Lecture 3. Compactness.

Def. A space X is <u>compact</u> if every open <u>cover</u> of X has/admits finite open subcover.
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A cover is a collection $\{X_\alpha\}$ of subsets of X s.t. $X = \bigcup_\alpha X_\alpha$.

Prop. Let $f: X \to Y$ be a surjective map. If X is compact, then so is Y.
(the image of compact space is compact).

proof. Let $\{V_\alpha\}$ be an open cover of Y. Then $U_\alpha = f^{-1}(V_\alpha)$ cover X.

Since X is compact, $X = U_1 \cup \cdots \cup U_n$, $U_1, \cdots, U_n \in \{U_\alpha\}$.

Thus $Y = f(X) = f\left(\bigcup_{i=1}^n U_i\right) = \bigcup_{i=1}^n f(U_i) = \bigcup_{i=1}^n V_i$; i.e. Y is compact. $\square$

Cor. Compactness is a topological property.

Example. $\mathbb{R}^1$ is not compact.

$\mathbb{R}^1 = \bigcup_{i=1}^\infty (-i, i)$ doesn't admit finite subcover.

Heine-Borel thm. (a,b) is not compact.

Fact. $[a,b]$ is a compact subset of $\mathbb{R}^1$, for any $a<b$.

Rmk. A subset $A \subseteq X$ is <u>compact</u> if every open cover $\{U_\alpha\}$ of A in X has a finite subcover $\{U_i \mid i=1,\cdots,n\}$, s.t. $A \subseteq \bigcup_{i=1}^n U_i$. $(n<\infty)$.

$$A = \bigcup_\alpha (V_\alpha \cap A),\ V_\alpha \subseteq X \text{ open}.$$
$$= \bigcup_{i=1}^n (V_i \cap A) = \left(\bigcup_{i=1}^n V_i\right) \cap A$$
$$\Leftrightarrow A \subseteq \bigcup_{i=1}^n V_i.$$

Prop. <u>Closed</u> subsets of a compact space are compact, as subspaces.

Proof. Let X be a compact space and let $A \subseteq X$ be closed. $X-A$ is open.

Given any open cover $\{U_\alpha\}$ of A in X, then $\{U_\alpha, X-A\}$ is an open cover of X.

Since X is compact, $X = \left(\bigcup_{i=1}^n U_i\right) \cup (X-A)$, which implies $A \subseteq \bigcup_{i=1}^n U_i$. A is compact. $\square$

《Counterexamples in Topology》.

Prop. Compact subsets of a Hausdorff space are closed.

proof. Let X be a Hausdorff space and let $A \subseteq X$ be compact.

Need to show A is closed: $\overline{A} \subseteq A$.

We may assume ~~that~~ $\phi \neq A \subsetneq X$. Then $\exists\, x \in X \setminus A$.

Since X is Hausdorff, $\forall\, a \in A$, $\exists\, U_x$ and V_a in X s.t. $U_x \cap V_a = \phi$.

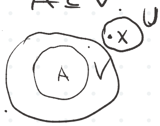
Let $V' = \bigcup_{a \in A} V_a$. Then V is an open cover of A.

Since A is compact, $A \subseteq V = \bigcup_{i=1}^{n} V_i$.

Let $U_1, \cdots, U_n$ be the corresponding nbhds of x such that $V_i \cap U_j = \phi$.

Form $U = \bigcap_{j=1}^{n} U_j$, then $x \in U$, $U \cap V = \phi$.

$A \subseteq V$. Thus every point out of A is not a limit point of A.



That is, every limit point of A lies in A: $\overline{A} \subseteq A$.

$\square$

Cor. Let $f: X \to Y$ be a surjective map with X compact and Y Hausdorff.

Then f is an open/closed map.

Moreover, if f is a bijective, then f is a homeomorphism.

proof. Exercise.

prop. Let X and Y be compact space, then $X \times Y$ is compact.

proof. Let $\{O_\alpha\}$ be an open cover of $X \times Y$. $\forall\, (x, y) \in X \times Y$,

$\exists\, (x, y) \in U_{xy} \times V_{xy} \subseteq O_\alpha$ for some α.

Fix x and let y vary, $\{U_{xy} \times V_{xy} \mid y \in Y\}$ is an open cover of $\{x\} \times Y$.

Since Y is compact, $Y \xrightarrow{\cong} \{x\} \times Y$, $\{x\} \times Y \subseteq \bigcup_{i=1}^{n} (U_{xy_i} \times V_{xy_i})$.

Let $U_x = \bigcap_{j=1}^{n} U_{xy_j}$ be the nbhd of x. Then $U_x \times V_{xy_1}, \cdots, U_x \times V_{xy_n}$ cover $\{x\} \times Y$.

Let x vary. Then $\{U_x\}$ is an open cover of X.

Since X is compact, $X = \bigcup_{i=1}^{m} U_i$, $U_i \in \{U_x\}$. Let $V_j = V_{x y_j}$.

Then $\{ U_i \times V_j \mid i=1,\dots,m;\ j=1,\dots,n \}$ is an open cover of $X \times Y$.

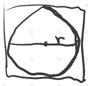
Thus $X \times Y$ is compact. $\square$

Rmk. If X_α are compact spaces, then $\prod_\alpha X_\alpha$ is compact.

Theorem. A subspace $X \subseteq \mathbb{R}^n$ is compact $\Leftrightarrow$ X is bounded and closed.

proof. $\Leftarrow$: X is bounded: $\exists\, r > 0$ and $x \in X$, such that $X \subseteq B(x,r) \subseteq \mathbb{R}^n$.

$X \subseteq B(x,r) \subseteq [-r,r] \times \cdots \times [-r,r]$.



By the above Proposition and the fact that $[-r,r]$ is compact. X is a closed subset of a compact space, X is compact.

$\Rightarrow$ Suppose $X \subseteq \mathbb{R}^n$ is compact.

$$\mathbb{R}^n = \bigcup_{r=1}^{\infty} B(0,r).$$

X is compact in $\mathbb{R}^n$ $\Rightarrow$ $X \subseteq \bigcup_{i=1}^{n} B(0,r_i) = B(0,r_n) \subseteq \mathbb{R}^n$.

Hence X is bounded.

Since $\mathbb{R}^n$ is Hausdorff, we derive that X is closed. $\square$

Cor. Let $f: X \to \mathbb{R}$ be a continuous function. If X is compact then $\mathrm{Im} f = f(X) = [f(x_1), f(x_2)]$ for $x_1, x_2 \in X$.

proof. Since X is compact, $\mathrm{Im} f = f(X) \subseteq \mathbb{R}^1$ is compact. $\square$

Lebesgue number.

Let X be a metric space, the <u>Lebesgue number for an open cover</u> $\{A_\alpha\}$ of X is a number $\delta > 0$ s.t. any open ball $B(x,r)$ with $r < \delta$ is contained in some A_α.

Theorem: Every open cover of a compact metric space has a Lebesgue number. $\square$