BDC5101

Deterministic Operations Research Models

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Duality Theory

Motivating Example

- **Consider a nonlinear optimization problem:**

$$z = \min_{x,y} x^2 + y^2$$
$$s.t.\ x + y = 1$$

- **Method of Lagrangian relaxation**

$$g(p) = \min_{x,y} x^2 + y^2 + p(1 - x - y)$$

  - p **is called the Lagrangian multiplier.**
  - **The function** $L(x, y, p) = x^2 + y^2 + p(1 - x - y)$ **is called the Lagrangian.**
  - **For any** p**, the relaxed problem provides a lower bound to the optimal value of the original problem.**
  - **If** $p > 0$**, it penalizes the case when** $x + y < 1$**; if** $p < 0$**, it penalizes the case when** $x + y > 1$**.**

Motivating Example

- **Relaxed problem**

$$\min_{x,y} x^2 + y^2 + p(1 - x - y)$$

  - $\frac{\partial L}{\partial x} = 2x - p = 0, \frac{\partial L}{\partial y} = 2y - p = 0.$
  - $x = y = \frac{p}{2}$ and the optimal value is $g(p) = p - \frac{p^2}{2}$.
  - For any p, $g(p) \leq z$.
- **We want to make $g(p)$ as close to z as possible:**

$$\max_{p} g(p)$$

  - Optimal solution is $p = 1$, and consequently $x = y = \frac{1}{2}$

Summary

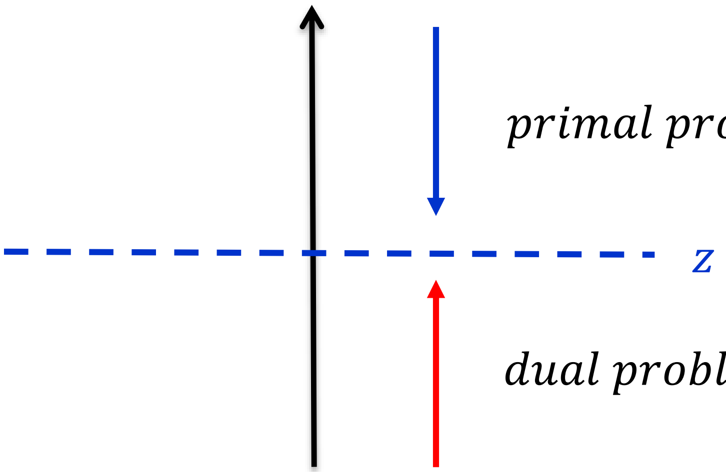
- **Primal problem**

$$z = \min_{x,y} x^2 + y^2$$
$$s.t.\ x + y = 1$$

- **We call the problem of finding the tightest lower bound as the dual problem:**

$$w = \max_{p} g(p)$$

  - $w \leq z$
  - $w = z$?



Duality in LP: Standard Form

- **Primal problem**

$$\begin{aligned} \min \quad & c'x \\ s.t. \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- **Relaxed problem:**

$$\begin{aligned} g(p) &= \min_{x \geq 0} c'x + p'(b - Ax) \\ &= p'b + \min_{x \geq 0}(c' - p'A)x \end{aligned}$$

  - **If** $c' - p'A \geq 0'$, $\min_{x \geq 0}(c' - p'A)x = 0$;
  - **Else,** $\min_{x \geq 0}(c' - p'A)x = -\infty$.

Duality in LP: Standard Form

- **Dual problem**

$$\max_{\boldsymbol{p}} \boldsymbol{g}(\boldsymbol{p})$$

  - **We want to enforce $\boldsymbol{c}' - \boldsymbol{p}'\boldsymbol{A} \geq \mathbf{0}'$, since otherwise $\boldsymbol{g}(\boldsymbol{p}) = -\infty$.**
  - **Under $\boldsymbol{c}' - \boldsymbol{p}'\boldsymbol{A} \geq \mathbf{0}'$, $\boldsymbol{g}(\boldsymbol{p}) = \boldsymbol{p}'\boldsymbol{b}$.**
  - **The problem can be rewritten as**

$$\begin{aligned} &\boldsymbol{max} \quad \boldsymbol{p}'\boldsymbol{b} \\ &\boldsymbol{s.t.} \quad \boldsymbol{p}'\boldsymbol{A} \leq \boldsymbol{c}', \end{aligned}$$

with decision variables $\boldsymbol{p}' = (p_1, \dots, p_m)$.

