Lecture 08. Simplicial / Singular Homology Groups

triangulation

n-Simplex $\Delta^n = [v_0, v_1, \cdots, v_n] = \{(t_0, t_1, \cdots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0,\ \sum_{i=0}^n t_i = 1\}$.

$\sum_{i=0}^n t_i v_i$

= the smallest convex subset in $\mathbb{R}^{n+1}$ with vertices v_i.

Δ^n is ordered: $v_0, v_1, \cdots, v_n$

$[v_1, v_0, v_2, \cdots, v_n] \neq [v_0, v_1, v_2, \cdots, v_n]$

i-face of Δ^n: $[v_0, v_1, \cdots \hat{v}_i, \cdots, v_n]$ $\quad \hat{v}_i$

$= \{(t_0, t_1, \cdots, t_{i-1}, 0, t_{i+1}, \cdots, t_n) \in \Delta^n\}$

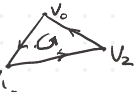
$\partial \Delta^n = \bigcup_{i=0}^n [v_0, v_1, \cdots, \hat{v}_i, \cdots, v_n]$

Example. $n=0$, $\Delta^0 = [v_0]$ $\qquad \partial \Delta^0 = \phi$

$n=1$:

$[v_0, v_1]$ $\qquad \partial [v_0, v_1] = [v_1] - [v_0]$

$n=2$:

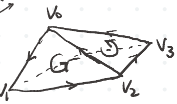


$[v_0, v_1, v_2]$

$\partial [v_0, v_1, v_2] = [v_1, v_2] + [v_0, v_2] + [v_0, v_1]$

$= [v_1, v_2] - [v_2, v_0] + [v_0, v_1]$.

$n=3$



$[v_0, v_1, v_2, v_3]$

$\partial [v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$.

In general, $\partial [v_0, v_1, \cdots, v_n] = \sum_{i=0}^n (-1)^i [v_0, v_1, \cdots, \hat{v}_i, \cdots, v_n]$

formal sum

Lemma. $\partial \circ \partial = 0$

proof. $\partial\partial [v_0, v_1, \cdots, v_n] = \partial\left(\sum_{i=0}^n (-1)^i [v_0, v_1, \cdots, \hat{v}_i, \cdots, v_n]\right)$

$= \sum_{i=0}^n (-1)^i \partial [v_0, v_1, \cdots, \hat{v}_i, \cdots, v_n]$

$= \sum_{i=0}^n (-1)^i \left(\sum_{j=0}^{i-1} (-1)^j [v_0, \cdots, \hat{v}_j, \cdots \hat{v}_i, \cdots, v_n] + \sum_{j=i+1}^n (-1)^{j-1} [v_0, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_n] \right)$

$= \sum_{0 \leq j < i} (-1)^{i+j} [v_0, \cdots, \hat{v}_j, \cdots, \hat{v}_i, \cdots, v_n] - \sum_{0 \leq i < j \leq n} (-1)^{i+j} [v_0, \cdots \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_n]$

$= 0$. $\qquad \square$

Simplicial Complexes

A simplicial complex K is a finite collection of simplexes such that

(i) If a simplex σ belongs to K (denoted by $\sigma \in K$), then all of its faces belong to K: $\sigma \in K \Rightarrow \sigma_i = [v_0, \cdots, \hat{v}_i, \cdots, v_n] \in K$.

(ii) If two simplexes σ, τ satisfy $\sigma \cap \tau \neq \phi$, then $\sigma \cap \tau$ is a common face of σ and τ.

Examples.

Denote $|K| = \bigcup_{\sigma \in K} \sigma$, viewed as a subspace of $\mathbb{R}^N$, $N \gg 0$.

is called the <u>polyhedron</u> associated to K.

$\dim K = \dim |K| = \max_{\sigma \in K} \{\dim \sigma\}$ is called the <u>dimension of</u> K.

- A space X is called <u>triangulable</u> if there is a homeomorphism $\varphi : |K| \longrightarrow X$.

Examples. ① polyhedra are triangulable.

② All closed/compact surfaces are triangulable.

 $\cong S^2$

<u>Lemma</u> ① $|K| \subseteq \mathbb{R}^N$ is bounded and closed $\Rightarrow$ $|K|$ is compact.

