\overset{\wr\wr}{\widetilde{H}_i(S^n)}}{\widetilde{H}_i(D^n/S^{n-1})} \xrightarrow[\cong]{\partial} \widetilde{H}_{i-1}(S^{n-1}) \xrightarrow{0} \underset{\overset{\parallel}{0}}{\widetilde{H}_{i-1}(D^n)} \longrightarrow \cdots$$

$\partial D^1 = S^0$

Thus $\widetilde{H}_i(S^n) \cong \widetilde{H}_{i-1}(S^{n-1}) \cong \cdots \cong \widetilde{H}_{i-n}(S^0) \cong \begin{cases} \mathbb{Z} & i=n \\ 0 & i \neq n \end{cases}$

Relative homology groups $H_n(X,A)$

Given a pair of spaces (X,A), $A \subseteq X$ is a subspace.

$$\begin{tikzcd}
0 \arrow[r] & S_n(A) \arrow[r, "i_\#"] \arrow[d, "\partial_A"] & S_n(X) \arrow[r] \arrow[d, "\partial_X"] & S_n(X)/S_n(A) =: S_n(X,A) \arrow[r] \arrow[d, "\tilde\partial"] & 0 \\
0 \arrow[r] & S_{n-1}(A) \arrow[r, "i_\#"] & S_{n-1}(X) \arrow[r] & S_{n-1}(X)/S_{n-1}(A) = S_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

exact

$$\partial_X \partial_X = 0,\ \partial_A \partial_A = 0 \Rightarrow \tilde\partial\tilde\partial = 0.$$

The quotient group $\operatorname{Ker}\tilde\partial / \operatorname{Im}\tilde\partial =: H_n(X,A)$ is called the n-th relative homology group of the pair (X,A).

- If $A = \phi$, then $H_n(X,\phi) = H_n(X)$.
- Exercise: describe elements of $\operatorname{Ker}\tilde\partial$ and $H_n(X,A)$.
- $H_n(X,A) = 0,\ \forall n \Leftrightarrow H_n(A) \cong H_n(X),\ \forall n.$

Prop. For "good pair" (X,A), there holds an isomorphism

$$H_n(X,A) \cong \tilde H_n(X/A),\ \forall n \geq 0.$$

Cor. For good pair (X,A), there is an exact sequence.

$$\to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to \cdots \to H_0(X,A) \to 0$$

Example (Exercise) ① Compute $H_k(D^n, S^{n-1})$.

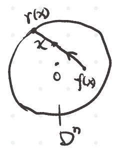
② proof of Brouwer's fixed point theorem.

Every map $f: D^n \to D^n$ $(n \geq 1)$ has a fixed point $(f(x) = x,\ x \in D^n)$.

$n=2$, proved by fundamental groups $\pi_1(S^1) \cong \mathbb{Z}$.

Recall that we constructed a map $r: D^n \to \partial D^n = S^{n-1}$ s.t. the composition

$S^{n-1} \xrightarrow{i} D^n \xrightarrow{r} S^{n-1}$ is the identity: $ri = id_{S^{n-1}}$.



$(ri)_* = r_* i_* = \underline{id}$:

$$\begin{array}{ccccc}
\tilde H_{n-1}(S^{n-1}) & \longrightarrow & \tilde H_{n-1}(D^n) & \longrightarrow & \tilde H_{n-1}(S^{n-1}) \\
\| & & \| & & \| \\
\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}
\end{array}$$

Contraction: $\mathbb{Z} \to 0 \to \mathbb{Z}$ cannot be the identity. $\square$

Long Exact Sequence II

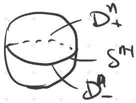
Let $X = \mathrm{int}(A) \cup \mathrm{int}(B)$, $A, B \subseteq X$.

Mayer–Vietoris Sequence: There is an exact seq:
(MV sequence)

$$\xrightarrow{\partial} \widetilde{H}_n(A\cap B) \xrightarrow{\Phi} \underset{(\alpha,\ \beta)}{\widetilde{H}_n(A)\oplus \widetilde{H}_n(B)} \xrightarrow{\Psi} \widetilde{H}_n(X) \xrightarrow{\partial} \widetilde{H}_{n-1}(A\cap B) \to \cdots$$

$$\Phi(\alpha) = (\alpha, -\alpha)$$
$$\Psi(\alpha, \beta) = \alpha + \beta.$$

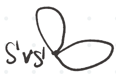
Example: ① $S^n = D^n_+ \cup_{S^{n-1}} D^n_-$ $\Rightarrow \widetilde{H}_i(S^n) \cong \widetilde{H}_{i-1}(S^{n-1}) \cong \cdots \cong \widetilde{H}_{i-n}(S^0) \cong \begin{cases} \mathbb{Z} & i=n \\ 0 & i\neq n \end{cases}$



② $\widetilde{H}_n(X_1 \vee X_2) \cong \widetilde{H}_n(X_1) \oplus \widetilde{H}_n(X_2) \Rightarrow \widetilde{H}_n(\vee_{i=1}^m X_i) \cong \oplus_{i=1}^m \widetilde{H}_i(X_i)$

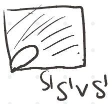
$$X_1 \vee X_2 = \frac{(X_1, x_1) \amalg (X_2, x_2)}{x_1 \sim x_2}$$

eg. $\widetilde{H}_1(S^1 \vee S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$.



