BDC5101: Deterministic Operations Research Models **Semester II, 2018/2019, NUS**

Assignment 2: Due on Mar.11, 2019

The solution is for your own reference only. **Do not CIRCULATE.**

1. (10') Consider the problem

$$\begin{aligned} \min \quad & 2x_1 + 15x_2 + 5x_3 + 6x_4 \\ s.t. \quad & x_1 + 6x_2 + 3x_3 + x_4 \geq 2 \\ & 2x_1 - 5x_2 + x_3 - 3x_4 \geq 3 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

(a) Give the dual linear problem.

(b) Solve the dual problem using any method you prefer.

(c) Use your solution for the dual problem in (b) and complementary slackness condition to solve the primal problem above. (**Hint: At the optimal solution, complementary slackness condition implies both (i): the dual solution multiplied by the slackness in the corresponding primal constraint equals to zero; (ii): the primal solution multiplied by the slackness in the corresponding dual constraint equals to zero.**)

Solution: **(a)** The dual problem is

$$\begin{aligned} \max \quad & 2p_1 + 3p_2 \\ s.t. \quad & p_1 + 2p_2 \leq 2 \\ & 6p_1 - 5p_2 \leq 15 \\ & 3p_1 + p_2 \leq 5 \\ & p_1 - 3p_2 \leq 6 \\ & p_1, p_2 \geq 0. \end{aligned}$$

(b) The optimal dual solution is $(8/5, 1/5)$ with optimal value $19/5$.

(c) From the dual solution, we know that in the dual problem the first and third constraints are binding, while the second and forth constraints are not binding at the optimal solution. By complementary slackness condition, we must have

$$x_2(15 - 6p_1 + 2p_2) = 0,\ x_4(6 - p_1 + 3p_2) = 0,$$

which implies $x_2 = x_4 = 0$. Again by complementary slackness condition,

$$[2 - (x_1 + 6x_2 + 3x_3 + x_4)]p_1 = 0,\ [3 - (2x_1 - 5x_2 + x_3 - 3x_4)]p_2 = 0.$$

Since $p_1, p_2 > 0$, we have

$$x_1 + 6x_2 + 3x_3 + x_4 = 2,\ 2x_1 - 5x_2 + x_3 - 3x_4 = 3.$$

Combined with $x_2, x_4 = 0$, we can solve $x_1 = 7/5$ and $x_3 = 1/5$, i.e., $\mathbf{x}^* = (7/5, 0, 1/5, 0)$ and the optimal value is $19/5$.

2. (10') **Portfolio Management Problem:** Consider the portfolio management problem discussed in class. There are 100 stocks. The expected return of stock i is r_i per dollar invested and the total budget is b dollars, where $b > 0$. Ignoring the risk of the return, the problem of maximizing the total expected return is

$$\begin{array}{ll} \max & \sum_{i=1}^{100} r_i x_i \\ \text{s.t.} & \sum_{i=1}^{100} x_i \leq b \\ & x_i \geq 0, \;\; i = 1, ..., 100, \end{array}$$

where x_i denotes the amount of money invested in stock i.

(a) Write down the dual problem to the above linear programming problem.

(b) What is the optimal solution to the primal and the dual respectively?

(c) Verify the solution you provided in (b) is indeed optimal using the weak duality.

Solution:

(a) Note that there is only one constraint (aside from nonnegativity constraints on variables) in the primal problem. We let p denote the dual variable associated with this constraint. The dual problem for the above LP is then the following simple one-dimensional linear programming:

$$\begin{array}{ll} \min & bp \\ \text{s.t.} & p \geq r_i, \;\; i = 1, ..., 100 \\ & p \geq 0. \end{array}$$

(b) For the primal problem, recall that the extreme points for the problem consist of the investment strategies that either put all the budget b in one stock or invest nothing. Putting all the budget b in stock i yields a total expected return $r_i b$ and investing nothing yields a total expected return 0. Note that *the sign of r_i is not specified in the problem, and it is possible that all the stocks have negative return.* As a result, let i^* be the index of the stock with maximum expected return, i.e., $r_{i^*} = \max_{1 \leq i \leq 100} r_i$, then the optimal solution to the primal is

$$\begin{cases} x_{i^*} = b, x_j = 0, \;\; j \neq i^*, & \text{if } r_{i^*} > 0, \\ x_i = 0, \;\; 1 \leq i \leq 100 & \text{if } r_{i^*} \leq 0. \end{cases}$$

For the dual problem, since $b > 0$, to minimize the objective, we would like to make p as small as possible. Hence, the optimal solution to the dual problem is $p^* = \max\{\max_{1 \leq i \leq 100} r_i, 0\}$.