② If $|K|$ is connected, then it is path-connected.

③ $\forall x \in |K|$, $\exists ! \, \sigma \in K$, s.t. $x \in \mathring{\sigma} = \mathrm{int}(\sigma)$. (Exercise)

Simplicial homology groups.

Let K be a simplicial complex.

Define $\Delta_n(K) = \mathbb{Z}\langle \text{oriented } n\text{-simplexes of } K \rangle$

$= \text{the free abelian group generated by oriented } n\text{-simplexes}$

$= \{\sum_i k_i \sigma_i \mid \sigma_i \text{ are oriented } n\text{-simplexes}\}$

$\partial_n : \Delta_n(K) \longrightarrow \Delta_{n-1}(K)$ is a homomorphism given by

$$\partial(\underbrace{\Delta^n = [v_0, v_1, \cdots, v_n]}_{\sigma}) = \sum_{i=0}^{n} (-1)^i [v_0, \cdots, \hat{v}_i, \cdots, v_n].$$

$$\partial\left(\sum_i k_i \sigma_i\right) = \sum_i k_i \partial(\sigma_i)$$

Recall

Lemma: $\partial_n \circ \partial_{n+1} = 0$.

$$0 \to \Delta_m(K) \xrightarrow{\partial_m} \cdots \to \Delta_{n+1}(K) \xrightarrow{\partial_{n+1}} \Delta_n(K) \xrightarrow{\partial_n} \Delta_{n-1}(K) \to \cdots \to \Delta_0(K) \to 0.$$

$m = \dim K$.

$\partial\partial = 0 \Rightarrow \operatorname{Im}\partial_{n+1} \subseteq \operatorname{Ker}\partial_n$.

The quotient group $H_n(K) := \dfrac{\operatorname{Ker}\partial_n}{\operatorname{Im}\partial_{n+1}} = \dfrac{Z_n(K)}{B_n(K)}$ is called the nth simplicial homology group of K. $H_n(K)$ is an abelian group.

$[\alpha] \in H_n(K)$, $[\alpha] = \alpha + \operatorname{Im}\partial_{n+1}$ is called a homology class represented by α.

Fact: $H_n(K)$ is a homotopy invariant of simplicial complexes: if $|K| \simeq |L|$, then $H_n(K) \cong H_n(L)$ for any $n \geq 0$.

$\Delta_0(K) = \mathbb{Z}\langle \text{vertices of } K \rangle$

$\Delta_1(K) = \mathbb{Z}\langle \text{edges of } K \rangle$

$\Delta_2(K) = \mathbb{Z}\langle \text{triangles of } K \rangle$

$$\Delta_2(K) \xrightarrow{\partial_2} \Delta_1(K) \xrightarrow{\partial_1} \Delta_0(K) \xrightarrow{\partial_0} 0$$

$$H_0(K) = \Delta_0(K) \big/ \partial_1(\Delta_1(K))$$

$\forall\, u, v \in \Delta_0(K)$, $[u] = [v] \Leftrightarrow u + \operatorname{Im}\partial_1 = v + \operatorname{Im}\partial_1$

$\Leftrightarrow u - v = \partial(\sigma)$, $\sigma \in \Delta_1(K)$.

$\Leftrightarrow u - v = \partial([v, v_1] + [v_1, v_2] + \cdots + [v_n, u])$

$\Leftrightarrow$ u and v are two end-vertices of an edge path

$\exists\, \gamma : [0,1]$, $\gamma(0) = u$, $\gamma(1) = v$,

$\Leftrightarrow$ u and v lie in the same (path-)component.

$\Rightarrow$ • if K is (path) connected, then $H_0(K) \cong \mathbb{Z}$.

• if K has m (path) components, then $H_0(K) \cong \mathbb{Z}^m$.

Singular Homology

Let X be a topological space.