③ T^2

$A \cap B = S^1$,
$A \cong \mathring{D}^2$
$B = T^2 \setminus D^2 \simeq S^1 \vee S^1$



$$0 \to \underset{\overset{\|}{0}}{\widetilde{H}_2(S^1)} \to \underset{\overset{\|}{0}}{\widetilde{H}_2(D^2)} \oplus \underset{\overset{\|}{\widetilde{H}_2(S^1)\oplus\widetilde{H}_2(S^1)} \overset{\|}{0}}{\widetilde{H}_2(S^1\vee S^1)} \to \underset{\overset{\|}{?}}{\widetilde{H}_2(T^2)} \xrightarrow{\partial} \underset{\overset{\|}{\mathbb{Z}}}{\widetilde{H}_1(S^1)} \to \underset{\overset{\|}{\mathbb{Z}\oplus\mathbb{Z}}}{\widetilde{H}_1(D^2)\oplus\widetilde{H}_1(S^1\vee S^1)} \to \underset{\overset{\|}{?}}{\widetilde{H}_1(T^2)} \to 0$$

$\therefore\ 0 \to \underset{\mathbb{Z}}{\widetilde{H}_2(T^2)} \xrightarrow{a} \mathbb{Z} \xrightarrow{b} \mathbb{Z}\oplus\mathbb{Z} \to \widetilde{H}_1(T^2) \to 0$

$0 \to \mathrm{Im}\,a \cong \mathbb{Z} \to 0$, $\mathrm{Im}\,b$

$0 \neq \widetilde{H}_2(T^2) \subseteq \mathbb{Z} \Rightarrow \widetilde{H}_2(T^2) \cong \mathbb{Z}$.

$0 \to \mathbb{Z} \cong \mathrm{Im}\,a \to \mathbb{Z} \to \mathrm{Im}\,b \to 0$ exact $\Rightarrow \mathrm{Im}\,b$ is finite $\xrightarrow{\mathrm{Hom}(\mathbb{Z}/n,\ \mathbb{Z}\oplus\mathbb{Z}) = 0}$ $0 \to \mathrm{Im}\,b \xrightarrow{0} \mathbb{Z}\oplus\mathbb{Z} \to \widetilde{H}_1(T^2) \to 0$
($\mathrm{Im}\,b \cong \mathbb{Z}/n\mathbb{Z}$ or $\mathrm{Im}\,b = 0$)

exact $\Rightarrow \widetilde{H}_1(T^2) \cong \mathbb{Z}\oplus\mathbb{Z}$.

⊕ Exercise. $\widetilde{H}_n(K)$, K is the Klein bottle.

$$\cong \begin{cases} \mathbb{Z}\oplus\mathbb{Z}/2 & n=1 \\ 0 & n\neq 1 \end{cases}$$

Exam Time: 8.5. Next Saturday

Lecture 10 Singular Homology Groups III

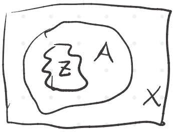
Recall $H_n(X,A)$. ① $H_n(X,\phi) = H_n(X)$

② $H_n(X, x_0) = \tilde{H}_n(X)$

③ Long exact seq: $\Rightarrow H_n(X,A)=0, \forall n \Leftrightarrow H_n(A) \xrightarrow[\cong]{i_*} H_n(X)$

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_{n*}} H_n(X,A) \xrightarrow{\partial_n} H_{n-1}(A) \to \cdots \quad \to H_0(X,A).$$

Excision Theorem Given subspaces $Z \subseteq A \subseteq X$, s.t. $\bar{Z} \subseteq \mathrm{int}(A)$, then the



inclusion $(X-Z, A-Z) \longrightarrow (X, A)$ induces an isomorphism $H_n(X-Z, A-Z) \xrightarrow{\cong} H_n(X,A)$, $\forall n \geq 0$.

Application

Theorem (Invariance of domains)

Let $\phi \neq U \subseteq \mathbb{R}^m$, $\phi \neq V \subseteq \mathbb{R}^n$ be open subsets.

If U is homeomorphic to V, then $m=n$.

In particular, $\mathbb{R}^m \cong \mathbb{R}^n \Rightarrow m=n$. $(m \neq n, \mathbb{R}^m \not\cong \mathbb{R}^n, \mathbb{R}^m \simeq \mathbb{R}^n)$

proof. Let $x \in U$. $(\underbrace{U}_{X-Z}, U-\{x\}) \longrightarrow (\mathbb{R}^m, \mathbb{R}^m - \{x\})$

$Z = \mathbb{R}^m - U$. $U = \mathbb{R}^m - Z$.

induces an isomorphism

$$\underline{H_k(U, U-\{x\})} \cong H_k(\mathbb{R}^m, \mathbb{R}^m - \{x\})$$

$\tilde{H}_k(\mathbb{R}^m) = 0, \forall k$. $\underset{\substack{\| \\ 0}}{\tilde{H}_k(\mathbb{R}^m)} \to H_k(\mathbb{R}^m, \mathbb{R}^m - \{x\}) \xrightarrow{\partial} \tilde{H}_{k-1}(\mathbb{R}^m - \{x\}) \to \underset{\substack{\| \\ 0}}{\tilde{H}_{k-1}(\mathbb{R}^m)}$

$$\therefore H_k(\mathbb{R}^m, \mathbb{R}^m - \{x\}) \cong \tilde{H}_{k-1}(\mathbb{R}^m - \{x\}) \cong \tilde{H}_{k-1}(S^{m-1}) \cong \begin{cases} \mathbb{Z} & k=m \\ 0 & k \neq m. \end{cases}$$

Let $h: U \to V$ be a homeomorphism, $h: (U, U-\{x\}) \xrightarrow{\cong} (V, V-\{h(x)\})$

$$h_*: \underset{\substack{\cong \\ \begin{cases} \mathbb{Z} & k=m \\ 0 & k\neq m \end{cases}}}{H_k(U, U-\{x\})} \xrightarrow{\cong} \underset{\substack{\cong \\ \begin{cases} \mathbb{Z} & k=n \\ 0 & k\neq n \end{cases}}}{H_k(V, V-\{h(x)\})}$$

Then h_* is an isomorphism implies that $m=n$. $\square$

• Lemma. Let $U \subseteq \mathbb{R}^m$ be a nonempty open subset. $x \in U$.

Then $H_k(U, U-x) \cong \begin{cases} \mathbb{Z} & k=m \\ 0 & k \neq m. \end{cases}$

Degree of selfmaps of S^n.

映射度

Let $f: S^n \to S^n$, $n \geq 0$, $H_n(f): \underset{\substack{\cong \\ \mathbb{Z}\ 1}}{\tilde{H}_n(S^n)} \longrightarrow \underset{\substack{\cong \\ \mathbb{Z}}}{\tilde{H}_n(S^n)}$

$H_n(f)(1) =: \deg(f)$, is called the degree of f.

