Optimization Problems, Lecture 2, Segment 1

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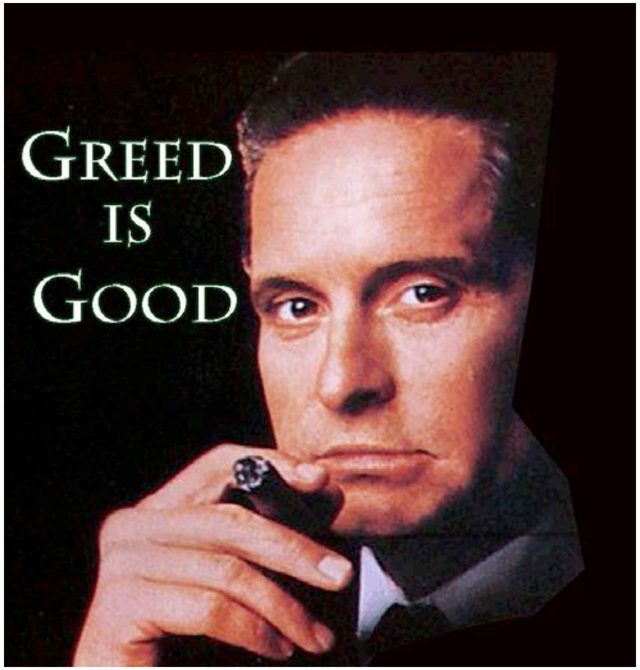
The Pros and Cons of Greedy

- Easy to implement
- Computationally efficient



- But does not always yield the best solution
  - Don't even know how good the approximation is
- On to finding optimal solutions

Brute Force Algorithm

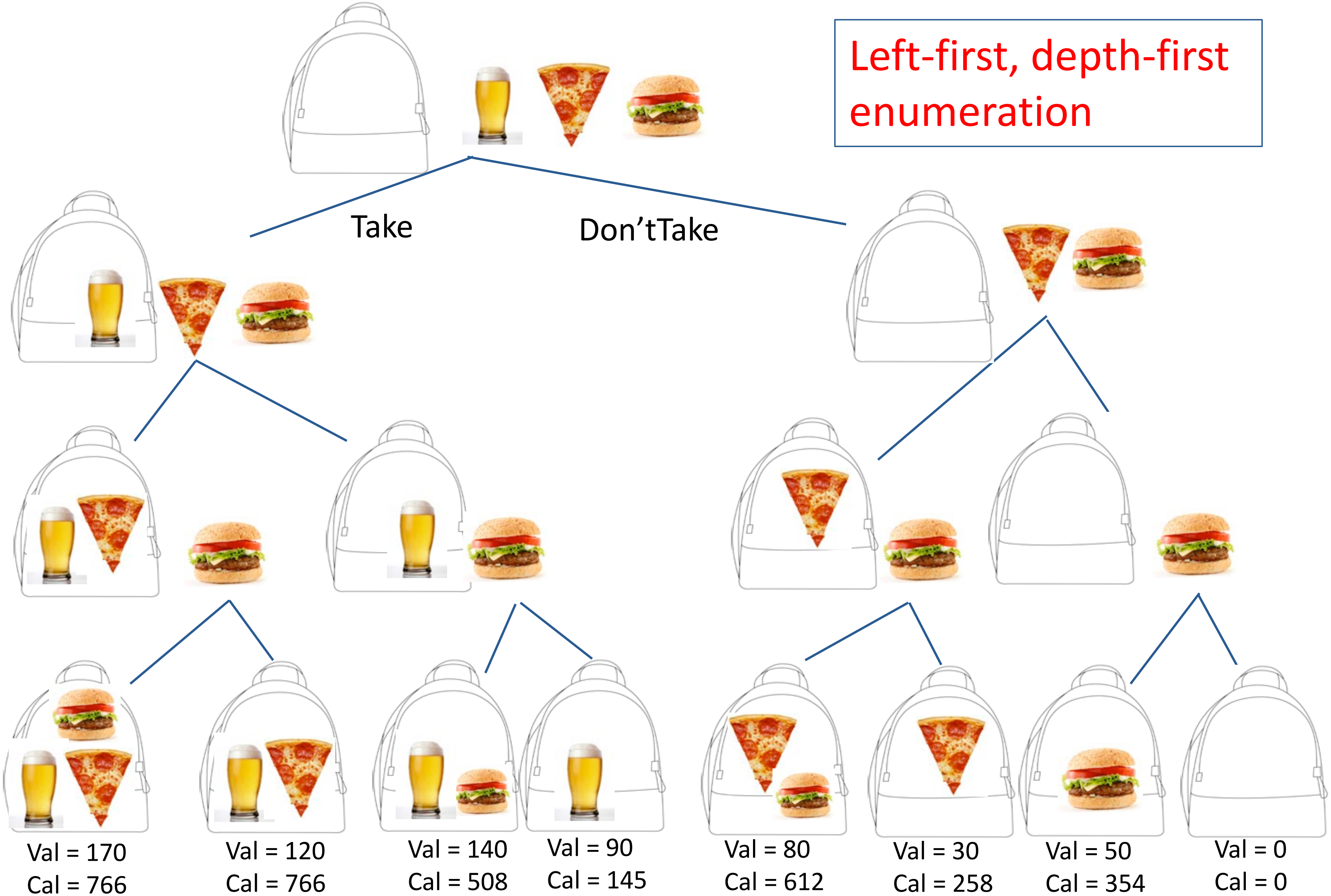
- 1. Enumerate all possible combinations of items.
- 2. Remove all of the combinations whose total units exceeds the allowed weight.
- 3. From the remaining combinations choose any one whose value is the largest.