  - **No constraint on p_j since we want penalize both the case $\boldsymbol{a}_j'\boldsymbol{x} > b_j$ and the case $\boldsymbol{a}_j'\boldsymbol{x} < b_j$**

Variants

- **Primal problem**

$$\begin{aligned} \min \quad & c'x \\ s.t. \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

- **Standard form:**

$$\begin{aligned} \min \quad & [c' \ 0'] \begin{bmatrix} x \\ s \end{bmatrix} \\ s.t. \quad & [A \ -I] \begin{bmatrix} x \\ s \end{bmatrix} = b \\ & x \geq 0 \end{aligned}$$

- **Dual problem**

$$\begin{aligned} \max \quad & p'b \\ s.t. \quad & p'[A \ -I] \leq [c' \ 0'], \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max \quad & p'b \\ s.t. \quad & p'A \leq c', p \geq 0 \end{aligned}$$

Variants: Intuition on $p \geq 0$

- **Primal problem**

$$\begin{aligned} \min \quad & c'x \\ s.t. \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

- **Relaxed problem:**

$$g(p) = \min_{x \geq 0} c'x + p'(b - Ax)$$

- **Since $b_j - a_j{}'x \leq 0$, we only need to penalize the case when $b_j - a_j'x > 0$. This can be achieved by restricting $p_j \geq 0$.**

General Relationship

$$
\begin{array}{lll}
\min & \mathbf{c}'\mathbf{x} & \\
s.t. & \mathbf{a}_i'\mathbf{x} \geq b_i, & i \in M_1 \\
 & \mathbf{a}_i'\mathbf{x} \leq b_i, & i \in M_2, \\
 & \mathbf{a}_i'\mathbf{x} = b_i, & i \in M_3, \\
 & x_j \geq 0, & j \in N_1, \\
 & x_j \leq 0, & j \in N_2, \\
 & x_j \quad \text{free}, & j \in N_3,
\end{array}
$$

$$
\begin{array}{lll}
\max & \mathbf{p}'\mathbf{b} & \\
s.t. & p_i \geq 0, & i \in M_1, \\
 & p_i \leq 0, & i \in M_2, \\
 & p_i \quad \text{free}, & i \in M_3, \\
 & \mathbf{p}'\mathbf{A}_j \leq c_j, & j \in N_1 \\
 & \mathbf{p}'\mathbf{A}_j \geq c_j, & j \in N_2 \\
 & \mathbf{p}'\mathbf{A}_j = c_j, & j \in N_3
\end{array}
$$

Duality Theory

- **Dual of the dual is primal:**

$$\begin{array}{ll} \min & c'x \\ s.t. & Ax \geq b, x \geq 0 \end{array} \Longleftrightarrow \begin{array}{ll} \max & p'b \\ s.t. & p'A \leq c', p \geq 0 \end{array}$$

- **Weak duality:**

$$p'b \leq c'x$$

  - **If the optimal value in the primal is $-\infty$, then the dual problem must be infeasible.**
  - **If the optimal value in the dual is $+\infty$, then the primal problem must be infeasible.**
  - **If x and p are feasible solutions to primal and dual and $p'b = c'x$, then they must be optimal.**

Duality Theory

- **Strong duality: If an LP has an optimal solution, then its dual also has a solution and the respective optimal value are equal.**

	Finite optimum	**Unbounded**	**Infeasible**
Finite optimum	Possible	Impossible	Impossible
Unbounded	Impossible	Impossible	Possible
Infeasible	Impossible	Possible	Possible