$H_n(f)(\alpha) = \deg(f) \cdot \alpha$, $\alpha \in \mathbb{Z}$ is a generator.

Proposition. The followings hold:

(a) $f = id: S^n \to S^n$, $\deg(id) = 1$

(b) $\deg(g \circ f) = (\deg g) \cdot (\deg f)$: $(gf)_* = g_* f_*$

(c) $\deg g = \deg f$ if $f \simeq g$: $f_* = g_*$ $\Rightarrow$ $\deg f$ is a homotopy invariant.

$\deg: [S^n, S^n] \longrightarrow \mathbb{Z}$, $[f] \mapsto \deg f$.

(4) $\deg f = -1$ if f is a reflection of S^n.

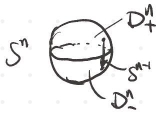
$$S^n = \{(x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \textstyle\sum_{i=1}^{n+1} x_i^2 = 1\}$$

A reflection $r_i: S^n \to S^n$ is defined by

$$r_i(x_1, \cdots, x_i, \cdots, x_{n+1}) = (x_1, \cdots, x_{i-1}, -x_i, x_{i+1}, \cdots, x_{n+1})$$

$r_{n+1}: S^n \to S^n$.

$\deg r_{n+1} = -1$ $\quad \underset{\substack{\cong \\ \mathbb{Z}\langle D^n_+ - D^n_- \rangle}}{H_n(S^n)} \longrightarrow H_n(S^n)$ (fact)



$$r_*(D^n_+ - D^n_-) = D^n_- - D^n_+ = -(D^n_+ - D^n_-)$$

(5) $-1: S^n \to S^n$, $(x_1, x_2, \cdots, x_{n+1}) \mapsto (-x_1, -x_2, \cdots, -x_{n+1})$

$-1 = r_1 \circ r_2 \circ \cdots \circ r_{n+1}$ $\quad \therefore \deg(-1) = (-1)^{n+1}$.

(6) If f is not surjective, then $\deg f = 0$. (exercise)

Local degree. $f: S^n \to S^n$, $n>0$

$\exists\, y \in S^n$, st $f^{-1}(y) = \{x_1, x_2, \cdots, x_m\}$

S^n is Hausdorff, let $U_1, U_2, \cdots, U_m$ be nbhds of $x_1, x_2, \cdots, x_m$, respectively

St. $U_i \cap U_j = \varnothing$, $\forall i \neq j$.

Let V be an nbhd of y st. $f(U_i) \subset V$, $i=1,2,\cdots,m$.

$1 \longmapsto \deg f|x_i$ — local degree

Consider the following commutative diagram:

$$\begin{tikzcd}
 & \mathbb{Z} \cong H_n(U_i, U_i - x_i) \arrow[r, "f_*"] \arrow[d, "k_i"] \arrow[dr, phantom, "①"] & H_n(V, V-y) \cong \mathbb{Z} \arrow[d, "\cong"', "\text{excision}"] \\
H_n(S^n, S^n - x_i) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f'_*"] & H_n(S^n, S^n - y) \cong \mathbb{Z} \\
 & H_n(S^n) \arrow[u, "j_*"] \arrow[r, "f_*"] \arrow[ur, phantom, "②"] & H_n(S^n) \arrow[u, "j_*"', "\cong"]
\end{tikzcd}$$

$H_n(S^n, S^n - f^{-1}(y)) \cong \mathbb{Z}^m$, $\quad j_*(1) = (1,1,\cdots,1)$, $\quad 1 = j_* p_i(1)$, $\quad S^n - y \cong \mathbb{R}^n$

$1 \longmapsto \deg f$

Lemma: $H_n(\amalg_{i=1}^m U_i, \amalg_{i=1}^m (U_i - x_i)) \xrightarrow{\cong} H_n(S^n, S^n - f^{-1}(y))$. (Excision theorem)

$$\begin{tikzcd}
H_n(\amalg_{i=1}^m U_i, \amalg_{i=1}^m (U_i - x_i)) \arrow[r, "\cong"] \arrow[d, phantom, "\cong"] & H_n(S^n, S^n - f^{-1}(y)) \arrow[d, "p_i'"] \arrow[dr, "p_i"] & \\
\oplus_{i=1}^m H_n(U_i, U_i - x_i) \arrow[ur, "\oplus_{i=1}^m k_i"', "\cong"] \arrow[d, phantom, "\cong"] & H_n(U_i, U_i - x_i) \arrow[r, "\cong"] & H_n(S^n, S^n - x_i) \\
\oplus_{i=1}^m \mathbb{Z} & &
\end{tikzcd}$$

By commutativity, $j_*(1) = (1,1,\cdots,1)$, $k_i(1) = 1$, $i=1,\cdots,m$.

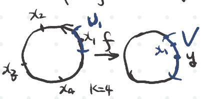
By ①, $f'_* k_i(1) = \underline{\deg f|x_i}$

By ②, $\deg f = f'_* j_*(1) = f'_*(1,1,\cdots,1) = f'_*(k_1(1), k_2(1), \cdots, k_m(1)) = \sum_{i=1}^m \deg f|x_i$

$$\boxed{\text{Prop. } \deg f = \sum_i \deg f|x_i}$$

Example. $f: S^1 \to S^1$, $z \mapsto z^k$, $k \in \mathbb{Z}$, has degree k.

proof. It suffices to prove it when $k>0$.



$\deg f = \sum_{i=1}^k \deg f|x_i$.

$\forall y \in S^1$, $\exists\, x_1, \cdots x_k \in S^1$ st. $f(x_i) = y$, $i=1,2,\cdots,k$.

$U_i \xrightarrow{f} V$ 拉伸 k 倍

$f|U_i : U_i \to V$ is a homeomorphism preserving orientation.

