International Journal of Non-Linear Mechanics 45 (2010) 301–305

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

An integral approach for large deflection cantilever beams

Li Chen

Division of Hydrologic Sciences, Desert Research Institute, Las Vegas, NV 89119, USA

ARTICLE INFO

Article history:
Received 6 July 2009
Received in revised form
17 November 2009
Accepted 9 December 2009

Keywords:
Large deflection
Cantilever beam
Integral
Numerical solution
Complex load
Varying beam properties

ABSTRACT

A new integral approach is proposed to solve the large deflection cantilever beam problems. By using the moment integral treatment, this approach can be applied to problems of complex load and varying beam properties. This versatile approach generally requires only simple numerical techniques thus is easy for application. Treatment for typical loading and beam property conditions are presented to demonstrate the capability of this approach.

1. Formulation of large deflection cantilever beams

The bending problem of a cantilever beam is schematically defined in Fig. 1. Under the assumption that the material of beam remains linearly elastic, the relationship of bending moment and beam deformation reads ([1])

$$\frac{d\theta}{ds} = \frac{M(s)}{EI} \tag{1}$$

where θ is the angle of rotation of the deflection curve, s is the distance measured along the beam, M is the bending moment, E is the module of elasticity of the material and I is the moment of inertia of the cross-sectional area of the beam about the axis of bending.

Dado and Al-sadder [2] summarized several major approaches used for large deflection problems: the elliptic integral approach developed by Bisshopp and Drucker [3] and is still widely used at present (e.g., [4,5]); the numerical integration approach with iterative shooting techniques (e.g., [6,7]); and the incremental finite element or finite difference method with Newton–Rhapson iteration techniques (e.g., [8,9]). The drawbacks of these approaches, mainly including applicability, complexity and stability, were analyzed in the same article. Dado and Al-sadder [2] developed an approach that approximates the angle of rotation by a polynomial function and minimizes the residual of the governing equation caused by the approximation. This approach is effective for complex load and non-prismatic cantilever beam with very large deflection. However, the formulation of this approach is also complicated which increases the difficulty of application in practical problems.

Ang et al. [10] proposed a numerical method based on the form of Eq. (1) in Cartesian coordinates, which provided the basis of this study. The left-hand side of Eq. (1) is the curvature of the beam curve. Thus the equation can be reformed to

$$\frac{\frac{d^2x}{dy^2}}{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}} = \frac{M(y)}{EI} \tag{2}$$

where x and y are coordinates in which y is parallel to the original beam. Ang et al applied a search procedure to solve the cantilever beam bending problem defined by Eq. (2). Define

$$z = \frac{dx}{dy} \tag{3}$$

thus the curve length of the beam can be calculated with

$$s(l) = \int_0^l \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \tag{4}$$

And we also have

$$\frac{ds}{dy} = \sqrt{1+z^2} \tag{5}$$

Hence Eq. (2) can be converted to

$$\frac{dz}{dy} = \frac{M(y)}{EI}(1+z^2)^{3/2} \tag{6}$$

E-mail address: lchen@dri.edu

0020-7462/$ - see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijnonlinmec.2009.12.004

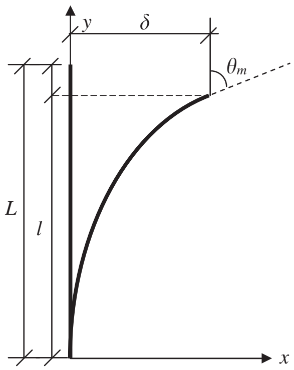

Fig. 1. Bending of a cantilever beam.

Eqs. (3), (5) and (6) can be solved numerically for a given projective length l with boundary conditions $z(0)=x(0)=s(0)=0$at the fixed end. The problem will then be solved by searching the projective length l until

$$s(l)=L \tag{7}$$

is satisfied. Ang et al. [10] proposed an approach using Runge–Kutta method to solve the Eqs. (6), (3), and (5).

2. Theory

We propose a new approach to solve the cantilever beam bending problem based on the formulation by Ang et al. [10].