Search Tree Implementation

- The tree is built top down starting with the root
- The first element is selected from the still to be considered items
  - If there is room for that item in the knapsack, a node is constructed that reflects the consequence of choosing to take that item. By convention, we draw that as the left child
  - We also explore the consequences of not taking that item. This is the right child
- The process is then applied recursively to non-leaf children
- Finally, chose a node with the highest value that meets constraints

A Search Tree Enumerates Possibilities

Left-first, depth-first enumeration



Computational Complexity

- Time based on number of nodes generated
- Number of levels is number of items to choose from
- Number of nodes at level *i* is 2^i
- So, if there are *n* items the number of nodes is
  - $\sum_{i=0}^{i=n} 2^i$
  - I.e., $O(2^{i+1})$
- An obvious optimization: don’t explore parts of tree that violate constraint (e.g., too many calories)
  - Doesn’t change complexity
- Does this mean that brute force is never useful?
  - Let’s give it a try

Header for Decision Tree Implementation

```
def maxVal(toConsider, avail):
    """Assumes toConsider a list of items,
               avail a weight
       Returns a tuple of the total value of a
           solution to 0/1 knapsack problem and
           the items of that solution"""
```

`toConsider`. Those items that nodes higher up in the tree (corresponding to earlier calls in the recursive call stack) have not yet considered

`avail`. The amount of space still available

Body of maxVal (without comments)

```
def maxVal(toConsider, avail):
"""Assumes toConsider a list of items, avail a weight
Returns a tuples of the total value of a solution to the 0/1 knapsack
problem and the items of that solution"""
if toConsider == [] or avail == 0:
    result = (0, ())
elif toConsider[0].getCost() > avail:
    # Explore right branch only
    result = maxVal(toConsider[1:], avail)
else:
    nextItem = toConsider[0]
    # Explore left branch
    withVal, withToTake = maxVal(toConsider[1:], avail - nextItem.getCost())
    withVal += nextItem.getValue()
    # Explore right branch
    withoutVal, withoutToTake = maxVal(toConsider[1:], avail)
    # Explore better branch
    if withVal > withoutVal:
        result = (withVal, withToTake + (nextItem,))
    else:
        result = (withoutVal, withoutToTake)
return result
```

Local variable `result` records best solution found so far

Try on Example from Lecture 1

- With calorie budget of 750 calories, chose an optimal set of foods from the menu