$f|U_i \simeq id$. $\deg f|U_i = 1$

$\therefore \deg f = \sum_{i=1}^k \deg f|x_i = k$. $\square$

Cone The cone CX on a topological space X is the quotient space
拓扑锥

$$CX = X\times I \big/ X\times\{1\}.$$


S^1

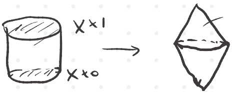
CX is contractible. $\tilde{H}_n(CX)=0,\ \forall n\geq 0.$

Suspension 双角锥 $\Sigma X = CX \cup_X CX$

$$= \frac{X\times I}{X\times\{0\},\ X\times\{1\}}$$

Examples: $CS^n \cong D^{n+1}$ (take $n=1$)

$\Sigma S^n \cong S^{n+1}$ (____)



$C_+X = A$

$C_-X = B$

$A\cap B = X.$

Exercise / Lemma. There is an isomorphism $H_{n+1}(\Sigma X) \xrightarrow[\cong]{\partial} H_n(X).$
(Mayer–Vietoris Sequence)

• Fact: Any $f: X\to Y$ induces maps $Cf: CX \to CY$
$\Sigma f: \Sigma X \to \Sigma Y.$

prop. Let $f: S^n \to S^n$. Then $\deg(\Sigma f) = \deg f$.

proof.

$$\begin{tikzcd} H_n(S^n) \arrow[r, "f_*"] & H_n(S^n) \\ H_n(\Sigma S^n) \arrow[u, "\cong"', "\partial"] \arrow[r, "(\Sigma f)_*"] & H_n(\Sigma S^n) \arrow[u, "\cong"', "\partial"] \end{tikzcd}$$

commutes.

$\therefore \deg(\Sigma f) = \deg f.$ $\square$

$S^1 \xrightarrow{f_k} S^1$, $z\mapsto z^k$.

$\Sigma f: S^2 \to S^2$, $\deg(\Sigma f) = k$

$\Sigma^2 f = \Sigma(\Sigma f): S^3 \to S^3$, $\deg(\Sigma^2 f) = k$.

$\Rightarrow$ $\deg: [S^n, S^n] \longrightarrow \mathbb{Z}$ is surjective.

CW complexes / Cell complexes $\longrightarrow$ for cellular homology. 胞腔同调

Def. A set X is a CW complex or cell complex if there is a chain of subsets

$$\phi = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq \cdots \subseteq X$$

s.t. (i) $X^0 = \{\text{discrete points of } X\}$ — 0-cells.

(ii) $X^n = X^{n-1} \sqcup_\alpha D^n_\alpha / x \sim \varphi_\alpha(x),\ \forall x \in \partial D^n_\alpha = S^{n-1}_\alpha$

$= X^{n-1} \cup_{\varphi_\alpha} D^n_\alpha$ $\qquad e^n_\alpha \cong \mathrm{int}(D^n)$ — n-cells of X.

$\varphi_\alpha : \partial D^n_\alpha = S^{n-1}_\alpha \longrightarrow X^{n-1}$ is called the attaching map of e^n_α.

φ_α can be extends to a map $\Phi_\alpha : (D^n_\alpha, \partial D^n_\alpha) \longrightarrow (X^n, X^{n-1})$

Φ_α is a homeomorphism when restricted to $\mathrm{int}(D^n_\alpha)$.

$e^n_\alpha = \Phi_\alpha(\mathrm{int}(D^n_\alpha))$.

(iii) $X = \bigcup_{i=0}^{n\leq\infty} X^n$. X^n is called the n-skeleton of X.

(iv) X has weak topology — W. $\qquad X^n = X^{n-1} \sqcup_\alpha e^n_\alpha$

$V \subseteq X$ is open (or closed)

$\Leftrightarrow$ $V \cap X^n \subseteq X^n$ is open (or closed).

$\Leftrightarrow$ $V \cap e^n_\alpha \subseteq e^n_\alpha$ is open (or $V \cap \overline{e^n_\alpha} \subseteq \overline{e^n_\alpha}$ is closed).

"C" — the closure of each e^n_α meets only finitely many other cells.

Closure-finiteness

Examples. ① $D^n = \mathrm{int}(D^n) \sqcup \partial D^n = e^n \cup S^{n-1} = e^n \cup e^{n-1} \cup e^0$

$S^n = e^0 \cup e^n$, $S^n \cong \{\infty\} \cup \mathbb{R}^n$ $\qquad \rightarrow D^n/\partial D^n = \dfrac{e^0 \cup e^{n-1} \cup e^n}{e^0 \cup e^{n-1}} \cong e^0 \cup e^n \cong S^n$.

② $nT^2 = T^2 \# \cdots \# T^2$ (n copies)

$= \left(\bigvee_{i=1}^n (S^1_{a_i} \vee S^1_{b_i})\right) \cup_\varphi D^2$

$= e^0 \cup (e^1_{a_1} \cup e^1_{b_1}) \cup \cdots \cup (e^1_{a_n} \cup e^1_{b_n}) \cup e^2$.

$\varphi_{nT^2} : S^1 \longrightarrow \bigvee_{i=1}^n (S^1_{a_i} \vee S^1_{b_i}) = X^1$

$S^1 \longmapsto a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1}$ $\qquad \varphi : S^1 \to a_1^2 a_2^2 \cdots a_m^2$.

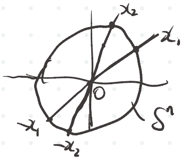
$mP^2 = \left(\bigvee_{i=1}^m S^1_{a_i}\right) \cup_\varphi D^2 = e^0 \cup (e^1_{a_1} \cup \cdots \cup e^1_{a_m}) \cup e^2$.

$X = \bigcup_{n,\alpha} e^n_\alpha$ is called a cell decomposition of X.