Rewrite Eq. (6) into

$$\frac{dz}{(1+z^2)^{3/2}}=\frac{M(y)}{EI}\,dy \tag{8}$$

This equation can be integrated and we obtain

$$\frac{z}{\sqrt{1+z^2}}=\int_0^y \frac{M(y)}{EI}\,dy=G(y) \tag{9}$$

Note that the boundary condition $z(0)=0$ is used here. Also it is worthy to point out that the left-hand side of the above equation is actually $\sin\theta$. Using Eq. (5), the above equation can be converted to

$$\frac{ds}{dy}=\frac{1}{\sqrt{1-G^2(y)}} \tag{10}$$

From Eq. (3), another equation for variable x can be obtained following the same procedure, which reads

$$\frac{dx}{dy}=\frac{G(y)}{\sqrt{1-G^2(y)}} \tag{11}$$

Eqs. (10) and (11) become new governing equations for cantilever beam bending problems. For simple loads and uniform beam properties, they are regular first order ODEs in which dependent variables do not appear in the right-hand side thus can be integrated straightforwardly. The problem can be solved numerically following the search procedure. With a given l value, the function $G(y)$ can be determined analytically or numerically by integrating the bending moment as defined in Eq. (9). Once the function $G(y)$ is determined, Eq. (10) can be integrated with respect to y from 0 to l to find the arc length

$$s(l)=\int_0^{s(l)} ds$$

If $s(l)\neq L$, a new l value will be assumed according to the calculated $s(l)$ value. The search procedure can be repeated until the correct l is found. Note that $G(y)$ is <1 by definition. If during the search process $G(y)>1$ is encountered, it means the assumed projective length l is too large and a smaller l value should be adopted. For more general cases, iteration can be applied to solve the problem.

This method is different from the traditional elliptic integral method ([3]) while can be converted to the elliptic integral in certain conditions (Section 3.1). The elliptic integral method is derived for a concentrated force and is not effective for complex load and variable stiffness ([11]). The proposed method uses the integral of the bending moment thus can be applied to arbitrary loads. This method can also be applied to variable beam properties such as changing cross section area or elasticity of material.

3. Applications for various types of loadings and beam properties

In this section the new approach is applied to treat some typical loading and beam property conditions.

3.1. Concentrated load

If a concentrated load P parallel to x axis is applied at the free end of a uniform cantilever beam, the function of bending moment can be expressed as

$$M(y)=P(l-y) \tag{12}$$

Hence $G(y)$ is found as

$$G(y)=\frac{P}{EI}\left(ly-\frac{y^2}{2}\right) \tag{13}$$

Eq. (10) is herein converted to

$$\frac{ds}{dy}=\frac{1}{\sqrt{1-\frac{P^2}{E^2I^2}\left(ly-\frac{y^2}{2}\right)^2}} \tag{14}$$

For a given l value, this equation can be integrated straightforwardly to solve s as a function of y. Thus the total arc length $s(l)$ can be found for each l value. It is easy to see that $s(l)$ is a monotonic function of l. By scanning the entire range of l an appropriate value can be found to satisfy $s(l)$=L (Fig. 1). Once the appropriate l value is obtained, the whole bending curve of the beam can be solved using the following equation based on (11)

$$\frac{dx}{dy}=\frac{\frac{P}{EI}\left(ly-\frac{y^2}{2}\right)}{\sqrt{1-\frac{P^2}{E^2I^2}\left(ly-\frac{y^2}{2}\right)^2}} \tag{15}$$

A solver for this approach can be easily coded. Applying the simple search procedure, Eqs. (14) and (15) can be numerically solved even with spreadsheet type software such as Microsoft Excel, which reduces the requirement on mathematic preparation. Generally, satisfactory accuracy can be achieved using simple numerical integration schemes such as trapezoidal integral with adequate number of integration elements.

On the other hand, from Eqs. (9) and (13) we have the following relationship:

$$\sin\theta_m=\frac{P}{EI}\frac{l^2}{2} \tag{16}$$

where θ_m is the θ value at the free end, or

$$\frac{l}{L}=\sqrt{2}\sin^{1/2}\theta_m/\left(\frac{PL^2}{EI}\right)^{1/2} \tag{17}$$

which is the same as that obtained by Bisshopp and Drucker [3].