A map $\sigma : \Delta^n \to X$ is called a <u>singular n-simplex</u>.

$$S_n(X) := \mathbb{Z}\langle \text{singular } n\text{-simplexes} \rangle = \{\textstyle\sum_i k_i \sigma_i \mid \sigma_i : \Delta^n \to X\}.$$

└ the group of <u>singular n-chains</u>.

$$S_{n+1}(X) \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X)$$

$$\tau = [v_0, \cdots, v_n] : \Delta^n \to X$$

$$\partial_n \tau = \sum_{i=0}^n (-1)^i \, \tau|_{[v_0, \cdots, \hat{v}_i, \cdots, v_n]}$$

$$\tau|_{[v_0,\cdots,\hat{v}_i,\cdots,v_n]} : \Delta^{n-1} = \Delta^n_i = [v_0,\cdots,\hat{v}_i,\cdots,v_n] \hookrightarrow \Delta^n \xrightarrow{\tau} X.$$

Lemma: $\partial_n \circ \partial_{n+1} = 0$ (Exercise)

$$H_n(X) := \frac{\operatorname{Ker}\partial_n}{\operatorname{Im}\partial_{n+1}} = \frac{\overset{n\text{-cycle}}{Z_n(X)}}{\underset{n\text{-boundary}}{B_n(X)}}$$

is called the <u>n-th singular homology group</u> of X.

$Z_n(X)$: elements of $\operatorname{Ker}\partial_n$ are called <u>n-cycles</u>

$B_n(X)$: - - - - - $\operatorname{Im}\partial_{n+1}$ - - - - - - <u>n-boundaries</u>.

$[\alpha] = \alpha + \operatorname{Im}\partial_{n+1} \in H_n(X)$ is called a <u>homology class</u> represented by a n-cycle α.

Example. $X = \{x_0\}$

$$S_n(X) = \langle \{\sigma : \Delta^n \to \{x_0\}\} \rangle = \mathbb{Z} e_n$$

$$S_n(X) \xrightarrow{\partial_n} S_{n-1}(X)$$

$$\partial e_n = \sum_{i=0}^n (-1)^i \underbrace{e_n|_{\Delta^n_i}}_{= e_{n-1}} = \left(\sum_{i=0}^n (-1)^i\right) \cdot e_{n-1} = \begin{cases} 0 & n \text{ is odd} \\ e_{n-1} & n \text{ is even.} \end{cases}$$

$$\begin{array}{ccccccccccc}
\cdots \to S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \to \cdots \to & S_2(X) & \xrightarrow{\partial_2} & S_1(X) & \xrightarrow{\partial_1 = 0} & S_0(X) & \to 0 \\
& \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} & \\
\xrightarrow{0} & \mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \to 0
\end{array}$$

Thus $H_0(X) \cong \mathbb{Z}$, $H_i(X) \cong \mathbb{Z}/\mathbb{Z} = 0$, $\forall\, i \geq 1$.

Example. If $X = \coprod_\alpha X_\alpha$, then $S_n(X) = S_n(\coprod_\alpha X_\alpha) \cong \bigoplus_\alpha S_n(X_\alpha)$.

$$\Delta^n \longrightarrow X = \coprod_\alpha X_\alpha$$

$$\begin{tikzcd} S_n(X) \arrow[r, "\partial_n"] \arrow[d, equal] & S_{n-1}(X) \arrow[d, equal] \\ \bigoplus_\alpha S_n(X_\alpha) \arrow[r, "\bigoplus_\alpha \partial_{n,\alpha}"] & \bigoplus_\alpha S_{n-1}(X_\alpha) \end{tikzcd}, \quad \partial_n = \bigoplus_\alpha \partial_{n,\alpha}$$

$$\therefore H_n(X) \cong \frac{\operatorname{Ker}\partial_n}{\operatorname{Im}\partial_{n+1}} = \frac{\bigoplus_\alpha \operatorname{Ker}\partial_{n,\alpha}}{\bigoplus_\alpha \operatorname{Im}\partial_{n+1,\alpha}} \cong \bigoplus_\alpha \frac{\operatorname{Ker}\partial_{n,\alpha}}{\operatorname{Im}\partial_{n+1,\alpha}} = \bigoplus_\alpha H_n(X_\alpha).$$

prop. If X is an nonempty path-connected space, then

$$H_0(X) \cong \mathbb{Z} \Leftrightarrow \widetilde{H}_0(X) = 0.$$

If $X = \coprod_{i=1}^m X_i$, X_i are path components, then $H_0(X) \cong \mathbb{Z}^m$

$\widetilde{H}_0(X) \cong \mathbb{Z}^{m-1}$.

proof. $\longrightarrow S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$

By definition, $H_0(X) = S_0(X) / \operatorname{Im}\partial_1$

Define $S_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$, $\varepsilon(\sum_i k_i \sigma_i) = \sum_i k_i$. $\varepsilon(\sigma) = 1$ for $\sigma: \Delta^0 \to X$.