Economic Interpretation of Duality

Production Example Revisited

TABLE 3.1 Data for the Wyndor Glass Co. problem

	Production Time per Batch, Hours		
	Product		
Plant	1	2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	$3,000	$5,000	

- **LP formulation**

$$\begin{aligned} \max \quad & 3x_1 + 5x_2 \\ s.t. \quad & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & \mathbf{c}'\mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Production with Resource Market

- **Suppose there is a market where plant hours can be sold and bought.**

- **Production hour at plant i is priced at $p_i \geq 0$.**

- **Consider plant 3, for example**
  - **If $3x_1 + 2x_2 < 18$, the firm can sell $18 - (3x_1 + 2x_2)$ hours at price p_3 to the market.**
  - **If $3x_1 + 2x_2 > 18$ The firm has to buy $3x_1 + 2x_2 - 18$ at price p_3 from the market.**

Production with Resource Market

$$\boldsymbol{max} \quad 3\,x_1 + 5x_2 + p_1(4 - x_1) + p_2(12 - 2x_2)\ + p_3\,[18 - (3x_1 + 2x_2)]$$

$$\boldsymbol{s.t.} \qquad x_1, x_2 \geq 0$$

$$\boldsymbol{max} \quad \boldsymbol{c'x} + \sum_{\boldsymbol{i=1}}^{\boldsymbol{m}} p_i(b_i - \sum_{j=1}^{n} a_{ij}x_j)$$

$$\boldsymbol{s.t.} \qquad x_j \geq 0,\ \ j = 1, \dots, n$$

Production with Resource Market

$$\boldsymbol{max} \quad 3\,x_1 + 5x_2 + p_1(4 - x_1) + p_2(12 - 2x_2)\ + p_3[18 - (3x_1 + 2x_2)]$$

- **Rewrite as**

$$\boldsymbol{max} \quad 4p_1 + 12p_2 + 18p_3 + (3\ - p_1 - 3p_3)x_1 + (5 - 2p_2 - 2p_3)x_2$$

- $p_1 + 3p_3$ **can be interpreted as the market cost of producing product 1.**
  - **Think about the case all the resources are bought from the market.**

At what price should the market price the plant hours?

Market Perspective

$$4p_1 + 12p_2 + 18p_3 + (3 - p_1 - 3p_3)x_1 + (5 - 2p_2 - 2p_3)x_2$$

- $4p_1 + 12p_2 + 18p_3$: **purchasing cost**
- $(3 - p_1 - 3p_3)x_1 + (5 - 2p_2 - 2p_3)x_2$: **opportunity cost**

- $\boldsymbol{min}_{p_1,p_2,p_3 \geq 0}\ [\boldsymbol{max}\quad 4p_1 + 12p_2 + 18p_3 + (3 - p_1 - 3p_3)x_1 + (5 - 2p_2 - 2p_3)x_2]$

Market Perspective

- **What happens if the market prices are such that**

$$3 - p_1 - 3p_3 > 0$$

In this case, there is an arbitrage opportunity, the firm would set $x_1 \to \infty$.

- **What happens if the market prices are such that**

$$3 - p_1 - 3p_3 \leq 0$$

In this case, there is no arbitrage opportunity in selling product 1: profit earned from arbitrage $(3 - p_1 - 3p_3)x_1 = 0$**.**

Market Perspective

- $\boldsymbol{min}\ [\boldsymbol{max}\quad 4p_1 + 12p_2 + 18p_3 + (3 - p_1 - 3p_3)x_1 + (5 - 2p_2 - 2p_3)x_2]$

$$\begin{aligned} \boldsymbol{min}\ \ & 4p_1 + 12p_2 + 18p_3 \\ & p_1 + 3p_3 \geq 3, \\ & 2p_2 + 2p_3 \geq 5, \\ & p_1, p_2, p_3 \geq 0 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \boldsymbol{max}\ \ & 3x_1 + 5x_2 \\ \boldsymbol{s.t.}\ \ & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

