BDC5101

Deterministic Operations Research Models

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Linear Programming
and
Its Applications

Prototype Example

- **Wyndor Glass Co.**
  - Produces windows and glass doors
  - Plant 1 makes aluminum frames and hardware
  - Plant 2 makes wood frames
  - Plant 3 produces glass and assembles products

Prototype Example

- **Company introducing two new products**
  - **Product 1: 8 ft. glass door with aluminum frame**
  - **Product 2: 4 x 6 ft. double-hung, wood-framed window**
- **Problem: What mix of products would be most profitable?**

Prototype Example

- **Data needed**
  - Number of hours of production time available per week in each plant for new products
  - Production time used in each plant for each batch of each new product
  - Profit per batch of each new product
  - Demand

Prototype Example

TABLE 3.1 Data for the Wyndor Glass Co. problem

	Production Time per Batch, Hours		
	Product		
Plant	**1**	**2**	**Production Time Available per Week, Hours**
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	\$3,000	\$5,000	

Demand: assume that all products produced can be sold.

Prototype Example

- **Formulating the model**

  x_1 = number of batches of product 1 produced per week

  x_2 = number of batches of product 2 produced per week

  Z = total profit per week (thousands of dollars) from producing these two products

- **From bottom row of Table 3.1**

$$Z = 3x_1 + 5x_2$$

Prototype Example

- **Constraints (see Table 3.1)**

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

- **Classic example of resource-allocation problem**
  - **Most common type of linear programming problem**

Prototype Example

- **Alternative formulation 1**

  x_1 = number of batches of product 1 produced per week

  s_1 = production time reserved at plant 1

  x_2 = number of batches of product 2 produced per week

- **Alternative formulation 2**

  x_i = number of batches of product i produced per week, i=1,2

  s_j = production time reserved at plant j, j=1,2,3

Graphical Approach/Contour Approach

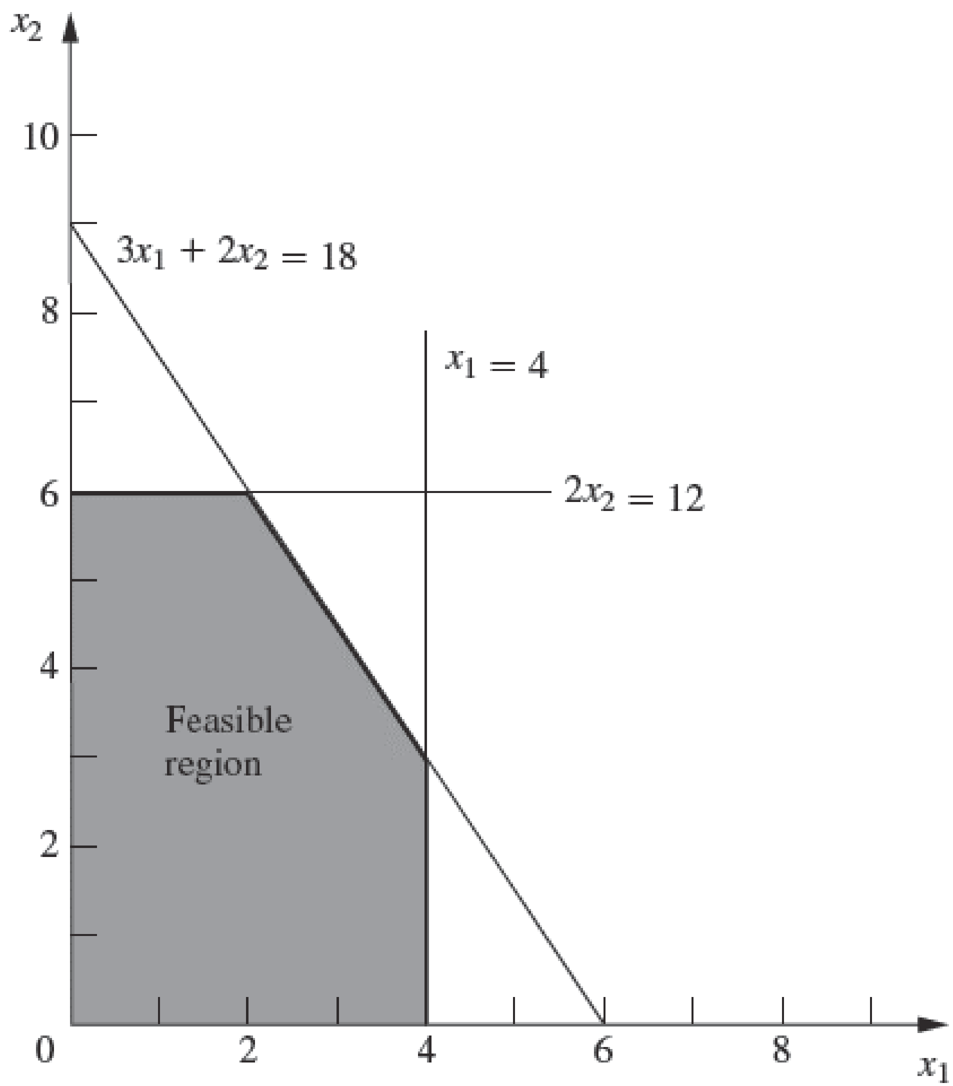


FIGURE 3.2
Shaded area shows the set of permissible values of (x_1, x_2), called the feasible region.