③ $\mathbb{R}P^n = \mathbb{R}^{n+1}\setminus\{0\} \Big/ x\sim\lambda x,\ \lambda\in\mathbb{R}\setminus\{0\},\ x\in\mathbb{R}^{n+1}$

real projective space $\quad = S^n \big/ x\sim -x,\ \forall x\in S^n$

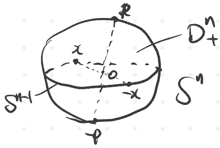
$P^2 = \mathbb{R}P^2.$



$= D^n_+ \big/ x\sim -x,\ x\in\partial D^n = S^{n-1}$

$= e^n \cup \mathbb{R}P^{n-1} = \cdots = e^n\cup e^{n-1}\cup\cdots\cup e^1\cup e^0 = \mathbb{R}P^n$

in particular, $\mathbb{R}P^0 = e^0$, $\mathbb{R}P^1 = e^1\cup e^0 = S^1$.



$D^n_+ = \{(x_1,\cdots,x_{n+1})\in S^n \mid x_{n+1}\geq 0\}.$

④ $\mathbb{C}P^n = \mathbb{C}^{n+1}\setminus\{0\} \Big/ z\sim\lambda z,\ z\in\mathbb{C}^{n+1}\setminus\{0\},\ \lambda\in\mathbb{C}\setminus\{0\}$

complex projective space $\quad (\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2})$

$= S^{2n+1} \big/ z\sim\lambda z,\ |\lambda|=1$

$= D^{2n}_+ \big/ z\sim\lambda z,\ |\lambda|=1,\ z\in\partial D^{2n} = S^{2n-1}$

$= e^{2n}\cup\mathbb{C}P^{n-1} = \cdots = e^{2n}\cup e^{2n-2}\cup\cdots\cup e^2\cup e^0 = \mathbb{C}P^n$

$\Rightarrow \mathbb{C}P^1 = S^2.$

$S^{2n+1} \subseteq \mathbb{C}^{n+1}$

$\{(z_1, z_2,\cdots, z_{n+1})\in\mathbb{C}^{n+1} \mid |z_1|^2+\cdots+|z_{n+1}|^2 = 1\}.$

$A = \{(z_1, z_2,\cdots, z_{n+1})\in S^{2n+1} \mid z_{n+1}\geq 0\}$

$= \{(z_1, z_2,\cdots, z_{n+1})\in S^{2n+1} \mid z_{n+1} = \sqrt{1-|z_1|^2-\cdots-|z_n|^2}\}$

$= \{(w, \sqrt{1-|w|^2})\in\mathbb{C}^n\times\mathbb{C} \mid |w|\leq 1\} = D^{2n}$

$\partial A = \{(w,\sqrt{1-|w|^2})\in\mathbb{C}^n\times\mathbb{C} \mid |w|=1\}$

$= \{(w,0)\in\mathbb{C}^n\times\mathbb{C} \mid |w|=1\} = S^{2n-1}$ $\quad (\mathbb{C}^n = \mathbb{R}^{2n})$

Time: 8.5. Saturday.

开卷. 2小时.

Lecture 11. Cellular Homology Groups 胞腔同调

CW complexes $X = \bigcup_{i=0}^{n\le\infty} X^n$, X^n — n-skeleton of X

$X = nT^2,\ mP^2,\ \mathbb{R}P^n,\ \mathbb{C}P^n$

$X^{n-1},\ X^1, \cdots$

Lemma. Let X be a CW complex. Then

(i) $H_k(X^n, X^{n-1}) = \begin{cases} \mathbb{Z}^{\phi(n)}, & k=n, \quad \phi(n) = \#\{e^n_\alpha\} \\ 0, & k\neq n. \end{cases}$

$$\left(\begin{array}{l} X^n = X^{n-1} \sqcup_\alpha e^n_\alpha \\ H_k(X^n, X^{n-1}) \cong \tilde{H}_k(X^n/X^{n-1}) \cong \tilde{H}_k(\vee_\alpha S^n_\alpha) \end{array} \right)$$

(ii) $H_k(X) = 0$ for $k > \dim X$ if X is of finite dimension. $X^{\dim X} = X$

(iii) the inclusion map $i_n : X^n \hookrightarrow X$ induces an isomorphism

$$i_{n*} : H_k(X^n) \to H_k(X) \text{ for } k<n,$$

and an epimorphism $i_{n*} : H_n(X^n) \to H_n X$.

proof. Exercise.

Rmk: By (i) of the Lemma, we usually say $H_n(X^n, X^{n-1})$ is generated by its n-cells.

Construction

$$\begin{tikzcd}
 & 0 \arrow[d] & & H_n(X) \arrow[r] & 0 \\
 & H_n(X^n) \arrow[urr] \arrow[dr, "j_n"] & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[u, "\partial_{n+1}"] \arrow[r, "d_{n+1} = j_n \circ \partial_{n+1}"] & H_n(X^n, X^{n-1}) \arrow[dr, "\partial_n"] \arrow[rr, "d_n = j_{n-1} \circ \partial_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"] \arrow[dr, two heads] & & \\
 & & 0 \arrow[ur] & & H_{n-1}(X) \arrow[r] & {}
\end{tikzcd}$$

$H_n(X) \cong H_n(X^n)/\mathrm{Im}\, \partial_{n+1}$

Observation: $d_n \circ d_{n+1} = 0$; $\partial_n j_n = 0$

Define $C_n(X) = H_n(X^n, X^{n-1}) \cong \mathbb{Z}\langle e^n_\alpha \rangle$

$$\cdots \to C_{n+1}(X) \xrightarrow{d_{n+1}} C_n(X) \xrightarrow{d_n} C_{n-1}(X) \to \cdots \xrightarrow{d_1} C_0(X) \to 0$$

$$H_n^{CW}(X) := \mathrm{Ker}\, d_n / \mathrm{Im}\, d_{n+1}$$ is called the n-th cellular homology group of the cell complex X.

prop. Let X be a CW complex. Then there is an isomorphism

$$H_n^{CW}(X) \cong H_n(X).$$

proof. By the exactness of " $\nearrow^0$ " we have an isomorphism

$$\begin{tikzcd} H_n(X^n)/\mathrm{Im}\,\partial_{n+1} \arrow[r, "\cong"] \arrow[d, "\cong"', "\tilde{j}_n"] & H_n X \\ \dfrac{\mathrm{Im}\, j_n = \mathrm{Ker}\,\partial_n = \mathrm{ker}\, d_n}{j_n(\mathrm{Im}\,\partial_{n+1}) = \mathrm{Im}\, d_{n+1}} = H_n^{CW}(X) & \end{tikzcd}$$