Solution

- **We know primal solution is (2, 6) with optimal profit 36K.**
- **Is (0, 1.5, 1) an optimal solution to the dual?**

$$\begin{aligned} \boldsymbol{min} \quad & 4p_1 + 12p_2 + 18p_3 \\ & p_1 + 3p_3 \geq 3, \\ & 2p_2 + 2p_3 \geq 5, \\ & p_1, p_2, p_3 \geq 0 \end{aligned}$$

$$\Leftrightarrow$$

$$\begin{aligned} \boldsymbol{max} \quad & 3x_1 + 5x_2 \\ \boldsymbol{s.t.} \quad & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

Firm and the Market

$$\begin{aligned} \max \quad & c'x \\ s.t. \quad & Ax \le b \\ & x \ge 0 \end{aligned} \qquad\qquad \begin{aligned} \min \quad & p'b \\ s.t. \quad & p'A \ge c' \\ & p' \ge 0' \end{aligned}$$

Summary

- **The primal problem: a firm seeks to maximize its profit by producing products from available resources.**

- **The dual problem: a market seeks to eliminate the arbitrage opportunities by choosing the right prices for resources.**

- **Dual variables are also called *shadow prices*.**

Profit Allocation Problem

TABLE 3.1 Data for the Wyndor Glass Co. problem

	Production Time per Batch, Hours		
	Product		
Plant	**1**	**2**	**Production Time Available per Week, Hours**
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	\$3,000	\$5,000	

$(x_1, x_2) = (2, 6)$, optimal profit = 36 K

How to allocate the 36 K? $(0.5, 0.5, 35)$? $(12, 12, 12)$? $\frac{36}{34}(4, 12, 18) \approx (4.2, 12.7, 19.1)$?

What if

- **Producing on your own…**
- **Plant 1 and Plant 2 produce together**

TABLE 3.1 Data for the Wyndor Glass Co. problem

Plant	Production Time per Batch, Hours — Product 1	Production Time per Batch, Hours — Product 2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	0 ~~18~~
Profit per batch	$3,000	$5,000	

$(x_1, x_2) = (0, 0)$, optimal profit = 0

What if

- **Plant 1 and Plant 3 produce together**

TABLE 3.1 Data for the Wyndor Glass Co. problem

	Production Time per Batch, Hours		
	Product		
Plant	1	2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	0 ~~12~~
3	3	2	18
Profit per batch	\$3,000	\$5,000	

$(x_1, x_2) = (4, 0)$, optimal profit = 12 K

What if

- **Plant 1 and Plant 3 produce together**

TABLE 3.1 Data for the Wyndor Glass Co. problem

	Production Time per Batch, Hours		
	Product		
Plant	1	2	Production Time Available per Week, Hours
1	1	0	0 ~~4~~
2	0	2	12
3	3	2	18
Profit per batch	$3,000	$5,000	

$(x_1, x_2) = (0, 6)$, optimal profit = 30 K

Cooperative Game: Stable Allocations

$$I_1 + I_2 + I_3 = 36$$
$$I_1 + I_2 \geq 0$$
$$I_1 + I_3 \geq 12$$
$$I_2 + I_3 \geq 30$$
$$I_1 \geq 0$$
$$I_2 \geq 0$$
$$I_3 \geq 0$$

$$(0.5, 0.5, 35)$$
$$(12, 12, 12)$$
$$\frac{36}{34}(4, 12, 18) \approx (4.2, 12.7, 19.1)$$

More Players

Table 3: Resource vectors for group with 5 members

Resource vectors	Plant 1	Plant 2	Plant 3
1	0	0	0
2	1	0	3
3	2	0	6
4	0	4	4
5	1	8	5

$$I_1 + I_2 + I_3 + I_4 + I_5 = 36$$
$$I_2 + I_3 + I_4 + I_5 \geq 36$$
$$I_1 + I_3 + I_4 + I_5 \geq 33$$
$$I_1 + I_2 + I_4 + I_5 \geq 30$$
$$I_1 + I_2 + I_3 + I_5 \geq 26$$
$$I_1 + I_2 + I_3 + I_4 \geq 19$$
$$I_1 + I_2 + I_3 \geq 9, \ldots$$
$$I_1 + I_2 \geq 3, \ldots$$
$$I_1 \geq 0,\ I_2 \geq 3,\ I_3 \geq 6,\ I_4 \geq 10,\ I_5 \geq 12.5$$

What is a stable allocation?