(c) To verify that the solutions derived above are indeed optimal, we check their corresponding objective values. For the primal solution provided above, the objective value is

$$\begin{cases} r_{i^*} b, & \text{if } r_{i^*} > 0, \\ 0, & \text{if } r_{i^*} \leq 0. \end{cases}$$

More compactly, we can write it as $\max\{r_{i^*} b, 0\}$. The objective value for the dual problem is clearly $bp^* = b \max\{\max_{1 \leq i \leq 100} r_i, 0\} = \max\{r_{i^*} b, 0\}$. The two problems yield exactly the same objective values and by weak duality, the two solutions must be optimal.

Table 2.1: Profit and resources consumed for each product

	E	**C**	**$\mathbf{P_1}$**	**$\mathbf{P_2}$**	**B**	**Capacity**
Clay (lbs)	10	15	10	10	20	130
Enamel (lbs)	1	2	2	1	1	13
Dry room (hours)	3	1	6	6	3	45
Klin (hours)	2	4	2	5	3	23
Profit	51	102	66	66	89	

3. (10') A potter manufacturer can make four different types of dining room service sets: JJP English, Currier, Primrose, and Bluetail. Furthermore, Primrose can be made by two different methods. Each set uses clay, enamel, dry room time, and kiln time, and results in a profit shown in Table 2.1.

   The manufacturer is currently committed to making the same amount of Primrose using method 1 and method 2. The formulation of the profit maximization problem is given below. The decision variables E, C, P_1, P_2, B are the number of sets of type English, Currier, Primrose Method 1, Primrose Method 2, and Bluetail, respectively. We assume, for the purposes of this problem, that the number of sets of each type can be fractional.

$$
\begin{aligned}
\max \quad & 51E + 102C + 66P_1 + 66P_2 + 89B \\
s.t. \quad & 10E + 15C + 10P_1 + 10P_2 + 20B \leq 130 \\
& E + 2C + 2P_1 + P_2 + B \leq 13 \\
& 3E + C + 6P_1 + 6P_2 + 3B \leq 45 \\
& 2E + 4C + 2P_1 + 5P_2 + 3B \leq 23 \\
& P_1 - P_2 = 0 \\
& E, C, P_1, P_2, B \geq 0.
\end{aligned}
$$

   The optimal solution to the primal is given in Table 2.2 and the optimal solution to the dual together with sensitivity information is given in Table 2.3. Use the information to answer the questions that follow.

Table 2.2: The optimal solution to the primal

	Optimal Value	**Reduced Cost**	**Objective Coefficient**
E	0	-3.571	51
C	2	0	102
$\mathbf{P_1}$	0	0	66
$\mathbf{P_2}$	0	-37.571	66
B	5	0	89

Table 2.3: The optimal solution to the dual and the sensitivity. The last two columns describe the allowed changes in the components of **b** for which the optimal dual solution remains the same

	Dual Variable	**Const. RHS**	**Allowable Increase**	**Allowable Decrease**
Clay	1.429	130	23.33	43.75
Enamel	0	13	∞	4
Dry room	0	45	∞	28
Klin	20.143	23	5.60	3.50
Primrose Methods	11.429	0	3.50	0

(a) What is the optimal quantity of each service set, and what is the total profit?

(b) Give an economic interpretation of the optimal dual variables appearing in the sensitivity report, for each of the five constraints.

(c) Should the manufacturer buy an additional 20 lbs. of Clay at \$1.1 per pound?

(d) In the current model, the number of Primrose produced using method 1 was required to be the same as the number of Primrose produced by method 2. Consider a revision of the model in which this constraint is replaced by the constraint $P_1 - P_2 \geq 0$. In the reformulated problem would the amount of Primrose made by method 1 be positive?

Solution:

(a) The optimal solution is (0, 2, 0, 0, 5). The optimal total profit is $10 \times 2 + 5 \times 89 = 649$.
(b) The optimal dual variables show how much the profit will increase by increasing one unit of the resources, e.g, one unit increase of 1 Clay will increase profit by 1.429.
(c) Yes. Since, $20 \leq 23.33$ which is the allowable increase for clay, and $1.429 > 1.1$, it is profitable.
(d) Yes. Note here that the dual variable associated with last constraint $P_1 - P_2 = 0$ is positive and the allowable increase of the right hand side is 3.5. That is, as long as

$$P_1 - P_2 = \delta, \ 0 < \delta \leq 3.5$$

the profit will increase while $P_1 = P_2 + \delta > 0$. Thus, if we relax the constraint to be $P_1 - P_2 \geq 0$, a positive amount of Primrose made by method 1 will increase the profit.

4. (10') **Diet Problem:** The "diet problem" concerns with finding a mix of foods that satisfies requirements on the amount of nutrition. We consider here the problem of choosing fruits to meet certain nutritional requirements. The nutrition information of five selected fruits is provided in Table 2.4 along with a minimum required and maximum allowed intake of each type of nutrition in the last two columns. Your goal is to find the cheapest combination of fruits that will meet nutritional requirements.

