**BDC5101: Deterministic Operations Research Models** **Semester II, 2018/2019, NUS**

# Assignment 1: Due on Feb.11, 2019

The solution is for your own reference only. **Do not CIRCULATE.**

1. (5') Consider the problem

$$\begin{aligned} \min \quad & 2x_1 + 3|x_2 - 10| \\ s.t. \quad & |x_1 + 2| + |x_2| \leq 5, \end{aligned}$$

and reformulate it as a linear programming problem.

**Solution:** Note that $|x_1 + 2| + |x_2| = \max\{x_1 + 2 + x_2, x_1 + 2 - x_2, -x_1 - 2 + x_2, -x_1 - 2 - x_2\}$, and we have

$$\begin{aligned} \min \quad & 2x_1 + 3x_3 \\ s.t. \quad & x_1 + 2 + x_2 \leq 5, \\ & x_1 + 2 - x_2 \leq 5, \\ & -x_1 - 2 + x_2 \leq 5, \\ & -x_1 - 2 - x_2 \leq 5, \\ & x_2 - 10 \leq x_3, \\ & -x_2 + 10 \leq x_3. \end{aligned}$$

An alternative (and better) formulation is

$$\begin{aligned} \min \quad & 2x_1 + 3x_3 \\ s.t. \quad & y_1 + y_2 \leq 5, \\ & x_1 + 2 \leq y_1, \\ & -x_1 - 2 \leq y_1, \\ & x_2 \leq y_2, \\ & -x_2 \leq y_2, \\ & x_2 - 10 \leq x_3, \\ & -x_2 + 10 \leq x_3. \end{aligned}$$

2. (15') The Primo Insurance Company is introducing two new product lines: special risk insurance and mortgages. The expected profit is \$5 per unit on special risk insurance and \$2 per unit on mortgages. Management wishes to establish sales quotas for the new product lines to maximize total expected profit. The work requirements are as follows:

   **(a)** (5') Formulate a linear programming model for this problem.

   **(b)** (5') Use the graphical method to solve this model. Numerically verify your solution using the software you prefer.

   **(c)** (5') Identify the two equations in the constraints, whose solution gives the optimal solution.

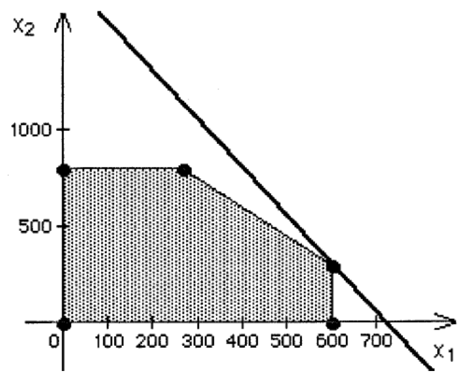
| | Work-Hours per Unit | | |
|---|---|---|---|
| **Department** | **Special Risk** | **Mortgage** | **Working-Hours Available** |
| Underwriting | 3 | 2 | 2400 |
| Administration | 0 | 1 | 800 |
| Claims | 2 | 0 | 1200 |

**Solution:**

**(a)** Let $x_1$ be the number of units on special risk insurance and $x_2$ be the number of units on mortgages.

$$\begin{array}{rl} \max & 5x_1 + 2x_2 \\ \text{s.t.} & 3x_1 + 2x_2 \le 2400 \\ & x_2 \le 800 \\ & 2x_1 \le 1200 \\ & x_1 \ge 0, x_2 \ge 0. \end{array}$$

**(b)** Optimal solution is $(x_1^*, x_2^*) = (600, 300)$ and the optimal profit is 3600. The figure below shows the feasible region and the contour line of the objective function.




**(c)** The relevant two equations are $3x_1 + 2x_2 = 2400$ and $2x_1 = 1200$, from which one can solve $(x_1^*, x_2^*) = (600, 300)$.

3. (10') Find all extreme points in the following polyhedra set:

   **(a)** $P = \{(x_1, x_2, x_3) | x_1 + x_2 + x_3 \le 1, x_1, x_2, x_3 \ge 0\}$.

   **(b)** $P = \{(x_1, x_2, x_3, x_4) | x_1 + x_2 + \frac{1}{2}x_3 \le 1, x_1, x_2, x_3, x_4 \ge 0\}$.

   **Solution:**

   **(a)** (0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0);
   **(b)** (0, 0, 0, 0), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 2, 0).