Graphical Approach/Contour Approach

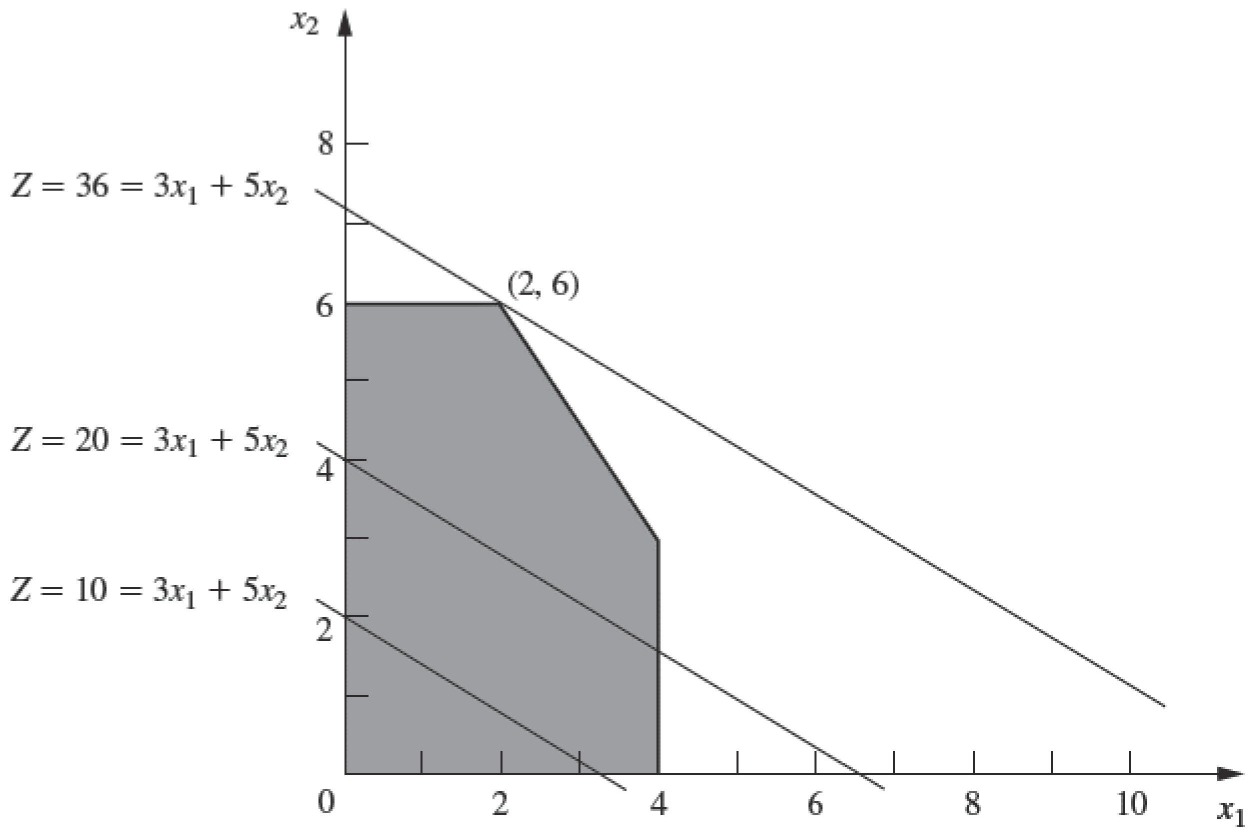


FIGURE 3.3
The value of (x_1, x_2) that maximizes $3x_1 + 5x_2$ is (2, 6).

A General Liner Programming Problem

$$\min \quad c_1x_1 + \cdots + c_nx_n$$

$$s.t. \quad a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i, i \in M_1$$

$$a_{j1}x_1 + \cdots + a_{jn}x_n \leq b_j, j \in M_2$$

$$a_{k1}x_1 + \cdots + a_{kn}x_n = b_k, k \in M_3$$

$$x_l \geq 0, l \in N_1$$

$$x_m \leq 0, m \in N_2$$

Linear functions

Standard Form

$$
\begin{aligned}
\min \quad & c_1x_1 + \cdots + c_nx_n \\
s.t. \quad & a_{i1}x_1 + \cdots + a_{in}x_n = b_i, i = 1, \ldots, m \\
& x_1 \geq 0, \ldots, x_n \geq 0
\end{aligned}
$$

- **Scalar**
  - **Variable described by a single number**

$$x \in \Re \qquad\qquad x = 3$$

- **Vector**
  - **Column of numbers**

$$\boldsymbol{x} \in \Re^n \qquad\qquad \boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 6 \\ 0 \end{pmatrix} \in \Re^5$$

MATH 1001

- **Rectangular display of vectors in rows and columns**
  - **Defined as rows x columns (R x C)**
- **Vector is just a n x 1 matrix**

$$\boldsymbol{A} \in \Re^{m \times n} \qquad \boldsymbol{A} = \begin{pmatrix} 1 & 4 & 6 \\ 2 & 8 & 4 \\ 7 & 3 & 4 \\ 6 & 4 & 2 \end{pmatrix} \in \Re^{4 \times 3}$$

Transposition

- **Reflect over its main diagonal:**
  - **Write the rows of A as the columns of A**

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 6 \\ 2 & 8 & 4 \\ 7 & 3 & 4 \\ 6 & 4 & 2 \end{pmatrix} \qquad \boldsymbol{A}' =$$

$$(\boldsymbol{A}')' = \boldsymbol{A}$$

Columns of Matrices

$$\boldsymbol{A} = [\boldsymbol{A}_1 \boldsymbol{A}_2 \ldots \boldsymbol{A}_n] \in \Re^{m \times n}$$

$$\boldsymbol{A}_i \in \Re^m, i = 1, \ldots, n$$

ith column of matrix $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 6 \\ 2 & 8 & 4 \\ 7 & 3 & 4 \\ 6 & 4 & 2 \end{pmatrix} \qquad \boldsymbol{A}_2 =$$

Rows of Matrices

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1' \\ \boldsymbol{a}_2' \\ \vdots \\ \boldsymbol{a}_m' \end{bmatrix} \in \Re^{m \times n}$$

$$\boldsymbol{a}_i \in \Re^n, i = 1, \ldots, m$$

ith row of matrix $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 6 \\ 2 & 8 & 4 \\ 7 & 3 & 4 \\ 6 & 4 & 2 \end{pmatrix} \qquad \boldsymbol{a}_2 =$$

