

## Lecture 0. Euler's formula for polyhedra.

- What's Topology?

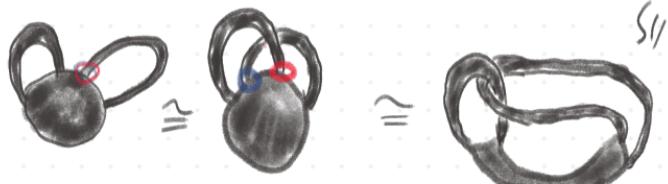
- Topology is a mathematical subject that studies properties of geometric objects that are preserved under continuous transformations.

Topology is a qualitative geometry.

If two geometric objects A and B can be continuously transformed from one to the other, then we say A and B are topologically equivalent.  
Denote by  $A \cong B$ .

Examples.  $\circ A \cong R$ ,  $D \cong O$ ,  $E \cong F \cong T \cong Y$

$$B \cong S$$



③ A knot is an embedding of  $S^1$  into  $\mathbb{R}^3$

$$S^1 \cong \text{knot}$$

isotopy

# Euler's formula for polyhedra (1750')

Polyhedron: A polyhedron  $P$  is a collection of ~~finitely many~~ plane polygons that fit together nicely:

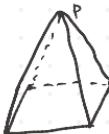
- (i) If two plane polygons meet, then they meet at one edge.
- (ii) For each edge, there are exactly two polygons containing it.
- (iii) Given a vertex  $p$  of  $P$ , the polygons containing  $p$  form a piece of surface of  $P$  around  $p$ .

Examples



$$v-e+f$$

$$4-6+4=2$$



$$5-8+5=2$$



$$6-12+6=2$$



$$7-10+$$



$$8-12+6=2$$

Given a polyhedron  $P$ ,  $v = \#\{\text{vertices of } P\} = v(P)$

$$e = \#\{\text{edges of } P\} = e(P)$$

$$f = \#\{\text{faces of } P\} = f(P)$$

$\chi(P) = v - e + f$  is called the Euler characteristic of  $P$ .

Theorem (Euler) Let  $P$  be a polyhedron such that

- (a) Any two vertices of  $P$  can be connected by a chain of edges.



- (b) Any loop in  $P$  that is made of line segments in  $P$

separates  $P$  into two pieces.

Then  $\chi(P) = 2$ .

proof. By graph theory.



(connected) graph: consists of vertices and edges, and any two vertices can be connected by a chain of edges.

A tree  $T$  is a graph that doesn't contain loops.



$$\text{Fact: } V(T) - e(T) = 1$$

Fact: every connected graph contains a tree.  
maximal

The assumption (a) tells that the vertices and edges of  $P$  form a graph.

Let  $T$  be a tree on  $P$  that contains all vertices of  $P$ , and partial edges. construct its dual graph  $T^*$ .

(i) For every face  $f$  of  $P$  corresponds to a vertex of  $T^*$ .

(ii) Two vertices of  $T^*$  are connected by an edge if the original faces meet in a common edge that is not in  $T$ .

By construction,

$$V(T^*) = f(P)$$

$$e(T^*) + e(T) = e(P)$$

$$V(T^*) = V(P)$$

$$T \cap T^* = \emptyset$$

$$\begin{aligned} X(P) &= V(P) - e(P) + f(P) = V(T^*) - e(T^*) - e(T) + V(T) \\ &= 1 + V(T) - e(T) \end{aligned}$$

$\therefore$  The proof completes by showing that  $T^*$  is a tree.

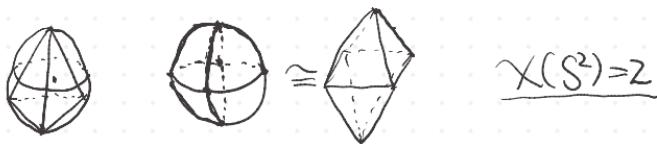
Assume that there is a loop in  $T^*$ . By (b), the loop separates  $P$  into two pieces, and these two pieces contain at least one vertex of  $P$ .

There exist a chain of edges in  $T$  that connects these two vertices.  
It follows that the chain intersects the loop.

However, we have  $T \cap \Gamma = \emptyset$ . Thus there are edges of the chain  
lying outside of  $T$ , contradiction.  $\square$

Therefore  $T$  is a tree.

Theorem. If  $P$  and  $K$  are two polyhedra, then  $X(P) = X(K)$ .