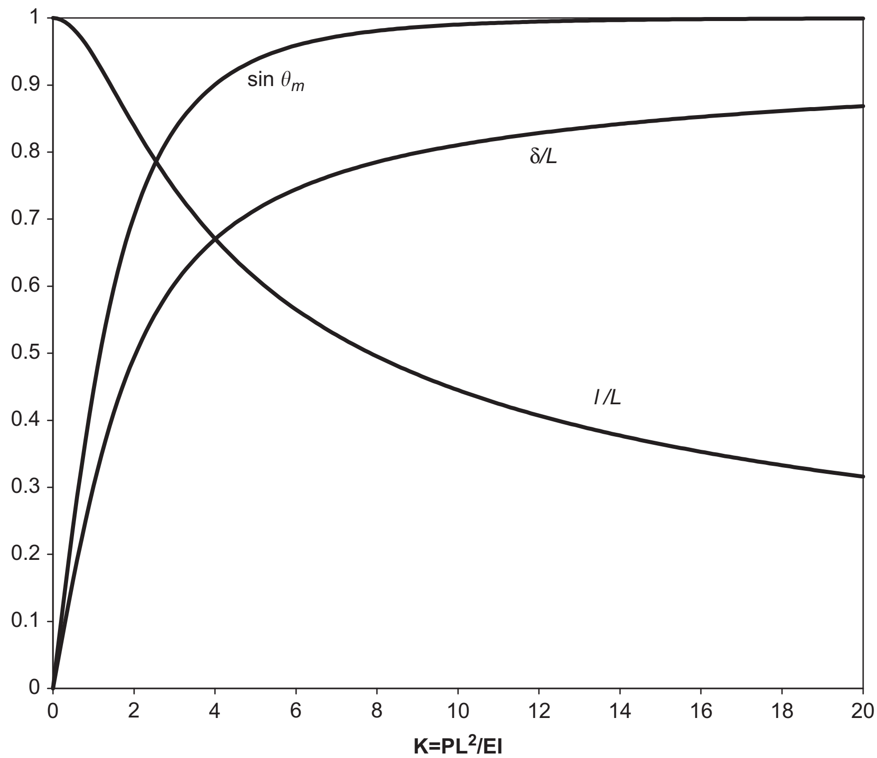

Fig. 2. Solution of Eqs. (18) and (19).

Eqs. (14) and (15) can be normalized into the following form:

$$\frac{ds^*}{dy^*}=\frac{1}{\sqrt{1-K^2\left(l^*y^*-\frac{y^{*2}}{2}\right)^2}} \tag{18}$$

and

$$\frac{dx^*}{dy^*}=\frac{K\left(l^*y^*-\frac{y^{*2}}{2}\right)}{\sqrt{1-K^2\left(l^*y^*-\frac{y^{*2}}{2}\right)^2}} \tag{19}$$

where $s^*=s/L$, $y^*=y/L$, $l^*=l/L$, $x^*=x/L$, and $K=PL^2/EI$. Applying the search procedure these equations can be solved for various K values and the results are shown in Fig. 2. These results are the same as that obtained by Bisshopp and Drucker [3], which shows the effectiveness of this approach.

Using relationships $\sin\theta_m=Kl^{*2}/2$ (from Eq. (17)) and $\sin\theta=K[l^{*2}-(l^*-y^*)^2]/2$ (from Eq. (13)), integrations of Eqs. (18) and (19), i.e.,

$$\int_0^1 ds^*=\int_0^{l^*} dy^*\frac{1}{\sqrt{1-K^2\left(l^*y^*-\frac{y^{*2}}{2}\right)^2}} \quad \text{and} \quad \frac{\delta}{L}=\int_0^{\delta} dx^*$$

$$=\int_0^{l^*} dy^*\frac{K\left(l^*y^*-\frac{y^{*2}}{2}\right)}{\sqrt{1-K^2\left(l^*y^*-\frac{y^{*2}}{2}\right)^2}},$$

can be converted into elliptic integrals in exactly the same form as that found by Bisshopp and Drucker [3]. However, this operation is not preferred since it is not a general approach for complex loading and beam properties. In addition, numerical solution of (18) and (19) can be obtained with simple approaches.

3.2. Distributed load

The proposed approach can easily handle arbitrarily distributed load that cannot be solved efficiently with the elliptic integral approach. For a uniformly distributed load, the approach can be formulated as follows.

If a uniformly distributed load parallel to x axis (normal to the original beam) is applied to the entire beam and the total load P over the beam is a known value, the function of bending moment can be expressed as

$$M(y)=\frac{\frac{P}{l}(l-y)^2}{2} \tag{20}$$

in which the total load P is uniformly distributed over the projective length l. Thus $G(y)$ is found to be

$$G(y)=\frac{\frac{P}{l}}{2EI}\left(\frac{y^3}{3}-ly^2+l^2y\right)=\frac{P}{2EI}\left(\frac{y^3}{3l}-y^2+ly^2\right) \tag{21}$$

Therefore we have the governing equations

$$\frac{ds}{dy}=\frac{1}{\sqrt{1-\left(\frac{P}{2EI}\right)^2\left(\frac{y^3}{3l}-y^2+ly\right)^2}} \tag{22}$$

and

$$\frac{dx}{dy}=\frac{\left(\frac{P}{2EI}\right)\left(\frac{y^3}{3l}-y^2+ly\right)}{\sqrt{1-\left(\frac{P}{2EI}\right)^2\left(\frac{y^3}{3l}-y^2+ly\right)^2}} \tag{23}$$

These equations can be solved with the same approach used for Eqs. (14) and (15).