Matrix Calculations

$$\begin{pmatrix} 1 & 4 \\ 2 & 8 \\ 7 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ -2 & 4 \\ 6 & 3 \end{pmatrix} =$$

$$\lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (1 - \lambda) \begin{pmatrix} 5 \\ 2 \end{pmatrix} =$$

Matrix Multiplication

- **Multiplication method:**

Sum over product of respective rows and columns

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} =$$

$$\mathbf{A} \qquad\qquad \mathbf{B}$$

Matrix Multiplication

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

$$\boldsymbol{a}'\boldsymbol{x} =$$

Matrix Multiplication

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1' \\ \boldsymbol{a}_2' \\ \vdots \\ \boldsymbol{a}_m' \end{bmatrix} \in \Re^{m \times n}$$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\boldsymbol{A}\boldsymbol{x} =$$

Matrix Multiplication

$$\boldsymbol{A} = [\boldsymbol{A}_1 \boldsymbol{A}_2 \dots \boldsymbol{A}_n] \in \Re^{m \times n}$$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\boldsymbol{A}\boldsymbol{x} =$$

Linear Independence

- **We say y is a linear combination of $A_1, \ldots, A_n$ if for some scalars $x_1, \ldots, x_n$**

$$y = \sum_{i=1}^{n} x_i A_i$$

- **A collection of m-dimensional vectors $A_1, \ldots, A_n$ is linearly independent if none of the vector is a linear combination of the rest of vectors.**

Identity Matrix

- **Is there a matrix which plays a similar role as the number 1 in number multiplication?**
- **Consider the nxn matrix:**

$$\boldsymbol{I}_n = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

For any nxn matrix ***A***, we have $\boldsymbol{A}\,\boldsymbol{I}_n = \boldsymbol{I}_n\,\boldsymbol{A} = \boldsymbol{A}$

Matrix inverse

- **Definition.** A matrix ***A*** is called **nonsingular** or **invertible** if there exists a matrix ***B*** such that:

$$\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}_n$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} + \frac{1}{3} & \frac{-1}{3} + \frac{1}{3} \\ \frac{-2}{3} + \frac{2}{3} & \frac{1}{3} + \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- **Notation.** A common notation for the inverse of a matrix ***A*** is ***A***$^{-1}$. So:

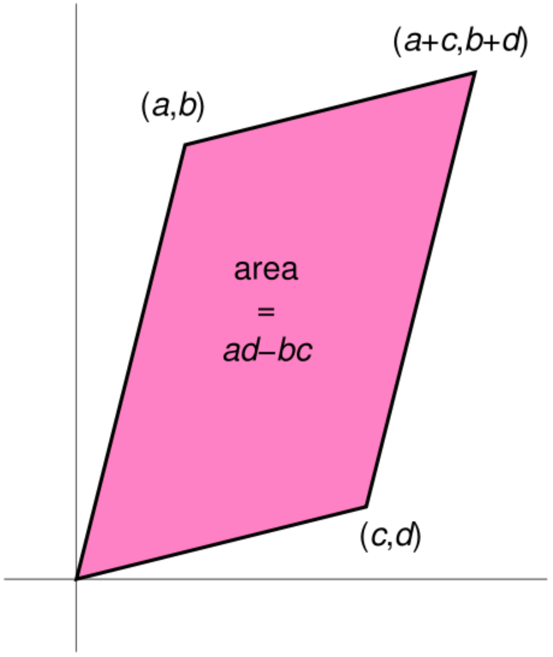
$$\boldsymbol{AA}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}_n$$

- The inverse matrix is unique when it exists.

Determinant

- The **determinant** is a function that associates a scalar det(*A*) to every square matrix *A*.
- Determinants can only be found for square matrices as it is related to the volume of a parallelepiped (think about diagonal matrix).

$$\det(A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$



- What happens when (a, b) and (c, d) are linearly dependent?
- If elements in A are integers, is det(A) an integer?

Solving simultaneous equations

For one linear equation *ax=b* where the unknown is *x* and *b* are constants, we have three possibilities:

- If $\boxed{a \neq 0}$ then $x = \dfrac{b}{a} \equiv a^{-1}b$ thus there is single solution
- If $\boxed{a = 0\,,\ b = 0}$ then the equation $ax = b$ becomes $0 = 0$ and any value of x will do
- If $\boxed{a = 0\,,\ b \neq 0}$ then $ax = b$ becomes $0 = b$ which is a contradiction

Solving simultaneous equations

- **Eg. Two equations, two unknowns**

$$2x_1 + 3x_2 = 5$$

$$x_1 + 2x_2 = 1$$

- **In matrix form**

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$\boldsymbol{Ax} = \boldsymbol{b}$$

- **What is $\boldsymbol{x}$?**

Solving simultaneous equations

- **If A is not invertible …**

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 40 \\ 20 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 40 \\ 30 \end{pmatrix}$$

Summary

- **If A is an n x n matrix, then the following are equivalent:**
  - A is invertible
  - The simultaneous equations Ax=b has a unique solution
  - $\boldsymbol{a_1}', \ldots, \boldsymbol{a_n}'$ are linearly independent
  - $\boldsymbol{A_1}, \ldots, \boldsymbol{A_n}$ are linearly independent
  - $\boldsymbol{det(A) \neq 0}$

Back to Standard Form

$$\begin{aligned} \min \quad & c_1x_1 + \cdots + c_nx_n \\ s.t. \quad & a_{i1}x_1 + \cdots + a_{in}x_n = b_i, i = 1, \dots, m \\ & x_1 \geq 0, \dots, x_n \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & c'x \\ s.t. \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Reduction to Standard Form - Inequalities

$$
\begin{array}{ll}
\min & x_1 + 2x_2 \\
s.t. & x_1 + x_2 \leq 1, \\
 & x_1 \geq 0, x_2 \geq 0
\end{array}
$$

$$
\begin{array}{ll}
\min & x_1 + 2x_2 + 0s_1 \\
s.t. & x_1 + x_2 + s_1 = 1, \\
 & x_1 \geq 0, x_2 \geq 0, s_1 \geq 0
\end{array}
$$