Application. If  $P$  and  $K$  are two orientable surfaces, then

$$P \cong K \Leftrightarrow X(P) = X(K).$$

# Lecture 1. Topological Spaces

$\mathbb{R}, \mathbb{R}^n$

A set  $X$  together with a collection  $\mathcal{T}$  of subsets of  $X$ , called open sets, satisfying the following three axioms:

$$(1) \quad \emptyset, X \in \mathcal{T}$$

$$(2) \quad \forall A_\alpha \in \mathcal{T}, \text{ then } \bigcup_\alpha A_\alpha \in \mathcal{T}$$

$$(3) \quad A_1, A_2 \in \mathcal{T} \Rightarrow A_1 \cap A_2 \in \mathcal{T}.$$

Example:  $\mathbb{R} \quad \mathcal{T} = \{(a, b) \mid a < b\}$   $\mathcal{T}(\mathbb{R}^n) = \left\{ B(x, r) \mid \begin{array}{l} y \in \mathbb{R} \\ |x - y| < r \end{array} \right\}$

The pair  $(X, \mathcal{T})$  is called a topological space.

Let  $(X, \mathcal{T})$  be a topological space. A subset  $U \subseteq X$  is closed if  $U^c = X - U \in \mathcal{T}$ .

Review: De Morgan's laws:  $(\bigcup_\alpha A_\alpha)^c = \bigcap_\alpha A_\alpha^c$  并之补=补之交

$(\bigcap_\alpha A_\alpha)^c = \bigcup_\alpha A_\alpha^c$  交之补=补之并

Distributive law:  $A \cup (\bigcap_\alpha B_\alpha) = \bigcap_\alpha (A \cup B_\alpha)$

$A \cap (\bigcup_\alpha B_\alpha) = \bigcup_\alpha (A \cap B_\alpha)$

Exercise. Define closed sets on  $X$  by 3 axioms

- Every set  $X$  has at least two topologies.

trivial topology:  $\mathcal{T}_0 = \{\emptyset, X\}$

discrete topology:  $\mathcal{T}_{\text{discrete}} = \{\text{subsets of } X\}$

$T_0 \subseteq T_{\text{co}}$

Every topology  $T$  on a set  $X$  satisfies  $T_0 \subseteq T \subseteq T_{\text{co}}$

Given a set  $X$  and two topologies  $T_1, T_2$  on  $X$ . We say that

$T_1$  is coarser than  $T_2$ , or equivalently,  $T_2$  is finer than  $T_1$ ,

if  $T_1 \subseteq T_2$ .

A neighborhood  $U_x$  of  $x \in X$  is an open set containing  $x$ .

( $x \in U_x \subseteq U_x \subseteq X$ , Somebody likes this definition)

Let  $A \subseteq X$  be a subset of a topological space  $X$ , and let  $x \in X$ .

There are exactly 3 possibilities for the positions of  $x$  relative to  $A$ :



- (1) There exists a  $U_x \subseteq A$ .
- (2) There exist a  $U_x \subseteq A^c = X - A$
- (3) Every  $U_x$  satisfies  $U_x \cap A \neq \emptyset, U_x \cap A^c \neq \emptyset$

Defn interior point:  $x \in A$  if  $\exists U_x \subseteq A$ .

$x \in \partial A$  if every  $U_x \subseteq X$  satisfies  $U_x \cap A \neq \emptyset, U_x \cap A^c \neq \emptyset$

$x$  is a limit point of  $A$  if every  $U_x \cap A \neq \emptyset$ .

$$x_1, x_2, \dots, x_n, \dots \rightarrow x \quad (A^\circ)$$

The closure  $\bar{A}$  is the set of all limit points of  $A$ .

The interior  $\text{int}(A) = A^\circ$  is the set of all interior point

There hold formulas:

$$\bar{A} = \partial A \sqcup \text{int}(A)$$

$$X = \text{int}(A) \sqcup \partial A \sqcup \text{int}(A^c)$$

<sup>= proposition</sup>  
prop. Let  $A$  be a subset of  $X$ . The followings hold:

- (1)  $\text{int}(A)$  is open.
- (2)  $\overline{A}$  is closed.
- (3)  $A$  is open iff  $A = \text{int}(A)$ .
- (4)  $A$  is closed iff  $A = \overline{A}$ .

Proof. Exercise.

### Topology basis

Let  $(X, \tau)$  be a topological space. A collection  $B$  of open sets of  $X$  ( $B \subseteq \tau$ ) is called a basis for  $\tau$  if every open set of  $X$  is a union of subsets of  $B$ .

If  $B$  is a basis for  $\tau$ , then

- (1)  $X = \bigcup_{\alpha} B_\alpha, B_\alpha \in B$
- (2)  $\forall B_1, B_2 \in B, B_1 \cap B_2 = \bigcup_{\lambda} B_\lambda, B_\lambda \in B$ .

prop. Let  $\mathcal{B}$  be a collection of subsets of  $X$  satisfying (1), (2) above.

Then  $\mathcal{B}$  is a basis for a topology on  $X$ .