- ε is a group homomorphism.
- ε is surjective. $\forall k \in \mathbb{Z}$, take $\sigma: \Delta^0 \to X$, $\varepsilon(k\sigma) = k$.

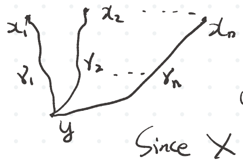
Claim: $\operatorname{Im}\partial_1 = \operatorname{Ker}\varepsilon$ if X is path connected.

$\operatorname{Im}\partial_1 \subseteq \operatorname{Ker}\varepsilon \Leftrightarrow \varepsilon\partial_1 = 0$: $\varepsilon(\partial_1\sigma) = \varepsilon(\sigma|_{[v_1]} - \sigma|_{[v_0]}) = 1 - 1 = 0$.

$\sigma = [v_0, v_1]: \Delta^1 \to X$

$\operatorname{Ker}\varepsilon \subseteq \operatorname{Im}\partial_1$: $\forall$ $\sum_i k_i \sigma_i \in \operatorname{Ker}\varepsilon$, $\sum_i k_i = 0$, $\sigma_i = [v_i]: \Delta^0 \to X$.

$x_i = \sigma_i(v_i)$, $\forall i$.



Take y to be a (base) point that is different from $x_1, x_2, \cdots, x_n$.

Since X is path-connected, there exist path γ_i s.t. $\gamma(0) = y$, $\gamma(1) = x_i$

$\gamma_i: I = \Delta^1 \to X$ can be viewed as 1-simplex in X

$\partial\gamma_i = x_i - y$.

Let $\gamma = \sum_i k_i \gamma_i$, then $\partial_1\gamma = \partial(\sum_i k_i \gamma_i) = \sum_i k_i x_i - (\sum_i k_i) y = \sum_i k_i \sigma_i$

Lemma 1. (Universal property of quotient groups).

Given a homomorphism $f: G \to H$ and a canonical projection $\pi: G \to G/N$.

If $N \subseteq \operatorname{Ker} f$, then there is a unique homomorphism $\bar{f}: G/N \to H$ s.t. $\bar{f} \circ \pi = f$

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & H \\ G/N \arrow[ur, dashed, "\exists ! \bar{f}"'] & \end{tikzcd}$$

Moreover, ① f is surjective $\Rightarrow$ $\bar{f}$ is surjective

② $\operatorname{Ker} \bar{f} = \operatorname{Ker} f / N$.

proof. Define $\bar{f}([x]) = f(x)$.

$N \subseteq \operatorname{Ker} f \Rightarrow \bar{f}$ is well-defined. $\square$

$$\begin{tikzcd} S_1(X) \arrow[r, "\partial_1"] & S_0(X) \arrow[r, "\varepsilon"] \arrow[d, "\pi"] & \mathbb{Z} \\ & H_0(X) = S_0(X)/\operatorname{Im}\partial_1 \arrow[ur, "\tilde{\varepsilon}"'] & \end{tikzcd}$$

$\operatorname{Im} \partial_1 = \operatorname{Ker} \varepsilon$ if X is path-connected

$\tilde{H}_0(X) = \operatorname{Ker} \tilde{\varepsilon} = \operatorname{Ker} \varepsilon / \operatorname{Im} \partial_1$ $(= 0$ if X is path connected$)$

is called the reduced 0-th singular homology group of X.

$H_0(X) / \tilde{H}_0(X) \cong \mathbb{Z} \Leftrightarrow \boxed{H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}.}$

$\tilde{H}_0(X) = 0$ if X is path-connected.

Lemma 2. If $G/N \cong \mathbb{Z}$, G and N are abelian groups, then $G \cong N \oplus \mathbb{Z}$

proof. $0 \to N \xrightarrow{i} G \underset{s}{\overset{\pi}{\rightleftarrows}} \mathbb{Z} \to 0$

$\mathbb{Z} = \langle 1 \rangle$, fix $g_0 \in \pi^{-1}(1)$, define $s(1) = g_0$. $\Rightarrow \pi s = id_{\mathbb{Z}}$.