Reduction to Standard Form - Inequalities

- $a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i$

$\Longrightarrow\ a_{i1}\, x_1 + \cdots + a_{in}x_n - s_i = b_i, s_i \geq 0$

- $a_{j1}x_1 + \cdots + a_{jn}x_n \leq b_j$

$\Longrightarrow\ a_{j1}\, x_1 + \cdots + a_{jn}x_n + s_j = b_j, s_j \geq 0$

Reduction to Standard Form - Variables

$$
\begin{aligned}
\min \quad & x_1 + 2x_2 \\
s.t. \quad & x_2 \leq 1, \\
& x_2 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\min \quad & x'_1 - x''_1 + 2x_2 + 0s_1 \\
s.t. \quad & x_2 + s_1 = 1, \\
& x'_1 \geq 0, x''_1 \geq 0, x_2 \geq 0, s_1 \geq 0
\end{aligned}
$$

Reduction to Standard Form - Variables

- **Free variables:**

  Replace x_j by $x'_j - x''_j$, and add constraints: $x'_j \geq 0, x''_j \geq 0$

- **Variable bounds:**

$$x_j \geq l_j \Longrightarrow s_j = x_j - l_j, s_j \geq 0$$

$$x_j \leq u_j \Longrightarrow s_j = u_j - x_j, s_j \geq 0$$

Reduction to Standard Form - Objective

- $max\ \ c'x$

$\Longrightarrow -min\ \ -c'x$

Wyndor Glass Co.

- **Original formulation**

$$\max \quad 3\,x_1 + 5x_2$$
$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0, x_2 \geq 0$$

- **Matrix form**

- **Standard form**

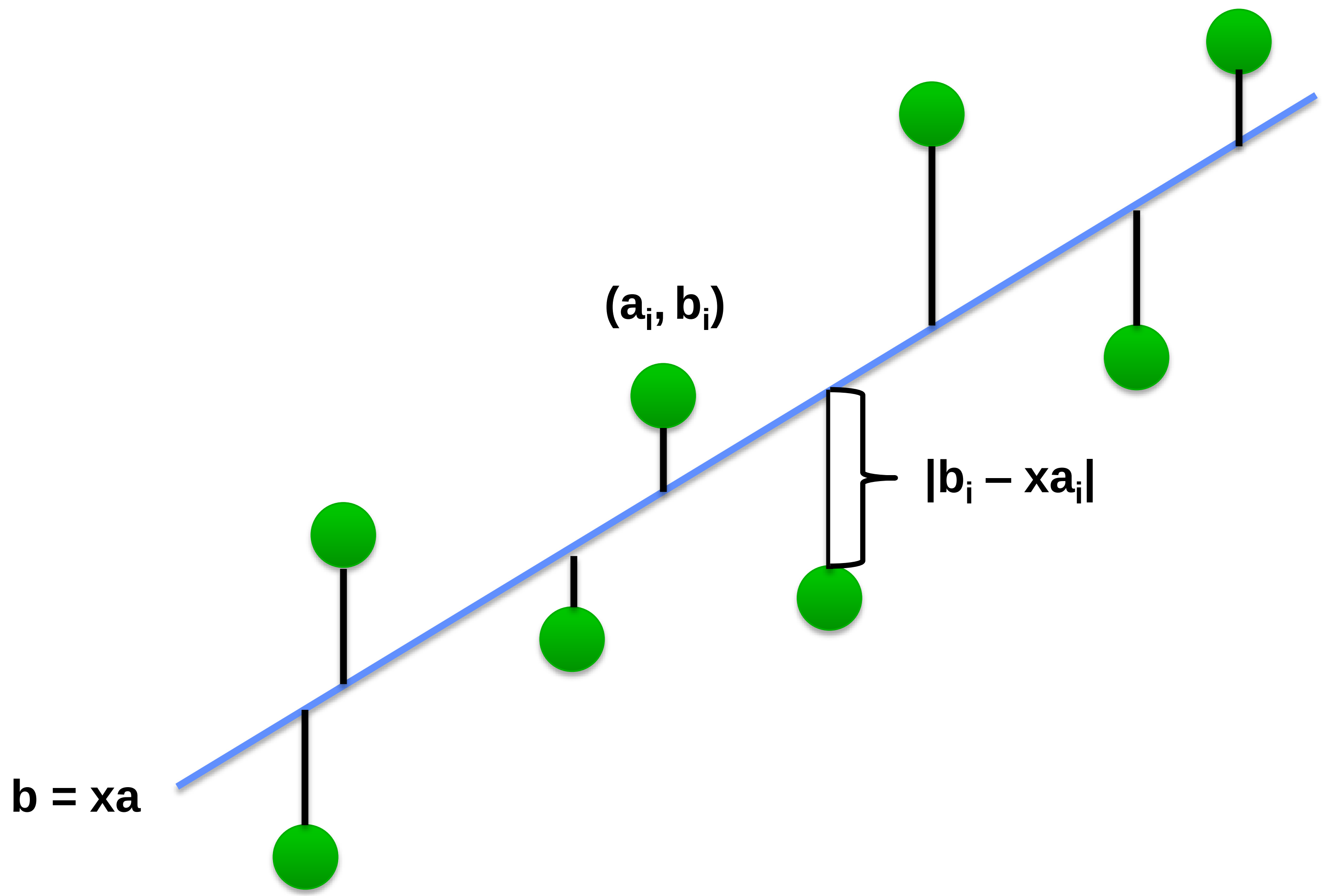
Reduction to Standard Form

$$\begin{aligned} \max \quad & c'x \\ s.t. \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} -\min \quad & -c'x \\ s.t. \quad & [A \quad I]\begin{bmatrix} x \\ s \end{bmatrix} = b \\ & x, s \geq 0 \end{aligned}$$