A more common situation for uniformly distributed load is that the intensity of the distributed load w is constant along arc length s rather than y axis. The gravity of the beam is an example of this kind of load. For such a load, the bending moment M and the function G cannot be expressed analytically because the load distribution along y is dependent on the deflection curve. The integral form of bending moment M can be written as follows:

$$M(s) = \int_s^L w \cdot [y(\eta) - y(s)]\, d\eta \tag{24}$$

This equation is also valid for arbitrarily distributed loads where the load w varies along s and y. Analytical approach is not feasible for a general treatment of this equation. For such a case, we can approximately calculate functions M and G in a discrete manner. The entire beam can be segmented into N elements by $N+1$ nodes s_0, $s_1, \ldots s_i, \ldots, s_N$ with $s_N=L$ at the free end. The corresponding coordinates of the $N+1$ nodes are denoted as $(x_0, y_0), \ldots, (x_i, y_i), \ldots, (x_N, y_N)$. Using Eq. (24), M and G at node i can thus be numerically calculated by

$$M_i = \sum_{j=i+1}^{N} w(s_j - s_{j-1}) \left(\frac{y_{j-1} + y_j}{2} - y_i \right), \quad i = 0, \ldots, N-1 \text{ and } M_N = 0 \tag{25}$$

$$G_i = G_{i-1} + \left(\frac{M_{i-1} + M_i}{2EI} \right) (y_i - y_{i-1}), \text{ with } G_0 = 0 \tag{26}$$

In this problem, y is also the variable to be determined at each node for M and G calculation while s is known. Therefore, an iteration procedure can be used for the solution. In each iteration step, we use y values obtained from the last step to calculate M and G and use the governing Eqs. (10) and (11) to solve the updated y. Repeat the procedure until the difference between two iterations is less than a specified small error limit and we can find the solution.

It is worth noting that arbitrarily distributed load can also be handled with exactly the same numerical approach, which is not simple for traditional approaches. A search procedure is not needed in this treatment because the bending moment M and the function G are calculated over the whole beam arc length L rather than the unknown projective length l. This approach can also be applied to concentrated loads but is not straightforward as the search method. An example of this type of bending problems is included in the next section (Fig. 3).

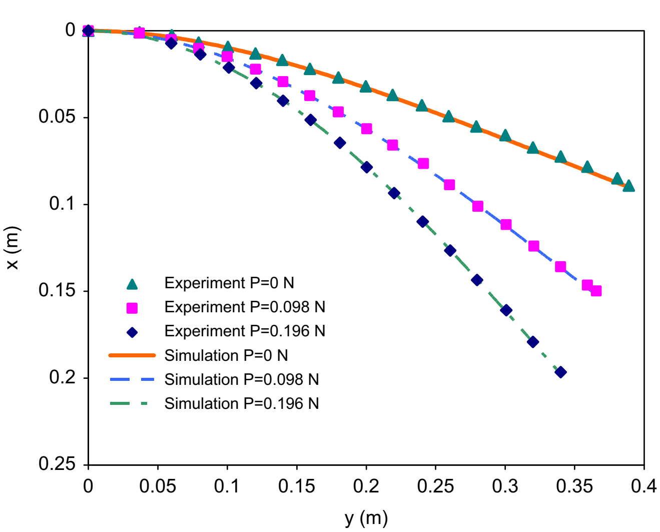

Fig. 3. Numerical simulation of the experiment by Beléndez et al. [13]. $P=0$ is a pure distributed load condition.