Since j_n is injective, we have

$$j_n(\mathrm{Im}\,\partial_{n+1}) = \mathrm{Im}(j_n \circ \partial_{n+1}) = \mathrm{Im}\, d_{n+1}$$

By the exactness of " $\searrow$ ", we have

$$\mathrm{Im}\, j_n = \ker \partial_n .$$

Since j_{n-1} is injective, we have

$$\ker d_n = \mathrm{Ker}(j_{n-1} \circ \partial_n) = \ker \partial_n .$$

Thus we get an isomorphism $H_n^{CW}(X) \xrightarrow{\cong} H_n(X)$. $\square$

Applications. $H_n^{CW}(X) \cong H_n(X)$, $C_n(X) \cong \mathbb{Z}\langle e_\alpha^n \rangle \xrightarrow{d_n} C_{n-1}(X) \cong \mathbb{Z}\langle e_\beta^n \rangle$

① If $k > \dim X$, $H_k(X) = 0$; or more general, if X has no n-cells, then $H_n(X) = 0$.

② If X is a CW complex having no two cells in adjacent dimensions, then $H_n(X)$ is either 0 or is isomorphic to $C_n(X)$.

Eg. $\mathbb{C}P^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$,

$$H_k(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & k = 0, 2, \cdots, 2n; \\ 0 & \text{otherwise.} \end{cases}$$

③ $H_0(X) \cong \mathbb{Z}$ if X is path-connected.

$$\underset{\displaystyle H_0^{CW}(X)}{\overset{\displaystyle \wr\!\wr}{}} = \frac{C_0(X) = \mathbb{Z}}{\mathrm{Im}\, d_1} \cong \mathbb{Z} \qquad C_1(X) \xrightarrow{d_1 = 0} \underset{\overset{\|}{\mathbb{Z}}}{C_0(X)} \to 0$$

Further assume that X has a unique 0-cell, then $H_0(X) \cong \mathbb{Z}$ implies that $\underline{d_1 = 0}$.

④ The Euler–Poincaré characteristic of a CW complex :

$$\chi(X) = \sum_n (-1)^n \#\{e_\alpha^n\} = \sum_n (-1)^n \operatorname{rank} C_n(X).$$

For an abelian group A, $\operatorname{rank}(A) = r$ if $\mathbb{Z}^r \hookrightarrow A$ and $\mathbb{Z}^{r+1} \not\hookrightarrow A$.
$\operatorname{rank}(\mathbb{Z}^n) = \operatorname{rank}(\mathbb{Z}^n \oplus \mathbb{Z}/m) = n$.

Lemma. (i) If B is a subgroup or a quotient group of an abelian group A,

then $\operatorname{rank}(B) \leq \operatorname{rank}(A)$

(ii) If there is a short exact seq. of abelian groups,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

if $\operatorname{rank}(B) < \infty$, then $\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C)$.

proof. Exercise.

Let X be a finite CW complex.

prop. $\chi(X) = \sum_n (-1)^n \operatorname{rank} C_n(X) = \sum_n (-1)^n \underline{\operatorname{rank} H_n(X)}$.

$\Rightarrow$ The Euler-Poincaré characteristic is a homotopy invariant.

proof of $\chi = \sum_n (-1)^n \operatorname{rank} C_n = \sum_n (-1)^n \operatorname{rank} H_n$.

$$C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}$$

Let $\ker d_n = Z_n$, $\operatorname{Im} d_{n+1} = B_n$. then $H_n = Z_n/B_n \Leftrightarrow 0 \to B_n \to Z_n \to H_n \to 0$ is exact.

$$0 \to \underset{\substack{\| \\ Z_n}}{\ker d_n} \longrightarrow C_n \xrightarrow{d_n} \operatorname{Im} d_n = B_{n-1} \to 0 \quad \text{is exact.}$$

By the Lemma above, $\operatorname{rank} Z_n = \operatorname{rank} B_n + \operatorname{rank} H_n$

$\operatorname{rank} C_n = \operatorname{rank} Z_n + \operatorname{rank} B_{n-1}$

$\Rightarrow \operatorname{rank} C_n = \operatorname{rank} H_n + \operatorname{rank} B_n + \operatorname{rank} B_{n-1}$.

$$\sum_n (-1)^n \operatorname{rank} C_n = \sum_n (-1)^n \operatorname{rank} H_n. \qquad \square$$

prop. (Cellular Boundary Formula)

Let $d_n : C_n(X) \longrightarrow C_{n-1}(X)$ be the boundary homomorphism. Then

$$\boxed{d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta}$$

where $d_{\alpha\beta}$ is the degree of the composition

$$\underline{\Delta_{\alpha\beta} : \partial D^n_\alpha = S^{n-1}_\alpha \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q} X^{n-1}/X^{n-2} = \vee_\beta S^{n-1}_\beta \xrightarrow{q_\beta} S^{n-1}_\beta}.$$

recall e^n_α has the characteristic map $\Phi_\alpha : D^n_\alpha \longrightarrow X^n$, $\Phi_\alpha|_{\partial D^n_\alpha} = \varphi_\alpha$.

proof. Consider the following comm. diagram induced by Φ_α:

$$\begin{tikzcd}
H_n(D^n_\alpha, \partial D^n_\alpha) \arrow[r, "\partial_n"] \arrow[d, "\Phi_{\alpha*}"] & \widetilde{H}_{n-1}(\partial D^n_\alpha = S^{n-1}_\alpha) \arrow[r, "\Delta_{\alpha\beta*}"] \arrow[d, "\varphi_{\alpha*}"] & \widetilde{H}_{n-1}(S^{n-1}_\beta) \\
H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"] & \widetilde{H}_{n-1}(X^{n-1}) \arrow[r, "q_*"] \arrow[d, "j_{n*}"] & \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}) \arrow[u, "q_{\beta*}"] \\
& H_{n-1}(X^{n-1}, X^{n-2}) &
\end{tikzcd}$$