More Applications

Simple Regression



Data Fitting Problem: L2 Criterion

- **Minimize the sum of square of residual (least square):**

$$\sum_{i=1}^{m}(b_i - a_i x)^2$$

- **LP?**

Data Fitting Problem: L1 Criterion

- **Minimize the sum of residual (least absolute deviations):**

$$\sum_{i=1}^{m} |b_i - a_i x|$$

- **LP?**

Data Fitting Problem: L1 Criterion

- **Consider:**

$$\min |x|$$

- **LP formulation**

Data Fitting Problem: L1 Criterion

- **Least absolute deviations :**

$$\boldsymbol{min} \sum_{i=1}^{m} |b_i - a_i x|$$

- **LP formulation**

$$\min_{z_i, x} \sum_{i=1}^{m} z_i$$

$$b_i - a_i x \leq z_i, \; i = 1, \dots, m$$

$$-b_i + a_i x \leq z_i, \; i = 1, \dots, m$$

L1 vs. L2

- Difficulty
- Sensitivity to outliers
- Solution uniqueness

Data Fitting Problem: Chebyshev Criterion

- **Minimize the largest residual:**

$$\max_{i} |\boldsymbol{b_i} - \boldsymbol{a_i x}|$$

- **LP formulation**

$$\min_{z,x} \boldsymbol{z}$$

$$\boldsymbol{b_i} - \boldsymbol{a_i x} \leq \boldsymbol{z}, \;\; \boldsymbol{i} = \mathbf{1}, \dots, \boldsymbol{m}$$

$$-\boldsymbol{b_i} + \boldsymbol{a_i x} \leq \boldsymbol{z}, \;\; \boldsymbol{i} = \mathbf{1}, \dots, \boldsymbol{m}$$

