Optimization Problems, Lecture 1, Segment 1

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Computational Models

- Using computation to help understand the world in which we live
- Experimental devices that help us to understand something that has happened or to predict the future

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- *Optimization models*
- Statistical models
- Simulation models

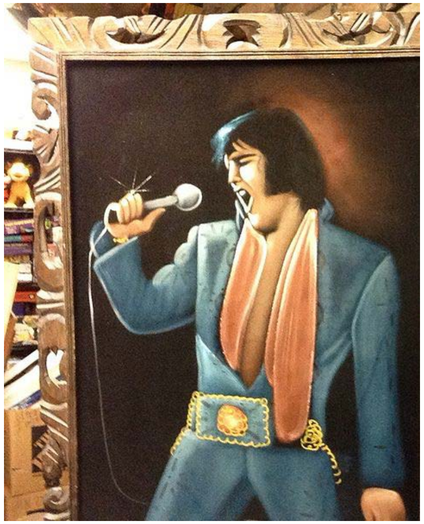
What Is an Optimization Model?

- An objective function that is to be maximized or minimized, e.g.,
  - Minimize time spent traveling from New York to Boston
- A set of constraints (possibly empty) that must be honored, e.g.,
  - Cannot spend more than $100
  - Must be in Boston before 5:00PM

Takeaways

- Many problems of real importance can be formulated as an optimization problem
- Reducing a seemingly new problem to an instance of a well-known problem allows one to use pre-existing methods for solving them
- Solving optimization problems is computationally challenging
- A greedy algorithm is often a practical approach to finding a pretty good approximate solution to an optimization problem

Knapsack and Bin-packing Problems



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Knapsack Problem

- You have limited strength, so there is a maximum weight knapsack that you can carry
- You would like to take more stuff than you can carry
- How do you choose which stuff to take and which to leave behind?
- Two variants
  - 0/1 knapsack problem
  - Continuous or fractional knapsack problem

versus

My Least-favorite Knapsack Problem

0/1 Knapsack Problem, Formalized

- Each item is represented by a pair, *<value, weight>*
- The knapsack can accommodate items with a total weight of no more than *w*
- A vector, *L*, of length *n*, represents the set of available items. Each element of the vector is an item
- A vector, *V*, of length *n*, is used to indicate whether or not items are taken. If *V[i] = 1*, item *I[i]* is taken. If *V[i] = 0*, item *I[i]* is not taken

0/1 Knapsack Problem, Formalized

Find a V that maximizes

$$\sum_{i=0}^{n-1} V[i] * I[i].value$$

subject to the constraint that

$$\sum_{i=0}^{n-1} V[i] * I[i].weight \le w$$

Brute Force Algorithm

- 1. Enumerate all possible combinations of items. That is to say, generate all subsets of the set of subjects. This is called the power set.
- 2. Remove all of the combinations whose total units exceeds the allowed weight.
- 3. From the remaining combinations choose any one whose value is the largest.

Often Not Practical

- How big is power set?
- Recall
  - A vector, *V*, of length *n*, is used to indicate whether or not items are taken. If *V[i] = 1*, item *I[i]* is taken. If *V[i] = 0*, item *I[i]* is not taken
- How many possible different values can *V* have?
  - As many different binary numbers as can be represented in *n* bits
- For example, if there are 100 items to choose from, the power set is of size 126,765,060,022,822,940,149,670,320,5376

Are We Just Being Stupid?

- Alas, no
- 0/1 knapsack problem is inherently exponential
- But don’t despair

Optimization Problems, Lecture 1, Segment 2

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0/1 Knapsack Inherently Exponential

Give up

Approximate solution

Exact solution that is often fast

Greedy Algorithm a Practical Alternative

- ```
  while knapsack not full
        put "best" available item in knapsack
  ```

- But what does best mean?
  - Most valuable
  - Least expensive
  - Highest value/units
  ```

An Example

- You are about to sit down to a meal
- You know how much you value different foods, e.g., you like donuts more than apples
- But you have a calorie budget, e.g., you don't want to consume more than 800 calories
- Choosing what to eat is a knapsack problem

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A Menu

Food	wine	beer	pizza	burger	fries	coke	apple	donut
Value	89	90	30	50	90	79	90	10
calories	123	154	258	354	365	150	95	195

- Let’s look at a program that we can use to decide what to order

Class Food

```
class Food(object):
    def __init__(self, n, v, w):
        self.name = n
        self.value = v
        self.calories = w

    def getValue(self):
        return self.value

    def getCost(self):
        return self.calories

    def density(self):
        return self.getValue()/self.getCost()

    def __str__(self):
        return self.name + ': <' + str(self.value)\
                 + ', ' + str(self.calories) + '>'
```

Build Menu of Foods

```
def buildMenu(names, values, calories):
    """names, values, calories lists of same length.
       name a list of strings
       values and calories lists of numbers
       returns list of Foods"""
    menu = []
    for i in range(len(values)):
        menu.append(Food(names[i], values[i],
                              calories[i]))
    return menu
```

Implementation of Flexible Greedy