$[D^n_\alpha] \mapsto e^n_\alpha$, $\quad H_n(X^n, X^{n-1}) = C_n(X) \ni e^n_\alpha$, $\quad \partial_n[D^n_\alpha] = [\partial D^n_\alpha]$

$\widetilde{H}_{n-1}(X^{n-1}/X^{n-2}) \cong \widetilde{H}_{n-1}(\vee_\beta S^{n-1}_\beta)$, $\quad H_{n-1}(X^{n-1}, X^{n-2}) = C_{n-1}(X)$

By the commutativity of the left square,

$$d_n(e^n_\alpha) = d_n \Phi_{\alpha*}[D^n_\alpha] = j_{n*}\varphi_{\alpha*}\partial_n[D^n_\alpha]$$

Since $q_{\beta*}$ is the projection, we get

$$\varphi_* \partial_n [D^n_\alpha] = \Sigma_\beta \Delta_{\alpha\beta*}(\partial[D^n_\alpha])$$
$$= \Sigma_\beta d_{\alpha\beta} e^{n-1}_\beta. \qquad \square$$

Examples. ① $nT^2 = T^2 \# \cdots \# T^2$ (n copies)

$$X^1 = V_n(S^1 \vee S^1) = e^0 \cup \bigcup_{i=1}^n (e^1_{a_i} \cup e^1_{b_i})$$
$$X^2 = nT^2.$$

$H_0(nT^2) \cong \mathbb{Z}$, $H_k(nT^2) = 0$ for $k > 2$.

$$0 \to C_2(nT^2) \xrightarrow{d_2} C_1(nT^2) \xrightarrow{d_1 = 0} C_0(nT^2)$$
$$0 \to \mathbb{Z}e^2 \xrightarrow{d_2} \mathbb{Z}^{2n}\langle e^1_{a_1}, e^1_{b_1}, \cdots, e^1_{a_n}, e^1_{b_n}\rangle$$

(with $C_2(nT^2) = \mathbb{Z}e^2$ and $C_1(nT^2) \cong \mathbb{Z}^{2n}\langle \cdots \rangle$)

$$\partial D^2 = S^1 \xrightarrow{\varphi} X^1 = V_{i=1}^n (S^1_{a_i} \vee S^1_{b_i}) \xrightarrow{q_{a_i}} S^1_{a_i}$$
$$\varphi = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1}.$$

Since $\pi_1 S^1 \cong \mathbb{Z}$, $q_{a_i} \circ \varphi$ is nullhomotopic (零伦: homotopic to the constant map).

$$d_{a_i} = 0$$
$$\therefore d_2 = 0. \quad \Rightarrow H_2(nT^2) \cong \mathbb{Z}\langle e^2\rangle$$
$$H_1(nT^2) \cong \mathbb{Z}^{2n}.$$

② Compute $H_i(mP^2)$. Exercise.

③ $H_i(\mathbb{RP}^n)$.

$\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup e^n = e^0 \cup e^1 \cup \cdots \cup e^{n-1} \cup e^n$.

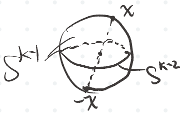
$C_i(\mathbb{RP}^n) \cong \mathbb{Z}$, $i=0,1,\dots,n$; $C_i(\mathbb{RP}^n) = 0$ for $i>n$.

$H_0(\mathbb{RP}^n) \cong \mathbb{Z}$, $H_k(\mathbb{RP}^n) = 0$ for $k>n$.

$$0 \to \underset{\mathbb{Z}}{\underset{\|}{C_n}} \xrightarrow{d_n} \underset{\mathbb{Z}}{\underset{\|}{C_{n-1}}} \xrightarrow{d_{n-1}} \underset{\mathbb{Z}}{\underset{\|}{C_{n-2}}} \to \cdots \to \underset{\mathbb{Z}}{\underset{\|}{C_2}} \xrightarrow{d_2} \underset{\mathbb{Z}}{\underset{\|}{C_1}} \xrightarrow{d_1=0} \underset{\mathbb{Z}}{\underset{\|}{C_0}}$$

For $1 \le k \le n$, $d_k(e^k) = y_k e^{k-1}$, y_k is the degree of the composition

$$S^{k-1} \xrightarrow{\varphi_k} \mathbb{RP}^{k-1} \xrightarrow{q} \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} = S^{k-1}.$$



Recall $\mathbb{RP}^{k-1} = S^{k-1}/x\sim -x$. $[x] = \{x, -x\}$.

$$\deg(q\varphi_k) = \deg(q\varphi_k)|_x + \deg(q\varphi_k)|_{-x}.$$

$S^{k-1} - S^{k-2} = \mathring{D}^{k-1}_+ \sqcup \mathring{D}^{k-1}_-$

- The composition $q\varphi_k$ restricted to $\mathring{D}^{k-1}_+$ or $\mathring{D}^{k-1}_-$ is a homeomorphism

For a homeomorphism, its degree is ± 1.

Assume $\deg q\varphi_k|_{\mathring{D}^{k-1}_+} = 1$, $(q\varphi_k)|_{\mathring{D}^{k-1}_-} = (q\varphi_k)|_{\mathring{D}^{k-1}_+} \circ (-1)$

$$S^{k-1} \xrightarrow{-1} S^{k-1} \xrightarrow{\varphi_k} \mathbb{RP}^{k-1} \xrightarrow{q} S^{k-1} \qquad \mathring{D}^{k-1}_- \xrightarrow{-1} \mathring{D}^{k-1}_+$$

$$x \mapsto -x$$

$q\varphi_k|_{\mathring{D}^{k-1}_-}$ is the restriction of the composition $(q\varphi_k \circ -1)$ to $\mathring{D}^{k-1}_+$

$$\deg(q\varphi_k)|_{\mathring{D}^{k-1}_-} = \deg q\varphi_k|_{\mathring{D}^{k-1}_+} \cdot \underbrace{\deg(-1)}_{(-1)^k} = (-1)^k.$$