Problems Involving Absolute Values

- **Consider a problem:**

$$\min_{x_i} \sum_{i=1}^{n} c_i |x_i|$$

$$Ax \geq b$$

- **Cost coefficients c_i are assumed to be nonnegative**
- **LP formulation**

$$\min_{z_i} \sum_{i=1}^{n} c_i z_i$$

$$Ax \geq b$$

$$x_i \leq z_i, \;\; i = 1, \dots, n$$

$$-x_i \leq z_i, \;\; i = 1, \dots, n$$

Problems Involving Absolute Values

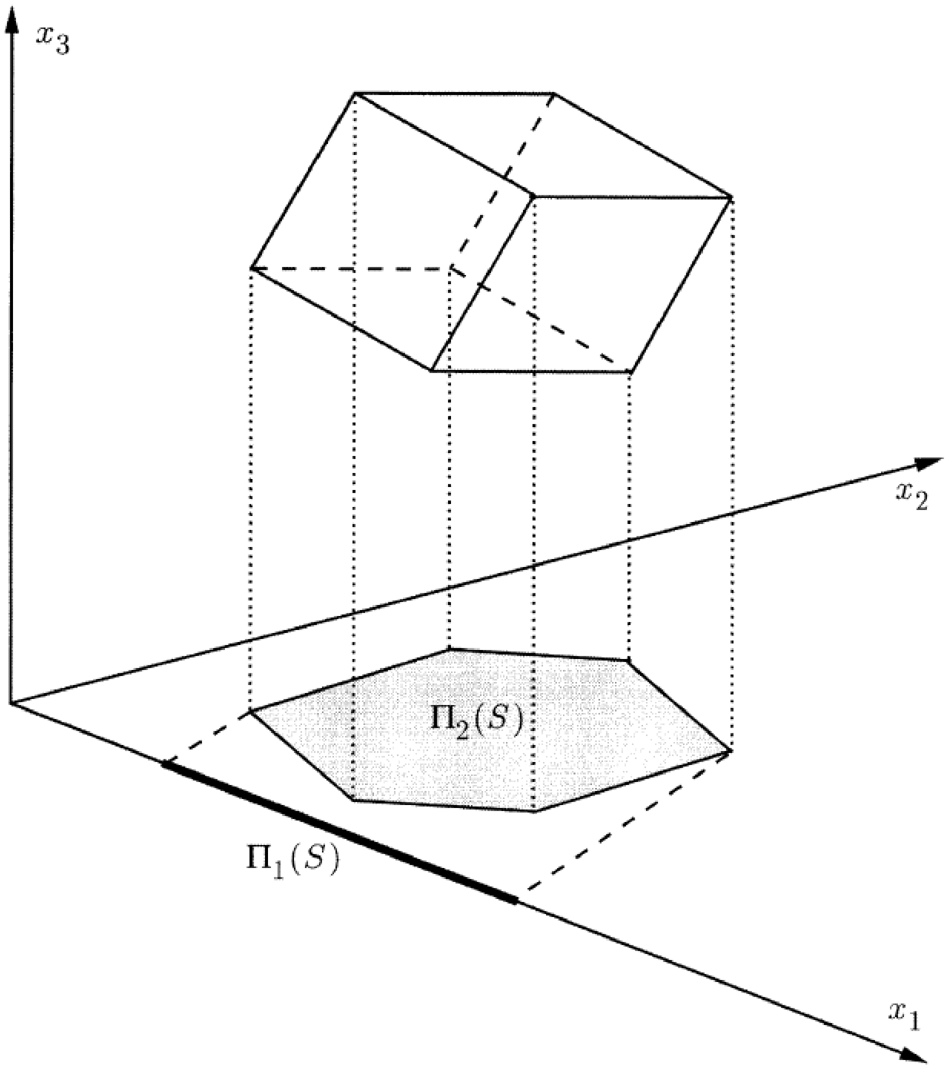
- **Consider a problem:**

$$\min_{x_i} \sum_{i=1}^{n} c_i x_i$$

$$\sum_{i=1}^{n} |x_i| \leq b$$

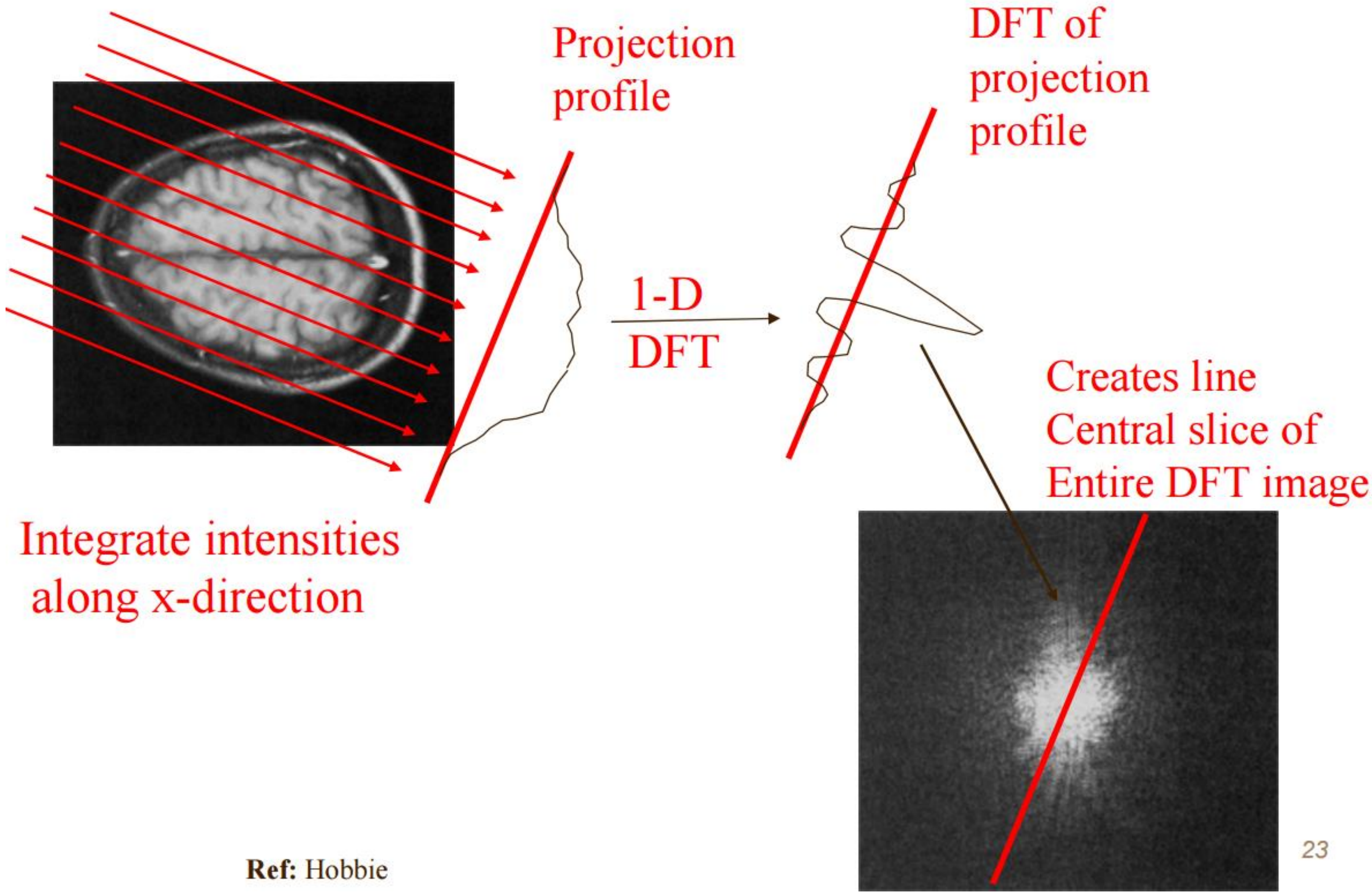
- **LP formulation**

Projection Idea



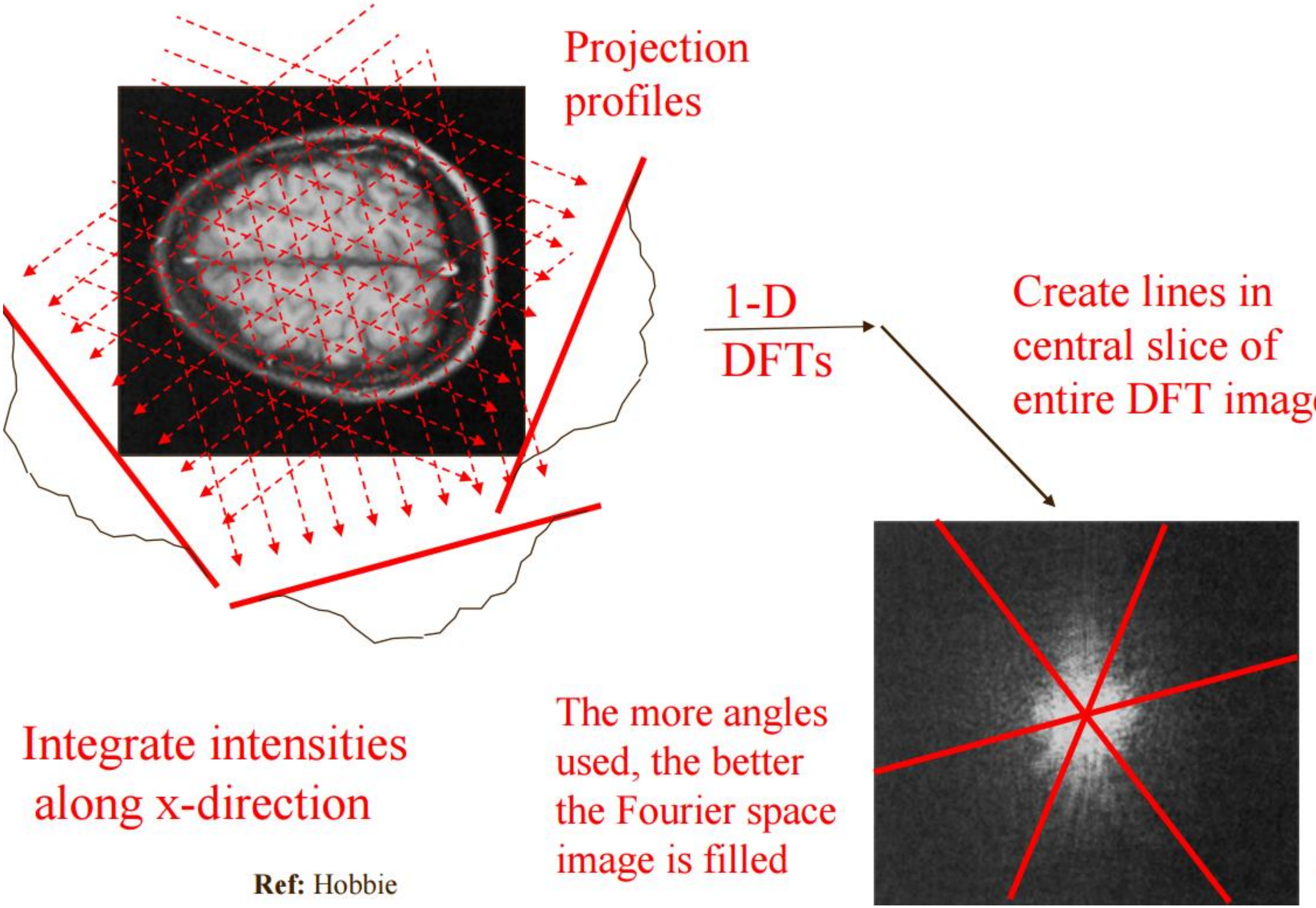
CT/MRI Scan

CT/MRI Scan



Source: http://engineering.dartmouth.edu/courses/engs167/12%20Image%20reconstruction.pdf

CT/MRI Scan



Source: http://engineering.dartmouth.edu/courses/engs167/12%20Image%20reconstruction.pdf

Compressed Sensing

- **Consider a signal (e.g. sound, images):**

$$x = (x_1, \dots, x_n)',$$

where n is very large (e.g. around 1,000,000).

- **The signal is not directly observable. Instead, one observes a measurement**

$$b = Ax \in R^m,$$

where m is small and A is an mxn matrix.

- **The problem is then to find x given b.**

Compressed Sensing: A Simple Example

- Consider an unknown signal:

$$\boldsymbol{x} = (\mathbf{0}, \mathbf{1})'.$$

- The measurement matrix is:

$$\boldsymbol{A} = [\mathbf{1} \quad \mathbf{2}].$$

- The observable output is then

$$\boldsymbol{b} = \mathbf{2}.$$

- If one solves the underdetermined system $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}$, then any $\boldsymbol{x}$ satisfying the following is a solution

$$\boldsymbol{x}_1 + \mathbf{2}\boldsymbol{x}_2 = \mathbf{2}.$$

- What if we know the solution is sparse (most components are zero)?