3.3. Combined load

Combined loading of uniformly distributed load and concentrated load has been studied previously with complex approaches. For instance, Lee [12] used the revised Bernoulli–Euler equation including two first order ODEs and an advanced numerical approach (Butcher's fifth order Runge–Kutta method) to solve the equations. The proposed new approach can be used to study this type of problems and basically only one first order ODE will be solved (Eqs. (10) and (11) can be transformed between each other and can be solved separately). Here we use the proposed approach to simulate an experiment by Beléndez et al. [13] of cantilever beam bending under a concentrated load and gravity. In this case, $L=0.4$ m, $E=1.943\times10^{11}$ N/m^2, $I=1.333\times10^{-13}$ m^4, the weight of the beam is 0.3032 N, i.e., $w=0.758$ N/m. Seven concentrated loads, $P=0$, 0.098, 0.196, 0.294, 0.392, 0.490, and 0.588 N were applied at the free end of the beam, respectively, and the deflections were measured. The $P=0$ case only involved the uniformly distributed load and is an example for the last section. We modeled this experiment following the procedure described in the last section, except using the combination of Eqs. (25) and (12) to calculate the bending moment M. The results of the simulated and observed deflection curve for three loads presented in Beléndez et al. [13] are shown in Fig. 3. Simulated results all match the measured curves of deflection very well. Beléndez et al also applied ANSYS model to simulate the experiment and presented the modeling results of vertical displacement at the free end for all seven loads which closely matched the experiment. Comparison of our modeling results and the ANSYS results showed that our results are almost identical to ANSYS results for all seven cases. The largest absolute error of vertical displacement compared to ANSYS results is 0.2 mm, or 0.088% in relative error.

3.4. Changing cross section with concentrated load

In this case we study a tapered cantilever beam with circular cross section. The radius of the cross section changes linearly along the arc length, following the relation:

$$r = as + b \tag{27}$$

where a and b are coefficients defining the linear relationship. Therefore the area moment of inertia for the cross section is a function of s, i.e.,

$$I(s) = \frac{\pi}{4} r^4 \tag{28}$$

For a concentrated load P at the free end, the function G can be expressed as

$$G(y) = \int_0^{y_l} \frac{M(y)}{EI}\, dy = \int_0^{y_l} \frac{P(l-y)}{EI(s)}\, dy \tag{29}$$

Because I cannot be directly expressed as a function of y, the simple search procedure is not directly applicable for this case. We use a combined numerical approach of iteration and search to solve this problem.

For an assumed projective length l, the entire beam can be discretized into N elements by $N+1$ nodes $y_0, y_1, \ldots, y_i, \ldots, y_N$ with $y_N = l$ at the free end. For each element, the relation between s and y can be locally approximated by a linear function. Thus the radius of cross section can be approximately calculated by

$$r(y) = a_i y + b_i \tag{30}$$

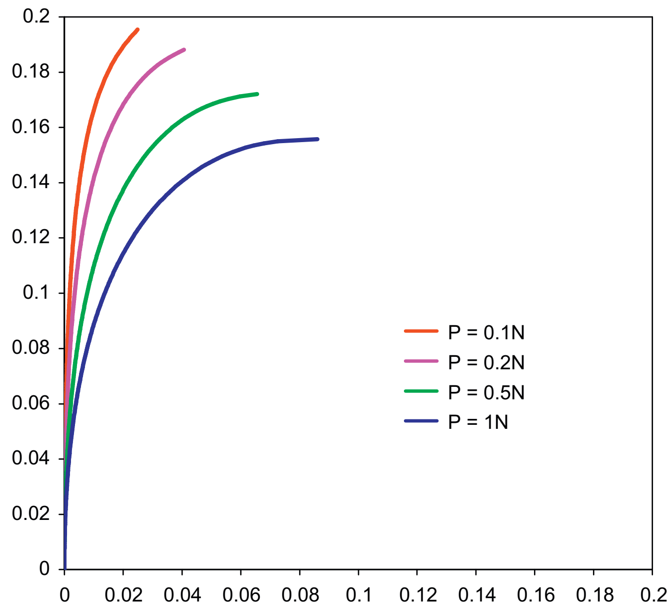

Fig. 4. Deflection curves for a beam with changing cross section.

Table 1
Bending characteristics of a tapered beam under concentrated loads.