Allocation Based on Shadow Price: 3 players

$$\begin{aligned} \boldsymbol{min}\ \ 4p_1 + 12p_2 + 18p_3 \\ p_1 + 3p_3 \geq 3, \\ 2p_2 + 2p_3 \geq 5, \\ p_1, p_2, p_3 \geq 0 \end{aligned}$$

Table 1: Resource vectors for group with 3 members

Resource vectors	Plant 1	Plant 2	Plant 3
1	4	0	0
2	0	12	0
3	0	0	18

- $(0, 1.5, 1)$ **is the shadow price.**
  - **Player 1:** $4 \times 0 = 0$
  - **Player 2:** $12 \times 1.5 = 18$
  - **Player 3:** $18 \times 1 = 18$
  - **Allocation:** $(0, 18, 18)$

Is it stable?

$(I_1, I_2, I_3) = (0, 18, 18)$

$$I_1 + I_2 + I_3 = 36$$
$$I_1 + I_2 \geq 0$$
$$I_1 + I_3 \geq 12$$
$$I_2 + I_3 \geq 30$$
$$I_1 \geq 0$$
$$I_2 \geq 0$$
$$I_3 \geq 0$$

Allocation Based on Shadow Price: 5 players

$$\begin{aligned} \boldsymbol{min}\ \ & 4p_1 + 12p_2 + 18p_3 \\ & p_1 + 3p_3 \geq 3, \\ & 2p_2 + 2p_3 \geq 5, \\ & p_1, p_2, p_3 \geq 0 \end{aligned}$$

Table 3: Resource vectors for group with 5 members

Resource vectors	Plant 1	Plant 2	Plant 3
1	0	0	0
2	1	0	3
3	2	0	6
4	0	4	4
5	1	8	5

- $(0, 1.5, 1)$ **is the shadow price.**
  - **Player 1:** $0 \times 0 + 1.5 \times 0 + 1 \times 0 = 0$
  - **Player 2:** $0 \times 1 + 1.5 \times 0 + 1 \times 3 = 3$
  - **Player 3:** $0 \times 2 + 1.5 \times 0 + 1 \times 6 = 6$
  - **Player 4:** $0 \times 0 + 1.5 \times 4 + 1 \times 4 = 10$
  - **Player 5:** $0 \times 1 + 1.5 \times 8 + 1 \times 5 = 17$
  - **Allocation:** $(0, 3, 6, 10, 17)$

Sensitivity Analysis

Motivation

- In practice, it is rarely sufficient to solve a single LP to arrive at good decisions.
  - The problem data may depend on some higher level decisions (strategic-level rather than operational-level)
  - Part of the problem data maybe controllable at an additional cost
  - Incomplete knowledge of problem data (randomness)

Sensitivity Analysis

- **Possible changes in data require sensitivity analysis.**
- **Sensitivity analysis: “What if” analysis**
- **We put more focus on the optimal value**

Production Example Revisited

- **LP formulation**

$$
\begin{gathered}
v^* = max\ 3x_1 + 5x_2 \\
x_1 \leq 4, \\
2x_2 \leq 12, \\
3x_1 + 2x_2 \leq 18, \\
x_1 \geq 0, x_2 \geq 0
\end{gathered}
$$

Adding a Constraint

- **What happens to v^* if we add a constraint?**

$$v^* = max\ 3x_1 + 5x_2$$
$$x_1 \leq 4,$$
$$2x_2 \leq 12,$$
$$3x_1 + 2x_2 \leq 18,$$
$$x_1 + x_2 \leq 10,$$
$$x_1 \geq 0, x_2 \geq 0$$

A new plant dedicated for packaging → $x_1 + x_2 \leq 10$

- **When will v^* not be affected by the additional constraint?**

Adding a New Variable

- **What happens to v^* if we add a variable?**

$$v^* = max\ 3x_1 + 5x_2 + 6x_3$$
$$x_1 + 2x_3 \leq 4,$$
$$2x_2 + 4x_3 \leq 12,$$
$$3x_1 + 2x_2 + 6x_3 \leq 18,$$
$$x_1 \geq 0, x_2 \geq 0,\ x_3 \geq 0$$

Consider $(x_1, x_2, x_3) = (2,6,0)$. Is it a BFS?