```
def greedy(items, maxCost, keyFunction):
    """Assumes items a list, maxCost >= 0,
         keyFunction maps elements of items to numbers"""
    itemsCopy = sorted(items, key = keyFunction,
                              reverse = True)
    result = []
    totalValue, totalCost = 0.0, 0.0

    for i in range(len(itemsCopy)):
        if (totalCost+itemsCopy[i].getCost()) <= maxCost:
            result.append(itemsCopy[i])
            totalCost += itemsCopy[i].getCost()
            totalValue += itemsCopy[i].getValue()

    return (result, totalValue)
```

Algorithmic Efficiency

```
def greedy(items, maxCost, keyFunction):
    itemsCopy = sorted(items, key = keyFunction,
                       reverse = True)
    result = []
    totalValue, totalCost = 0.0, 0.0

    for i in range(len(itemsCopy)):
        if (totalCost+itemsCopy[i].getCost()) <= maxCost:
            result.append(itemsCopy[i])
            totalCost += itemsCopy[i].getCost()
            totalValue += itemsCopy[i].getValue()

    return (result, totalValue)
```

n log n   where n = len(items)

\+

n

---

n log n

Using greedy

```
def testGreedy(items, constraint, keyFunction):
    taken, val = greedy(items, constraint, keyFunction)
    print('Total value of items taken =', val)
    for item in taken:
        print('   ', item)
```

Using greedy

```
def testGreedys(maxUnits):
    print('Use greedy by value to allocate', maxUnits,
          'calories')
    testGreedy(foods, maxUnits, Food.getValue)
    print('\nUse greedy by cost to allocate', maxUnits,
          'calories')
    testGreedy(foods, maxUnits,
               lambda x: 1/Food.getCost(x))
    print('\nUse greedy by density to allocate', maxUnits,
          'calories')
    testGreedy(foods, maxUnits, Food.density)

testGreedys(800)
```

lambda

- lambda used to create anonymous functions
  - `lambda` <id_1, id_2, … id_n>: <expression>
  - Returns a function of n arguments

lambda

- lambda used to create anonymous functions
  - `lambda` <id_1, id_2, … id_n>: <expression>
  - Returns a function of n arguments
- Possible to write amazing complicated lambda expressions
- Don't—use `def` instead

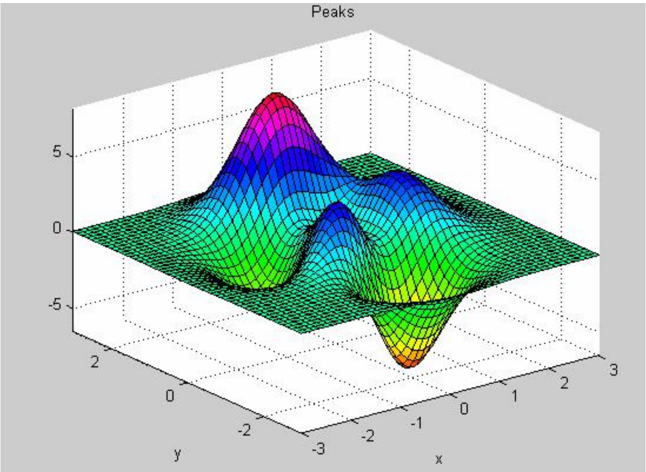
Using greedy

```
def testGreedys(foods, maxUnits):
    print('Use greedy by value to allocate', maxUnits,
          'calories')
    testGreedy(foods, maxUnits, Food.getValue)
    print('\nUse greedy by cost to allocate', maxUnits,
          'calories')
    testGreedy(foods, maxUnits,
               lambda x: 1/Food.getCost(x))
    print('\nUse greedy by density to allocate', maxUnits,
          'calories')
    testGreedy(foods, maxUnits, Food.density)

names = ['wine', 'beer', 'pizza', 'burger', 'fries',
         'cola', 'apple', 'donut', 'cake']
values = [89,90,95,100,90,79,50,10]
calories = [123,154,258,354,365,150,95,195]
foods = buildMenu(names, values, calories)
testGreedys(foods, 750)
```

Why Different Answers?

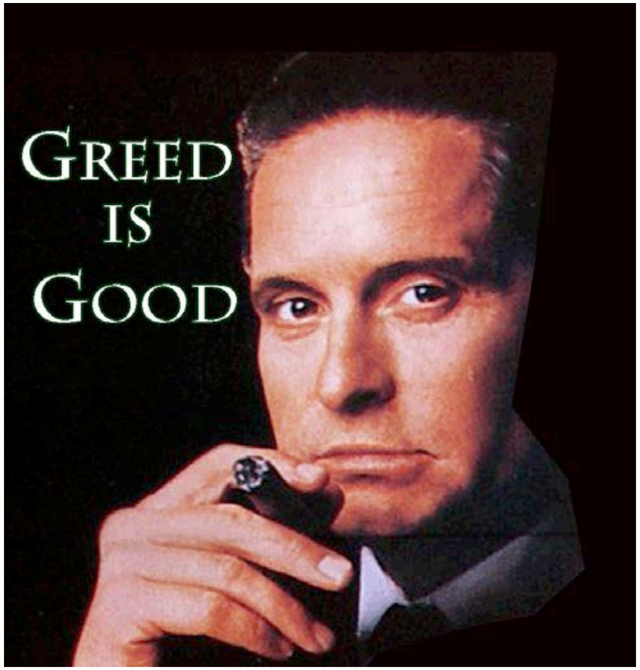
- Sequence of locally "optimal" choices don't always yield a globally optimal solution



- Is greedy by density always a winner?
  - Try testGreedys(foods, 1000)

The Pros and Cons of Greedy

- Easy to implement
- Computationally efficient



- But does not always yield the best solution
  - Don't even know how good the approximation is
- In the next lecture we'll look at finding truly optimal solutions