Thus $d_k e^k = (1+(-1)^k) \cdot e^{k-1}$.

$$0 \to \underset{C_n}{\mathbb{Z}} \xrightarrow{d_n} \mathbb{Z} \to \cdots \to \underset{C_2}{\mathbb{Z}} \xrightarrow{2} \underset{C_1}{\mathbb{Z}} \xrightarrow{d_1=0} \mathbb{Z} \to 0$$

$$\Rightarrow H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}/2 & \text{for } 1 \le k \le n,\ k \text{ odd} \\ \mathbb{Z} & \text{for } k=0, \text{ or } k=n \text{ is even.} \\ 0 & \text{otherwise.} \end{cases}$$

Cor. $n=2$, $\mathbb{RP}^2$ $\quad H_k(\mathbb{RP}^2) = \begin{cases} \mathbb{Z}/2 & k=1 \\ \mathbb{Z} & k=0 \\ 0 & k\ge 2 \end{cases}$

④ $M(\mathbb{Z}/m, n)$, $m, n \in \mathbb{N}$.

if $m=0$, $M(\mathbb{Z}, n) = S^n$

if $m>0$, $M(\mathbb{Z}/m, n) = S^n \cup_{\varphi_m} e^{n+1}$, $\varphi_m : S^n \to S^n$ has degree m.

$$\tilde{H}_k(M(\mathbb{Z}/m, n)) = \begin{cases} \mathbb{Z}/m, & k=n \\ 0 & k \neq n \end{cases}$$

proof : Exercise.

For a finitely generated abelian group $G \cong \mathbb{Z}^k \oplus \bigoplus_{i=1}^{l} \mathbb{Z}/p_i^{r_i}$,

p_i is a prime.

Let $X = \left(\bigvee_{j=1}^{k} S_j^n\right) \vee \left(\bigvee_{i=1}^{l} M(\mathbb{Z}/p_i^{r_i}, n)\right) = M(G, n)$

Moore space of type (G, n).

Then $\tilde{H}_m(X) \cong \bigoplus_{j=1}^{k} \tilde{H}_m(S_j^n) \oplus \bigoplus_{i=1}^{l} \tilde{H}_m(M(\mathbb{Z}/p_i^{r_i}, n))$

$$\cong \begin{cases} \mathbb{Z}^k \oplus \bigoplus_{i=1}^{l} \mathbb{Z}/p_i^{r_i} = G, & m=n \\ 0, & m \neq n. \end{cases}$$

Thus Every finitely generated abelian group can be realised as the reduced homology group of a CW complex.

Moore Space.

Homology groups with coefficients

$H_k(X; G)$, G abelian group

$C_k(X) = \{ \sum_i k_i \cdot \sigma_i \mid \sigma_i : \Delta^k \to X, k_i \in \mathbb{Z} \}$ $\mathbb{Z}$ module

$C_k(X; G) = \{ \sum_i g_i \sigma_i \mid \sigma_i : \Delta^k \to X, g_i \in G \} \cong C_k(X) \otimes G$

$\partial_k(\sum_i k_i \sigma_i) = \sum_i k_i (\partial_k \sigma_i)$, $\partial_k^G (\sum_i g_i \sigma_i) = \sum_i g_i \cdot \partial_k \sigma_i$.

$\partial_k \partial_{k+1} = 0 \to \partial_k^G \partial_{k+1}^G = 0 \implies H_k(X; G) = \operatorname{Ker} \partial_k^G / \operatorname{Im} \partial_{k+1}^G$.

- All other properties hold for $H_k(X, A; G)$ (with $H_k(X,A) = H_k(X, A; \mathbb{Z})$).

The 7 Eilenberg–Steenrod axioms still hold for $H_k(X,A;G)$.

- Cellular boundary formula

$$d_n e^n_\beta = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta, \quad d_{\alpha\beta} = \deg \Delta_{\alpha\beta}:$$

$$\Delta_{\alpha\beta}: S^{n-1}_\alpha \longrightarrow S^{n-1}_\beta.$$

Lemma. If $f: S^n \to S^n$ has degree m, then

$f_*: H_n(S^n; G) \to H_n(S^n; G)$ is the multiplication by m.

$(f_* = \mu_m: g \mapsto mg)$.

proof. A homomorphism $\varphi: G \to H$ induces a homomorphism

$$\varphi_\#: C_k(X;G) \longrightarrow C_k(X;H)$$

$$\sum_i g_i \sigma_i \longmapsto \sum_i \varphi(g_i)\sigma_i$$

and hence $\partial\varphi_\# = \varphi_\#\partial$ induces a homomorphism $\varphi_*: H_k(X;G) \to H_k(X;H)$.

$\varphi: \mathbb{Z} \to G$, $1 \mapsto g_0$

$$\begin{array}{ccccccc} H_n(S^n;\mathbb{Z}) & \xrightarrow{f_*} & H_n(S^n;\mathbb{Z}) & & 1 & \longmapsto & \deg f = m \\ \downarrow \varphi_* & & \downarrow \varphi_* & & \downarrow & & \downarrow \\ H_n(S^n;G) & \xrightarrow{f_*} & H_n(S^n;G) \cong & & g & \xrightarrow{f_* = \mu_m} & m \cdot g \end{array}$$

$\square$

Cor. $d_n: C_n(X;G) \to C_{n-1}(X;G)$

$d_n(e^n_\alpha) = \sum_\beta d^G_{\alpha\beta} e^{n-1}_\beta$, $d^G_{\alpha\beta} = \mu_{d_{\alpha\beta}}$, $d_{\alpha\beta}$ is the degree of $\Delta_{\alpha\beta}: S^{n-1}_\alpha \to S^{n-1}_\beta$. $\square$

Exercise. Compute $H_k(\mathbb{R}P^n; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & k=0, 1, \dots, n; \\ 0 & k>n. \end{cases}$

$\square$

Example. $H_k(S^n; G) = \begin{cases} G & k=0, n \\ 0 & k\neq 0, n \end{cases}$.