Table 2.4: Nutrition and price information per 100 grams

	Apple	**Banana**	**Blueberries**	**Durian**	**Tangerine**	**Min Reqd**	**Max Allowed**
Calories	52	89	57	147	53	500	3000
Carbohydrate (g)	14	23	14	27	13	50	400
Fiber (g)	2.4	2.6	2.4	3.8	1.8	20	30
Vitamin A (IU)	54	64	54	44	681	2000	3500
Vitamin C (mg)	4.6	8.7	9.7	19.7	26.7	75	150
Price (S\$/100g)	0.5	0.3	2.5	10	0.5		

(a) Formulate a mathematical model for this problem. Solve the problem using the software you prefer and attach your code. Present and interpret the results.

(b) Without solving the problem using the software, can you determine how your optimal solution and optimal spending in (a) will change if you are also concerned with the total fat intake. The fat contained in each of the fruit and your nutrition requirement on fat is provided in the table below. Verify your solution by solving the problem numerically.

	Apple	**Banana**	**Blueberries**	**Durian**	**Tangerine**	**Min Reqd**	**Max Allowed**
Fat (g)	0.2	0.3	0.3	5	0.3	0	10

Solution:

Students are referred to python code on IVLE for the implementation of this problem.

(a) The optimal solution is to buy 605 grams of banana and 237 grams of tangerine. The total spending is around 3 S$.

(b) The solution in (a) satisfies the new constraint. Hence, both the optimal solution and optimal spending in (a) will not change.

5. (10') **Retailer Game:** Consider the benchmark model and solution for the retailer game discussed in class. The initial inventory is 2000 units. The retailer needs to sell his inventory over a time horizon of 15 weeks by employing in order the four price levels: \$60, \$54, \$48, \$36 over time. The estimated average demands under these price levels are summarized in the following table: We have shown in

Table 2.5: Benchmark model

Price	\$60	\$54	\$48	\$36
Demand	125	162.5	217.5	348.8

class the optimal policy is to use for the first 11.67 weeks of full price \$60 and then 3.33 weeks of \$54 (ignoring integrality constraints). The revenue collected under this policy is \$116750. Consider the following three scenarios **separately** and answer the corresponding questions. Attach the codes you used to derive the solutions.

(a) Suppose in order to maintain the brand reputation, the retailer is only allowed to discount from week 4. Will the retailer change his markdown strategy in this case?

(b) Due to bad weather, only 25 units were sold in the first week. How should the retailer adjust his markdown policy? What is the improvement in revenue compared to the case when he sticks with the original markdown policy (specified above)?

(c) In the traditional brick-and-mortar business, menu cost and consumer price impression are the two major reasons behind the very few number of price levels. With the rise of e-commerce and other technologies, it is possible for the retailer to employ more price levels and change the price more frequently. In the retailer game, what happens if the retailer employs the following price grids (with a grid of \$2) instead of a few discount levels? For simplicity, the demand estimates at the additional price levels are taken as a linear interpolation of the original demands. What is the improvement in revenue compared to the benchmark? What is the new markdown policy? How do you compare with the original policy?

Table 2.6: Price grids model

Price	\$60	\$58	\$56	\$54	\$52	\$50	\$48	\$46	\$44	\$42	\$40	\$38	\$36
Demand	125	137.5	150	162.5	180.8	199.1	217.5	239.4	261.3	283.2	305.1	327	348.8

Solution:

Students are referred to the python code on IVLE for the implementation of this problem.

(a) Clearly, in the current solution, the retailer is already employing the full price for more than 11 weeks. So requiring the retailer to employ at least 3 weeks of full price will not the change the optimality of the current optimal solution. Hence, no change to the markdown policy.

(b) First, if the retailer sticks with his original markdown policy, then the total revenue would be

$$25\times 60+10.67\times 60\times 125+3.33\times 54\times 162.5+[2000-(60+10.67\times 125+3.33\times 162.5)]\times 25=\$113250.$$

If, on the other hand, the retailer adjusts his markdown strategy, then he has to resolve the problem with the week = 14 and initial inventory 1975. The resulting total revenue (remember to add back the revenue obtained in the first period: 25×60) is \$114150. The improvement is \$900 or 0.79%.
(c) One can implement this by replacing the original data on demand and price by the new data in the table. The new markdown policy is to implement full price \$60 for 5 weeks and then give a \$2 discount for the remaining 10 weeks. One can see that even though the retailer is provided a lot of options in prices, only very few price points will be used.

The optimal profit here is \$117250, which is \$500 higher than the one with only 4 price points. The implication here is that there maybe not much value in considering a complicated pricing scheme, say, giving one dollar discount per week.