   *Note: The problem, say, part (a) is asking for the extreme points of* $P = \{(x_1, x_2, x_3) | x_1 + x_2 + x_3 \le 1, x_1, x_2, x_3 \ge 0\}$ *not the extreme point of* $Q = \{(x_1, x_2, x_3, x_4) | x_1 + x_2 + x_3 + x_4 = 1, x_1, x_2, x_3, x_4 \ge 0\}$ *after standardization, which are two different polyhedrons.*

4. (5') Consider the problem

$$\begin{aligned} \min \quad & x_1 \\ s.t. \quad & x_1 = 1, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Find all the extreme points and optimal solutions to the above problem.

**Solution:** The only extreme point for this problem is $(x_1, x_2) = (1, 0)$. Any feasible solution in this case is optimal and the set of optimal solutions can be written as $\{(x_1, x_2) | x_1 = 1, x_2 \geq 0\}$.

*Note: In this problem, there is an extreme point, i.e., $(x_1, x_2) = (1, 0)$, which is optimal. However, there are also points, say $(x_1, x_2) = (1, 1)$ which is not an extreme points, that are optimal as well.*

5. (5') Consider the problem

$$\begin{aligned} \min \quad & -x_1 - x_2 \\ s.t. \quad & x_1 - x_2 \leq 3 \\ & x_1 + x_2 \leq 6 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Convert the problem into standard form and construct a basic feasible solution at which $(x_1, x_2) = (0, 0)$. Is this basic feasible solution optimal?

**Solution:** The standard form is

$$\begin{aligned} \min \quad & -x_1 - x_2 + 0 \times x_3 + 0 \times x_4 \\ \text{s.t.} \quad & x_1 - x_2 + x_3 + 0 \times x_4 = 3 \\ & x_1 + x_2 + 0 \times x_3 + x_4 = 6 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

and the associated BFS is (0, 0, 3, 6).

This BFS is not optimal. You can simply compare the objectives by choosing a feasible solution, e.g. (1, 0, 2, 5), where $-1 - 0 + 0 + 0 < -0 - 0 + 0 + 0$.

Alternatively, you should be able to compute the reduced cost along the nonbasic variables $x_1, x_2$:

$$(\bar{c}_1, \bar{c}_2) = (c_1, c_2) - c_B' \mathbf{B}^{-1}[\mathbf{A}_1, \mathbf{A}_2] = (-1, -1) - (0, 0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = (-1, -1),$$

the reduced costs are negative along both directions, and hence the current BFS is not optimal.

6. **Investment under Taxation:** (10') An investor has a portfolio of $n$ different stocks. He has bought $s_i$ shares of stock $i$ at price $p_i, i = 1, ..., n$. The current price of one share of stock $i$ is $q_i$. The investor expects that the price of one share of stock $i$ in one year will be $r_i$. If he sells shares, the investor pays transaction costs at the rate of 1% of the amount transacted. In addition, the investor pays taxes at the rate of 30% on capital gains. For example, suppose that the investor sells $1,000$ shares of a stock at \$50 per share. He has bought these shares at \$30 per share. Upon selling, he receives $1,000 \times 50 = \$50,000$. However, he owes $0.30 \times (50,000 - 30,000) = \$6,000$ on capital gain taxes and $0.01 \times 50,000 = \$500$ on transaction costs. So, by selling $1,000$ shares of this stock he nets $50,000 - 6,000 - 500 = \$43,500$.

   **(a)** Formulate the problem of selecting how many shares the investor needs to sell in order to raise an amount of money at least $K$, net of capital gains and transaction costs, while maximizing the expected value of his (remaining) portfolio next year.

**(b)** Using the data for the portfolio in investment.csv, solve the problem for $K = \$9,000$ and attach your code.

**Solution:** **(a)** Let $x_i$ be the number of shares of stock $i$ to be sold. The amount of money raised from selling $x_i$ shares of stock $i$ is then $q_i x_i - 0.3 \times \max\{q_i - p_i, 0\}x_i - 0.01 q_i x_i)$. The LP formulation of the problem is then

$$\begin{array}{ll} \max & \sum_{i=1}^{n} r_i(s_i - x_i) \\ s.t. & \sum_{i=1}^{n}(q_i x_i - 0.3 \max\{q_i - p_i, 0\}x_i - 0.01 x_i q_i) \geq K \\ & 0 \leq x_i \leq s_i, \forall i = 1, ..., n \end{array}$$

**(b)** Please refer to investment_tax_code.ipynb for the code. The maximum expected value of the portfolio in the next year is \$15356.8 and the optimal selling strategy is

| | S1 | S2 | S3 | S4 | S5 |
|---|---|---|---|---|---|
| Number of shares | 1,000 | 1,000 | 1,000 | 118.762 | 0 |