Define $N \oplus \mathbb{Z} \xrightarrow{(i,s)} G$

$(i,s)(x,y) = i(x) + s(y)$.

(i,s) is injective: If $i(x) + s(y) = 0$, then

$0 = \pi \circ i(x) + \pi \circ s(y) = 0 + y \Rightarrow y = 0$

$\therefore i(x) = 0$. Since i is injective, we have $x = 0$

(i,s) is surjective: $\forall g \in G$, $s \circ \pi(g) \in G$.

$g - s \circ \pi(g) \in \operatorname{Ker}(\pi)$: $\pi(g - s \circ \pi(g)) = 0$

$x = g - s \circ \pi(g) \in N$

$y = \pi(g)$. then $(i,s)(x,y) = g$. $\square$

$$S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial_0 = \varepsilon} \mathbb{Z}$$

reduced homology groups $\tilde{H}_n(X)$ = the homology groups of $\left\{ S_n(X), \partial_n ; \begin{matrix} S_{-1}(X) = \mathbb{Z} \\ \partial_0 = \varepsilon \end{matrix} \right\}$

$$\begin{cases} \tilde{H}_n(X) \cong H_n(X) & \text{for } n > 0. \\ \tilde{H}_0(X) \cong H_0(X)/\mathbb{Z}. \end{cases}$$

· Induced homomorphisms

Let $f: X \to Y$ be map. $\quad \Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y$

Define $f_\#: S_n(X) \to S_n(Y)$, $f(\sigma) = f \circ \sigma$.

Lemma: $\partial f_\# = f_\# \partial$:

$$\begin{tikzcd} S_n(X) \arrow[r, "f_\#"] \arrow[d, "\partial"] & S_n(Y) \arrow[d, "\partial"] \\ S_{n-1}(X) \arrow[r, "f_\#"] & S_{n-1}(Y) \end{tikzcd}$$

proof $\quad (\partial f_\#)(\sigma) = \partial(f \circ \sigma) = \sum_{i=0}^n (-1)^i (f \circ \sigma)|_{[v_0, \cdots, \hat{v}_i, \cdots, v_n]}$

$$= \sum_{i=0}^n (-1)^i f(\sigma|_{[v_0, \cdots, \hat{v}_i, \cdots, v_n]})$$

$$= f \circ \left(\sum_{i=0}^n (-1)^i \sigma|_{[v_0, \cdots, \hat{v}_i, \cdots, v_n]} \right)$$

$$= f_\#(\partial \sigma) = (f_\# \circ \partial)(\sigma). \qquad \square$$

· $f_\#: \operatorname{Ker} \partial_n^X \to \operatorname{Ker} \partial_n^Y$, $\quad \partial \sigma = 0 \Rightarrow \partial(f_\# \sigma) = f_\#(\partial \sigma) = 0$

$\quad \operatorname{Im} \partial_{n+1}^X \to \operatorname{Im} \partial_{n+1}^Y$, $f_\#(\partial \sigma) = \partial(f_\# \sigma)$

Thus $f: X \to Y$ induces a homomorphism $f_*: H_n(X) \to H_n(Y)$, $\forall n \geq 0$.

Moreover, ① $(g \circ f)_\# = g_\# \circ f_\#: S_n(X) \to S_n(Y) \to S_n(Z)$

$\quad (id_X)_\# = id_{S_n(X)} : S_n(X) \to S_n(X)$

$\quad \Rightarrow (g \circ f)_* = g_* \circ f_*$, $(id_X)_* = id_{H_*(X)}$.

④ If X is contractible, then $\tilde{H}_n(X) = 0$, $\forall n \geq 0$.

② $f \simeq g: X \to Y$, $\quad f \circ \sigma \simeq g \circ \sigma \Rightarrow f_\# = g_\# \Rightarrow f_* = g_*: H_n(X) \to H_n(Y)$, $\forall n$

③ If $X \simeq Y$, then $H_n(X) \cong H_n(Y)$. (H_*(X) is a htp invariant.) $\square$