Adding a New Variable

- **When will v^* remain unchanged?**

$$v^* = -min - 3x_1 - 5x_2 - 6x_3$$
$$\begin{aligned} x_1 + \quad\quad 2x_3 + x_4 \quad\quad &= 4, \\ 2x_2 + 4x_3 + \quad\quad x_5 \quad &= 12, \\ 3x_1 + 2x_2 + 6x_3 + \quad\quad\quad x_6 &= 18, \end{aligned}$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

- **Compute the reduced cost for nonbasic variable x_3 at $(x_1, x_2, x_3\ x_4, x_5, x_6) = (2,6,0,2,0,0)$**

$\Uparrow$

Controllable Data Inputs

- Available resources

- Prices

Change in resource

- **What happens to v^* if plant 1 has one more capacity?**

$$
\begin{gathered}
v^* = max\ 3x_1 + 5x_2 \\
x_1 \leq 4 + 1, \\
2x_2 \leq 12, \\
3x_1 + 2x_2 \leq 18, \\
x_1 \geq 0, x_2 \geq 0
\end{gathered}
$$

Change in resource

- **What happens to v^* if plant 3 has one more capacity?**

$$
\begin{gathered}
v^* = max\ 3x_1 + 5x_2 \\
x_1 \leq 4, \\
2x_2 \leq 12, \\
3x_1 + 2x_2 \leq 18 + 1, \\
x_1 \geq 0, x_2 \geq 0
\end{gathered}
$$

- **How about 3 more capacity?**

Change in resource

- **What happens to v^* if plant 3 has 9 more capacity?**

$$
\begin{gathered}
v^* = max\ 3x_1 + 5x_2 \\
x_1 \leq 4, \\
2x_2 \leq 12, \\
3x_1 + 2x_2 \leq 18 + 9, \\
x_1 \geq 0, x_2 \geq 0
\end{gathered}
$$

Relation with Shadow Price

- **Shadow price for the plant constraints**

$$v^* = max\ 3x_1 + 5x_2$$

$$x_1 \le 4, \quad \Longleftrightarrow \quad p_1 = 0$$

$$2x_2 \le 12, \quad \Longleftrightarrow \quad p_2 = 1.5$$

$$3x_1 + 2x_2 \le 18, \quad \Longleftrightarrow \quad p_3 = 1$$

$$x_1 \ge 0, x_2 \ge 0$$

Complementary slackness condition:

$$(b_j - \boldsymbol{a_j}'\boldsymbol{x})p_j = 0$$

Change in Price

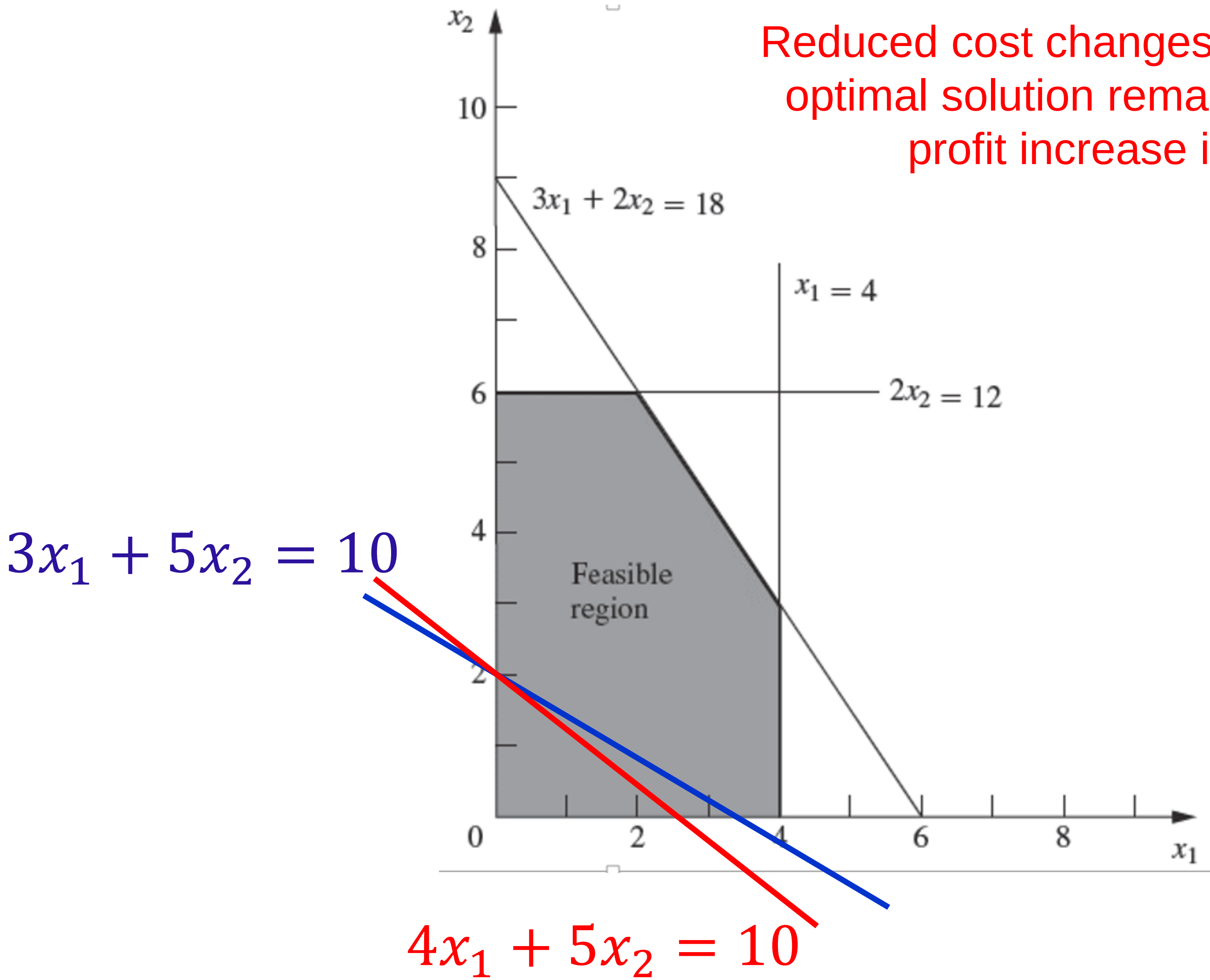
- **What happens to v^* if the profit per batch for product 1 increases by 1K?**

$$v^* = max\ (3+1)x_1 + 5x_2$$
$$x_1 \le 4,$$
$$2x_2 \le 12,$$
$$3x_1 + 2x_2 \le 18,$$
$$x_1 \ge 0, x_2 \ge 0$$

At least increase by 2K.

Change in Price

Reduced cost changes but still all positive: optimal solution remains unchanged and profit increase is exactly 2K.