Here a "topology" is the topology generated by  $\mathcal{B}$ , denoted by  $\tau_{\mathcal{B}}$ .

$U \subseteq X$  is open  $\Leftrightarrow U = \bigcup_{\alpha} B_\alpha, B_\alpha \in \mathcal{B}$ .

Proof. Exercise.

$$U \cap V = (\bigcup_{\alpha} B_\alpha) \cap (\bigcup_{\lambda} B_\lambda) = \bigcup_{\alpha, \lambda} B_\alpha \cap B_\lambda \in \tau_{\mathcal{B}}. \quad \square$$

metric spaces Def. A set  $X$  is called a metric space if there is a

Eg.  $\mathbb{R}^n$  metric  $d: X \times X \rightarrow \mathbb{R}$  satisfying the following 3 axioms.

distance

(i)  $d(x, y) \geq 0$ , and  $d(x, y) = 0$  iff  $x = y$ .

(ii)  $d(x, y) = d(y, x)$ ; or equivalently  $d \circ T = d$ ,  $T: X \times X \rightarrow X \times X$ ,  $T(x, y) = (y, x)$

(iii)  $d(x, y) + d(y, z) \geq d(x, z)$ .



Example:  $(\mathbb{R}^n, d)$ .  $d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$

$$d_1(x, y) = \sum_{i=1}^n |x_i - y_i|$$

$$d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$$

Define  $B(x, r) = \{y \in X \mid d(x, y) < r\}$ ,  $x \in X, r > 0$ .

prop. Let  $(X, d)$  be a metric space. Then

the collection  $B = \{B(x, r) \mid x \in X, r > 0\}$  is a basis for a topology on  $X$ . the generating topology is called the metric topology, and is denoted by  $T_d$ .

proof. It suffices to show that  $\forall x_1, x_2$  two different points in  $X$ , and  $r_1, r_2 > 0$ ,

$$B(x_1, r_1) \cap B(x_2, r_2) = \bigcup_{x, r} B(x, r).$$

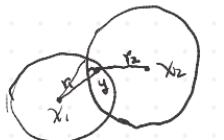
$\forall y \in B(x_1, r_1) \cap B(x_2, r_2)$ ,

(claim. For  $s < r - d(x, y)$ , there holds  $B(y, s) \subseteq B(x, r)$ .)

Take  $s < \min\{r_1 - d(y, x_1), r_2 - d(y, x_2)\}$ , then by the claim,

we have  $B(y, s) \subseteq B(x_1, r_1) \cap B(x_2, r_2)$ .

$$\bigcup_{y, s} B(y, s) = B(x_1, r_1) \cap B(x_2, r_2). \quad \square$$



Exercise. Show the above claim.

## Subspace

prop. Let  $(X, \tau)$  be a topological space, and let  $A \subseteq X$  be a subset.

Then  $A$  inherits a topology from  $\tau$ .

proof. Let  $\tau_A = A \cap \tau = \{A \cap U \mid U \in \tau\}$ .

Check that  $\tau_A$  is a topology on  $A$ .  $\square$

The topology  $\tau_A$  is called the subspace topology.  $(A, \tau_A) \subseteq (X, \tau)$ .

If  $B$  is a topology basis for  $(X, \tau)$ , then  $B_A = A \cap B$  is a basis for  $\tau_A$ .

If  $(X, d)$  is a metric space and  $A \subseteq X$ , then

$$\tau = \tau_d, \quad d_A : A \times A \xrightarrow{i \times i} X \times X \xrightarrow{d} \mathbb{R}$$

is a metric on  $A$ .

Exercise.  $\tau_{d_A} = \tau_A$ : the induced metric topology by  $\tau_d$  coincides with the subspace topology  $\tau_A$ .

Let  $X \subseteq Y$  be a subspace and let  $A \subseteq X$  be a subset. Then

(1) If  $X$  is open (resp. closed), then  $A \subseteq X$  is open (resp. closed) implies  $A \subseteq Y$  is open (resp. closed).

$$(2) \quad \overline{A} \text{ in } X = (\overline{A} \text{ in } Y) \cap X.$$

proof. Exercise.

## product Spaces

Let  $(X, \tau_1)$  and  $(Y, \tau_2)$  be topological spaces. The Cartesian product

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$$

$X \times Y$  inherits a topology, called the product topology,

$$U_{(a,b)} \times (c,d)$$

$$\tau = \bigcup_{\alpha, \beta} U_\alpha \times V_\beta, \quad U_\alpha \in \tau_1, V_\beta \in \tau_2.$$

$$\begin{cases} \text{If } \tau_1 = \tau_{B_1}, \\ \tau_2 = \tau_{B_2} \\ \text{then } \tau = \tau_{B_1 \times B_2} \end{cases}$$