	P=0.1 N	P=0.2 N	P=0.5 N	P=1 N
θ_m	41.3°	63.0°	83.3°	87.9°
δ/L	0.124	0.203	0.328	0.430
l/L	0.977	0.941	0.860	0.779

Based on this approximation the function G can be analytically integrated for a small element $[y_{i-1}, y_i]$:

$$G(y_i)-G(y_{i-1}) = \frac{4P}{\pi E}\int_{y_{i-1}}^{y_i} \frac{l-y}{(a_i y+b_i)^4}\,dy = \frac{4P}{\pi E}\,\frac{3a_i y-2a_i l+b_i}{6a_i^2(a_i y+b_i)^3}\Big|_{y_{i-1}}^{y_i} \quad (31)$$

The coefficient a_i and b_i will be determined in an iterative procedure. Starting from initial guess values of s_i (i=1,…,N), a_i and b_i can be calculated using Eqs. (27) and (30). We can then use Eqs. (31) and (10) to solve the updated s_i. Repeat this process we will be able to obtain the converged solution of s_i (i=1,…,N) corresponding to the assumed l. Following the search procedure we can find the ultimate solution of the problem.

We show an example of application of this approach. A 0.2 m long cantilever beam has a radius r_0=0.001 m at the fixed end and a radius r_e=0.0001 m at the free end. The modulus elasticity $E=1.2\times10^{11}$ N/m^2. A concentrated load P=0.1, 0.2, 0.5 and 1 N is horizontally exerted at the free end, respectively. Using the above approach, the deflection curves of the four concentrated loads are solved and shown in Fig. 4. Characteristics of deflection curves for the four cases are presented in Table 1.

The same approach can be applied to more general problems involve arbitrarily varying beam properties and distributed load.

4. Concluding remarks

The proposed approach uses the moment integral treatment thus can be applied to general problems of bending cantilever beam such as arbitrarily distributed loads and non-uniform beam properties. For concentrated loads on uniform beams, theoretical and numerical analyses showed that this approach is equivalent to the Bisshopp and Drucker [3] elliptic integral approach. When applied to more complicated loading this approach accurately repeated experimental results and agreed very well with sophisticate engineering software. Application of this approach usually requires only simple numerical techniques and straightforward solution procedures. Only one first order ODE needs to be solved which is easier than traditional approaches. Standard search or iterative procedures or combination of them are usually adequate for the solution. With simple coding work and fast solution process, this approach can be effectively applied in practices.

Acknowledgments

This study is partially supported by the UFDP program sponsored by US Army Corps of Engineers under Contract number W912HZ-08-2-0021. The author is grateful to the comments from two anonymous reviewers, which greatly helped the improvement of the presentation.

References

[1] J.M. Gere, S.P. Timoshenko, Mechanics of Materials, fourth ed., PWS-KENT Publishing, 1997.
[2] M. Dado, S. Al-Sadder, A new technique for large deflection analysis of non-prismatic cantilever beams, Mech. Res. Commun. 32 (2005) 692–703.
[3] K.E. Bisshopp, D.C. Drucker, Large deflections of cantilever beams, Q. Appl. Math. (1945) 272–275.
[4] A. Ohtsuki, F. Ellyin, Analytical approach to large deformation problems of frame structures (in case of a square frame with rigid joints), JSME Int. J. 44 (2001) 89–93.
[5] T. Beléndez, C. Neipp, A. Beléndez, Large and small deflections of a cantilever beam, Eur. J. Phys. 23 (2002) 371–379.
[6] K. Lee, Post-buckling of uniform cantilever column under a combined load, Int. J. Non-linear Mech. 36 (2001) 813–816.
[7] A. Magnusson, M. Ristinmaa, C. Ljung, Behavior of the extensible elastica column, Int. J. Solids Struct. 38 (2001) 8441–8457.
[8] B.W. Kooi, M. Kuipers, A unilateral contact problem with the heavy elastica, Int. J. Non-Linear Mech. 19 (1984) 309–321.
[9] B.W. Golley, The solution of open and closed elasticas using intrinsic coordinate finite elements, Comput. Methods Appl. Mech. Eng. 146 (1997) 127–134.
[10] M.H. Ang Jr., W. Wang, T. Low, On the estimation of the large deflection of a cantilever beam, in: Proceedings of International Conference on Industrial Electronics, Control, and Instrumentation (IECON), Maui, HI, USA, November 15–19 1993, IEEE, vol. 3, 1993, pp. 1604–1609.
[11] D.G. Fertis, Nonlinear Mechanics, second ed., CRC Press, 1999.
[12] K. Lee, Large deflections of cantilever beams of non-linear elastic material under a combined loading, Int. J. Non-Linear Mech. 37 (3) (2002) 439–443.
[13] T. Beléndez, C. Neipp, A. Beléndez, Numerical and experimental analysis of a cantilever beam: a laboratory project to introduce geometric nonlinearity in mechanics of materials, Int. J. Eng. Educ. 19 (6) (2003) 885–892.