Food	wine	beer	pizza	burger	fries	coke	apple	donut
Value	89	90	30	50	90	79	90	10
calories	123	154	258	354	365	150	95	195

Search Tree Worked Great

- Gave us a better answer
- Finished quickly
- But 2^8 is not a large number
  - We should look at what happens when we have a more extensive menu to choose from

Optimization Problems, Lecture 2, Segment 2

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Search Tree Algorithm

- Gave us a better answer than any of the greedy solutions
- Finished quickly
- But 2^8 is not a large number
- Let's look at what happens when we have a more extensive menu to choose from

Code to Try Larger Examples

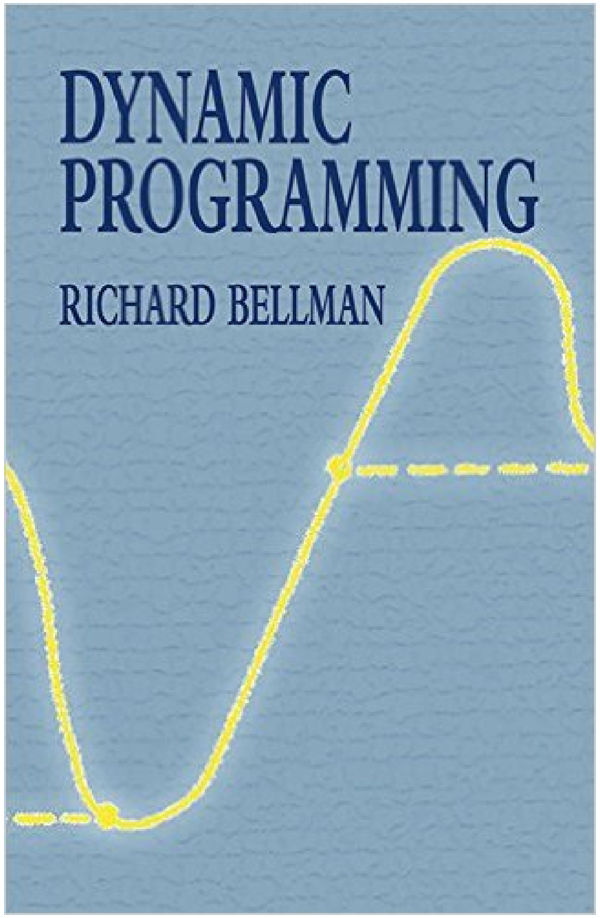
```
import random

def buildLargeMenu(numItems, maxVal, maxCost):
    items = []
    for i in range(numItems):
        items.append(Food(str(i),
                          random.randint(1, maxVal),
                          random.randint(1, maxCost)))
    return items


for numItems in (5, 10, 15, 20, 25, 30, 35, 40, 45):
    items = buildLargeMenu(numItems, 90, 250)
    testMaxVal(items, 750, False)
```

Is It Hopeless?

- In theory, yes
- In practice, no!
- Dynamic programming to the rescue



Dynamic Programming?

Sometimes a name is just a name

“The 1950s were not good years for mathematical research… I felt I had to do something to shield Wilson and the Air Force from the fact that I was really doing mathematics... What title, what name, could I choose? ... It's impossible to use the word dynamic in a pejorative sense. Try thinking of some combination that will possibly give it a pejorative meaning. It's impossible. Thus, I thought dynamic programming was a good name. It was something not even a Congressman could object to. So I used it as an umbrella for my activities.