Compressed Sensing

- **Ideally, one want to solve**

$$\min_{x} \#\{1 \le i \le n\colon\ x_i \neq 0\}$$
$$Ax = b$$

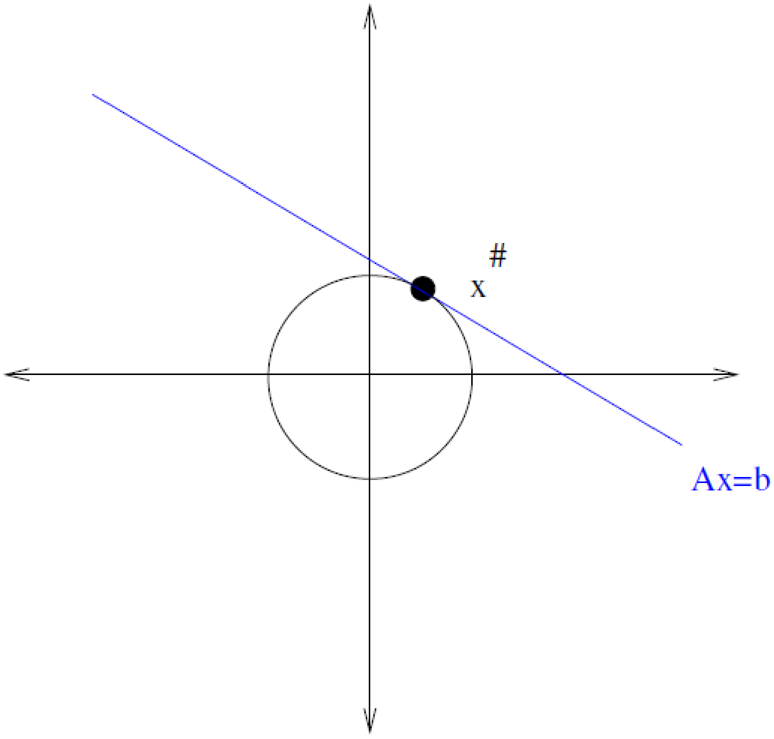
- **The L1 Approach?**

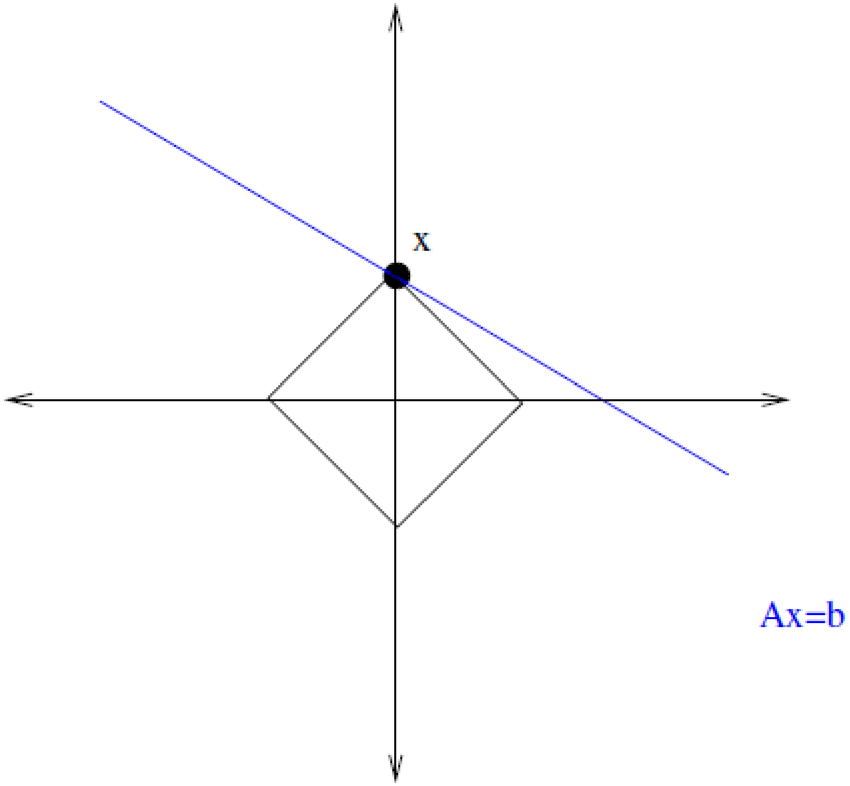
$$\min_{x} \sum_{i=1}^{n} |x_i|$$
$$Ax = b$$

- **The L2 Approach?**

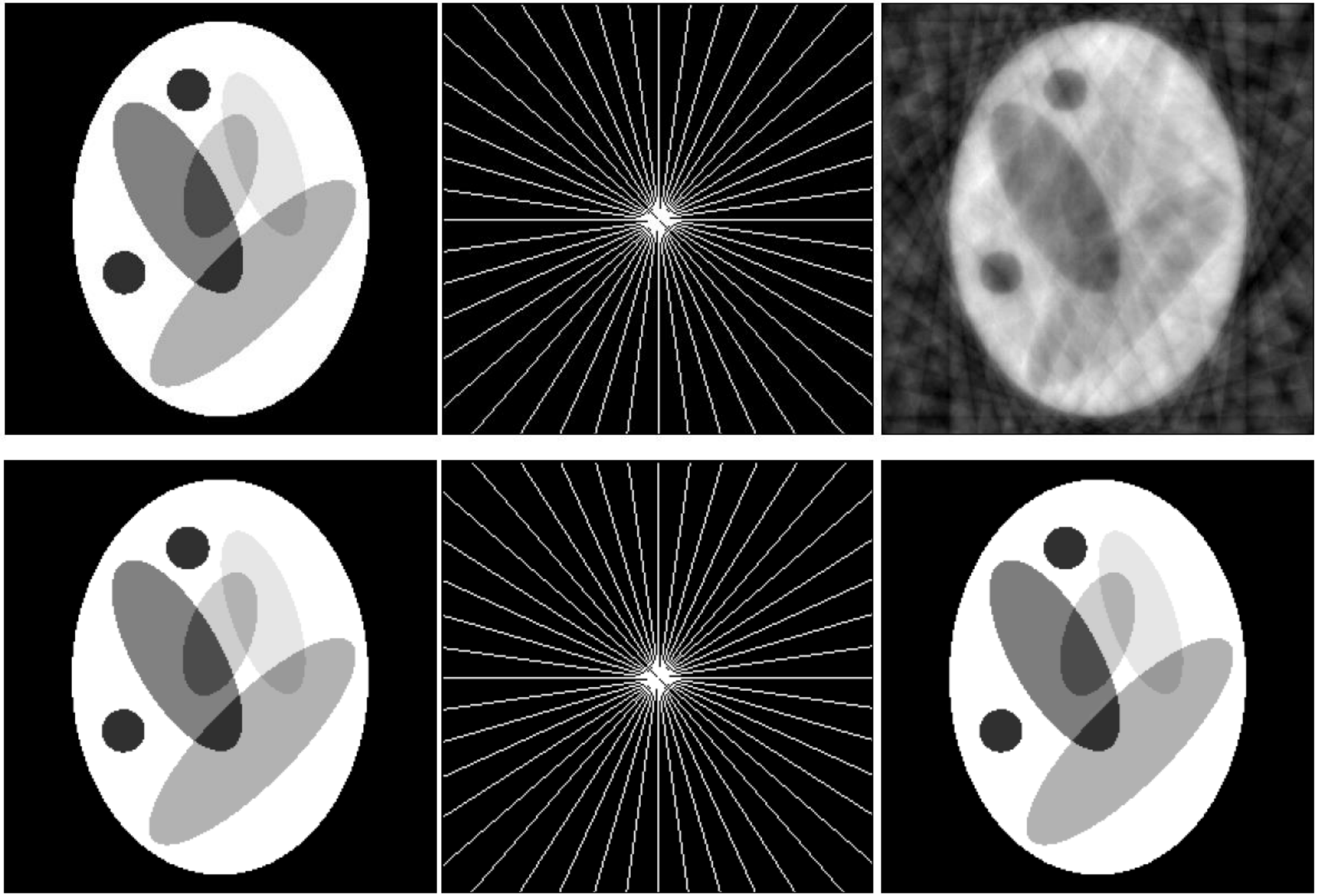
$$\min_{x} \sum_{i=1}^{n} x_i^2$$
$$Ax = b$$

Compressed Sensing





Compressed Sensing



Source: Terrence Tao, “Compressed sensing Or: the equation Ax=b, revisited”, UCLA, Mahler Lecture Series