Change in Price

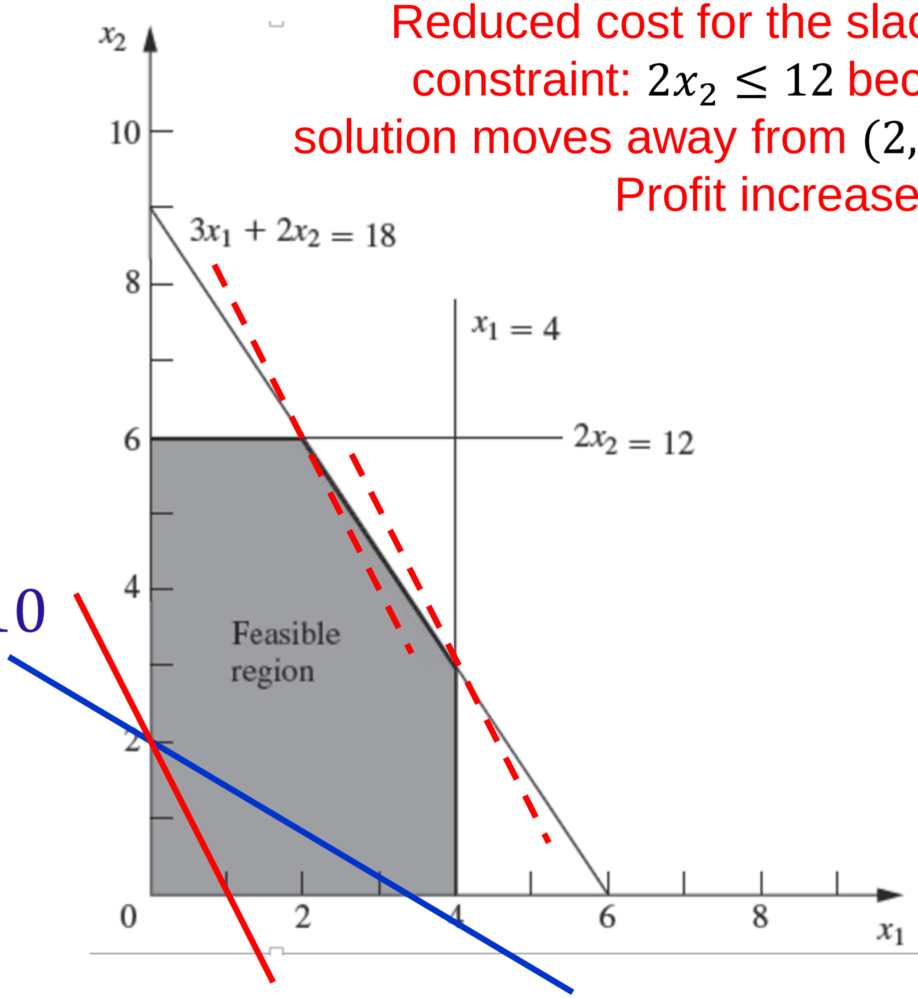
- **What happens to v^* if the profit per batch for product 1 increases by 7K?**

$$v^* = max\ (3+7)x_1 + 5x_2$$
$$x_1 \le 4,$$
$$2x_2 \le 12,$$
$$3x_1 + 2x_2 \le 18,$$
$$x_1 \ge 0, x_2 \ge 0$$

At least increase by 14K.

Change in Price

Reduced cost for the slack variable of the constraint: $2x_2 \leq 12$ becomes negative: solution moves away from $(2,6,2,0,0)$ to $(4,3,0,6,0)$. Profit increase: 19K.



$3x_1 + 5x_2 = 10$

$10x_1 + 5x_2 = 10$

Sensitivity Report (Excel)

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C10	Batches Produced Hours Used Per Batch Produced	2	0	3	4.5	3
D10	Batches Produced Windows	6	0	5	1E+30	3

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
E6	Plant 1 Hours Used	2	0	4	1E+30	2
E7	Plant 2 Hours Used	12	1.5	12	6	6
E8	Plant 3 Hours Used	18	1	18	6	6

Sensitivity Report (Gurobi)

```python
#Print sensitivity information

print("\n Sensitivity information:")
for d in m.getConstrs():
    print(d.ConstrName, d.Pi, d.SARHSUp, d.SARHSLow)


```

```
 Sensitivity information:
Plant[0] 0.0 1e+100 2.0
Plant[1] 1.5 18.0 6.0
Plant[2] 1.0 24.0 12.0
```

Reference: http://www.gurobi.com/documentation/8.1/refman/linear_constraint_attribut.html