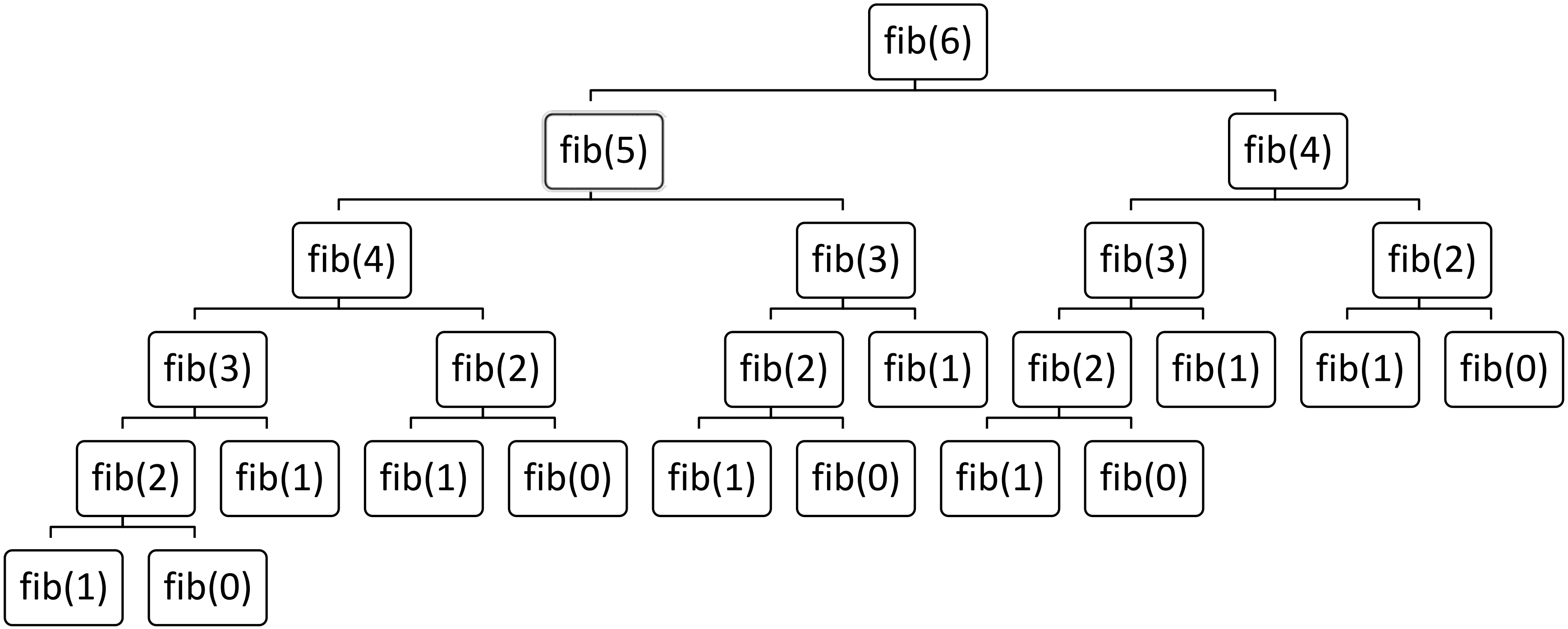
-- Richard Bellman

Recursive Implementation of Fibonnaci

```
def fib(n):
    if n == 0 or n == 1:
        return 1
    else:
        return fib(n - 1) + fib(n - 2)
```

```
fib(120) = 8,670,007,398,507,948,658,051,921
```

Call Tree for Recursive Fibonnaci(6) = 13



Clearly a Bad Idea to Repeat Work

- Trade a time for space
- Create a table to record what we've done
  - Before computing fib(x), check if value of fib(x) already stored in the table
    - If so, look it up
    - If not, compute it and then add it to table
  - Called memoization

Using a Memo to Compute Fibonnaci

```
def fastFib(n, memo = {}):
    """Assumes n is an int >= 0, memo used only by
        recursive calls
       Returns Fibonacci of n"""
    if n == 0 or n == 1:
        return 1
    try:
        return memo[n]
    except KeyError:
        result = fastFib(n-1, memo) +\
                 fastFib(n-2, memo)
        memo[n] = result
        return result
```

When Does It Work?

- Optimal substructure: a globally optimal solution can be found by combining optimal solutions to local subproblems
  - For $x > 1$, $fib(x) = fib(x - 1) + fib(x – 2)$

- Overlapping subproblems: finding an optimal solution involves solving the same problem multiple times
  - Compute fib(x) or many times

What About 0/1 Knapsack Problem?

- Do these conditions hold?